Review

Types of Submanifolds in Metallic Riemannian Manifolds: A Short Survey

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Abstract: We provide a brief survey on the properties of submanifolds in metallic Riemannian manifolds. We focus on slant, semi-slant and hemi-slant submanifolds in metallic Riemannian manifolds and, in particular, on invariant, anti-invariant and semi-invariant submanifolds. We also describe the warped product bi-slant and, in particular, warped product semi-slant and warped product hemi-slant submanifolds in locally metallic Riemannian manifolds, obtaining some results regarding the existence and nonexistence of non-trivial semi-invariant, semi-slant and hemi-slant warped product submanifolds. We illustrate all these by suitable examples.

Keywords: metallic Riemannian manifold; warped product submanifold; slant; semi-slant; hemi-slant; bi-slant submanifold

MSC: 53B20; 53B25; 53C42; 53C15

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1. Introduction

The notion of Golden structure on a Riemannian manifold was introduced by C. E. Hretcanu and M. Crasmareanu in ([1]). Then, the properties of submanifolds in Golden Riemannian manifolds were studied in ([2,3]) using the correspondents of a Golden structure related to an almost product structure. The metallic structure introduced in ([4]) is a generalization of the Golden structure. Different types of submanifolds in metallic and Golden Riemannian manifolds were studied in ([5–7]), obtaining different integrability conditions for the distributions involved in these types of submanifolds. The metallic (and in particular Golden) warped product Riemannian manifold was studied in ([8–10]).

In this paper we provide a brief survey on the properties of metallic structures defined on Riemannian manifolds, stating the definitions and some properties of these structures, related to generalized secondary Fibonacci sequences (in Section 2). Then, in Section 3, we discuss about some properties of the structures induced on submanifolds, called by us, Σ -metallic Riemannian structures, especially regarding the normality of these types of structures and, in particular, we present some properties of invariant and anti-invariant submanifolds in metallic Riemannian manifolds ([11]). In Section 4, we define and discuss about properties of slant submanifolds in metallic Riemannian manifolds. In Section 5, we treat bi-slant submanifolds in metallic Riemannian manifolds and, in particular, for semi-slant and hemi-slant submanifolds, we obtain some integrability conditions of the distributions involved. In Section 6, we give some properties of warped product submanifolds in metallic Riemannian manifolds and provide examples of warped product semi-invariant, semi-slant and respectively hemi-slant submanifolds in these types of manifolds.

2. Preliminaries

We recall the definitions and some basic formulas for metallic structures on a Riemannian manifold and, in particular, for Golden Riemannian structures.

The name of “metallic means family” (MMF) (or metallic proportions) was introduced by Vera W. de Spinadel ([12]) as the set of positive solutions $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ of the equation $x^2 - px - q = 0$, where p and q are positive integer values. These $\sigma_{p,q}$ numbers can be seen as generalizations of the Golden number $\phi = \frac{1+\sqrt{5}}{2} = 1.618\dots$ (the positive solution of the equation $x^2 - x - 1 = 0$). The name metallic means family (MMF) was explained by Vera W. de Spinadel in ([13]) where she affirmed that “besides carrying the name of a metals, they have common mathematical properties that attach a fundamental importance to them in modern investigations about the search of universal roads to chaos.” These metallic numbers found many applications in researches that “analyze the behavior of non linear dynamical systems when they proceed from a periodic regime to a chaotic one” ([14]). Moreover, Vera W. de Spinadel ([15]) remarked that the metallic means family is the set of positive eigenvalues of the matrix $\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$ for different values of natural number p and integer q , with $p^2 + 4q > 0$.

The generalized secondary Fibonacci sequence ([12,16,17]) are given by relations of the type

$$G_{n+1} = p \cdot G_n + q \cdot G_{n-1}, \quad n \geq 1,$$

where $p, q, G_0 = a$ and $G_1 = b$ are real numbers. The ratio G_{n+1}/G_n of two consecutive generalized secondary Fibonacci numbers converges to the metallic numbers $\sigma_{p,q}$ which in particular are called: the Golden mean $\phi = \frac{1+\sqrt{5}}{2}$ (for $a = b = p = q = 1$), the silver mean $\sigma_{2,1} = 1 + \sqrt{2}$, the bronze mean $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$, the subtle mean $\sigma_{4,1} = 2 + \sqrt{5} = \phi^3$, the copper mean $\sigma_{1,2} = 2$, the nickel mean $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$ and so on.

The most remarkable element from (MMF) is the Golden mean, known from ancient times as an expression of harmony of many constructions, paintings and in music. It also appears as an expression of the objects from the natural world (flowers, trees, fruits) that possess pentagonal symmetry ([18]).

One can remark that Golden mean is determined by the ratio of two consecutive classical Fibonacci numbers and the silver mean is determined by the ratio of two consecutive Pell numbers ([19]). The bronze mean plays an important role in studying topics such as dynamical systems and quasicrystals and the subtle mean is significant in the theory of Cantorian fractal-like micro-space-time E^∞ , being involved, in a fundamental way, in noncommutative geometry and four manifold theory ([20,21]).

The members of (MMF), named also metallic numbers, have found many applications in differential geometry, as it can be seen in the next properties.

A metallic structure on a differentiable manifold is a particular case of a polynomial structure introduced by S. I. Goldberg, K. Yano and N. C. Petridis in ([22,23]).

Precisely, a polynomial structure J of degree 2, defined on a differentiable manifold $\bar{M}$, is called a metallic structure if it satisfies the equality $J^2 = p \cdot J + q \cdot I$, where I is the identity endomorphism on $T\bar{M}$ and $p, q \in \mathbb{N}^*$. The pair $(\bar{M}, J)$ is called a metallic manifold.

Another important (for our study) is the polynomial structure F of degree 2 on $\bar{M}$, i.e., $F^2 = I$, called an almost product structure. In this case, $(\bar{M}, F)$ is called an almost product manifold.

The metallic structure and the almost product structure are closely related shown below.

Proposition 1 ([4]). (i) Any metallic structure J induces two almost product structures, given by

$$F_{1,2} = \pm \frac{2}{2\sigma_{p,q} - p} \cdot J - \frac{p}{2\sigma_{p,q} - p} \cdot I.$$

(ii) Any almost product structure F induces two metallic structures, given by

$$J_{1,2} = \pm \frac{2\sigma_{p,q} - p}{2} \cdot F + \frac{p}{2} \cdot I.$$

Definition 1. A triple $(\overline{M}, \overline{g}, J)$ is called a metallic Riemannian manifold if the differentiable manifold $\overline{M}$ is endowed with a metallic structure J and a Riemannian metric $\overline{g}$ such that

$$\overline{g}(JX, Y) = \overline{g}(X, JY),$$

for any $X, Y \in \Gamma(T\overline{M})$.

We remark that in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, we have

$$\overline{g}(JX, JY) = p\overline{g}(JX, Y) + q\overline{g}(X, Y),$$

for any $X, Y \in \Gamma(T\overline{M})$.

Recall that an almost product Riemannian structure $(\overline{g}, F)$ on $\overline{M}$ is a pair $(\overline{g}, F)$, where $\overline{g}$ is a Riemannian metric and F is a $\overline{g}$ -compatible almost product structure on $\overline{M}$, i.e.,

$$\overline{g}(FX, Y) = \overline{g}(X, FY),$$

for any $X, Y \in \Gamma(T\overline{M})$. In this case, $(\overline{M}, \overline{g}, F)$ is called almost product Riemannian manifold.

If the almost product structure F is a Riemannian one, then the induced metallic structures J_1 and J_2 are metallic Riemannian structures.

On a metallic manifold $(\overline{M}, J)$ there are two complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ corresponding to the projection operators π_1 and π_2 ([4])

$$\pi_1 = -\frac{1}{2\sigma_{p,q} - p} \cdot J + \frac{\sigma_{p,q}}{2\sigma_{p,q} - p} \cdot I, \quad \pi_2 = \frac{1}{2\sigma_{p,q} - p} \cdot J + \frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p} \cdot I,$$

which verify the following relations

$$\pi_1 + \pi_2 = I, \quad \pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2, \quad \pi_1\pi_2 = \pi_2\pi_1 = 0,$$

$$J\pi_1 = \pi_1J = (p - \sigma_{p,q}) \cdot \pi_1, \quad J\pi_2 = \pi_2J = \sigma_{p,q} \cdot \pi_2.$$

Thus

$$J = (p - \sigma_{p,q}) \cdot \pi_1 + \sigma_{p,q} \cdot \pi_2.$$

Proposition 2 ([4]). A metallic structure J on $\overline{M}$ has the following properties:

(i) for every integer number $n \geq 1$:

$$J^n = G_n \cdot J + qG_{n-1} \cdot I,$$

where $(G_n)_{n \in \mathbb{N}^*}$ is the generalized secondary Fibonacci sequence with $G_0 = 0$ and $G_1 = 1$;

(ii) J is an isomorphism on $T_x\overline{M}$, for every $x \in \overline{M}$. It follows that J is invertible and its inverse $\hat{J} = J^{-1} = \frac{1}{q} \cdot J - \frac{p}{q} \cdot I$ is not a metallic structure, but it is still polynomial, more precisely, a quadratic one:

$$q \cdot \hat{J}^2 + p \cdot \hat{J} - I = 0;$$

(iii) the eigenvalues of J are the metallic number $\sigma_{p,q}$ and $p - \sigma_{p,q}$.

In particular, for $p = q = 1$ one gets the Golden structure ([24,25]), namely a $(1, 1)$ -tensor field J which satisfies $J^2 = J + I$. In this case, $(\overline{M}, J)$ is called a Golden manifold. Every Golden structure defines two almost product structures and any almost product structure defines two Golden structures ([2,24]). The eigenvalues of J are the Golden Ratio

ϕ and $1 - \phi$. Notice that the Golden structures appear in pairs, i.e., if J is a Golden structure, then $\tilde{J} = I - J$ is also a Golden structure.

Moreover, the power $n \geq 1$ of the Golden structure J on $\overline{M}$ can be written using the Fibonacci sequence $(f_n)_{n \in \mathbb{N}^*}$ ([24])

$$J^n = f_n \cdot J + f_{n-1} \cdot I.$$

By using an explicit expression for the Fibonacci sequence, namely, the Binet's formula from ([18])

$$f_n = \frac{\phi^n - (1 - \phi)^n}{\sqrt{5}},$$

we can obtain an expression of the power $n \geq 1$ of the Golden structure J by means of the Golden number ϕ ([24]).

3. Submanifolds in Metallic Riemannian Manifolds

In ([11]), C. E. Hretcanu and A. M. Blaga discussed about some properties of the structure induced by a metallic Riemannian structure on a submanifold, called Σ -metallic Riemannian structure.

Let M^m be an m -dimensional Riemannian manifold with the metric g and let $\Gamma(TM)$ be the set of all vector fields on M .

Definition 2. If on M^m there exist a tensor field T of type $(1, 1)$, r vector fields $\xi_1, \dots, \xi_r$ ($0 < r < m$), r 1-forms $u_1, \dots, u_r$ and a $r \times r$ matrix $(a_{\alpha\beta})_r$ ($\alpha, \beta \in \{1, \dots, r\}$) of differentiable functions which verify the relations

$$u_\alpha(X) = g(X, \xi_\alpha), \quad (1)$$

$$T^2X = pTX + qX - \sum_{\alpha=1}^r u_\alpha(X)\xi_\alpha, \quad (2)$$

$$u_\alpha(TX) = pu_\alpha(X) - \sum_{\beta=1}^r a_{\alpha\beta}u_\beta(X), \quad a_{\alpha\beta} = a_{\beta\alpha}, \quad (3)$$

$$u_\beta(\xi_\alpha) = q\delta_{\alpha\beta} + pa_{\alpha\beta} - \sum_{\gamma=1}^r a_{\alpha\gamma}a_{\gamma\beta}, \quad T\xi_\alpha = p\xi_\alpha - \sum_{\beta=1}^r a_{\alpha\beta}\xi_\beta, \quad (4)$$

for any $X \in \Gamma(TM)$, where $\delta_{\alpha\beta}$ is the Kronecker symbol and p, q are positive integers, then the structure $\Sigma = (T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is called a Σ -metallic Riemannian structure ([11]).

In particular, for $p = q = 1$, the structure $\Sigma = (T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is called a Σ -Golden Riemannian structure.

If M is a submanifold of dimension $m \in \mathbb{N}^*$, isometrically immersed in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, $T_x M$ is the tangent space of M in a point $x \in M$ and $T_x^\perp M$ is the normal space of M in x , then $T_x \overline{M} = T_x M \oplus T_x^\perp M$, for any $x \in M$.

Denoting by i_* the differential of the immersion $i : M \rightarrow \overline{M}$ and by X the vector field $i_* X$, for $X \in \Gamma(TM)$, the induced Riemannian metric g on M is given by $g(X, Y) = \overline{g}(i_* X, i_* Y)$, for any $X, Y \in \Gamma(TM)$.

Let us consider, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, the decomposition of JX and JV into the tangential and normal components

$$JX = TX + NX, \quad JV = tV + nV,$$

where $TX := (JX)^\top$, $NX := (JX)^\perp$, $tV := (JV)^\top$ and $nV := (JV)^\perp$.

It can be easily checked that the maps T and n are $\bar{g}$ -symmetric ([5])

$$\bar{g}(TX, Y) = \bar{g}(X, TY), \quad \bar{g}(nU, V) = \bar{g}(U, nV),$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^\perp M)$. Moreover:

$$\bar{g}(NX, U) = \bar{g}(X, tU),$$

for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$.

If $r \in N^*$ is the codimension of the submanifold M in the Riemannian manifold $(\bar{M}, \bar{g})$, then we can fix a local orthonormal basis $\{N_1, \dots, N_r\}$ of $T^\perp M$. Hereafter we assume that the indices α, β, γ run over the range $\{1, \dots, r\}$.

Then the vectors JX and JN_α can be decomposed into the tangential and normal components ([4])

$$JX = TX + \sum_{\alpha=1}^r u_\alpha(X)N_\alpha, \quad JN_\alpha = \xi_\alpha + \sum_{\beta=1}^r a_{\alpha\beta}N_\beta,$$

for any $X \in \Gamma(TM)$, where $\xi_\alpha =: tN_\alpha$ are vector fields on M , u_α are 1-forms on M , $(a_{\alpha\beta})_r$ is an $r \times r$ matrix of differentiable functions on M , $\alpha, \beta \in \{1, \dots, r\}$, and

$$NX = \sum_{\alpha=1}^r u_\alpha(X)N_\alpha, \quad nN_\alpha = \sum_{\beta=1}^r a_{\alpha\beta}N_\beta.$$

We can remark that

Theorem 1 ([11]). *The structure $(T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on a submanifold M of codimension r by the metallic Riemannian structure $(\bar{g}, J)$ on $\bar{M}$ satisfies the relations (1)–(4). Thus, it is a Σ -metallic Riemannian structure.*

On the Riemannian manifold $(\bar{M}, \bar{g})$ and its submanifold (M, g) , we consider the Levi-Civita connections $\bar{\nabla}$ and ∇ , respectively. For any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

where h is the second fundamental form, A_V is the shape operator of M and $\nabla^\perp$ is the normal connection. Moreover:

$$\bar{g}(h(X, Y), V) = \bar{g}(A_V X, Y),$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

We denote by $A_{N_\alpha} =: A_\alpha$ and remark that ([11])

$$\bar{\nabla}_X N_\alpha = -A_\alpha X + \nabla_X^\perp N_\alpha, \quad h_\alpha(X, Y) = g(A_\alpha X, Y),$$

for any $X, Y \in \Gamma(TM)$, $\alpha \in \{1, \dots, r\}$, where $\{N_1, \dots, N_r\}$ is a local orthonormal basis of the normal space $T_x^\perp M$ and r is the codimension of M in $\bar{M}$.

For $\alpha \in \{1, \dots, r\}$, $\nabla_X^\perp N_\alpha = \sum_{\beta=1}^r l_{\alpha\beta}(X)N_\beta$, for any $X \in \Gamma(TM)$, where $(l_{\alpha\beta})_r$ is an $r \times r$ matrix of 1-forms on M . Since $\bar{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}$, we obtain ([11])

$$\bar{g}(\nabla_X^\perp N_\alpha, N_\beta) + \bar{g}(N_\alpha, \nabla_X^\perp N_\beta) = 0,$$

which is equivalent to $l_{\alpha\beta} = -l_{\beta\alpha}$, for any $\alpha, \beta \in \{1, \dots, r\}$ and $X \in \Gamma(TM)$. From $\bar{g}(JX, Y) = \bar{g}(X, JY)$, it follows

$$\bar{g}((\bar{\nabla}_X J)Y, Z) = \bar{g}(Y, (\bar{\nabla}_X J)Z),$$

for any $X, Y, Z \in \Gamma(T\bar{M})$, where $\bar{\nabla}J$ is the covariant derivative of J . Moreover, if M is an isometrically immersed submanifold in the $(\bar{M}, \bar{g}, J)$, then ([9])

$$\bar{g}((\nabla_X T)Y, Z) = \bar{g}(Y, (\nabla_X T)Z),$$

for any $X, Y, Z \in \Gamma(TM)$, where ∇T is the covariant derivative of T , given by

$$(\nabla_X T)Y := \nabla_X TY - T(\nabla_X Y),$$

for any $X, Y \in \Gamma(TM)$.

Moreover, the covariant derivatives of N , t and n are, respectively, given by

$$\begin{aligned} (\bar{\nabla}_X N)Y &:= \nabla_X^\perp NY - N(\nabla_X Y), & (\nabla_X t)V &:= \nabla_X tV - t(\nabla_X^\perp V), \\ (\bar{\nabla}_X n)V &:= \nabla_X^\perp nV - n(\nabla_X^\perp V), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

We can define the notion of locally metallic Riemannian manifold by analogy with a locally product manifold, as follows.

Definition 3. We say that a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ is a locally metallic Riemannian manifold if J is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$, i.e., $\bar{\nabla}J = 0$.

Proposition 3 ([11]). If M is a submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ with ∇ the Levi-Civita connection on M and $\nabla^\perp$ the normal connection, then, for any $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} T([X, Y]) &= \nabla_X TY - \nabla_Y TX - A_{NY}X + A_{NX}Y, \\ N([X, Y]) &= h(X, TY) - h(TX, Y) + \nabla_X^\perp NY - \nabla_Y^\perp NX. \end{aligned}$$

Proposition 4 ([11]). If M is a submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then, for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we have

$$\begin{aligned} (\nabla_X T)Y &= A_{NY}X + th(X, Y), & (\bar{\nabla}_X N)Y &= nh(X, Y) - h(X, TY), \\ (\nabla_X t)V &= A_{nV}X - TA_VX, & (\bar{\nabla}_X n)V &= -h(X, tV) - NA_VX. \end{aligned}$$

Proposition 5 ([11]). Let M be a submanifold of codimension r in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ and let $(T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the Σ -metallic Riemannian structure induced by the metallic Riemannian structure $(\bar{g}, J)$. Then, for any $X, Y \in \Gamma(TM)$, we get

$$\begin{aligned} T([X, Y]) &= \nabla_X TY - \nabla_Y TX - \sum_{i=1}^r [u_\alpha(Y)A_\alpha X - u_\alpha(X)A_\alpha Y], \\ N([X, Y]) &= \sum_{\alpha=1}^n [(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y + u_\alpha(X)l_{\alpha\beta}(Y) - u_\alpha(Y)l_{\alpha\beta}(X)]N_\alpha. \end{aligned}$$

Proposition 6 ([11]). Let M be a submanifold of codimension r in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ and let $(T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the Σ -metallic Riemannian structure induced by the metallic Riemannian structure $(\overline{g}, J)$. Then, for any $X, Y \in \Gamma(TM)$, we get

$$(\nabla_X T)Y = \sum_{\alpha=1}^r h_\alpha(X, Y)\xi_\alpha + \sum_{\alpha=1}^r u_\alpha(Y)A_\alpha X,$$

$$(\nabla_X u_\alpha)Y = -h_\alpha(X, TY) + \sum_{\beta=1}^r [u_\beta(Y)l_{\alpha\beta}(X) + h_\beta(X, Y)a_{\beta\alpha}].$$

If N_J is the Nijenhuis tensor field of J , defined for any $X, Y \in \Gamma(T\overline{M})$, by

$$N_J(X, Y) := [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY],$$

then it verifies ([26])

$$N_J(X, Y) = (\overline{\nabla}_{JX}J)Y - (\overline{\nabla}_{JY}J)X - J[(\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X].$$

Thus, we remark that in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, we have $N_J = 0$ and the metallic structure $(\overline{g}, J)$ can be called integrable.

Now we shall define a normal Σ -metallic Riemannian structure.

Definition 4. A Σ -metallic Riemannian structure induced on a submanifold M of codimension r in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ is said to be normal if

$$N_T = 2 \sum_{\alpha=1}^r du_\alpha \otimes \xi_\alpha.$$

We denote by $B_\alpha := TA_\alpha - A_\alpha T$ and remark that $g(B_\alpha X, Y) = -g(X, B_\alpha Y)$, for any $X, Y \in \Gamma(TM)$.

Theorem 2 ([11]). If $(T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is the Σ -metallic Riemannian structure induced on a submanifold M of codimension r in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then, for any $X, Y \in \Gamma(TM)$, we have

$$N_T(X, Y) = \sum_{\alpha=1}^r [g(X, \xi_\alpha)B_\alpha Y - g(Y, \xi_\alpha)B_\alpha X - g(B_\alpha X, Y)\xi_\alpha],$$

$$2du_\alpha(X, Y) = -g(B_\alpha X, Y) + \sum_{\beta=1}^r [l_{\alpha\beta}(X)g(Y, \xi_\beta) - l_{\alpha\beta}(Y)g(X, \xi_\beta)],$$

where $l_{\alpha\beta}$ are the coefficients of the normal connection $\nabla^\perp$.

Remark 1 ([11]). In the conditions of the previous theorem, if T commutes with the Weingarten operators A_α , for any $\alpha \in \{1, \dots, r\}$, i.e., $B_\alpha = 0$, then the Nijenhuis tensor field of T vanishes on M .

By a direct computation, for any $X, Y \in \Gamma(TM)$, we get

$$\begin{aligned} N_T(X, Y) - 2 \sum_{\alpha=1}^r du_\alpha(X, Y)\xi_\alpha &= \sum_{\alpha=1}^r [g(X, \xi_\alpha)B_\alpha Y - g(Y, \xi_\alpha)B_\alpha X] \\ &\quad - \sum_{\alpha=1}^r \sum_{\beta=1}^r [l_{\alpha\beta}(X)g(Y, \xi_\beta) - l_{\alpha\beta}(Y)g(X, \xi_\beta)]\xi_\alpha. \end{aligned}$$

The components $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ of the Nijenhuis tensor field of T can be computed using the similar idea from ([27]), as follows.

Remark 2 ([5]). Let M be a submanifold of codimension r in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ and let $(T, g, \eta_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the Σ -metallic Riemannian structure induced on M . Then, for any $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \{1, \dots, r\}$, we obtain

1. $N^{(1)}(X, Y) = N_T(X, Y) - 2 \sum_{\alpha=1}^r du_\alpha(X, Y) \xi_\alpha$;
2. $N_\alpha^{(2)}(X, Y) = (\mathcal{L}_{TX} u_\alpha)Y - (\mathcal{L}_{TY} u_\alpha)X$;
3. $N_\alpha^{(3)}(X) = (\mathcal{L}_{\xi_\alpha} T)X$;
4. $N_{\alpha\beta}^{(4)}(X) = (\mathcal{L}_{\xi_\alpha} u_\beta)X$,

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X .

We shall further provide conditions such that the induced Σ -metallic Riemannian structure to be normal.

We remark that, if the Σ -metallic Riemannian structure induced on M is normal and the normal connection $\nabla^\perp$ of M vanishes identically, i.e., $l_{\alpha\beta} = 0$, then ([11])

$$\sum_{\alpha=1}^r g(X, \xi_\alpha)(TA_\alpha - A_\alpha T)(Y) = \sum_{\alpha=1}^r g(Y, \xi_\alpha)(TA_\alpha - A_\alpha T)(X)$$

for any $X, Y \in \Gamma(TM)$.

Theorem 3 ([11]). Let M be a submanifold of codimension $r \geq 2$ in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. If the normal connection $\nabla^\perp$ vanishes identically and M is a non-invariant submanifold with respect to the metallic structure J , then the vector fields $\{\xi_1, \dots, \xi_r\}$ are linearly independent.

Theorem 4 ([11]). Let M be a submanifold of codimension $r \geq 1$ in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. If the normal connection $\nabla^\perp$ vanishes identically and T commutes with every Weingarten operator A_α , then the induced Σ -metallic Riemannian structure on M is normal.

Theorem 5 ([11]). Let M be a submanifold of codimension $r \geq 1$ in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. If the normal connection $\nabla^\perp$ vanishes identically and M is a non-invariant submanifold with respect to the metallic structure J , then the induced Σ -metallic Riemannian structure on M is normal if and only if T commutes with the Weingarten operator A_α , for any $\alpha \in \{1, \dots, r\}$.

Corollary 1 ([11]). If M is a non-invariant totally umbilical (or totally geodesic) submanifold of codimension $r \geq 1$ in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ such that the normal connection $\nabla^\perp$ vanishes identically, then the Σ -metallic Riemannian structure induced on M is normal.

We can observe that the matrix $\mathcal{A} := (a_{\alpha\beta})_r$ of the Σ -structure induced on an invariant submanifold M by the metallic Riemannian structure $(\overline{g}, J)$ from $\overline{M}$ is a metallic matrix, that is a matrix which verifies

$$\mathcal{A}^2 = p \cdot \mathcal{A} + q \cdot I_r,$$

where I_r is the identically matrix of order r .

If $\mathcal{A} := (a_{\alpha\beta})_r$ is a metallic matrix, then $\sum_{\gamma=1}^r a_{\alpha\gamma} a_{\gamma\beta} = p a_{\alpha\beta} + q \delta_{\alpha\beta}$ and we obtain $u_\beta(\xi_\alpha) = 0$, which implies that $T^2 \xi_\alpha = p T \xi_\alpha + q \xi_\alpha$ and $J \xi_\alpha = T \xi_\alpha$, for any $\alpha \in \{1, \dots, r\}$.

3.1. Invariant Submanifolds in a Metallic Riemannian Manifold $(\overline{M}, \overline{g}, J)$

A submanifold M of $\overline{M}$ is called invariant if $J(T_x M) \subseteq T_x M$, for any $x \in M$. In this case, $J(T_x^\perp M) \subseteq T_x^\perp M$, for any $x \in M$.

Proposition 7 ([4]). Let M be an isometrically immersed submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then M is invariant if and only if (M, g, T) is a metallic Riemannian manifold, whenever T is non-trivial.

Proposition 8 ([4]). Let M be an invariant submanifold of codimension r isometrically immersed in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then, for any $X, Y \in \Gamma(TM)$:

$$\sum_{\alpha=1}^r h_\alpha(X, JY)N_\alpha = \sum_{\alpha=1}^r h_\alpha(X, Y)JN_\alpha = \sum_{\alpha=1}^r h_\alpha(JX, Y)N_\alpha,$$

and for any $1 \leq \alpha \leq r$:

$$h_\alpha(JX, JY) = ph_\alpha(X, JY) + qh_\alpha(X, Y).$$

Remark 3 ([4]). Let M be an invariant submanifold of codimension r isometrically immersed in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ and let $(T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the Σ -metallic Riemannian structure induced on M . Then ξ_α are zero vector fields, the 1-forms u_α vanish identically on M , for any $1 \leq \alpha \leq r$, and for any $X \in \Gamma(TM)$:

$$JX = TX, \quad JN_\alpha = \sum_{\beta=1}^r a_{\alpha\beta}N_\beta, \quad \text{for any } 1 \leq \alpha \leq r.$$

Also, the Σ -metallic Riemannian structure satisfies, for any $X, Y \in \Gamma(TM)$ and $1 \leq \alpha, \beta \leq r$:

$$T^2 = p \cdot T + q \cdot I,$$

$$a_{\alpha\beta} = a_{\beta\alpha}, \quad \sum_{\gamma=1}^r a_{\alpha\gamma}a_{\gamma\beta} = q\delta_{\alpha\beta} + pa_{\alpha\beta},$$

$$X(a_{\alpha\beta}) = g((\overline{\nabla}_X J)N_\alpha, N_\beta) - \sum_{\gamma=1}^r [a_{\alpha\gamma}\lambda_{\gamma\beta}(X) + a_{\beta\gamma}\lambda_{\gamma\alpha}(X)],$$

$$g(TX, Y) = g(X, TY), \quad g(TX, TY) = pg(X, TY) + qg(X, Y).$$

Proposition 9 ([4]). If M is an invariant submanifold of codimension r isometrically immersed in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then the Σ -metallic Riemannian structure induced on M is $(T, g, u_\alpha = 0, \xi_\alpha = 0, (a_{\alpha\beta})_r)$ and we get

$$((\overline{\nabla}_X J)Y)^\top = (\nabla_X T)Y,$$

$$((\overline{\nabla}_X J)Y)^\perp = \sum_{\alpha=1}^r [h_\alpha(X, TY)N_\alpha - h_\alpha(X, Y)nN_\alpha],$$

$$((\overline{\nabla}_X J)N_\alpha)^\top = T(A_\alpha X) - \sum_{\beta=1}^r a_{\alpha\beta}A_\beta X,$$

$$((\overline{\nabla}_X J)N_\alpha)^\perp = \sum_{\beta=1}^r [X(a_{\alpha\beta}) + \sum_{\gamma=1}^r (a_{\alpha\gamma}\lambda_{\gamma\beta}(X) + a_{\beta\gamma}\lambda_{\gamma\alpha}(X))]N_\beta,$$

for any $X, Y \in \Gamma(TM)$ and $1 \leq \alpha \leq r$.

Corollary 2 ([4]). If $(T, g, u_\alpha = 0, \xi_\alpha = 0, (a_{\alpha\beta})_r)$ is the Σ -metallic Riemannian structure induced on an invariant submanifold M of codimension r isometrically immersed in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then

$$\nabla T = 0, \quad \sum_{\alpha=1}^r [h_\alpha(X, TY)N_\alpha - h_\alpha(X, Y)nN_\alpha] = 0,$$

$$T(A_\alpha X) - \sum_{\beta=1}^r a_{\alpha\beta} A_\beta X = 0,$$

$$X(a_{\alpha\beta}) + \sum_{\gamma=1}^r [a_{\alpha\gamma} \lambda_{\gamma\beta}(X) + a_{\beta\gamma} \lambda_{\gamma\alpha}(X)] = 0,$$

for any $X, Y \in \Gamma(TM)$ and $1 \leq \alpha, \beta \leq r$.

Proposition 10 ([4]). If $(T, g, u_\alpha = 0, \xi_\alpha = 0, (a_{\alpha\beta})_r)$ is the Σ -metallic Riemannian structure induced on an invariant submanifold M of codimension r isometrically immersed in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then the Nijenhuis tensor field of T vanishes identically on M and T commutes by A_α , i.e., $TA_\alpha = A_\alpha T$, for any $1 \leq \alpha \leq r$.

3.2. Anti-Invariant Submanifolds a Metallic Riemannian Manifold $(\bar{M}, \bar{g}, J)$

A submanifold M of $\bar{M}$ is called anti-invariant if $J(T_x M) \subseteq T_x^\perp M$, for any $x \in M$.

Proposition 11 ([5]). Let M be an anti-invariant submanifold of codimension r isometrically immersed in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$. Then, for any $X, Y \in \Gamma(TM)$:

$$\sum_{\alpha=1}^r h_\alpha(X, Y)tN_\alpha = - \sum_{\alpha=1}^r g(JY, N_\alpha)A_\alpha X,$$

$$\sum_{\alpha=1}^r h_\alpha(X, Y)nN_\alpha = \sum_{\alpha=1}^r g(JY, N_\alpha)\nabla_X^\perp N_\alpha + \sum_{\alpha=1}^r X(g(JY, N_\alpha))N_\alpha - J(\nabla_X Y).$$

Remark 4 ([5]). Let M be an anti-invariant submanifold of codimension r isometrically immersed in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ and let $(T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the Σ -metallic Riemannian structure induced on M . Then T vanishes identically on M and, for any $X \in \Gamma(TM)$:

$$JX = \sum_{\alpha=1}^r u_\alpha(X)N_\alpha.$$

Also, the Σ -metallic Riemannian structure satisfies, for any $X \in \Gamma(TM)$ and $1 \leq \alpha, \beta \leq r$:

$$\sum_{\alpha=1}^r u_\alpha \otimes \xi_\alpha = q \cdot I, \quad \sum_{\beta=1}^r a_{\alpha\beta} u_\beta(X) = p u_\alpha(X), \quad a_{\alpha\beta} = a_{\beta\alpha},$$

$$u_\beta(\xi_\alpha) = q\delta_{\alpha\beta} + p a_{\alpha\beta} - \sum_{\gamma=1}^r a_{\alpha\gamma} a_{\gamma\beta}, \quad \sum_{\beta=1}^r a_{\alpha\beta} \xi_\beta = p \xi_\alpha, \quad u_\alpha(X) = g(X, \xi_\alpha),$$

$$X(a_{\alpha\beta}) = g((\nabla_X J)N_\alpha, N_\beta) - [h_\alpha(X, \xi_\beta) + h_\beta(X, \xi_\alpha)] - \sum_{\gamma=1}^r [a_{\alpha\gamma} \lambda_{\gamma\beta}(X) + a_{\beta\gamma} \lambda_{\gamma\alpha}(X)].$$

Proposition 12 ([5]). If M is an anti-invariant submanifold of codimension r isometrically immersed in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then the Σ -metallic Riemannian structure induced on M is $(T = 0, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ and we get

$$\begin{aligned} ((\bar{\nabla}_X J)Y)^\top &= -\sum_{\alpha=1}^r u_\alpha(Y)A_\alpha X - \sum_{\alpha=1}^r h_\alpha(X, Y)\xi_\alpha, \\ ((\bar{\nabla}_X J)Y)^\perp &= \sum_{\alpha=1}^r X(u_\alpha(Y))N_\alpha - \sum_{1 \leq \alpha, \beta \leq r} \lambda_{\alpha\beta}(X)u_\beta(Y)N_\alpha \\ &\quad - \sum_{\alpha=1}^r h_\alpha(X, Y)nN_\alpha - N(\nabla_X Y), \\ ((\bar{\nabla}_X J)N_\alpha)^\top &= \nabla_X \xi_\alpha - \sum_{\beta=1}^r a_{\alpha\beta}A_\beta X - \sum_{\beta=1}^r \lambda_{\alpha\beta}(X)\xi_\beta, \\ ((\bar{\nabla}_X J)N_\alpha)^\perp &= \sum_{\beta=1}^r [X(a_{\alpha\beta}) + h_\beta(X, \xi_\alpha) \\ &\quad + \sum_{\gamma=1}^r (a_{\alpha\gamma}\lambda_{\gamma\beta}(X) + a_{\beta\gamma}\lambda_{\gamma\alpha}(X))]N_\beta + N(A_\alpha X), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $1 \leq \alpha \leq r$.

Corollary 3 ([5]). If $(T = 0, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is the Σ -metallic Riemannian structure induced on an anti-invariant submanifold M of codimension r isometrically immersed in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then

$$\begin{aligned} \sum_{\alpha=1}^r u_\alpha(Y)A_\alpha X + \sum_{\alpha=1}^r h_\alpha(X, Y)\xi_\alpha &= 0, \\ \sum_{\alpha=1}^r X(u_\alpha(Y))N_\alpha &= -\sum_{1 \leq \alpha, \beta \leq r} \lambda_{\alpha\beta}(X)u_\beta(Y)N_\alpha - \sum_{\alpha=1}^r h_\alpha(X, Y)nN_\alpha - N(\nabla_X Y) = 0, \\ \nabla_X \xi_\alpha - \sum_{\beta=1}^r a_{\alpha\beta}A_\beta X - \sum_{\beta=1}^r \lambda_{\alpha\beta}(X)\xi_\beta &= 0, \\ X(a_{\alpha\beta}) + h_\alpha(X, \xi_\beta) + h_\beta(X, \xi_\alpha) + \sum_{\gamma=1}^r [a_{\alpha\gamma}\lambda_{\gamma\beta}(X) + a_{\beta\gamma}\lambda_{\gamma\alpha}(X)] &= 0, \\ \sum_{\beta=1}^r h_\alpha(X, \xi_\beta)N_\beta &= N(A_\alpha X), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $1 \leq \alpha \leq r$.

Proposition 13 ([5]). If $(T = 0, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is the Σ -metallic Riemannian structure induced on an anti-invariant submanifold M of codimension r isometrically immersed in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then $N^{(2)}$ and $N^{(3)}$ vanish identically on M . Moreover, if ξ_α are parallel with respect to a symmetric linear connection, for any $1 \leq \alpha \leq r$, then $N^{(1)}$ and $N^{(4)}$ vanish, too, on M .

4. Slant Submanifolds in Metallic Riemannian Manifolds

Let M be a submanifold of codimension $r \in N^*$ isometrically immersed in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$. For any $X \in \Gamma(TM)$, one obtains (from the Cauchy-Schwartz inequality)

$$\bar{g}(JX, TX) \leq \|JX\| \cdot \|TX\|.$$

Therefore, we can consider a function $\theta : \Gamma(TM) \rightarrow [0, \frac{\pi}{2}]$, such that:

$$\bar{g}(JX_x, TX_x) = \cos \theta(X_x) \cdot \|JX_x\| \cdot \|TX_x\|,$$

for any $x \in M$ and any nonzero $X_x \in T_x M$. If $JX_x \neq 0$, then the angle $\theta(X_x)$ between JX_x and $T_x M$ is called the Wirtinger angle of X .

Definition 5 ([5]). A slant submanifold M in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ is a submanifold having the angle $\theta(X_x) = \text{constant}$ (where $\theta(X_x)$ is the angle between JX_x and $T_x M$, for any $x \in M$ and $X_x \in T_x M$, whenever $JX_x \neq 0$). Then $\theta =: \theta(X_x)$ (called the slant angle of M in $\bar{M}$) has the property

$$\cos \theta = \frac{\|TX\|}{\|JX\|}.$$

In this case, $i : M \rightarrow \bar{M}$ is called the slant immersion of M in $\bar{M}$.

Remark 5. For $\theta = 0$ (or $\theta = \frac{\pi}{2}$) we obtain the particular cases of invariant (or respectively, anti-invariant) submanifolds M in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$. A proper slant submanifold is a slant submanifold which is neither invariant nor anti-invariant and $i : M \rightarrow \bar{M}$ is called a proper slant immersion.

Proposition 14 ([5]). If M is a slant submanifold with the slant angle θ , isometrically immersed in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then, for any $X, Y \in \Gamma(TM)$:

$$\bar{g}(TX, TY) = \cos^2 \theta \cdot \bar{g}(X, p \cdot TY + q \cdot Y),$$

$$\bar{g}(NX, NY) = \sin^2 \theta \cdot \bar{g}(X, p \cdot TY + q \cdot Y).$$

We also have

$$T^2 = \cos^2 \theta \cdot (p \cdot T + q \cdot I), \quad \nabla T^2 = p \cos^2 \theta \cdot \nabla T,$$

where I is the identity on $\Gamma(TM)$.

Remark 6. From ([5]) we obtain

$$tN = \sin^2 \theta \cdot (p \cdot T + q \cdot I) = \sum_{\alpha=1}^r u_{\alpha} \otimes \xi_{\alpha}.$$

Like in the Riemannian product case ([28]), we can define a slant distribution in a metallic Riemannian manifold.

Definition 6 ([5]). If M is an immersed submanifold in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then a differentiable distribution D on M is called a slant distribution if the angle θ_D between JX_x and the vector subspace D_x is constant, for any $x \in M$ and any nonzero $X_x \in D_x$. The constant angle θ_D is called the slant angle of the distribution D .

Proposition 15 ([6]). If D is a differentiable distribution on a submanifold M of a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ and P_D is the orthogonal projection on D , then D is a slant distribution if and only if there exists a constant $\lambda \in [0, 1]$ such that:

$$(P_D T)^2 X = \lambda \cdot (p \cdot P_D TX + q \cdot X),$$

for any $X \in \Gamma(D)$. Moreover, the slant angle θ_D of D satisfies $\lambda = \cos^2 \theta_D$.

5. Bi-Slant Submanifolds in Metallic Riemannian Manifolds

The differential geometry of slant submanifolds has shown an increasing development in the early 1990's when B.-Y. Chen defined slant submanifolds in complex manifolds ([29]).

Particular cases of bi-slant submanifolds, such as semi-invariant submanifolds in locally product Riemannian manifolds were studied in ([30,31]), semi-slant submanifolds were studied by J. L. Cabrerizo and A. Carriazo ([32–34]). Moreover, slant and semi-slant submanifolds in almost product Riemannian manifolds were studied in ([28,30]). The hemi-slant submanifolds (called, also, pseudo-slant submanifolds) in locally decomposable Riemannian manifolds were studied by M. Ateken et al. ([35]) and in locally product Riemannian manifolds were studied by H. M. Tařtan and F. Ozdem in ([36]).

Definition 7 ([10]). If M is an immersed submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then M is called a bi-slant submanifold if there exist two orthogonal differentiable distributions D_1 and D_2 on M such that:

1. TM admits the orthogonal decomposition $TM = D_1 \oplus D_2$;
2. $J(D_1) \perp D_2$ and $J(D_2) \perp D_1$;
3. the distributions D_1 and D_2 are slant with angles $\theta_1 \neq \theta_2$.

If M is a bi-slant submanifold of $(\overline{M}, \overline{g}, J)$, then $T(D_1) \subseteq D_1$ and $T(D_2) \subseteq D_2$.

We provide an example of a bi-slant submanifold in a metallic Riemannian manifold.

Example 1. We consider the Euclidean space R^4 endowed with the Euclidean metric $\langle \cdot, \cdot \rangle$ and the immersion $i : M \rightarrow R^4$, given by

$$i(\alpha, \beta) := \left(\alpha \cos t, \frac{\sigma}{\sqrt{q}} \alpha \sin t, \beta, \beta \right),$$

where $M := \{(\alpha, \beta) \mid \alpha, \beta > 0\}$, $t \in [0, \frac{\pi}{2}]$ and $\sigma := \sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$ is a metallic number. The local orthogonal frame on TM is

$$Z_1 = \cos t \frac{\partial}{\partial x_1} + \frac{\sigma}{\sqrt{q}} \sin t \frac{\partial}{\partial x_2}, \quad Z_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}.$$

The structure $J : R^4 \rightarrow R^4$ given by

$$J\left(\frac{\partial}{\partial x_k}\right) := \begin{cases} \sigma \frac{\partial}{\partial x_k}, & k \in \{1, 3\} \\ \bar{\sigma} \frac{\partial}{\partial x_k}, & k \in \{2, 4\} \end{cases}$$

is a metallic structure, where $\bar{\sigma} := p - \sigma$. Since

$$JZ_1 = \sigma \cos t \frac{\partial}{\partial x_1} - \sqrt{q} \sin t \frac{\partial}{\partial x_2}, \quad JZ_2 = \sigma \frac{\partial}{\partial x_3} + \bar{\sigma} \frac{\partial}{\partial x_4},$$

we remark that $\langle JZ_1, Z_1 \rangle = \sigma \cos 2t$, $\langle JZ_2, Z_2 \rangle = \sigma + \bar{\sigma}$ and

$$\|Z_1\|^2 = \frac{p\sigma}{q} \sin^2 t + 1, \quad \|Z_2\|^2 = 2,$$

$$\|JZ_1\|^2 = \sigma(\sigma - p \sin^2 t), \quad \|JZ_2\|^2 = p^2 + 2q.$$

If $D_1 := \text{span}\{Z_1\}$ and $D_2 := \text{span}\{Z_2\}$, then $\cos \theta_1 = \frac{\sqrt{q} \cos 2t}{\sqrt{(\sigma - p \cos^2 t)(\sigma - p \sin^2 t)}}$ and $\cos \theta_2 = \frac{p}{\sqrt{2(p^2 + 2q)}}$. Thus, D_1 and D_2 are the slant distributions, with the slant angles θ_1 and θ_2 , respectively.

Let M_{θ_1} and M_{θ_2} be the integral manifolds of the distributions D_1 and D_2 , respectively. Thus, $M_{\theta_1} \times M_{\theta_2}$ is a bi-slant submanifold in the metallic Riemannian manifold $(R^4, \langle \cdot, \cdot \rangle, J)$, with the metric

$$g := \left(\frac{p\sigma}{q} \sin^2 t + 1 \right) g_1 + 2g_2.$$

5.1. Semi-Slant Submanifolds in Metallic Riemannian Manifolds

Semi-slant submanifolds in a metallic Riemannian manifold are particular cases of bi-slant submanifolds, which can be defined in a similar manner as semi-slant submanifolds in a locally product Riemannian manifold ([28]).

Definition 8 ([6]). A semi-slant submanifold M in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ is a submanifold which has two orthogonal differentiable distributions D and D^θ , such that:

1. TM admits the orthogonal decomposition $TM = D \oplus D^\theta$;
2. the distribution D is invariant;
3. the distribution D^θ is slant with angle $\theta \neq 0$, and $J(D^\theta) \perp D$.

Moreover, M is called a proper semi-slant submanifold if $\theta \in (0, \frac{\pi}{2})$.

Remark 7. Let M be a semi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with $TM = D \oplus D^\theta$. If $\theta = \frac{\pi}{2}$, then M is a semi-invariant submanifold of $\overline{M}$.

Let P_1 and P_2 be the orthogonal projections on D and D^θ . The decomposition of $X \in \Gamma(TM)$ is given by $X = P_1X + P_2X$, where $P_1X \in \Gamma(D)$ and $P_2X \in \Gamma(D^\theta)$.

Proposition 16 ([6]). Let M be a semi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then, for any $X \in \Gamma(TM)$:

$$JX = TP_1X + TP_2X + NP_2X = P_1TX + P_2TX + NP_2X,$$

$$JP_1X = TP_1X, \quad NP_1X = 0, \quad TP_2X \in \Gamma(D^\theta).$$

Remark 8 ([6]). Let M be a semi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then, for any $X \in \Gamma(TM)$, we get

$$\overline{g}(JP_2X, TP_2X) = \cos \theta(X) \cdot \|TP_2X\| \cdot \|JP_2X\|$$

and for $P_2X \neq 0$, the angle θ of the distribution D^θ is constant, given by

$$\cos \theta(X) = \frac{\|TP_2X\|}{\|JP_2X\|}.$$

Proposition 17 ([6]). If M is a semi-slant submanifold with the slant angle θ , isometrically immersed in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then, for any $X, Y \in \Gamma(TM)$:

$$\overline{g}(TP_2X, TP_2Y) = \cos^2 \theta \cdot \overline{g}(p \cdot TP_2X + q \cdot P_2X, P_2Y),$$

$$\overline{g}(NX, NY) = \sin^2 \theta \cdot \overline{g}(p \cdot TP_2X + q \cdot P_2X, P_2Y).$$

We also have

$$(TP_2)^2 = \cos^2 \theta \cdot (p \cdot TP_2 + q \cdot I), \quad \nabla (TP_2)^2 = p \cos^2 \theta \cdot \nabla (TP_2),$$

where I is the identity on $\Gamma(D^\theta)$.

Proposition 18 ([6]). A necessary and sufficient condition for a submanifold M in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ to be a semi-slant submanifold in $\bar{M}$ is to exist a constant $\lambda \in [0, 1)$, such that

$$D_0 = \{X \in \Gamma(TM) \mid T^2X = \lambda(p \cdot TX + q \cdot X)\}$$

is a differentiable distribution, and $NX = 0$, for any $X \in \Gamma(TM)$ orthogonal to D_0 .

Proposition 19 ([6]). Let M be a semi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$. Then:

(i) a necessary and sufficient condition for the integrability of the distribution D is:

$$h(X, TY) = h(TX, Y) \text{ or } JA_VX = A_VJX,$$

for any $X, Y \in \Gamma(D)$ and $V \in \Gamma(T^\perp M)$;

(ii) a necessary and sufficient condition for the integrability of the distribution D^θ is:

$$P_1(\nabla_Z TW - \nabla_W TZ) = P_1(A_{NW}Z - A_{NZ}W),$$

for any $Z, W \in \Gamma(D^\theta)$.

Remark 9 ([6]). The condition $\nabla T = 0$ implies the integrability of the distributions D and D^θ on a semi-slant submanifold M in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$.

Proposition 20 ([6]). If M is a semi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then $\nabla N = 0$ if and only if, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$:

$$A_{nV}X = TA_VX = A_VTX.$$

Definition 9. Let M be a semi-slant submanifold in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$. We say that M is a $D - D^\theta$ mixed totally geodesic if $h(X, Z) = 0$, for any $X \in \Gamma(D)$ and any $Z \in \Gamma(D^\theta)$.

Proposition 21 ([6]). A necessary and sufficient condition for the semi-slant submanifold M in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ to be a $D - D^\theta$ mixed totally geodesic submanifold is $A_VX \in \Gamma(D)$ and $A_VZ \in \Gamma(D^\theta)$, for any $X \in \Gamma(D)$, $Z \in \Gamma(D^\theta)$ and $V \in \Gamma(T^\perp M)$.

Proposition 22 ([6]). If M is a proper semi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then M is a $D - D^\theta$ mixed totally geodesic submanifold if one of the following conditions are true, for any $X \in \Gamma(D)$, $Z \in \Gamma(D^\theta)$ and $V \in \Gamma(T^\perp M)$:

- (i) $(\bar{\nabla}_X N)Z = 0$ and $h(X, Z)$ is not an eigenvector of the tensor field n with the eigenvalue $-\frac{q}{p}$;
- (ii) $A_{nV}X = TA_VX = A_VTX$ and $h(X, Z)$ is not an eigenvector of the tensor field n .

Proposition 23 ([6]). If M is a $D - D^\theta$ mixed totally geodesic proper semi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then $(\bar{\nabla}_X N)Y = 0$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\theta)$.

Example 2. For $t = 0$ in the Example 1, we obtain $\theta_1 = 0$ and the submanifold $M_0 \times M_{\theta_2}$ is a semi-slant submanifold in the metallic Riemannian manifold $(R^4, \langle \cdot, \cdot \rangle, J)$.

5.2. Hemi-Slant Submanifolds in Metallic Riemannian Manifolds

Hemi-slant submanifolds in a metallic Riemannian manifold are particular cases of bi-slant submanifolds, which can be defined in a similar manner as hemi-slant submanifolds in a locally product Riemannian manifold ([36]).

Definition 10 ([7]). A hemi-slant submanifold M in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ is a submanifold which has two orthogonal differentiable distributions D^θ and $D^\perp$, such that:

1. TM admits the orthogonal decomposition $TM = D^\theta \oplus D^\perp$;
2. the distribution $D^\perp$ is anti-invariant;
3. the distribution D^θ is slant with angle $\theta \neq \frac{\pi}{2}$, and $J(D^\theta) \perp D^\perp$.

Moreover, M is called a proper hemi-slant submanifold if $\theta \in (0, \frac{\pi}{2})$.

Remark 10. Let M be a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with $TM = D^\theta \oplus D^\perp$. If $\theta = 0$, then M is a semi-invariant submanifold of $\overline{M}$.

Let P_1 and P_2 be the orthogonal projections on D^θ and $D^\perp$. The decomposition of $X \in \Gamma(TM)$ is given by $X = P_1X + P_2X$, where $P_1X \in \Gamma(D^\theta)$ and $P_2X \in \Gamma(D^\perp)$.

Proposition 24 ([7]). Let M be a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then, for any $X \in \Gamma(TM)$:

$$JX = TP_1X + NP_1X + NP_2X = TP_1X + NX,$$

$$JP_2X = NP_2X, \quad TP_2X = 0, \quad TP_1X \in \Gamma(D^\theta).$$

Remark 11 ([7]). Let M be a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then, for any $X \in \Gamma(TM)$, we get

$$\overline{g}(JP_1X, TP_1X) = \cos \theta(X) \cdot \|TP_1X\| \cdot \|JP_1X\|$$

and for $P_1X \neq 0$, the angle θ of the distribution D^θ is constant, given by

$$\cos \theta(X) = \frac{\|TP_1X\|}{\|JP_1X\|}.$$

Proposition 25 ([7]). If M is a hemi-slant submanifold with the slant angle θ , isometrically immersed in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then, for any $X, Y \in \Gamma(TM)$:

$$\overline{g}(TP_1X, TP_1Y) = \cos^2 \theta \cdot \overline{g}(p \cdot TP_1X + q \cdot P_1X, P_1Y),$$

$$\overline{g}(NX, NY) = \sin^2 \theta \cdot \overline{g}(p \cdot TP_1X + q \cdot P_1X, P_1Y).$$

We also have

$$(TP_1)^2 = \cos^2 \theta \cdot (p \cdot TP_1 + q \cdot I), \quad \nabla(TP_1)^2 = p \cos^2 \theta \cdot \nabla(TP_1),$$

where I is the identity on $\Gamma(D^\theta)$.

Definition 11. Let M be a hemi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. We say that M is a $D^\theta - D^\perp$ mixed totally geodesic if $h(X, Z) = 0$, for any $X \in \Gamma(D^\theta)$ and any $Z \in \Gamma(D^\perp)$.

Proposition 26 ([7]). A necessary and sufficient condition for a submanifold M in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ to be a hemi-slant submanifold in $\overline{M}$ is to exist a constant $\lambda \in [0, 1)$, such that

$$D_0 = \{X \in \Gamma(TM) \mid T^2X = \lambda(p \cdot TX + q \cdot X)\}$$

is a differentiable distribution, and $TX = 0$, for any $X \in \Gamma(TM)$ orthogonal to D_0 .

Proposition 27 ([7]). Let M be a hemi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$. Then:

(i) the distribution D^θ is integrable and, for any $X, Y \in \Gamma(D^\theta)$, we get

$$\nabla_X TY - \nabla_Y TX - A_{NY}X + A_{NX}Y \in \Gamma(D^\theta);$$

(ii) a necessary and sufficient condition for the integrability of the distribution $D^\perp$ is:

$$A_{NZ}W = 0 \text{ or } (\nabla_Z T)W = (\nabla_W T)Z,$$

for any $Z, W \in \Gamma(D^\perp)$.

Remark 12 ([7]). The condition $(\nabla_Z T)W = 0$, for any $Z, W \in \Gamma(D^\perp)$, implies the integrability of the distribution $D^\perp$ on a semi-slant submanifold M in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$.

Proposition 28 ([7]). If M is a hemi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then $\nabla N = 0$ if and only if, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$:

$$A_{nV}X = TA_VX = A_VTX.$$

Proposition 29 ([7]). A necessary and sufficient condition for the hemi-slant submanifold M in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ to be a $D^\theta - D^\perp$ mixed totally geodesic submanifold is $A_VX \in \Gamma(D^\theta)$ and $A_VY \in \Gamma(D^\perp)$, for any $X \in \Gamma(D^\theta)$, $Y \in \Gamma(D^\perp)$ and $V \in \Gamma(T^\perp M)$.

Proposition 30 ([7]). If M is a proper hemi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ and $(\nabla_X N)Y = 0$, for any $X, Y \in \Gamma(TM)$, then M is a $D^\theta - D^\perp$ mixed totally geodesic submanifold in $\bar{M}$.

Example 3. For $t = \frac{\pi}{4}$ in the Example 1, we obtain $\theta_1 = \frac{\pi}{2}$ and the submanifold $M_{\frac{\pi}{2}} \times M_{\theta_2}$ is a hemi-slant submanifold in the metallic Riemannian manifold $(R^4, \langle \cdot, \cdot \rangle, J)$.

6. Warped Product Bi-Slant Submanifolds in Metallic Riemannian Manifolds

Many properties for warped product manifolds and submanifolds were presented by B.-Y. Chen in his book ([37]). Warped product submanifolds in locally Riemannian product manifolds were studied by F. R. Al-Solamy and S. Uddin ([38–40]), B. Şahin ([41]), M. Atçeken ([42–44]).

In this section we present some results regarding the existence and nonexistence of non-trivial semi-invariant, semi-slant and hemi-slant warped product submanifolds in locally metallic Riemannian manifolds and we provide examples.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and denote by π_1 and π_2 the projection maps from the product manifold $M_1 \times M_2$ onto M_1 and M_2 , respectively.

Definition 12 ([45]). If g is the Riemannian metric on $M_1 \times M_2$ defined by

$$g := \pi_1^*g_1 + (f \circ \pi_1)^2\pi_2^*g_2,$$

where $f : M_1 \rightarrow (0, \infty)$ is a differentiable function on M_1 , then $M_1 \times_f M_2 =: (M_1 \times M_2, g)$ is called the warped product Riemannian manifold of M_1 and M_2 , having the warping function f . Moreover, $M_1 \times_f M_2$ is called trivial if f is constant. In this case, it is just a Riemannian product $M_1 \times M_2$, where M_2 is equipped with the metric f^2g_2 (which is homothetic to g_2).

For simplification, we will use $(f \circ \pi_1)^2 =: f^2$, $\pi_1^*g_1 =: g_1$ and $\pi_2^*g_2 =: g_2$.

It is known that M_1 is a totally geodesic and M_2 is a totally umbilical submanifold of the warped product manifold $M_1 \times_f M_2$ ([45]).

Definition 13. If M_1 and M_2 are slant submanifolds in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then the warped product $M_1 \times_f M_2$ is called warped product bi-slant submanifold. Moreover, $M_1 \times_f M_2$ is called a proper warped product bi-slant submanifold if $\theta_1 \neq \theta_2$.

Definition 14. Let $M_1 \times_f M_2$ be a warped product submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. If one of the components M_i ($i \in \{1, 2\}$) is invariant (respectively, anti-invariant) submanifold in $\overline{M}$ and the other one is slant having the slant angle $\theta \in (0, \frac{\pi}{2}]$ (or $\theta \in [0, \frac{\pi}{2})$, respectively), then the submanifold M is called a warped product semi-slant (respectively, warped product hemi-slant) submanifold.

Definition 15. Let $M_1 \times_f M_2$ be a warped product semi-slant (respectively, hemi-slant) submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ such that the slant angle is $\theta = \frac{\pi}{2}$ (respectively, $\theta = 0$). Then the warped products $M_\top \times_f M_\perp$ and $M_\perp \times_f M_\top$ are called warped product semi-invariant submanifolds.

Proposition 31 ([10]). If $M_1 \times_f M_2$ is a warped product bi-slant submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then

$$\overline{g}(h(X, Y), NZ) = -\overline{g}(h(X, Z), NY), \quad \overline{g}(h(X, Z), NW) = 0,$$

$$\overline{g}(h(Z, W), NX) = TX(\ln f)\overline{g}(Z, W) - X(\ln f)\overline{g}(Z, TW),$$

for any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$.

Proposition 32 ([10]). If $M_\top \times_f M_\perp$ is a warped product semi-invariant submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then it is a trivial warped product Riemannian manifold.

Proposition 33 ([10]). If $M_\perp \times_f M_\top$ is a warped product semi-invariant submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then it is a trivial warped product Riemannian manifold if and only if

$$t\nabla_Z^\perp JX = -(T - pI)A_{JX}Z,$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\top)$.

Proposition 34 ([10]). If $M_{1\top} \times_f M_{2\top}$ is a warped product submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ and $X(\ln f) \neq 0$, for any $X \in \Gamma(TM_{1\top})$, then

$$TX(\ln f) = \tilde{\sigma}X(\ln f),$$

for any $X \in \Gamma(TM_{1\top})$ and $Z \in \Gamma(TM_{2\top})$, where $\tilde{\sigma} \in \{\sigma, \bar{\sigma}\}$, $\sigma := \sigma_{p,q}$ is a metallic number and $\bar{\sigma} := p - \sigma$.

Proposition 35 ([10]). If $M_{1\perp} \times_f M_{2\perp}$ is a warped product submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then it is a trivial warped product Riemannian manifold if and only if

$$t\nabla_Z^\perp NX = pth(X, Z),$$

for any $X \in \Gamma(TM_{1\perp})$ and $Z \in \Gamma(TM_{2\perp})$.

We provide an example of a non-trivial warped product semi-invariant submanifold in a metallic Riemannian manifold.

Example 4. We consider the Euclidean space R^5 endowed with the Euclidean metric $\langle \cdot, \cdot \rangle$ and the immersion $i : M \rightarrow R^5$, given by

$$i(f, \alpha, \beta) := \left(f \sin \alpha, f \cos \alpha, f \sin \beta, f \cos \beta, \sqrt{\frac{p\sigma}{q}} f \right),$$

where $M := \{(f, \alpha, \beta) \mid f > 0, \alpha, \beta \in (0, \frac{\pi}{2})\}$ and $\sigma := \sigma_{p,q}$ is a metallic number.

The local orthogonal frame on TM is

$$Z_1 = \sin \alpha \frac{\partial}{\partial x_1} + \cos \alpha \frac{\partial}{\partial x_2} + \sin \beta \frac{\partial}{\partial x_3} + \cos \beta \frac{\partial}{\partial x_4} + \sqrt{\frac{p\sigma}{q}} \frac{\partial}{\partial x_5},$$

$$Z_2 = f \cos \alpha \frac{\partial}{\partial x_1} - f \sin \alpha \frac{\partial}{\partial x_2}, \quad Z_3 = f \cos \beta \frac{\partial}{\partial x_3} - f \sin \beta \frac{\partial}{\partial x_4}.$$

The structure $J : R^5 \rightarrow R^5$ given by

$$J\left(\frac{\partial}{\partial x_k}\right) := \begin{cases} \sigma \frac{\partial}{\partial x_k}, & k \in \{1, 2\} \\ \bar{\sigma} \frac{\partial}{\partial x_k}, & k \in \{3, 4, 5\} \end{cases}$$

is a metallic structure, where $\bar{\sigma} := 1 - \sigma$. Since

$$JZ_1 = \sigma \sin \alpha \frac{\partial}{\partial x_1} + \sigma \cos \alpha \frac{\partial}{\partial x_2} + \bar{\sigma} \sin \beta \frac{\partial}{\partial x_3} + \bar{\sigma} \cos \beta \frac{\partial}{\partial x_4} + \bar{\sigma} \sqrt{\frac{p\sigma}{q}} \frac{\partial}{\partial x_5},$$

$$JZ_2 = \sigma Z_2, \quad JZ_3 = \bar{\sigma} Z_3,$$

we remark that $\langle JZ_1, Z_k \rangle = 0$, for any $k \in \{1, 2, 3\}$, $JZ_2, JZ_3 \subseteq \text{span}\{Z_2, Z_3\}$ and

$$\|Z_1\|^2 = \frac{p\sigma}{q} + 2, \quad \|Z_2\|^2 = \|Z_3\|^2 = f^2,$$

$$\|JZ_1\|^2 = p^2 + 2q - p\bar{\sigma}, \quad \|JZ_2\|^2 = f^2(p\sigma + q), \quad \|JZ_3\|^2 = f^2(p\bar{\sigma} + q).$$

If $D_1 := \text{span}\{Z_1\}$ and $D_2 := \text{span}\{Z_2, Z_3\}$, then D_1 is an anti-invariant and D_2 is an invariant distribution.

Let $M_\perp$ and $M_\top$ be the integral manifolds of the distributions D_1 and D_2 , respectively. Thus, $M_\perp \times_f M_\top$ is a warped product semi-invariant submanifold in the metallic Riemannian manifold $(R^5, \langle \cdot, \cdot \rangle, J)$, with the metric

$$g := g_{M_\perp} + f^2 g_{M_\top},$$

where $g_{M_\perp} := \left(\frac{p\sigma}{q} + 2\right)df^2$ and $g_{M_\top} := d\alpha^2 + d\beta^2$.

Proposition 36 ([10]). If $M_\top \times_f M_\theta$ is a proper warped product semi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then it is a trivial warped product Riemannian manifold.

We provide an example of a non-trivial warped product semi-slant submanifold $M_\theta \times_f M_\top$ in a metallic Riemannian manifold.

Example 5. We consider the Euclidean space R^7 endowed with the Euclidean metric $\langle \cdot, \cdot \rangle$ and the immersion $i : M \rightarrow R^7$, given by

$$i(f, \alpha, \beta) := (f \cos \alpha, f \sin \alpha, f \cos \beta, f \sin \beta, f, \alpha, \beta),$$

where $M := \{(f, \alpha, \beta) \mid f > 0, \alpha, \beta \in [0, \frac{\pi}{2}]\}$.

The local orthogonal frame on TM is

$$Z_1 = \cos \alpha \frac{\partial}{\partial x_1} + \sin \alpha \frac{\partial}{\partial x_2} + \cos \beta \frac{\partial}{\partial x_3} + \sin \beta \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5},$$

$$Z_2 = -f \sin \alpha \frac{\partial}{\partial x_1} + f \cos \alpha \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, \quad Z_3 = -f \sin \beta \frac{\partial}{\partial x_3} + f \cos \beta \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_7}.$$

The structure $J : R^7 \rightarrow R^7$ given by

$$J\left(\frac{\partial}{\partial x_k}\right) := \begin{cases} \sigma \frac{\partial}{\partial x_k}, & k \in \{1, 2, 6\} \\ \bar{\sigma} \frac{\partial}{\partial x_k}, & k \in \{3, 4, 5, 7\} \end{cases}$$

is a metallic structure, where $\sigma := \sigma_{p,q}$ is a metallic number and $\bar{\sigma} := p - \sigma$. Since

$$JZ_1 = \sigma \cos \alpha \frac{\partial}{\partial x_1} + \sigma \sin \alpha \frac{\partial}{\partial x_2} + \bar{\sigma} \cos \beta \frac{\partial}{\partial x_3} + \bar{\sigma} \sin \beta \frac{\partial}{\partial x_4} + \bar{\sigma} \frac{\partial}{\partial x_5},$$

$$JZ_2 = \sigma Z_2, \quad JZ_3 = \bar{\sigma} Z_3,$$

we remark that $\langle JZ_k, Z_l \rangle = 0$, for any $k \neq l$, where $k, l \in \{1, 2, 3\}$, $\langle JZ_1, Z_1 \rangle = \sigma + 2\bar{\sigma}$ and

$$\|Z_1\|^2 = 3, \quad \|Z_2\|^2 = \|Z_3\|^2 = f^2 + 1,$$

$$\|JZ_1\|^2 = \sigma^2 + 2\bar{\sigma}^2, \quad \|JZ_2\|^2 = \sigma^2(f^2 + 1), \quad \|JZ_3\|^2 = \bar{\sigma}^2(f^2 + 1).$$

If $D_1 := \text{span}\{Z_1\}$ and $D_2 := \text{span}\{Z_2, Z_3\}$, then D_1 is a slant distribution with the slant angle θ with $\cos \theta = \frac{\sigma + 2\bar{\sigma}}{\sqrt{3(\sigma^2 + 2\bar{\sigma}^2)}}$, and D_2 is an invariant distribution.

Let M_θ and $M_\top$ be the integral manifolds of the distributions D_1 and D_2 , respectively. Thus, $M_\theta \times_{\sqrt{f^2+1}} M_\top$ is a warped product semi-slant submanifold in the metallic Riemannian manifold $(R^7, \langle \cdot, \cdot \rangle, J)$, with the metric

$$g := g_{M_\theta} + (f^2 + 1)g_{M_\top},$$

where $g_{M_\theta} := 3df^2$ and $g_{M_\top} := d\alpha^2 + d\beta^2$.

Proposition 37 ([10]). If $M_\perp \times_f M_\theta$ (or $M_\theta \times_f M_\perp$) is a proper warped product hemi-slant submanifold in a locally metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then it is a trivial warped product Riemannian manifold if and only if

$$A_{NZ}X = A_{NX}Z,$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\theta)$ (or $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, respectively).

We provide an example of a non-trivial warped product hemi-slant submanifold in a metallic Riemannian manifold.

Example 6. We consider the Euclidean space R^5 endowed with the Euclidean metric $\langle \cdot, \cdot \rangle$ and let $i : M \rightarrow R^5$ be the immersion, given by

$$i(f, \alpha) := (\sqrt{q}f \sin \alpha, \sqrt{q}f \cos \alpha, \sigma f \sin \alpha, \sigma f \cos \alpha, -\sqrt{q}f),$$

where $M := \{(f, \alpha) \mid f > 0, \alpha \in (0, \frac{\pi}{2})\}$ and $\sigma := \sigma_{p,q}$ is a metallic number.

The local orthogonal frame on TM is

$$Z_1 = \sqrt{q} \sin \alpha \frac{\partial}{\partial x_1} + \sqrt{q} \cos \alpha \frac{\partial}{\partial x_2} + \sigma \sin \alpha \frac{\partial}{\partial x_3} + \sigma \cos \alpha \frac{\partial}{\partial x_4} - \sqrt{q} \frac{\partial}{\partial x_5},$$

$$Z_2 = \sqrt{q}f \cos \alpha \frac{\partial}{\partial x_1} - \sqrt{q}f \sin \alpha \frac{\partial}{\partial x_2} + \sigma f \cos \alpha \frac{\partial}{\partial x_3} - \sigma f \sin \alpha \frac{\partial}{\partial x_4}.$$

The structure $J : R^5 \rightarrow R^5$ given by

$$J\left(\frac{\partial}{\partial x_k}\right) := \begin{cases} \sigma \frac{\partial}{\partial x_k}, & k \in \{1, 2, 5\} \\ \bar{\sigma} \frac{\partial}{\partial x_k}, & k \in \{3, 4\} \end{cases}$$

is a metallic structure, where $\bar{\sigma} := 1 - \sigma$. Since

$$JZ_1 = \sqrt{q}\sigma \left(\sin \alpha \frac{\partial}{\partial x_1} + \cos \alpha \frac{\partial}{\partial x_2} \right) - q \left(\sin \alpha \frac{\partial}{\partial x_3} + \cos \alpha \frac{\partial}{\partial x_4} \right) - \sqrt{q}\sigma \frac{\partial}{\partial x_5},$$

$$JZ_2 = \sqrt{q}\sigma f \left(\cos \alpha \frac{\partial}{\partial x_1} - \sin \alpha \frac{\partial}{\partial x_2} \right) - qf \left(\cos \alpha \frac{\partial}{\partial x_3} - \sin \alpha \frac{\partial}{\partial x_4} \right),$$

we remark that $\langle JZ_2, Z_k \rangle = 0$, for any $k \in \{1, 2\}$, $\langle JZ_1, Z_1 \rangle = q\sigma$ and

$$\|Z_1\|^2 = \sigma^2 + 2q, \quad \|Z_2\|^2 = f^2(\sigma^2 + q),$$

$$\|JZ_1\|^2 = q(2\sigma^2 + q), \quad \|JZ_2\|^2 = qf^2(\sigma^2 + q).$$

If $D_1 := \text{span}\{Z_1\}$ and $D_2 := \text{span}\{Z_2\}$, then D_1 is a slant distribution with the slant angle θ with $\cos \theta = \frac{\sqrt{q}\sigma}{\sqrt{(2\sigma^2+q)(\sigma^2+q)}}$, and D_2 is an anti-invariant distribution.

Let M_θ and $M_\perp$ be the integral manifolds of the distributions D_1 and D_2 , respectively. Thus, $M_\theta \times_f \sqrt{\sigma^2+q} M_\perp$ is a warped product hemi-slant submanifold in the metallic Riemannian manifold $(R^5, \langle \cdot, \cdot \rangle, J)$, with the metric

$$g = g_{M_\theta} + f^2(\sigma^2 + q)g_{M_\perp},$$

where $g_{M_\theta} := (\sigma^2 + 2q)d\alpha^2$ and $g_{M_\perp} := d\alpha^2$.

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