mathematics

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Cohomologies of n-Lie Algebras with Derivations

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Abstract: The goal of this paper is to study cohomological theory of n-Lie algebras with derivations. We define the representation of an n-LieDer pair and consider its cohomology. Likewise, we verify that a cohomology of an n-LieDer pair could be derived from the cohomology of associated LeibDer pair. Furthermore, we discuss the $(n-1)$-order deformations and the Nijenhuis operator of n-LieDer pairs. The central extensions of n-LieDer pairs are also investigated in terms of the first cohomology groups with coefficients in the trivial representation.

Keywords: n-Lie algebra; representation; n-LieDer pair; cohomology; deformation; central extension

MSC: 17B60; 17A30; 81R12

Citation: Sun, Q.; Wu, Z. Cohomologies of n-Lie Algebras with Derivations. *Mathematics* **2021**, *9*, 2452. https://doi.org/10.3390/math9192452

Academic Editor: Mancho Manev

Received: 12 July 2021
Accepted: 28 September 2021
Published: 2 October 2021

1. Introduction

The notion of n-Lie algebras was introduced by Filippov [1] in 1985. It is the algebraic structure corresponding to Nambu mechanics [2–4]. If $n = 2$, then we get Lie algebra structure. Nanmbu's used 3-Lie algebras to describe simultaneous classical dynamics of three particles in [3]. Takhtajan [5] systematically developed a foundemental theory of n-Poisson or Nambu–Poisson manifolds, and established a connection between Nambu mechanics and Filippov algebras. Numerous works have been devoted to various aspects of n-Lie algebras in both mathematics and physics, see [6–14] and their references.

The method of deformation is ubiquitous in mathematics and physics. Gerstenhaber developed a deformation theory of associative algebras in [15,16]. Subsequently, Nijenhuis and Richardson generalized it to Lie algebras in [17,18]. Both associative algebras and Lie algebras are algebras over some quadratic operads. Balavoine [19] investigated a deformation theory of quadratic operads. Cohomology and deformation of n-Lie algebras have been studied from several directions. In [7], central trivial extensions and infinitesimal deformations are considered. The two-order deformations of 3-Lie algebras are discussed in [1]. In general, $(n-1)$-order deformations of n-Lie algebras were studied in [12], meanwhile, Nijenhuis operators were obtained from a trivial deformation.

A Nijenhuis algebra is a nonunitary associative algebra A with a linear endomorphism N satisfying the Nijenhuis equation: $N(x)N(y) = N(N(x)y + xN(y)) + N^2(xy), \forall x, y \in A$, where N is called a Nijenhuis operator. The concept of a Nijenhuis operator on a Lie algebra originated from the important concept of a Nijenhuis tensor that was introduced by Nijenhuis [20] in the study of pseudo-complex manifolds in the 1950s and was related to the well-known concepts of the Schouten–Nijenhuis bracket, the Frölicher–Nijenhuis bracket [21], and the Nijenhuis–Richardson bracket. Nijenhuis operators on a Lie algebra appeared in a more general study of Poisson–Nijenhuis manifolds [22] and in the context of the classical Yang–Baxter equation [23,24]. A Nijenhuis operator on a Lie algebra is related to its deformation. Nijenhuis operators on n-Lie algebras have been studied in [12]. They can be used to construct a deformation of n-Lie algebras. As a generalization of the classical Yang–Baxter equation (CYBE) on Lie algebras [25], the $\mathcal{O}$-operator provides a solution of the CYBE on a Lie algebra [26]. The $\mathcal{O}$-operator on Lie algebras was generalized to n-Lie algebras in [12].

Deformations of algebras are described by cohomology groups. Derivations of algebras are also controlled by a cohomological group. Derivations are a basic concept in homotopy Lie algebras [27], deformation formulas [28] and differential Galois theory [29]. They are of vital importance to control theories and gauge theories in quantum field theory [30,31]. In [32,33], the authors study algebras with derivations, which are a kind of homotopy algebra, from the operadic point of view. Lie algebras with derivations are usually called LieDer pairs. Recently, LieDer pairs have been studied from the cohomological point of view. Extensions and deformations of LieDer pairs are considered in [34]. These results have been extended to associative algebras with derivations [35], Leibniz algebras with derivations [36] and Pre-Lie algebras with derivations [4].

Inspired by the previous works, we would like to investigate the cohomological theory of n-Lie algebras with derivations.

The paper is organized as follows. In Section 2, we introduce the notion of an n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ and its representation (V, φ_V). In Section 3, we consider the cohomology theory of n-LieDer pairs. The relation between the cohomogy of n-LieDer pair and associated LeibDer pair is also characterized. In Section 4, we study $(n-1)$-order deformations of n-LieDer pairs. We also describe the notions of a Nijenhuis operator and an $\mathcal{O}$-operator on n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$. Moreover, we show that (V, φ_V) becomes an n-LieDer pair and $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ is a representation of (V, φ_V). Finally, we discuss central extensions of an n-LieDer pair in terms of the first cohomology group with coefficients in its trivial representation.

Unless otherwise specified, all vector spaces, linear maps, and tensor products are studied over an algebraically closed field $\mathbf{k}$.

2. n-LieDer Pair and Representation

In this section, we introduce the concept of n-LieDer pairs and representations of an n-LieDer pair. An n-LieDer pair is an n-Lie algebra with a derivation, namely, we have the following

Definition 1. *A derivation of an n-Lie algebra $\mathfrak{g}$ is a linear map $\varphi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ satisfying*

$$\varphi_{\mathfrak{g}}[x_1, \cdots, x_n] = \sum_{i=1}^{n} [x_1, \cdots, x_{i-1}, \varphi_{\mathfrak{g}}(x_i), x_{i+1}, \cdots, x_n]$$

for any $x_i \in \mathfrak{g}$. Suppose that $\varphi_{\mathfrak{g}}$ is a derivation of an n-Lie algebra $\mathfrak{g}$. Then we call the datum $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ an n-LieDer pair.

For any n-Lie algebra $\mathfrak{g}$, $L = \wedge^{n-1}\mathfrak{g}$ is a Leibniz algebra with Leibniz bracket $[\ ,\ ]_F$ given by

$$[X, Y]_F = \sum_{i=1}^{n-1} y_1 \wedge y_2 \wedge \cdots \wedge y_{i-1} \wedge [x_1, x_2, \cdots x_{n-1}, y_i] \wedge y_{i+1} \wedge \cdots \wedge y_{n-1}$$

for all $X = x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1}$ and $Y = y_1 \wedge y_2 \wedge \cdots \wedge y_{n-1}$. Furthermore, if $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ is an n-LieDer pair, then (L, φ_L) is a LeibDer pair (see [36]), where

$$\varphi_L = \sum_{i=1}^{n-1} \underbrace{I \otimes I \otimes \cdots \otimes I}_{i-1} \otimes \varphi_{\mathfrak{g}} \otimes I \otimes \cdots \otimes I$$

and I is the identity endomorphism of $\mathfrak{g}$.

A representation of an n-Lie algebra $\mathfrak{g}$ consists of a vector space V together with a linear map $\rho : \wedge^{n-1}\mathfrak{g} \longrightarrow gl(V)$ such that

$$\rho(X)\rho(Y) - \rho(Y)\rho(X) = \rho([X, Y]_F)$$

and

$$\begin{aligned}&\rho(x_1 \wedge \cdots \wedge x_{n-2} \wedge [y_1, \cdots, y_n]) \\ &= \textstyle\sum_{i=1}^{n} (-1)^{n-i} \rho(y_1 \wedge \cdots \wedge y_{i-1} \wedge y_{i+1} \wedge \cdots \wedge y_n) \rho(x_1 \wedge \cdots \wedge x_{n-2} \wedge y_i).\end{aligned}$$

for all $X, Y \in \wedge^{n-1}\mathfrak{g}$ *and* $x_i, y_i \in \mathfrak{g}$.

Next, we give the definition of representations of an n-LieDer pair.

Definition 2. *Let* $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ *be an n-LieDer pair and* (V, ρ) *be a representation of the n-Lie algebra* $\mathfrak{g}$. *Suppose* φ_V *is an endomorphism of* V. *Then* (V, ρ, φ_V) *is called a representation of* $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ *if*

$$\varphi_V \rho(X) = \rho(\varphi_{\mathfrak{g}}(X)) + \rho(X)\varphi_V, \tag{1}$$

for any $X = (x_1, x_2, \cdots, x_{n-1}) \in \wedge^{n-1}\mathfrak{g}$, *where*

$$\rho(\varphi_{\mathfrak{g}}(X)) = \sum_{i=1}^{n-1} \rho(x_1, x_2, \cdots, \varphi_{\mathfrak{g}}(x_i), x_{i+1}, \cdots, x_{n-1}).$$

Suppose (V, φ_V) *is a representation of an n-LieDer pair* $(\mathfrak{g}, \rho_{\mathfrak{g}})$ *and* $V^* := Hom(V, \mathbf{k})$ *is the dual space of* V. *Define* $\rho^* : \wedge^{n-1}\mathfrak{g} \to gl(V^*)$ *and* $\varphi_V^* : V^* \to V^*$ *via*

$$\langle \rho^*(X)u^*, v \rangle = -\langle \rho(X)v, u^* \rangle, \quad \varphi_V^*(u^*)(y) = u^*(\varphi_V(y)) \tag{2}$$

for any $u^* \in V^*$, $v \in V$ *and* $X \in \wedge^{n-1}\mathfrak{g}$ *respectively. Then* V^* *can be endowed with a representation of the n-LieDer pair* $(\mathfrak{g}, \rho_{\mathfrak{g}})$, *as follows.*

Proposition 1. *Let* (V, ρ, φ_V) *be a representation of an n-LieDer pair* $(\mathfrak{g}, \varphi_{\mathfrak{g}})$. *Then* $(V^*, \rho^*, -\varphi_V^*)$ *is also a representation of* $(\mathfrak{g}, \varphi_{\mathfrak{g}})$.

Proof. We only need to check that (1) holds for ρ^* and $-\varphi_V^*$. In fact, for any $u^* \in V^*$, $v \in V$ and $X \in \wedge^{n-1}\mathfrak{g}$, in view of (1) and (2), we have

$$\begin{aligned} & \langle -\varphi_V^* \rho^*(X)u^*, v \rangle + \langle \rho^*(X)\varphi_V^* u^*, v \rangle - \langle \rho^*(\varphi_{\mathfrak{g}}(X))u^*, v \rangle \\ = \ & -\langle \rho^*(X)u^*, \varphi_V v \rangle - \langle \rho(X)v, \varphi_V^* u^* \rangle + \langle \rho(\varphi_{\mathfrak{g}}(X))v, u^* \rangle \\ = \ & \langle u^*, \rho(X)\varphi_V v \rangle - \langle \varphi_V \rho(X)v, u^* \rangle + \langle \rho(\varphi_{\mathfrak{g}}(X))v, u^* \rangle \\ = \ & 0. \end{aligned}$$

It follows that

$$-\varphi_V^* \rho^*(X) = \rho^*(\varphi_{\mathfrak{g}}(X)) - \rho^*(X)\varphi_V^*.$$

Hence $(V^*, \rho^*, -\varphi_V^*)$ is also a representation of $(\mathfrak{g}, \varphi_{\mathfrak{g}})$. □

Example 1. *Let* $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ *be an n-LieDer pair and define linear map* $ad : \wedge^{n-1}\mathfrak{g} \longrightarrow gl(\mathfrak{g})$ *by* $ad(X)(y) = [X, y]$ *for any* $X \in \wedge^{n-1}\mathfrak{g}$, $y \in \mathfrak{g}$. *Then* $(\mathfrak{g}, ad, \varphi_{\mathfrak{g}})$ *is a representation and* $(\mathfrak{g}^*, -ad^*, -\varphi_{\mathfrak{g}}^*)$ *is a dual representation.*

Similar to the trivial extension of a Lie algebra by its representation, we can check the following proposition.

Proposition 2. *Let (V, ρ, φ_V) be a representation of n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$. Given operations $[\cdot, \cdots, \cdot]_\rho : \wedge^n(\mathfrak{g} \oplus V) \longrightarrow \mathfrak{g} \oplus V$ by*

$$[x_1 + v_1, \cdots, x_n + v_n]_\rho = [x_1, \cdots, x_n]_{\mathfrak{g}} + \sum_{i=1}^{n} (-1)^{n-i} \rho(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) v_i$$

and $\varphi_{\mathfrak{g}} + \varphi_V : \mathfrak{g} \oplus V \longrightarrow \mathfrak{g} \oplus V$ by

$$(\varphi_{\mathfrak{g}} + \varphi_V)(x + v) = \varphi_{\mathfrak{g}}(x) + \varphi_V(v).$$

Then $(\mathfrak{g} \oplus V, \varphi_{\mathfrak{g}} + \varphi_V)$ is an n-LieDer pair.

Suppose that A, A' are 3-Lie algebras, and $\rho : \wedge^2 A \longrightarrow gl(A')$ and $\varrho : \wedge^2 A' \longrightarrow gl(A)$ are two linear mappings. Recall that (A, A', ρ, ϱ) is said to be a matched pair of 3-Lie algebras [37] if (A', ρ) is a representation of A and (A, ϱ) is a representation of A' and satisfying the following:

$$\varrho(a_1, a_2)[x_1, x_2, x_3] = [\varrho(a_1, a_2)x_1, x_2, x_3] + [x_1, \varrho(a_1, a_2)x_2, x_3] + [x_1, x_2, \varrho(a_1, a_2)x_3],$$

$$-\varrho(\rho(x_1, x_2)a_1, a_2)x_3 = -\varrho(\rho(x_1, x_3)a_2, a_1)x_2 + \varrho(\rho(x_2, x_3)a_2, a_1)x_1 - [x_1, x_2, \varrho(a_1, a_2)x_3],$$

$$[\varrho(a_1, a_2)x_1, x_2, x_3] = \varrho(a_1, a_2)[x_1, x_2, x_3] + \varrho(\rho(x_1, x_2)a_1, a_2)x_1 + \varrho(a_1, \rho(x_2, x_3)a_2)x_1,$$

$$\rho(x_1, x_2)[a_1, a_2, a_3] = [\rho(x_1, x_2)a_1, a_2, a_3] + [a_1, \rho(x_1, x_2)a_2, a_3] + [a_1, a_2, \rho(x_1, x_2)a_3],$$

$$-\rho(\varrho(a_1, a_2)x_1, x_2)a_3 = -\rho(\varrho(a_1, a_3)x_2, x_1)a_2 + \rho(\varrho(a_2, a_3)x_2, x_1)a_1 - [a_1, a_2, \rho(x_1, x_2)a_3],$$

$$[\rho(x_1, x_2)a_1, a_2, a_3] = \rho(x_1, x_2)[a_1, a_2, a_3] + \rho(\varrho(a_1, a_2)x_1, x_2)a_1 + \rho(x_1, \varrho(a_2, a_3)x_2)a_1$$

for any $x_1, x_2, x_3 \in A$ and $a_1, a_2, a_3 \in A'$. Then, there is a 3-Lie algebra structure on $A \oplus A'$ given by

$$\begin{aligned} [x_1 + a_1, x_2 + a_2, x_3 + a_3] &= [x_1, x_2, x_3] + \varrho(a_1, a_2)x_3 + \varrho(a_3, a_1)x_2 + \varrho(a_2, a_3)x_1 \\ &\quad + [a_1, a_2, a_3] + \rho(x_1, x_2)a_3 + \rho(x_3, x_1)a_2 + \rho(x_2, x_3)a_1. \end{aligned}$$

Then we have the following result.

Proposition 3. *Let (A, φ_A) and $(A', \varphi_{A'})$ be two 3-LieDer pairs such that (A, A', ρ, ϱ) is a matched pair of 3-Lie algebras with $\rho : \wedge^2 A \longrightarrow gl(A')$ and $\varrho : \wedge^2 A' \longrightarrow gl(A)$. Furthermore, if $(A', \rho, \varphi_{A'})$ is a representation of 3-LieDer pair (A, φ_A) and (A, ϱ, φ_A) is a representation of 3-LieDer pair $(A', \varphi_{A'})$. Define*

$$(\varphi_A + \varphi_{A'})(x + a) = \varphi_A(x) + \varphi_{A'}(a)$$

for any $x, y \in A, a, b \in A'$. Then $(A \oplus A', \varphi_A + \varphi_{A'})$ is a 3-LieDer pair with We call $(A, A', \varphi_A, \varphi_{A'}, \rho, \varrho)$ the matched pair of 3-LieDer pairs.

Proof. It is straightforward. □

3. Cohomology of *n*-LieDer Pair

In this section, we will define the cohomology of an n-LieDer pair with coefficients in its representation. For this purpose, let us recall the cohomology of Leibniz algebras and LeibDer pairs in [19,36,38]. Let (V, ρ^L, ρ^R) be a representation of a Leibniz algebra $\mathfrak{l}$, and $C^p_{\text{Leib}}(\mathfrak{l}, V) = \text{Hom}(\mathfrak{l}^{\otimes p}, V)$ for any $p \geq 0$, the p-cochains group. Suppose $d : C^p_{\text{Leib}}(\mathfrak{l}, V) \longrightarrow C^{p+1}_{\text{Leib}}(\mathfrak{l}, V)$ is given by

$$\begin{aligned} & (df)(x_1, \cdots, x_{p+1}) \\ = & \sum_{i=1}^{p}(-1)^{i+1}\rho^L(x_i)f(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{p+1}) + (-1)^{p+1}\rho^R(x_{p+1})f(x_1, \cdots, x_p) \\ & + \sum_{1 \leq i < j \leq p+1} (-1)^i f(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, [x_i, x_j], x_{j+1}, \cdots, x_{p+1}) \end{aligned}$$

for any $x_1, x_2, \cdots, x_{p+1} \in \mathfrak{l}$. Then d is a coboundary operator and cohomological groups of $\mathfrak{l}$ with coefficients in V is determined by the coboundary operator d.

Next recall the definition of the cohomology of LeibDer pairs. Let (ρ, V, φ_V) be a representation of a LeibDer pair $(\mathfrak{l}, \varphi_{\mathfrak{l}})$ and $C^p_{\text{LeibDer}}(\mathfrak{l}, V) = C^p_{\text{Leib}}(\mathfrak{l}, V) \times C^{p-1}_{\text{Leib}}(\mathfrak{l}, V)$. Define a linear map $\delta : C^p_{\text{Leib}}(\mathfrak{l}, V) \longrightarrow C^p_{\text{Leib}}(\mathfrak{l}, V)$ by

$$\begin{aligned} & \delta f_p(x_1 \otimes \cdots \otimes x_p) \\ = & \sum_{k=1}^{p} f_p(x_1 \otimes \cdots \otimes x_{k-1} \otimes \varphi_{\mathfrak{l}}(x_k) \otimes x_{k+1} \otimes \cdots \otimes x_p) \\ & -\varphi_V f_p(x_1 \otimes \cdots \otimes x_p) \end{aligned}$$

for $x_i \in \mathfrak{l}$. Then $\partial_{\mathfrak{l}} : C^p_{\text{LeibDer}}(\mathfrak{l}, V) \longrightarrow C^{p+1}_{\text{LeibDer}}(\mathfrak{l}, V)$, $\partial_{\mathfrak{l}}(f_p, g_{p-1}) \mapsto (\mathrm{d}f_p, \mathrm{d}g_{p-1} + (-1)^p \delta f_p)$, is a coboundary operator.

Finally, let us recall the cohomology of n-Lie algebras. Let $\mathfrak{g}$ be an n-Lie algebra and $L = \wedge^{n-1}\mathfrak{g}$ be the associated Leibniz algebra. Suppose that (ρ, V) is a representation of $\mathfrak{g}$, the space $C^p_{n-\text{Lie}}(\mathfrak{g}, V)$ of p-cochains ($p \geq 0$) is the set of multilinear maps of the form

$$f : \wedge^p L \wedge \mathfrak{g} \longrightarrow V,$$

and the coboundary operator $\mathrm{d} : C^p_{n-\text{Lie}}(\mathfrak{g}, V) \longrightarrow C^{p+1}_{n-\text{Lie}}(\mathfrak{g}, V)$ is as follows:

$$\begin{aligned} & (\mathrm{d}f)(X_1, \cdots, X_{p+1}, z) \\ = & \sum_{1 \leq i < k \leq p+1} (-1)^i f(X_1, \cdots, \hat{X}_i, \cdots, X_{k-1}, [X_i, X_k]_F, X_{k+1}, \cdots, X_{p+1}, z) \\ & + \sum_{i=1}^{p+1}(-1)^i f(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1}, [X_i, z]) + \sum_{i=1}^{p+1}(-1)^{i+1}\rho(X_i)f(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1}, z) \\ & + \sum_{k=1}^{n-1}(-1)^{n+p-k}\rho(x^1_{p+1}, x^2_{p+1}, \cdots, x^{k-1}_{p+1}, x^{k+1}_{p+1}, \cdots, x^{n-1}_{p+1}, z)f(X_1, \cdots, X_p, x^k_{p+1}), \end{aligned}$$

for all $X_i = (x^1_i, x^2_i, \cdots, x^{n-1}_i) \in L$ and $z \in \mathfrak{g}$.

Based on the previous cohomologies, we introduce a cohomology of an n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$. Let (V, ρ, φ_V) be a representation of n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ and

$$C^p_{n-\text{LieDer}}(\mathfrak{g}, V) := C^p_{n-\text{Lie}}(\mathfrak{g}, V) \times C^{p-1}_{n-\text{Lie}}(\mathfrak{g}, V),$$

which is called the p-cochain group. Define a map $\delta : C^p_{n-\mathrm{Lie}}(\mathfrak{g}, V) \longrightarrow C^p_{n-\mathrm{Lie}}(\mathfrak{g}, V)$ by

$$\begin{aligned} & \delta f_p(X_1 \otimes \cdots \otimes X_p \otimes z) \\ = & \sum_{k=1}^{p} \sum_{i=1}^{n-1} f_p(X_1 \otimes \cdots \otimes X_{k-1} \otimes \varphi_{\mathfrak{g}}(x_k^i) \otimes x_k^{i+1} \otimes \cdots \otimes x_k^{n-1} \otimes X_{k+1} \cdots \otimes X_p \otimes z) \\ & -\varphi_V f_p(X_1 \otimes \cdots \otimes X_p \otimes z). \end{aligned}$$

Then we have the following.

Theorem 1. *The map $\partial_{\mathfrak{g}} : C^p_{n-LieDer}(\mathfrak{g}, V) \to C^{p+1}_{n-LieDer}(\mathfrak{g}, V)$ given by*

$$\partial_{\mathfrak{g}}(f_p, g_{p-1}) \mapsto (df_p, dg_{p-1} + (-1)^{p+1}\delta f_p),$$

is a coboundary operator, that is, $\partial_{\mathfrak{g}} \cdot \partial_{\mathfrak{g}} = 0$.

Proof. By direct calculation, we get the required result. □

By Theorem 1, we can obtain cohomological groups of $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ with the coefficients in (ρ, V, φ_V). It is well-know that any n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ associates a LeibDer pair $(L = \wedge^{n-1}\mathfrak{g}, \varphi_L)$. Then it has the cohomology of the LeibDer pair $(L = \wedge^{n-1}\mathfrak{g}, \varphi_L)$. Are there some relations between these two cohomologies? Suppose that $C^p_{n-\mathrm{LieDer}}(\mathfrak{g}, \mathfrak{g})$ (resp. $C^p_{\mathrm{LeibDer}}(L, L)$) is the set of p-cochains of n-LieDer pair (resp. LeibDer pair). Define $\Theta : C^p_{n-\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}) \to C^{p+1}_{\mathrm{Leib}}(L, L)$ as follows. For $p = 0$, $f_0 \in C^0_{n-\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$ and $X_1 = x_1^1 \otimes x_1^2 \otimes \cdots \otimes x_1^{n-1} \in L$,

$$\Theta(f_0)(X_1) = \sum_{i=0}^{n-1} x_1^1 \otimes x_1^2 \otimes \cdots \otimes x_1^{i-1} \otimes f_0(x_1^i) \otimes \cdots \otimes x_1^{n-1}.$$

If $p > 0$,$f_p \in C^p_{n-\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$ and $X_i = x_i^1 \otimes x_i^2 \otimes \cdots \otimes x_i^{n-1} \in L$,

$$\Theta(f_p)(X_1 \otimes X_2 \otimes \cdots \otimes X_{p+1}) = \sum_{i=1}^{n-1} x_{p+1}^1 \otimes x_{p+1}^2 \otimes \cdots \otimes x_{p+1}^{i-1} \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes x_{p+1}^i) \otimes \cdots \otimes x_{p+1}^{n-1}.$$

With these notations, we have an important result. However, let us prove the following result first.

Theorem 2. *The linear map $\bar{\Theta} : C^p_{n-LieDer}(\mathfrak{g}, \mathfrak{g}) \longrightarrow C^{p+1}_{LeibDer}(L, L)$ given by*

$$\bar{\Theta}(f, g) = (\Theta f, \Theta g) \textit{ for any } (f, g) \in C^p_{n-LieDer}(\mathfrak{g}, \mathfrak{g})$$

is a cochain map, that is, $\partial_{\mathfrak{g}}\bar{\Theta} = \bar{\Theta}\partial_L$.

Proof. For any $(f_p, g_{p-1}) \in C^p_{n-\mathrm{LieDer}}(\mathfrak{g}, \mathfrak{g})$, we get

$$\partial_{\mathfrak{g}}\bar{\Theta}(f_p, g_{p-1}) = \partial_{\mathfrak{g}}(\Theta f_p, \Theta g_{p-1}) = (d\Theta f_p, d\Theta g_{p-1} + (-1)^{p+1}\delta\Theta f_p)$$

and

$$\bar{\Theta}\partial_L(f_p, g_{p-1}) = (\Theta df_p, \Theta dg_{p-1} + (-1)^{p+1}\Theta\delta f_p).$$

See ref. [13] (Theorem 3), we only need to check that $\delta\Theta = \Theta\delta$. In fact, for the case of $p = 0$, it is clear. When $p \geq 1$, for any $X_1, X_2, \cdots, X_{p+1} \in L$, $f_p \in C^p_{n-\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$,

$$
\begin{aligned}
&\delta\Theta(f_p)(X_1 \otimes X_2 \otimes \cdots \otimes X_{p+1}) \\
= \; & \sum_{i=1}^{p+1}\sum_{j=1}^{n-1} \Theta(f_p)(X_1 \otimes X_2 \otimes \cdots \otimes X_{i-1} \otimes x_i^1 \otimes x_i^2 \otimes \cdots \otimes \varphi_{\mathfrak{g}}(x_i^j) \otimes \cdots \otimes x_i^{n-1} \otimes X_{i+1} \\
& \otimes \cdots \otimes X_{p+1}) - \varphi_L\Theta(f_p)(X_1 \otimes X_2 \otimes \cdots \otimes X_{p+1}) \\
= \; & \sum_{i=1}^{p}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{k-1} \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_{i-1} \otimes x_i^1 \otimes x_i^2 \otimes \cdots \otimes \varphi_{\mathfrak{g}}(x_i^j) \\
& \otimes \cdots \otimes x_i^{n-1} \otimes X_{i+1} \otimes \cdots \otimes X_p \otimes x_{p+1}^k) \otimes x_{p+1}^{k+1} \otimes \cdots \otimes x_{p+1}^{n-1} \\
& + \sum_{j=1}^{n-1}\sum_{k=1, k\neq j}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{k-1} \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes x_{p+1}^k \otimes \cdots \otimes \varphi_{\mathfrak{g}}(x_{p+1}^j) \\
& \otimes \cdots \otimes x_{p+1}^{n-1}) + \sum_{j=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{j-1} \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes \varphi_{\mathfrak{g}}(x_{p+1}^j) \\
& \otimes x_{p+1}^{j+1} \otimes \cdots \otimes x_{p+1}^{n-1}) - \sum_{k=1}^{n-1}\sum_{j=1, j\neq k}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{j-1} \otimes \varphi_{\mathfrak{g}}(x_{p+1}^j) \otimes x_{p+1}^{j+1} \otimes \cdots \otimes x_{p+1}^{k-1} \\
& \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes x_{p+1}^k) \otimes x_{p+1}^{k+1} \otimes \cdots \otimes x_{p+1}^{n-1} - \sum_{k=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{k-1} \\
& \otimes \varphi_{\mathfrak{g}} f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes x_{p+1}^k) \otimes x_{p+1}^{k+1} \otimes \cdots \otimes x_{p+1}^{n-1} \\
= \; & \sum_{i=1}^{p}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{k-1} \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_{i-1} \otimes x_i^1 \otimes x_i^2 \otimes \cdots \otimes \varphi_{\mathfrak{g}}(x_i^j) \\
& \otimes \cdots \otimes x_i^{n-1} \otimes X_{i+1} \otimes \cdots \otimes X_p \otimes x_{p+1}^k) \otimes x_{p+1}^{k+1} \otimes \cdots \otimes x_{p+1}^{n-1} \\
& + \sum_{j=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{j-1} \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes \varphi_{\mathfrak{g}}(x_{p+1}^j) \otimes x_{p+1}^{j+1} \otimes \cdots \otimes x_{p+1}^{n-1} \\
& - \sum_{k=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{k-1} \otimes \varphi_{\mathfrak{g}} f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes x_{p+1}^k) \otimes x_{p+1}^{k+1} \otimes \cdots \otimes x_{p+1}^{n-1}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\Theta\delta(f_p)(X_1 \otimes X_2 \otimes \cdots \otimes X_{p+1}) \\
= \; & \sum_{i=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{i-1} \otimes \delta(f_p)(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes x_{p+1}^i) \otimes x_{p+1}^{i+1} \otimes \cdots \otimes x_{p+1}^{n-1} \\
= \; & \sum_{i=1}^{n-1}\sum_{k=1}^{p}\sum_{j=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{i-1} \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_{k-1} \otimes x_k^1 \otimes x_k^2 \otimes \cdots \otimes \varphi_{\mathfrak{g}}(x_k^j) \\
& \otimes x_k^{j+1} \otimes \cdots \otimes x_k^{n-1} \otimes X_{k+1} \otimes \cdots \otimes X_p \otimes x_{p+1}^i) \otimes x_{p+1}^{i+1} \otimes \cdots \otimes x_p^{n-1} \\
& + \sum_{i=1}^{n-1} x_{p+1}^1 \otimes \cdots \otimes x_{p+1}^{i-1} \otimes f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes \varphi_{\mathfrak{g}}(x_{p+1}^i)) \otimes x_{p+1}^{i+1} \otimes \cdots \otimes x_p^{n-1} \\
& - \sum_{i=1}^{n-1} x_p^1 \otimes \cdots \otimes x_{p+1}^{i-1} \otimes \varphi_{\mathfrak{g}} f_p(X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes x_{p+1}^i) \otimes x_{p+1}^{i+1} \otimes \cdots \otimes x_{p+1}^{n-1}.
\end{aligned}
$$

Hence, $\Theta\delta = \delta\Theta$ and thus $\bar{\Theta}\partial_L = \partial_{\mathfrak{g}}\bar{\Theta}$. □

The following corollary gives another proof of Theorem 1.

Corollary 1. *If* $\partial_L^2 = 0$, *then* $\partial_{\mathfrak{g}}^2 = 0$.

Proof. In view of $\bar{\Theta}\partial_L = \partial_{\mathfrak{g}}\bar{\Theta}$, we have $\bar{\Theta}\partial_{\mathfrak{g}}^2 = \partial_L\bar{\Theta}\partial_{\mathfrak{g}} = \partial_L\partial_L\bar{\Theta} = 0$. □

Let $L^p(\mathfrak{g}) = \mathrm{Hom}(\wedge^{p+1}L \otimes \mathfrak{g}, \mathfrak{g})$, $M^p(L) = \mathrm{Hom}(L^{\otimes(p+1)}, L)$, $L^p_{n-\mathrm{LieDer}}(\mathfrak{g}) = L^p(\mathfrak{g}) \times L^{p-1}(\mathfrak{g})$, and $M^p_{\mathrm{LeibDer}}(L) = M^p(L) \times M^{p-1}(L)$. In the light of [13], we know that both $M(L) = \oplus_{p=0}^{\infty} M^p(L)$ and $L(\mathfrak{g}) = \oplus_{p=0}^{\infty} L^p(\mathfrak{g})$ have graded Lie algebra structures, we denote them by $[\ ,\ ]^L$ and $[\ ,\ ]^{nL}$ respectively. In view of [36], $M_{\mathrm{LeibDer}}(L) = \oplus_{p=0}^{\infty} M^p_{\mathrm{LeibDer}}(L)$ is a graded Lie algebra with the Lie algebra structure

$$[[\ ,\ ]]^L : M^p_{\mathrm{LeibDer}}(L) \otimes M^q_{\mathrm{LeibDer}}(L) \longrightarrow M^{p+q}_{\mathrm{LeibDer}}(L)$$

given by

$$[[(f_{p+1}, g_p), (f_{q+1}, g_q)]]^L = ([f_{p+1}, f_{q+1}]^L, [g_p, f_{q+1}]^L + (-1)^{p+1}[f_{p+1}, g_q]^L).$$

Define the linear map

$$[[\ ,\ ]]^{nL} : L^p_{n-\mathrm{LieDer}}(\mathfrak{g}) \otimes L^q_{n-\mathrm{LieDer}}(\mathfrak{g}) \longrightarrow L^{p+q}_{n-\mathrm{LieDer}}(L)$$

by

$$[[(f_{p+1}, g_p), (f_{q+1}, g_q)]]^{nL} = ([f_{p+1}, f_{q+1}]^{nL}, [g_p, f_{q+1}]^{nL} + (-1)^{p+1}[f_{p+1}, g_q]^{nL}).$$

Then we have the following.

Proposition 4. $[[\ ,\ ]]^L(\bar{\Theta} \otimes \bar{\Theta}) = \bar{\Theta}[[\ ,\ ]]^{nL}$.

Proof. It can be induced directly from Corollary 1 and [13] (Lemma 1). □

From the previous proposition, we obtain the following theorem immediately.

Theorem 3. *$L_{n-LieDer}(\mathfrak{g}) = (\oplus_{p=0}^{\infty} L^p_{n-LieDer}(\mathfrak{g}), [[\ ,\ ]]^{nL})$ is a graded Lie algebra. Its Maurer–Cartan elements are n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$.*

Proof. According to Proposition 4, $(L_{n-\mathrm{LieDer}}(\mathfrak{g}), [[\ ,\ ]]^{nL})$ is a graded Lie algebra.

For any $(\pi, \varphi) \in L^1_{n-\mathrm{LieDer}}(\mathfrak{g})$,

$$[[(\pi, \varphi), (\pi, \varphi)]]^{nL} = ([\pi, \pi]^{nL}, [\pi, \varphi]^{nL} + [\pi, \varphi]^{nL}).$$

So (π, φ) is a Maurer–Cartan element if and only if $[\pi, \pi]^{nL} = 0$ and $2[\pi, \varphi]^{nL} = 0$. In the light of [13], $[\pi, \pi]^{nL} = 0$ if and only if $(\mathfrak{g}, \pi)$ is an n-Lie algebra.

At the same time, for any $X \otimes z \in L \otimes \mathfrak{g}$, suppose $X = x_1 \otimes \cdots \otimes x_{n-1}$, we get

$$\begin{aligned}
[\pi, \varphi]^{nL}(X \otimes z) &= [\pi, \varphi]^{nL}(x_1 \otimes \cdots \otimes x_{n-1} \otimes z) \\
&= \sum_{i=1}^{n} \pi(x_1 \otimes \cdots \otimes \varphi(x_i) \otimes \cdots \otimes x_{n-1} \otimes z) + \pi(x_1 \otimes \cdots \otimes x_{n-1} \otimes \varphi(z)) \\
&\quad - \varphi\pi(x_1 \otimes \cdots \otimes x_{n-1} \otimes z).
\end{aligned}$$

It follows that (π, φ) is a Maurer–Cartan element if and only if (π, φ) is an n-LieDer pair. □

Let $(\mathfrak{g}, \pi, \varphi_{\mathfrak{g}})$ be an n-LieDer pair. Define the linear map

$$\partial_{(\pi, \varphi_{\mathfrak{g}})} : L^p_{n-\mathrm{LieDer}}(\mathfrak{g}) \longrightarrow L^{p+1}_{n-\mathrm{LieDer}}(\mathfrak{g})$$

by

$$\partial_{(\pi,\varphi_{\mathfrak{g}})}((f_{p+1}, g_p)) = [[(\pi, \varphi_{\mathfrak{g}}), (f_{p+1}, g_p)]]^{nL},$$

then $\partial_{(\pi,\varphi_{\mathfrak{g}})}\partial_{(\pi,\varphi_{\mathfrak{g}})} = 0$. Hence, we get the following.

Proposition 5. *$(L_{n-LieDer}(\mathfrak{g}), [[\ ,\ ]]^{nL}, \partial_{(\pi,\varphi_{\mathfrak{g}})})$ is a differential graded Lie algebra.*

4. Deformation of *n*-LieDer Pairs

In the next two sections, we give some applications of the cohomology of an n-LieDer pair. In this section we use it to study the deformation of n-Lie Der pairs. First of all, let us introduce the deformations of an n-LieDer pair.

Let $(\mathfrak{g}, [\ ,\cdots,\ ]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ be an n-LieDer pair. Denote $\pi_0(x_1, x_2, \cdots, x_n) = [x_1, x_2, \cdots, x_n]_{\mathfrak{g}}$. Let $\pi_i : \wedge^n\mathfrak{g} \to \mathfrak{g}$ be skew-symmetric multilinear maps and linear maps $\varphi_i : \mathfrak{g} \to \mathfrak{g}$ for any $0 \leq i \leq n-1$. Consider the space $\mathfrak{g}[[t]]$ of formal power series in t with coefficients in $\mathfrak{g}$ and a family of linear maps:

$$\pi_t(x_1, x_2, \cdots, x_n) = \sum_{i=0}^{n-1} t^n \pi_i(x_1, x_2, \cdots, x_n), \tag{3}$$

and

$$\varphi_t(x) = \sum_{i=0}^{n-1} t^n \varphi_i(x), \tag{4}$$

where $\varphi_0 = \varphi_{\mathfrak{g}}$.

If all $(\mathfrak{g}[[t]], \pi_t, \varphi_t)$ are n-LieDer pairs, we say that (π_i, φ_i) $(i = 0, 1, \cdots, n-1)$ generate an $(n-1)$-order deformation of the n-LieDer pair $(\mathfrak{g}, [\ ,\cdots,\ ]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$. We also denote by $[x_1, x_2, \cdots, x_n]_t = \pi_t(x_1, x_2, \cdots, x_n)$. For any $X, Y \in \wedge^{n-1}\mathfrak{g}$, in the next proposition, we use $\pi_j(X, \cdot) \circ Y \in \wedge^{n-1}\mathfrak{g}$ to denote the element

$$\pi_j(X, \cdot) \circ Y = \sum_{i=1}^{n-1} y_1 \wedge \cdots \wedge y_{i-1} \wedge \pi_j(X, y_i) \wedge y_{i+1} \wedge \cdots \wedge y_{n-1}.$$

Proposition 6. *(π_i, φ_i) $(i = 0, 1, \cdots, n-1)$ generate an $(n-1)$-order deformation of the n-LieDer pair $(\mathfrak{g}, [\ ,\cdots,\ ]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ if and only if the following holds for any $i, j, k = 0, 1, \cdots, n-1$ and $\sum_{i+j=k} \pi_i\pi_j = 0$, and*

$$\sum_{i+j=k} (\varphi_i\pi_j - \pi_j(\sum_{l=1}^{n} I \otimes \cdots \otimes \varphi_l \otimes \cdots \otimes I)) = 0, \tag{5}$$

where $\pi_i\pi_j : \wedge^{n-1}\mathfrak{g} \otimes \wedge^{n-1}\mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$ is given by

$$\pi_i\pi_j(X, Y, z) = \pi_i(\pi_j(X, \cdot) \circ Y, z) - \pi_i(X, \pi_j(Y, z)) + \pi_i(Y, \pi_j(X, z)). \tag{6}$$

Proof. $(\mathfrak{g}, \pi_t, \varphi_t)$ is an n-LieDer pair if and only if

$$\pi_t(X, \pi_t(Y, z)) = \pi_t(\pi_t(X, \cdot) \circ Y, z) + \pi_t(Y, \pi_t(X, z))$$

and

$$\varphi_t\pi_t = \sum_{i=1}^{n} \pi_t(I \otimes \cdots \otimes \varphi_t \otimes I \otimes \cdots \otimes I). \tag{7}$$

In the light of [12] (Proposition 1]), we only need to check that (7) is equivalent to (5). Combining (4) and (7), we achieve that (5) holds. Hence, we get the results. □

Corollary 2. *If (π_i, φ_i) $(i = 0, 1, \cdots, n-1)$ generate an $(n-1)$-order deformation of the n-LieDer pair $(\mathfrak{g}, [\,, \cdots, \,]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$, then (π_1, φ_1) is a one-cocycle of the n-LieDer pair $(\mathfrak{g}, [\,, \cdots, \,]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ with the coefficients in the adjoint representation $(\mathfrak{g}, ad, \varphi_{\mathfrak{g}})$.*

Proof. For any $(\pi_1, \varphi_1) \in C^1_{n-\mathrm{LieDer}}(\mathfrak{g}, \mathfrak{g}) = \mathrm{Hom}(L \otimes \mathfrak{g}, \mathfrak{g}) \times \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$,

$$\partial_{\mathfrak{g}}(\pi_1, \varphi_1) = (\mathrm{d}\pi_1, \mathrm{d}\varphi_1 + \delta\pi_1).$$

For $k = 1$, (6) is equivalent to $\mathrm{d}\pi_1 = 0$, and $\mathrm{d}\varphi_1 + \delta\pi_1 = 0$ is equivalent to (5). So the conclusion holds. □

Definition 3. *An $(n-1)$-order deformation of the n-LieDer pair $(\mathfrak{g}, [\,, \cdots, \,]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ is said to be trivial if there is a linear map $N : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $T_t = I + tN$ $(\forall\ t)$ satisfies*

$$T_t \varphi_{\mathfrak{g}} = \varphi_{\mathfrak{g}} T_t$$

and

$$T_t[x_1, x_2, \cdots, x_n]_t = [T_t x_1, T_t x_2, \cdots, T_t x_n]_{\mathfrak{g}}.$$

Similarly to the case of n-Lie algebra [12], we can define the Nijenhuis operator of an n-LieDer pair.

Definition 4. *Let $(\mathfrak{g}, [\,, \cdots, \,]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ be an n-LieDer pair. A linear map $N : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a Nijenhuis operator if the following holds:*

$$N\varphi_{\mathfrak{g}} = \varphi_{\mathfrak{g}} N$$

and

$$[Nx_1, Nx_2, \cdots, Nx_n]_{\mathfrak{g}} = N([x_1, x_2, \cdots, x_n]_N^{n-1})$$

for any $x_1, x_2, \cdots, x_n \in \mathfrak{g}$, where

$$[x_1, x_2, \cdots, x_n]_N^{n-1} = \sum_{i_1 < i_2 < \cdots < i_{n-1}} [\cdots, Nx_{i_1}, \cdots, Nx_{i_{n-1}}, \cdots] - N[x_1, x_2, \cdots, x_n]_N^{n-2}.$$

Definition 5. *Let $(\mathfrak{g}, [\,, \cdots, \,]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ be an n-LieDer pair and (V, ρ, φ_V) be its representation. A linear map $T : V \longrightarrow \mathfrak{g}$ is called an $\mathcal{O}$-operator if it satisfies:*

$$T\varphi_V = \varphi_{\mathfrak{g}} T$$

and

$$[Tv_1, Tv_2, \cdots, Tv_n]_{\mathfrak{g}} = T(\sum_{i=1}^{n} (-1)^{n-i} \rho(Tv_1, \cdots, Tv_{i-1}, Tv_{i+1}, \cdots, Tv_n)v_i)$$

for any $v_1, v_2, \cdots, v_n \in V$.

Using the above concepts, we can define some new n-LieDer pairs.

Proposition 7. *Let (V, ρ, φ_V) be a representation of n-LieDer pair $(\mathfrak{g}, [\,, \cdots, \,]_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ and $T : V \to \mathfrak{g}$ be an $\mathcal{O}$-operator. Then $(V, [\cdots, \,]_T, \varphi_V)$ is an n-LieDer pair, where*

$$[v_1, \cdots, v_n]_T = \sum_{i=1}^{n} (-1)^{n-i} \rho(Tv_1, \cdots, Tv_{i-1}, Tv_{i+1}, \cdots, Tv_n)v_i. \quad (8)$$

Proof. We only need to check that φ_V is a derivation of V. In fact, according to (1) and (8), we have

$$\begin{aligned}
& \varphi_V[u_1,\cdots,u_n]_T \\
= & \varphi_V \sum_{i=1}^{n}(-1)^{n-i}\rho(Tu_1,\cdots,Tu_{i-1},Tu_{i+1},\cdots,Tu_n)u_i \\
= & \sum_{i=1}^{n}(-1)^{n-i}\varphi_V\rho(Tu_1,\cdots,Tu_{i-1},Tu_{i+1},\cdots,Tu_n)u_i \\
= & \sum_{i=1}^{n}(-1)^{n-i}(\rho(\sum_{k=1,k\neq i}^{n} Tu_1,\cdots,Tu_{k-1},\varphi_{\mathfrak{g}}Tu_k,Tu_{k+1},\cdots,Tu_n)u_i \\
& +\rho(Tu_1,\cdots,Tu_{i-1},Tu_{i+1},\cdots,Tu_n)\varphi_V(u_i)),
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=1}^{n}[u_1,\cdots,\varphi_V(u_i),\cdots,u_n]_T \\
= & \sum_{i=1}^{n}(\sum_{k=1,k\neq i}^{n}(-1)^{n-k}\rho(Tu_1,\cdots,Tu_{i-1},T\varphi_V(u_i),Tu_{i+1},\cdots,Tu_n)u_k \\
& +(-1)^{n-i}\rho(Tu_1,\cdots,Tu_{i-1},Tu_{i+1},\cdots,Tu_n)\varphi_V(u_i)).
\end{aligned}$$

Therefore,

$$\varphi_V[u_1,\cdots,u_n]_T = \sum_{i=1}^{n}[u_1,\cdots,\varphi_V(u_i),\cdots,u_n]_T,$$

that is, φ_V is a derivation of V. □

Similarly to the case of an n-Lie algebra [12], we have the following.

Proposition 8. *Let (V,ρ,φ_V) a representation of an n-LieDer pair $(\mathfrak{g},[\ ,\cdots,\ ]_{\mathfrak{g}},\varphi_{\mathfrak{g}})$. Then, a linear map $T : V \to \mathfrak{g}$ is an $\mathcal{O}$-operator operator if and only if $\tilde{T} : \mathfrak{g}\oplus V \to \mathfrak{g}\oplus V$ with $\tilde{T}(x+u) = Tu$ is a Nijenhuis operator on semidirect product $(\mathfrak{g}\oplus V,\varphi_{\mathfrak{g}}+\varphi_V)$.*

Let $\varrho_T : \wedge^{n-1}V \to gl(\mathfrak{g})$ be a linear map given by

$$\begin{aligned}
& \varrho_T(u_1,\cdots,u_{n-1})(x) \\
= & [Tu_1,\cdots,Tu_{n-1},x] - T\sum_{i=1}^{n-1}(-1)^{n-i}\rho(Tu_1,\cdots,Tu_{i-1},Tu_{i+1},\cdots,Tu_{n-1},x)u_i. \qquad (9)
\end{aligned}$$

Then $(\mathfrak{g},\varrho_T)$ is a representation of $(V,[\cdots,\ ]_T)$ by Proposition 8. Furthermore, we have the following result.

Proposition 9. *Let $T : V \to \mathfrak{g}$ be an $\mathcal{O}$-operator of n-LieDer pair $(\mathfrak{g},\varphi_{\mathfrak{g}})$ with representation (V,ρ,φ_V). Then $(\mathfrak{g},\varrho_T,\varphi_{\mathfrak{g}})$ is a representation of n-LieDer pair $(V,[\ ,\cdots,\ ]_T,\varphi_V)$.*

Proof. In view of (1) and (9), for any $x \in \mathfrak{g}$ and $u_1, u_2, \cdots,, u_{n-1} \in V$, we have

$$\begin{aligned}
& \varphi_{\mathfrak{g}} \varrho_T(u_1, u_2, \cdots,, u_{n-1})x \\
= \ & \varphi_{\mathfrak{g}}([Tu_1, \cdots, Tu_{n-1}, x] - T\sum_{i=1}^{n-1}(-1)^{n-i}\rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, x)u_i) \\
= \ & \sum_{i=1}^{n-1}[Tu_1, \cdots, Tu_{i-1}, \varphi_{\mathfrak{g}} Tu_i, Tu_{i+1}, \cdots, Tu_{n-1}, x] + [Tu_1, \cdots, Tu_{n-1}, \varphi_{\mathfrak{g}}(x)] \\
& -T\sum_{i=1}^{n-1}(-1)^{n-i}\varphi_V \rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, x)u_i \\
= \ & -\sum_{i=1}^{n-1}\sum_{k=1,k\neq i}^{n-1}(-1)^{n-i}T\rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, \varphi_{\mathfrak{g}} T(u_k), \cdots, Tu_{n-1}, x)u_i \\
& -\sum_{i=1}^{n-1}(-1)^{n-i}T\rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, \varphi_{\mathfrak{g}} x)u_i \\
& -\sum_{i=1}^{n-1}(-1)^{n-i}T\rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, x)\varphi_V(u_i)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=1}^{n-1}\varrho_T(u_1, u_2, \cdots, \varphi_V(u_i), u_{i+1}, \cdots, u_{n-1})x + \varrho_T(u_1, u_2, \cdots, u_{n-1})\varphi_{\mathfrak{g}} x \\
= \ & \sum_{i=1}^{n-1}[Tu_1, \cdots, Tu_{i-1}, T\varphi_V(u_i), Tu_{i+1}, \cdots, Tu_{n-1}, x] \\
& -\sum_{i=1}^{n-1}\sum_{k=1,k\neq i}^{n-1}(-1)^{n-k}T\rho(Tu_1, \cdots, Tu_{i-1}, T\varphi_V(u_i), Tu_{i+1}, \cdots, Tu_{n-1}, x)u_k \\
& -\sum_{i=1}^{n-1}(-1)^{n-i}T\rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, x)\varphi_V(u_i) \\
& +[Tu_1, \cdots, Tu_{n-1}, \varphi_{\mathfrak{g}} x] - \sum_{i=1}^{n-1}(-1)^{n-i}T\rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, \varphi_{\mathfrak{g}} x)u_i.
\end{aligned}$$

Hence,

$$\varphi_{\mathfrak{g}} \varrho_T(u_1, \cdots,, u_{n-1}) = \sum_{i=1}^{n-1}\varrho_T(u_1, u_2, \cdots, \varphi_V(u_i), u_{i+1}, \cdots, u_{n-1}) + \varrho_T(u_1, u_2, \cdots, u_{n-1})\varphi_{\mathfrak{g}}.$$

□

Finally, in this section, let us study the cohomology of the new n-LieDer pair $(V, [\ ,\cdots\cdot,\ ]_T, \varphi_V)$ with coefficients in the representation $(\mathfrak{g}, \varrho_T, \varphi_{\mathfrak{g}})$. Suppose that $L(V) = \wedge^{n-1}V$ is the associated Leibniz algebra. Then the space $C^p_{n-\mathrm{Lie}}(V, \mathfrak{g})$ of p-cochains ($p \geq 0$) is the set of multilinear maps of the form $f : \wedge^p L(V) \wedge V \longrightarrow \mathfrak{g}$, and the coboundary operator $d : C^p_{n-\mathrm{Lie}}(V, \mathfrak{g}) \longrightarrow C^{p+1}_{n-\mathrm{Lie}}(V, \mathfrak{g})$ is given by:

$$
\begin{aligned}
& (\mathrm{d}f)(U_1,\cdots,U_{p+1},w)\\
= & \sum_{1\le i<k\le p+1}(-1)^i f(U_1,\cdots,\hat{U}_i,\cdots,U_{k-1},[U_i,U_k]_F,U_{k+1},\cdots,U_{p+1},w)\\
& +\sum_{i=1}^{p+1}(-1)^i f(U_1,\cdots,\hat{U}_i,\cdots,U_{p+1},[U_i,w]_T)+\sum_{i=1}^{p+1}(-1)^{i+1}\varrho_T(U_i)f(U_1,\cdots,\hat{U}_i,\cdots,U_{p+1},w)\\
& +\sum_{k=1}^{n-1}(-1)^{n+p-k}\varrho_T(u^1_{p+1},u^2_{p+1},\cdots,u^{k-1}_{p+1},u^{k+1}_{p+1},\cdots,u^{n-1}_{p+1},w)f(U_1,\cdots,U_p,u^k_{p+1}),
\end{aligned}
$$

for all $U_i=(u_i^1,u_i^2,\cdots,u_i^{n-1})\in L(V)$ and $w\in V$.

Define a map $\delta: C^p_{n-\mathrm{Lie}}(V,\mathfrak{g})\longrightarrow C^p_{n-\mathrm{Lie}}(V,\mathfrak{g})$ by

$$
\begin{aligned}
& \delta\phi_p(U_1\otimes\cdots\otimes U_p\otimes w)\\
= & \sum_{k=1}^{p}\sum_{i=1}^{n-1}\phi_p(U_1\otimes\cdots\otimes U_{k-1}\otimes\varphi_V(u_k^i)\otimes u_k^{i+1}\otimes\cdots\otimes u_k^{n-1}\otimes U_{k+1}\cdots\otimes U_p\otimes w)\\
& -\varphi_{\mathfrak{g}}\phi_p(U_1\otimes\cdots\otimes U_p\otimes w).
\end{aligned}
$$

Now we are ready to define cohomology of the n-LieDer pair $(V,[\ ,\cdots,\ ]_T,\varphi_V)$ with coefficients in the representation $(\mathfrak{g},\varrho_T,\varphi_{\mathfrak{g}})$. Denote the set of p-cochains ($p\ge 0$) by $C^p_{n-\mathrm{LieDer}}(V,\mathfrak{g}) := C^p_{n-\mathrm{Lie}}(V,\mathfrak{g})\times C^{p-1}_{n-\mathrm{Lie}}(V,\mathfrak{g})$, and the coboundary operator

$$\partial_V: C^p_{n-\mathrm{LieDer}}(V,\mathfrak{g})\longrightarrow C^{p+1}_{n-\mathrm{LieDer}}(V,\mathfrak{g})$$

is given by

$$\partial_V(\phi_p,\psi_{p-1})\mapsto(\mathrm{d}\phi_p,\mathrm{d}\psi_{p-1}+(-1)^{p+1}\delta\phi_p).$$

Denote by $\mathcal{H}^*_{n-Lie}(V,\mathfrak{g})$ the cohomology group of this cochain complex, which is called the cohomology of the n-LieDer pair $(V,[\ ,\cdots,\ ]_T,\varphi_V)$ with coefficients in the representation $(\mathfrak{g},\varrho_T,\varphi_{\mathfrak{g}})$.

We calculate the 0-cocycle.

For any $\phi\in\mathrm{Hom}(V,\mathfrak{g})$, $\partial_V(\phi)=(d\phi,-\delta\phi)$. By direct computation,

$$
\begin{aligned}
& d(\phi)(u_1,\cdots,u_{n-1},w)\\
= & -\phi([u_1,\cdots,u_{n-1},w]_T)+\varrho_T(u_1,\cdots,u_{n-1})\phi(w)\\
& +\sum_{k=1}^{n-1}(-1)^{n-k}\varrho_T(u_1,\cdots u_{k-1},u_{k+1},\cdots,u_{n-1},w)\phi(u_k)\\
= & -\phi([u_1,\cdots,u_{n-1},w]_T)+[Tu_1,\cdots,Tu_{n-1},\phi(w)]\\
& -T\sum_{i=1}^{n-1}(-1)^{n-i}\rho(Tu_1,\cdots,Tu_{i-1},Tu_{i+1},\cdots,Tu_{n-1},\phi(w))u_i\\
& +\sum_{k=1}^{n-1}(-1)^{n-k}([Tu_1,\cdots,Tu_{i-1},Tu_{i+1},\cdots,Tu_{n-1},Tw,\phi(u_k)]\\
& -T\sum_{i=1}^{n-1}(-1)^{n-i}\rho(Tu_1,\cdots,Tu_{i-1},Tu_{i+1},\cdots,Tu_{n-1},Tw,\phi(u_k))u_i)
\end{aligned}
$$

and

$$\delta\phi(w)=\phi\varphi_V(w)-\varphi_{\mathfrak{g}}\phi(w)$$

for any $u_i,w\in V$.

It follows that $\phi \in \mathrm{Hom}(V, \mathfrak{g})$ is a 0-cocycle if and only

$$\begin{aligned} & -\phi([u_1, \cdots, u_{n-1}, w]_T) + [Tu_1, \cdots, Tu_{n-1}, \phi(w)] \\ & -T\sum_{i=1}^{n-1}(-1)^{n-i}\rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, \phi(w))u_i \\ & +\sum_{k=1}^{n-1}(-1)^{n-k}([Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, Tw, \phi(u_k)] \\ & -T\sum_{i=1}^{n-1}(-1)^{n-i}\rho(Tu_1, \cdots, Tu_{i-1}, Tu_{i+1}, \cdots, Tu_{n-1}, Tw, \phi(u_k))u_i) \\ = \quad & 0 \end{aligned}$$

and

$$\phi\varphi_V = \varphi_{\mathfrak{g}}\phi.$$

Clearly, $\mathcal{O}$-operator $T : V \longrightarrow \mathfrak{g}$ on n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ associated to the representation (V, ρ, φ_V) satisfying the above conditions. Hence, we have the following conclusion.

Proposition 10. *The $\mathcal{O}$-operator $T : V \longrightarrow \mathfrak{g}$ on n-LieDer pair $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ associated with the representation (V, ρ, φ_V) is a 0-cocycle of the cochain complex $C^p_{n-LieDer}(V, \mathfrak{g})$.*

5. Central Extension of *n*-LieDer Pairs

In this section, we use the result in Section 3 to describe the central extension of an n-LieDer pair. First of all, let us define a central extension of an n-LieDer pair.

Definition 6. *Let $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ be an n-LieDer pair, $\mathfrak{h} = \mathbf{k}c$ be a vector space with basis $\{c\}$, $\chi : \mathfrak{g} \to \mathfrak{h}$, and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Suppose that $\varphi_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$ are linear maps and $\psi \in \wedge^{n-1}\mathfrak{g}^* \wedge \mathfrak{g}^*$ is a skew symmetric n-linear map. Let $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{h}$ and $\varphi_{\tilde{\mathfrak{g}}}(x + ac) = \varphi_{\mathfrak{g}}(x) + \chi(x) + a\varphi_{\mathfrak{h}}(c)$ for any $x \in \mathfrak{g}$ and $a \in \mathbf{k}$. Then the n-LieDer pair $(\tilde{\mathfrak{g}}, \varphi_{\tilde{\mathfrak{g}}})$ is called a central extension of $(\mathfrak{g}, \varphi_{\mathfrak{g}})$ if*

$$[\tilde{x_1}, \tilde{x_2}, \cdots, \tilde{x_n}]_{\tilde{\mathfrak{g}}} = [x_1, x_2, \cdots, x_n]_{\mathfrak{g}} + \psi(x_1, x_2, \cdots, x_n)c,$$

and

$$[\tilde{x_1}, \tilde{x_2}, \cdots, \widetilde{x_{n-1}}, c]_{\tilde{\mathfrak{g}}} = 0$$

for all $x_i \in \mathfrak{g}$, where $\tilde{x_i} = x_i + a_i c$, $(i = 1, 2, \cdots, n)$ for some $a_i \in \mathbf{k}$.

Remark 1. *$\chi(x_i) + a_i\varphi_{\mathfrak{h}}(c)$ can be identified with $\lambda(x_i)c$, where $\lambda : \mathfrak{g} \to \mathbf{k}$ is a linear map.*

Proposition 11. *With above notations, $(\tilde{\mathfrak{g}}, \varphi_{\tilde{\mathfrak{g}}})$ is an n-LieDer pair if and only if (ψ, χ) is a 1-cocycle with coefficients in the trivial representation $(\mathfrak{h}, \varphi_{\mathfrak{h}})$.*

Proof. On the one hand, $(\tilde{\mathfrak{g}}, [\ ,\cdots,\ ]_{\tilde{\mathfrak{g}}})$ is an n-Lie algebra if and only if

$$[\tilde{x_1}, \tilde{x_2}, \cdots, \widetilde{x_{n-1}}, [\tilde{y_1}, \tilde{y_2}, \cdots, \tilde{y_n}]_{\tilde{\mathfrak{g}}}]_{\tilde{\mathfrak{g}}} = \sum_{i=1}^{n}[\tilde{y_1}, \tilde{y_2}, \cdots, \widetilde{y_{i-1}}, [\tilde{x_1}, \tilde{x_2}, \cdots, \widetilde{x_{n-1}}, \tilde{y_i}]_{\tilde{\mathfrak{g}}}, \widetilde{y_{i+1}}, \cdots, \tilde{y_n}]_{\tilde{\mathfrak{g}}}.$$

In fact,

$$\begin{aligned} & [\tilde{x_1}, \tilde{x_2}, \cdots, \widetilde{x_{n-1}}, [\tilde{y_1}, \tilde{y_2}, \cdots, \tilde{y_n}]_{\tilde{\mathfrak{g}}}]_{\tilde{\mathfrak{g}}} \\ = \quad & [\tilde{x_1}, \tilde{x_2}, \cdots, \widetilde{x_{n-1}}, [y_1, y_2, \cdots, y_n]_{\mathfrak{g}} + \psi(y_1, y_2, \cdots, y_n)c]_{\tilde{\mathfrak{g}}} \\ = \quad & [x_1, x_2, \cdots, x_{n-1}, [y_1, y_2, \cdots, y_n]_{\mathfrak{g}}]_{\mathfrak{g}} + \psi(x_1, x_2, \cdots, x_{n-1}, [y_1, y_2, \cdots, y_n]_{\mathfrak{g}})c \end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}[\tilde{y_1},\tilde{y_2},\cdots,\tilde{y_{i-1}},[\tilde{x_1},\tilde{x_2},\cdots,\tilde{x_{n-1}},\tilde{y_i}]_{\tilde{\mathfrak{g}}},\tilde{y_{i+1}},\cdots,\tilde{y_n}]_{\tilde{\mathfrak{g}}}\\
= &\sum_{i=1}^{n}[\tilde{y_1},\tilde{y_2},\cdots,\tilde{y_{i-1}},[x_1,x_2,\cdots,x_{n-1},y_i]_{\mathfrak{g}}+\psi(x_1,x_2,\cdots,x_{n-1},y_i)c,\tilde{y_{i+1}},\cdots,\tilde{y_n}]_{\tilde{\mathfrak{g}}}\\
= &\sum_{i=1}^{n}[y_1,y_2,\cdots,y_{i-1},[x_1,x_2,\cdots,x_{n-1},y_i]_{\mathfrak{g}},y_{i+1},\cdots,y_n]_{\mathfrak{g}}\\
&+\psi(y_1,y_2,\cdots,y_{i-1},[x_1,x_2,\cdots,x_{n-1},y_i]_{\mathfrak{g}},y_{i+1},\cdots,y_n)c.
\end{aligned}$$

So $(\tilde{\mathfrak{g}},[\ ,\cdots,\ ]_{\tilde{\mathfrak{g}}})$ is an n-Lie algebra if and only if

$$\psi(x_1,x_2,\cdots,x_{n-1},[y_1,y_2,\cdots,y_n]_{\mathfrak{g}})=\sum_{i=1}^{n}\psi(y_1,y_2,\cdots,y_{i-1},[x_1,x_2,\cdots,x_{n-1},y_i]_{\mathfrak{g}},y_{i+1},\cdots,y_n). \tag{10}$$

At the same time, $\varphi_{\tilde{\mathfrak{g}}}$ is a derivation of $\tilde{\mathfrak{g}}$ if and only if

$$\varphi_{\tilde{\mathfrak{g}}}([\tilde{x_1},\tilde{x_2},\cdots,\tilde{x_n}]_{\tilde{\mathfrak{g}}})=\sum_{i=1}^{n}[\tilde{x_1},\tilde{x_2},\cdots,\tilde{x_{i-1}},\varphi_{\tilde{\mathfrak{g}}}(\tilde{x_i}),\cdots,\tilde{x_n}]_{\tilde{\mathfrak{g}}}.$$

By direct calculation,

$$\begin{aligned}
&\varphi_{\tilde{\mathfrak{g}}}([\tilde{x_1},\tilde{x_2},\cdots,\tilde{x_n}]_{\tilde{\mathfrak{g}}})\\
= &\varphi_{\tilde{\mathfrak{g}}}([x_1,x_2,\cdots,x_n]_{\mathfrak{g}}+\psi(x_1,x_2,\cdots,x_n)c\\
= &\varphi_{\mathfrak{g}}([x_1,x_2,\cdots,x_n]_{\mathfrak{g}})+\chi([x_1,x_2,\cdots,x_n]_{\mathfrak{g}})+\psi(x_1,x_2,\cdots,x_n)\varphi_{\mathfrak{h}}(c),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}[\tilde{x_1},\tilde{x_2},\cdots,\tilde{x_{i-1}},\varphi_{\tilde{\mathfrak{g}}}(\tilde{x_i}),\cdots,\tilde{x_n}]_{\tilde{\mathfrak{g}}}\\
= &\sum_{i=1}^{n}[x_1,x_2,\cdots,x_{i-1},\varphi_{\mathfrak{g}}(x_i),\cdots,x_n]_{\mathfrak{g}}+\psi(x_1,x_2,\cdots,x_{i-1},\varphi_{\mathfrak{g}}(x_i),\cdots,x_n)c.
\end{aligned}$$

Therefore, $\varphi_{\tilde{\mathfrak{g}}}$ is a derivation of $\tilde{\mathfrak{g}}$ if and only if

$$\psi(x_1,x_2,\cdots,x_{i-1},\varphi_{\mathfrak{g}}(x_i),\cdots,x_n)=\chi([x_1,x_2,\cdots,x_n]_{\mathfrak{g}})+\psi(x_1,x_2,\cdots,x_n)\varphi_{\mathfrak{h}}. \tag{11}$$

On the other hand, for any $(\psi,\chi)\in\mathrm{Hom}(L\otimes\mathfrak{g},\mathfrak{h})\times\mathrm{Hom}(\mathfrak{g},\mathfrak{h})$, (ψ,χ) is a one-cocycle if and only if

$$\partial_{\mathfrak{g}}(\psi,\chi)=(\mathrm{d}\psi,\mathrm{d}\chi+\delta\psi)=0,$$

that is, for every $X\otimes Y\otimes z\in L\otimes L\otimes\mathfrak{g}$,

$$\begin{aligned}
&\mathrm{d}\psi(X\otimes Y\otimes z)\\
= &-\psi([X,Y]_F\otimes z)-\psi(Y\otimes[X,z]_{\mathfrak{g}})+\psi(X\otimes[Y,z]_{\mathfrak{g}})\\
= &-\psi(y_1\otimes\cdots\otimes y_{i-1}\otimes[x_1,\cdots,x_{n-1},y_i]_{\mathfrak{g}}\otimes y_{i+1}\otimes\cdots\otimes y_{n-1}\otimes z)\\
&-\psi(y_1\otimes\cdots\otimes y_{n-1}\otimes[x_1,x_2,\cdots,\otimes x_{n-1}\otimes z]_{\mathfrak{g}})\\
&+\psi(x_1\otimes\cdots\otimes x_{n-1}\otimes[y_1,y_2,\cdots,\otimes y_{n-1}\otimes z]_{\mathfrak{g}})\\
= &\ 0,
\end{aligned} \tag{12}$$

and

$$
\begin{aligned}
& (\mathrm{d}\chi + (-1)^2\delta\psi)(X \otimes z) \\
= \; & -\chi([X, z]_{\mathfrak{g}}) + \sum_{i=1}^{n-1} \psi(x_1 \otimes \cdots \otimes x_{i-1} \otimes \varphi_{\mathfrak{g}}(x_i) \otimes \cdots \otimes x_{n-1} \otimes z) \\
& +\psi(x_1 \otimes \cdots \otimes x_{n-1} \otimes \varphi_{\mathfrak{g}}(z)) - \varphi_{\mathfrak{h}}\psi(x_1 \otimes \cdots \otimes x_{n-1} \otimes z) \\
= \; & -\chi([x_1, \cdots, x_{n-1}, z]_{\mathfrak{g}}) + \sum_{i=1}^{n-1} \psi(x_1 \otimes \cdots \otimes x_{i-1} \otimes \varphi_{\mathfrak{g}}(x_i) \otimes \cdots \otimes x_{n-1} \otimes z) \\
& +\psi(x_1 \otimes \cdots \otimes x_{n-1} \otimes \varphi_{\mathfrak{g}}(z)) - \varphi_{\mathfrak{h}}\psi(x_1 \otimes \cdots \otimes x_{n-1} \otimes z) \\
= \; & 0.
\end{aligned} \tag{13}
$$

In the light of (12) and (13), (ψ, χ) is a 1-cocycle if and only if (10) and (11) hold. Hence, we get the conclusion. □

Author Contributions: Writing—review and editing, Q.S.; supervision and revision, Z.W. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (No. 11871421; No. 11401530) and the Natural Science Foundation of Zhejiang Province of China (No. LY19A010001).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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