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Article

Periodic Solutions in Slowly Varying Discontinuous Differential Equations: The Generic Case

Flaviano Battelli [1] and Michal Fečkan [2,3,*]

[1] Department of Industrial Engineering and Mathematics, Marche Polytecnic University, 60121 Ancona, Italy; battelli@dipmat.univpm.it
[2] Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská Dolina, 84248 Bratislava, Slovakia
[3] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 81473 Bratislava, Slovakia
* Correspondence: Michal.Feckan@fmph.uniba.sk

Abstract: We study persistence of periodic solutions of perturbed slowly varying discontinuous differential equations assuming that the unperturbed (frozen) equation has a non singular periodic solution. The results of this paper are motivated by a result of Holmes and Wiggins where the authors considered a two dimensional Hamiltonian family of smooth systems depending on a scalar variable which is the solution of a singularly perturbed equation.

Keywords: discontinuous differential equations; periodic solutions; persistence

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1. Introduction

In [1] a system like

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \varepsilon g(x,y,\varepsilon), \quad \varepsilon \in \mathbb{R} \end{aligned} \tag{1}$$

has been considered, where $x \in \mathbb{R}^2$, $\dot{x} = f(x,y)$ is Hamiltonian for any $y \in \mathbb{R}$ and has a one-parameter family of periodic solutions $q(t-\theta, y, \alpha)$ with period $T(y,\alpha)$ being C^1 in (y,α). As a matter of fact, in [1], $f(x,y)$ is allowed to depend on ε and t being like $f_0(x,y) + \varepsilon f_1(x,y,t,\varepsilon)$ and it is because of the t dependence of the perturbed equation that θ has been introduced. Indeed, introducing the variable $\theta = t$ mod T, the perturbed time dependent vector field is reduced to a time independent system on $\mathbb{R}^3 \times S^1$ where S^1 is the unit circle. Then, they answered the following question: do any of these periodic solutions persist for $\varepsilon \neq 0$? They constructed a vector valued function $M^{p/q}(y,\alpha,\theta)$ that they called *subharmonic Melnikov function* which is a measure of the difference between the starting value and the value of the solution at the time $\frac{p}{q}T$ in a direction transverse to the unperturbed vector field at the starting point. They proved that periodic solutions of the perturbed vector field arise near the simple zeros of $M^{p/q}(y,\alpha,\theta)$.

Motivated by [1], in this paper we study Equation (1) in higher dimension and allowing $f(x,y)$ to be more general than Hamiltonian and also discontinuous. As a matter of fact we assume that

$$f(x,y) := \begin{cases} f_-(x,y) & \text{if } h(x,y) < 0 \\ f_+(x,y) & \text{if } h(x,y) > 0. \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, all functions here considered (i.e., $f_\pm(x,y)$, $g(x,u,\varepsilon)$ and $h(x,y)$) are C^1 in their arguments, and $\varepsilon \in \mathbb{R}$ is a small parameter. In this paper we study a non degenerate case where the unperturbed discontinuous system $\dot{x} = f(x,y)$ has a periodic solution for $y = y_0$ and certain non degenerateness conditions are satisfied. We construct a Jacobian matrix and show that, if it is invertible, the perturbed system has a unique periodic solution near the periodic solution of the unperturbed system. The Jacobian matrix being invertible does not allow the system to have a smooth family of periodic solution $q(t,\alpha,y)$ since in this case $q_\alpha(0,\alpha,y)$ belongs to its kernel. We plan to consider this more degenerate case in a forthcoming paper.

We emphasize that the results of this paper easily extend to the case where $f_\pm(x,y)$ is replaced by $f_\pm(x,y,\varepsilon) = f_{0,\pm}(x,y) + \varepsilon f_{1,\pm}(x,y,\varepsilon)$ and $f_{0,\pm}(x,y)$, $f_{1,\pm}(x,y,\varepsilon)$ are smooth outside the singularity manifold $\{h(x,y) = 0\}$. In this case in the unperturbed system

$$\dot{x} = f_\pm(x,\eta) \tag{3}$$

the term $f_\pm(x,y)$ has to be replaced by $f_{0,\pm}(x,y)$. Finally, we observe that our results fit into a general theory of discontinuous differential equations presented in series of works [2–9].

2. Preliminary Results

We set

$$\Omega_\pm = \{(x,y) \mid \pm h(x,y) > 0\}$$
$$\Omega_0 = \{(x,y) \mid h(x,y) = 0\}.$$

In the whole paper, given a vector v or a matrix A with v^t, (resp. A^t) we denote the transpose of v (resp. A).

Let $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m$. We denote with $u_\pm(t,\xi,\eta)$ the solution of (3) such that $u(0) = \xi$. We assume that $(x_0,y_0) \in \Omega_+$ exists such that the following conditions hold:

(A_1) there exists $t_1 > 0$ such that $u_+(t_1,x_0,y_0) \in \Omega_0$ and $u_+(t,x_0,y_0) \in \Omega_+$ for $0 \le t < t_1$. Moreover,

$$h_x(u_+(t_1,x_0,y_0),y_0) f_\pm(u_+(t_1,x_0,y_0),y_0) < 0. \tag{4}$$

(A_2) there exists $t_2 > 0$ such that $u_-(t,u_+(t_1,x_0,y_0),y_0) \in \Omega_-$ for $0 < t < t_2$ and $u_-(t_2,u_+(t_1,x_0,y_0),y_0) \in \Omega_0$. Moreover,

$$h_x(u_-(t_2,u_+(t_1,x_0,y_0),y_0),y_0) f_\pm(u_-(t_2,u_+(t_1,x_0,y_0),y_0),y_0) > 0. \tag{5}$$

(A_3) there exists $t_3 > 0$ such that $u_+(t,u_-(t_2,u_+(t_1,x_0,y_0),y_0),y_0) \in \Omega_+$ for $0 < t \le t_3$ and $u_+(t_3,u_-(t_2,u_+(t_1,x_0,y_0),y_0),y_0) = x_0$.

Remark 1. *(i) We may as well consider $(x_0,y_0) \in \Omega_-$. As a matter of fact, changing $h(x,y)$ with $-h(x,y)$ the roles of Ω_+ and Ω_- are interchanged.*

(ii) The first part of condition (A_1) is equivalent to $h(u_+(t,x_0,y_0),y_0) > 0$ for $0 \le t < t_1$ and $h(u_+(t_1,x_0,y_0),y_0) = 0$. Similarly, the first part of condition (A_2) is equivalent to $h(u_-(t,u_+(t_1,x_0,y_0),y_0),y_0) < 0$ for $t_1 < t < t_2$ and $h(u_-(t_1,u_+(t_1,x_0,y_0),y_0),y_0) = 0$. Hence, being $u_-(t_1,u_+(t_1,x_0,y_0),y_0) = u_+(t_1,x_0,y_0)$, in general we have

$$h_x(u_+(t_1,x_0,y_0),y_0) f_\pm(u_+(t_1,x_0,y_0),y_0) \le 0. \tag{6}$$

Similarly,

$$h_x(u_-(t_2,u_+(t_1,x_0,y_0),y_0),y_0) f_\pm(u_-(t_2,u_+(t_1,x_0,y_0),y_0),y_0) \ge 0. \tag{7}$$

Hence (4) and (5) are stronger than the condition of existence of a continuous, piecewise C^1, solution of the discontinuous equation $\dot{x} = f(x,y_0)$ such that $u(t) \in \Omega_+$ for $0 \le t < t_1$ or $t_2 < t \le T$, $u(t) \in \Omega_-$ for $t_1 < t < t_2$ and $u(t_1), u(t_2) \in \Omega_0$. Moreover, they are generic conditions having the important consequence that we do not need to define the vector field on the discontinuity manifold Ω_0. Indeed, (A_1) and (A_2) imply transverse intersection of the solution with the discontinuity manifold Ω_0. Heuristically, (4) implies that when a solution in Ω_+, hits Ω_0, it immediately leaves Ω_0 and enters Ω_-. Similarly, condition (5) implies that when a solution in

Ω_- hits Ω_0, it immediately leaves Ω_0 and enters Ω_+. This case is referred to as the transverse *case. More generally, we have a* topologically transverse *case at $t = t_1$, when*

$$\begin{aligned} &h_x(u_+(t_1, x_0, y_0), y_0) f_+(u_+(t_1, x_0, y_0), y_0) = 0 \quad \text{and} \\ &h_x(u_+(t_1, x_0, y_0), y_0) f_-(u_+(t_1, x_0, y_0), y_0) < 0. \end{aligned}$$

Of course there are other important cases arising in the applications. For example, it may happen that $h(u_+(t, x_0, y_0), y_0)$ has a strong minimum at $t = t_1$ and

$$h_x(u_+(t_1, x_0, y_0), y_0) f_-(u_+(t_1, x_0, y_0), y_0) > 0.$$

In this case the solution of the discontinuous systems is tangent to Ω_0 at $u(t_1)$ and belongs to Ω_+ for $t \neq t_1$. This case is referred to as grazing*. Another important case arising in the applications is the* sliding *case. This happens when the inequalities*

$$\begin{aligned} &h_x(u_+(t_1, x_0, y_0), y_0) f_+(u_+(t_1, x_0, y_0), y_0) < 0 \textit{ and} \\ &h_x(u_+(t_1, x_0, y_0), y_0) f_-(u_+(t_1, x_0, y_0), y_0) > 0. \end{aligned}$$

hold. These conditions force the solution to remain in the discontinuity manifold Ω_0 until one of the two conditions

$$h_x(\bar{u}(t, u_+(t_1, x_0, y_0), y_0) f_+(\bar{u}(t, u_+(t_1, x_0, y_0), y_0), y_0) = 0 \tag{8}$$

or

$$h_x(\bar{u}(t, u_+(t_1, x_0, y_0), y_0) f_-(\bar{u}(t, u_+(t_1, x_0, y_0), y_0), y_0) = 0 \tag{9}$$

arises first (it is assumed that these two conditions do not happen simultaneously). Here $\bar{u}(t, u_+(t_1, x_0, y_0), y_0)$ is the solution of a continuous differential equation on Ω_0 defined by means of the Filippov's method [6] that takes into account the average of f_+ and f_- at the points of Ω_0. Then, if it is condition (8) that happens first, the solution re-enters into Ω_+, while if it is (9) that happen first, the solution enter into Ω_-.

In this paper we focus on the transverse case (A_1) and (A_2), leaving the other cases to forthcoming papers. As we have already observed in the transverse case, there is no need to know the Filippov equation on Ω_0.

For simplicity we set $t_* = t_1$, $t^* = t_1 + t_2$, $T = t_1 + t_2 + t_3$ and

$$u(t, x_0, y_0) := \begin{cases} u_+(t, x_0, y_0) & \text{if } 0 \leq t \leq t_* \\ u_-(t - t_*, u_+(t_*, x_0, y_0), y_0) & \text{if } t_* \leq t \leq t^* \\ u_+(t - t^*, u_-(t^* - t_*, u_+(t_*, x_0, y_0), y_0), y_0) & \text{if } t^* \leq t \leq T \end{cases}$$

Then using (A_3) it is easy to check that

$$u(T, x_0, y_0) = x_0. \tag{10}$$

Hence for $0 \leq t \leq T$, $u(t, x_0, y_0)$ is a T-periodic solution of Equation (1) with $\varepsilon = 0$, such that $u(t, x_0, y_0) \notin \Omega_0$ for all $t \in [0, T]$ with $t \neq t_*, t^*$ and the following hold:

$$\begin{aligned} &u(t_*, x_0, y_0), u(t^*, x_0, y_0) \in \Omega_0 \\ &h_x(u(t_*, x_0, y_0), y_0) f_+(u(t_*, x_0, y_0), y_0) < 0 \\ &h_x(u(t^*, x_0, y_0), y_0) f_-(u(t^*, x_0, y_0), y_0) > 0. \end{aligned} \tag{11}$$

Now, let $B(x_0, r) \subset \mathbb{R}^n$ be an open ball of radius r centered at x_0 and L be a local hyperplane in $\mathbb{R}^n$ passing through x_0 and transverse to $f_+(x_0, y_0)$. So

$$L = \{x_0\} + \{v_0\}^{\perp} \tag{12}$$

where $v_0^t f(x_0, y_0) \neq 0$. We have the following

Lemma 1. *Assume (A_1)–(A_3). Then there exist open balls $B(x_0, r_1) \subset \mathbb{R}^n$, $B(y_0, r_2) \subset \mathbb{R}^m$ such that for any $(\xi, \eta) \in B(x_0, r_1) \times B(y_0, r_2)$ there exist smooth functions $t_*(\xi, \eta)$, $t^*(\xi, \eta)$, $T(\xi, \eta)$ and a continuous, piecewise C^1 function $u(t, \xi, \eta)$ such that $u(0, \xi, \eta) = \xi$ and the following hold:*

(i) $|t_*(\xi, \eta) - t_*| + |t^*(\xi, \eta) - t^*| + |T(\xi, \eta) - T| \to 0$ *as* $(\xi, \eta) \to (x_0, y_0)$;

(ii) $u(t, \xi, \eta) \in \Omega_+$, *for* $0 \le t \le t_*(\xi, \eta)$, $u(t_*(\xi, \eta), \xi, \eta) \in \Omega_0$ *and*
$$h_x(u(t_*(\xi, \eta), \xi, \eta), \eta) f_\pm(u(t_*(\xi, \eta), \xi, \eta), \eta) < 0.$$

(iii) $u(t, \xi, \eta) \in \Omega_-$, *for* $t_*(\xi, \eta) \le t \le t^*(\xi, \eta)$, $u(t^*(\xi, \eta), \xi, \eta) \in \Omega_0$ *and*
$$h_x(u(t^*(\xi, \eta), \xi, \eta), \eta) f_\pm(u(t^*(\xi, \eta), \xi, \eta), \eta) > 0.$$

(iv) $u(t, \xi, \eta) \in \Omega_+$, *for* $t^*(\xi, \eta) \le t \le T(\xi, \eta)$, $u(T(\xi, \eta), \xi, \eta) \in L$

(v) *for* $0 \le t \le T(\xi, \eta)$, $t \ne t_*(\xi, \eta), t^*(\xi, \eta)$, $u(t, \xi, \eta)$ *satisfies the differential equation* $\dot{x} = f_\pm(x, \eta)$, *where the signs* $\pm$ *are taken accordingly to* $u(t, \xi, \eta) \in \Omega_+$ *or* $u(t, \xi, \eta) \in \Omega_-$.

Moreover, $(\xi, \eta) \mapsto u(t, \xi, \eta)$ *is a smooth map in the space of piecewise continuous functions and*
$$\sup_{0 \le t \le T(\xi, \eta)} |u(t, \xi, \eta) - u(t, x_0, y_0)| \to 0 \tag{13}$$
as $(\xi, \eta) \to (x_0, y_0)$.

Proof. Let $\rho_1 > 0$ and $\rho_2 > 0$ be two, sufficiently small, positive numbers such that $B(x_0, \rho_1) \times B(y_0, \rho_2) \subset \Omega_+$. For $(\xi, \eta) \in B(x_0, \rho_1) \times B(y_0, \rho_2)$ we consider the equation
$$h(u_+(t, \xi, \eta), \eta) = 0, \quad (\xi, \eta) \in B(x_0, \rho_1) \times B(x_0, \rho_2),$$
whose left-hand side vanish at $t = t_*$, $\xi = x_0$, $\eta = y_0$. Moreover, the derivative with respect to t of the left hand side at $\xi = x_0$, $\eta = y_0$ is
$$h_x(u_+(t, x_0, y_)), y_0)\dot{u}_+(t, x_0, y_0) = h_x(u_+(t, x_0, y_0), y_0) f_+(u_+(t, x_0, y_0), y_0).$$

According to (A_1), possibly changing ρ_1 and ρ_2, from the Implicit Function Theorem, it follows the existence of a smooth function $t_*(\xi, \eta)$ such that
$$t_*(x_0, y_0) = t_*$$
and
$$h(u_+(t_*(\xi, \eta), \xi, \eta), \eta) = 0.$$

Next, since $u(t_*(\xi, \eta), \xi, \eta) \to u(t_*, x_0, y_0)$ as $(\xi, \eta) \to (x_0, y_0)$ it follows by continuity that
$$h_x(u_+(t_*(\xi, \eta), \xi, \eta), \eta) f_\pm(u_+(t_*(\xi, \eta), \xi, \eta), \eta) \le -\delta < 0.$$
for some $\delta > 0$, uniformly with respect to $(\xi, \eta) \in B(x_0, \rho_1) \times B(x_0, \rho_2)$. Hence (ii) holds with $u(t, \xi, \eta) = u_+(t, \xi, \eta)$, for $0 \le t \le t_*(\xi, \eta)$.

Then we see that $\sigma > 0$ exists such that, for $t_*(\xi, \eta) - \sigma \le t < t_*(\xi, \eta)$, we have $h(u_+(t, \xi, \eta), \eta), \eta) > 0$. Using the continuous dependence on the data we also see that
$$\sup_{0 \le t \le t_*(\xi, \eta)} |u_+(t, \xi, \eta) - u(t, x_0, y_0)| \to 0$$
as $(\xi, \eta) \to (x_0, y_0)$ and then
$$h(u_+(t, \xi, \eta), \eta), \eta) > 0$$

for $0 \le t \le t_*(\xi,\eta) - \sigma$. Hence (i) holds. Now, consider the solution $\hat{u}_-(t,\xi,\eta)$ of the equation

$$\begin{aligned} &\dot{x} = f_-(x,\eta) \\ &x(t_*(\xi,\eta)) = u_+(t_*(\xi,\eta),\xi,\eta). \end{aligned}$$

Note that, using the previous notation, we have

$$\hat{u}_-(t,\xi,\eta) = u_-(t - t_*(\xi,\eta), u_+(t_*(\xi,\eta),\xi,\eta),\eta).$$

Note also that

$$\hat{u}_-(t,x_0,y_0) = u_-(t - t_*, u_+(t_*,x_0,y_0),y_0) = u(t,x_0,y_0) \tag{14}$$

for any $t_* \le t \le t^*$.

It follows from the continuous dependence on the data that $\hat{u}_-(t,\xi,\eta)$ tends to $u(t,x_0,y_0)$, as $(\xi,\eta) \to (x_0,y_0)$ together with its t-derivative, uniformly with respect to t in compact intervals such as $[t_* - \sigma, t^* + \sigma]$, with $\sigma > 0$ sufficiently small. Next we consider the equation

$$h(\hat{u}_-(t,\xi,\eta),\eta) = 0, \quad (\xi,\eta) \in B(x_0,\rho_1) \times B(x_0,\rho_2),$$

in a neighborhood of t^*. From (11)–(14) we get $h(\hat{u}_-(t^*,x_0,y_0),y_0) = 0$ and

$$h_x(\hat{u}_-(t^*,x_0,y_0),y_0)\frac{\partial \hat{u}_-}{\partial t}(t,x_0,y_0) > 0.$$

Then, the Implicit Function Theorem and an argument similar to the above imply that $\rho_1 > 0, \rho_2 > 0$ and a smooth function $t^*(\xi,\eta)$, with $(\xi,\eta) \in B(x_0,\rho_1) \times B(x_0,\rho_2)$ exist such that $t^*(x_0,y_0) = t^*$ and (iii) holds with $u(t,\xi,\eta) = \hat{u}_-(t,\xi,\eta) = u_-(t - t_*(\xi,\eta), u_+(t_*(\xi,\eta),\xi,\eta),\eta)$, $t_*(\xi,\eta) \le t \le t^*(\xi,\eta)$. Moreover, by continuity,

$$\sup_{t_*(\xi,\eta) \le t \le t^*(\xi,\eta)} |u(t,\xi,\eta) - u(t,x_0,y_0)| \to 0$$

Another argument of similar nature shows that iv) holds. Since all pieces of $u(t,\xi,\eta)$ in the intervals $[0,t_*(\xi,\eta))$, $(t_*(\xi,\eta),t^*(\xi,\eta))$ and $(t^*(\xi,\eta),T(\xi,\eta)]$ consist of solutions of equation $\dot{x} = f_\pm(x,\eta)$, it is easy to see that v) holds. The last conclusion follows from

$$\begin{aligned} &\sup\nolimits_{0 \le t \le t_*(\xi,\eta)} |u_+(t,\xi,\eta) - u_+(t,x_0,y_0)| \to 0 \\ &\sup\nolimits_{t_*(\xi,\eta) \le t \le t^*(\xi,\eta)} |u_+(t,\xi,\eta) - u_+(t,x_0,y_0)| \to 0 \\ &\sup\nolimits_{t^*(\xi,\eta) \le t \le T(\xi,\eta)} |u_+(t,\xi,\eta) - u_+(t,x_0,y_0)| \to 0 \end{aligned}$$

as $(\xi,\eta) \to (x_0,y_0)$. $\square$

Note that for $t \in [0,T(\xi,\eta)]$ it results $u(t,\xi,\eta) \in \Omega_-$ if $t_*(\xi,\eta) < t < t_*(\xi,\eta)$, $u(t,\xi,\eta) \in \Omega_0$ if $t = t_*(\xi,\eta)$ or $t = t_*(\xi,\eta)$ and $u(t,\xi,\eta) \in \Omega_+$ otherwise.

We set

$$\bar{T} := \sup\{T(\xi,\eta) \mid (\xi,\eta) \in B(x_0,r_1) \times B(y_0,r_2)\}.$$

We assume conditions (A_1)–(A_3) hold. We know that $u(t,x_0,y_0)$ is a T-periodic solution of Equation (1) with $\varepsilon = 0$:

$$\dot{x} = f(x,\eta) := \begin{cases} f_-(x,\eta) & \text{if } h(x,\eta) < 0 \\ f_+(x,\eta,) & \text{if } h(x,\eta) > 0. \end{cases} \tag{15}$$

Now, does this periodic, piecewise continuous solution persist when $\varepsilon \ne 0$? We have the following

Theorem 1. *Suppose (A_1)–(A_3) hold. Then there exist open balls $B(x_0, r_1) \subset \mathbb{R}^n$, $B(y_0, r_2) \subset \mathbb{R}^m$ and $\bar{\varepsilon} > 0$ such that for $(\xi, \eta) \in B(x_0, r_1) \times B(x_0, r_2)$ and $|\varepsilon| \le \varepsilon_0$ there exist smooth functions $t_*(\xi, \eta, \varepsilon)$, $t^*(\xi, \eta, \varepsilon)$, $T(\xi, \eta, \varepsilon)$ and continuous, piecewise C^1 functions $x(t, \xi, \eta, \varepsilon)$, $y(t, \xi, \eta, \varepsilon)$ such that $x(0, \xi, \eta, \varepsilon) = \xi$, $y(0, \xi, \eta, \varepsilon) = \eta$ and the following hold:*

(i) $|t_*(\xi, \eta, \varepsilon) - t_*(\xi, \eta)| + |t^*(\xi, \eta, \varepsilon) - t^*(\xi, \eta)| + |T(\xi, \eta, \varepsilon) - T(\xi, \eta)| \to 0$ *as* $\varepsilon \to 0$ *uniformly in* $(\xi, \eta) \in B(x_0, r_1) \times B(x_0, r_2)$;

(ii) $(x(t_*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon), y(t_*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon)) \in \Omega_0$, $(x(t, \xi, \eta, \varepsilon), y(t, \xi, \eta, \varepsilon)) \in \Omega_+$, *for* $0 \le t < t_*(\xi, \eta, \varepsilon)$ *and*

$$h_x(u(t_*(\xi, \eta, \varepsilon), \xi, \eta), \eta) f_\pm(u(t_*(\xi, \eta, \varepsilon), \xi, \eta), \eta) < 0;$$

(iii) $(x(t^*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon), y(t^*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon)) \in \Omega_0$, $(x(t, \xi, \eta, \varepsilon), y(t, \xi, \eta, \varepsilon)) \in \Omega_-$, *for* $t_*(\xi, \eta, \varepsilon) < t < t^*(\xi, \eta, \varepsilon)$ *and*

$$h_x(x(t^*(\xi, \eta, \varepsilon), \xi, \eta), \eta) f_\pm(x(t^*(\xi, \eta, \varepsilon), \xi, \eta), \eta) > 0;$$

(iv) $(x(t, \xi, \eta, \varepsilon), y(t, \xi, \eta, \varepsilon)) \in \Omega_+$, *for* $t^*(\xi, \eta, \varepsilon) < t \le T(\xi, \eta, \varepsilon)$ *and* $x(T(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon) \in L$;

(v) $y(t, \xi, \eta, \varepsilon) \in B(y_0, r_2)$ *for any* $0 \le t \le T(\xi, \eta, \varepsilon)$;

(vi) *for* $0 \le t \le T(\xi, \eta, \varepsilon)$, $t \ne t_*(\xi, \eta, \varepsilon), t^*(\xi, \eta, \varepsilon)$, $(x(t, \xi, \eta, \varepsilon), y(t, \xi, \eta, \varepsilon))$ *satisfies the differential Equation (1), where the signs $\pm$ are taken accordingly to* $x(t, \xi, \eta, \varepsilon) \in \Omega_+$ *or* $x(t, \xi, \eta, \varepsilon) \in \Omega_-$.

Moreover, $(\xi, \eta, \varepsilon) \mapsto (x(t, \xi, \eta, \varepsilon), y(t, \xi, \eta, \varepsilon))$ *is a smooth map in the space of piecewise continuous functions and*

$$\sup_{0 \le t \le T^*(\xi, \eta, \varepsilon)} |x(t, \xi, \eta, \varepsilon) - u(t, \xi, \eta)| + |y(t, \xi, \eta, \varepsilon) - \eta| \to 0$$

as $\varepsilon \to 0$, *uniformly with respect to* $(\xi, \eta) \in B(x_0, r_1) \times B(x_0, r_2)$.

Proof. Let r_1, r_2 be sufficiently small so that $B(x_0, r_1) \times B(y_0, r_2) \subset \Omega_+$. For $0 \le t \le t_* + 1$ and $(\xi, \eta) \in B(x_0, r_1) \times B(y_0, r_2)$, let $(x_+^1(t, \xi, \eta, \varepsilon), y_+^1(t, \xi, \eta, \varepsilon))$ be the solution of

$$\begin{aligned} \dot{x} &= f_+(x, y), & x(0) &= \xi \\ \dot{y} &= \varepsilon g(x, y, \varepsilon), & y(0) &= \eta. \end{aligned} \tag{16}$$

From the continuous dependence of the data we see that

$$\begin{aligned} &\sup\nolimits_{0 \le t \le t_* + 1} |x_+^1(t, \xi, \eta, \varepsilon) - u_+(t, \xi, \eta)| = O(\varepsilon) \\ &\sup\nolimits_{0 \le t \le t_* + 1} |y_+^1(t, \xi, \eta, \varepsilon) - \eta| = O(\varepsilon) \end{aligned} \tag{17}$$

as $\varepsilon \to 0$. So, taking ε sufficiently small we get $y_+^1(t, \xi, \eta, \varepsilon) \in B(y_0, r_2)$ for $0 \le t \le \bar{T} + 1$. As a consequence there exists a unique $t_*(\xi, \eta, \varepsilon)$ such that

$$\begin{aligned} &|t_*(\xi, \eta, \varepsilon) - t_*(\xi, \eta)| = O(\varepsilon), \\ &h(x_+^1(t_*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon), y_+^1(t_*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon)) = 0, \\ &h_x(x_+^1(t_*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon), y_+^1(t_*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon)) \dot{x}_+(t_*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon) < 0. \end{aligned} \tag{18}$$

By the Implicit Function Theorem $t_*(\xi, \eta, \varepsilon)$ is a smooth function of (ξ, η, ε). Moreover, from the last inequality in (18) we see that ii) holds and then $x_+^1(t, \xi, \eta, \varepsilon)$ intersects transversally Ω_0 at the point $x_+^1(t_*(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon)$.

Repeating the above argument we see that, for $t_* - 1 \le t \le t^* + 1$, the equation

$$\begin{aligned} &\dot{x} = f_-(x,y) \\ &\dot{y} = \varepsilon g(x,y) \\ &x(t_*(\xi,\eta,\varepsilon)) = x_+^1(t_*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon) \\ &y(t_*(\xi,\eta,\varepsilon)) = y_+^1(t_*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon) \end{aligned}$$

has a solution $(x_-(t,\xi,\eta,\varepsilon), y_-(t,\xi,\eta,\varepsilon))$ such that

$$\begin{aligned} &\sup_{t_*-1\le t\le t^*+1} |x_-(t,\xi,\eta,\varepsilon) - u_-(t,\xi,y_+(t_*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon))| = O(\varepsilon) \\ &\sup_{t_*-1\le t\le t^*+1} |y_-(t,\xi,\eta,\varepsilon) - y_+^1(t_*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon)| = O(\varepsilon) \end{aligned}$$

from which we also get, using (17)

$$\begin{aligned} &\sup_{t_*-1\le t\le t^*+1} |x_-(t,\xi,\eta,\varepsilon) - u_-(t,\xi,\eta)| = O(\varepsilon) \\ &\sup_{t_*-1\le t\le t^*+1} |y_-(t,\xi,\eta,\varepsilon) - \eta| = O(\varepsilon) \end{aligned} \tag{19}$$

Moreover, by the Implicit function Theorem, there exists $t^*(\xi,\eta,\varepsilon)$ such that

$$\begin{aligned} &|t^*(\xi,\eta,\varepsilon) - t^*(\xi,\eta)| = O(\varepsilon) \\ &h(x_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon), y_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon)) = 0 \end{aligned} \tag{20}$$

and

$$h_x(x_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon), y_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon))\dot{x}_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon) > 0.$$

Hence (iii) holds, i.e., at the point $(x_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon), y_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon)) \in \Omega_0$, $f_+(x,y)$ points inward Ω_+. Finally, by a similar argument we show that equation

$$\begin{aligned} &\dot{x} = f_+(x,y) \\ &\dot{y} = \varepsilon g(x,y) \\ &x(t^*(\xi,\eta,\varepsilon)) = x_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon) \\ &y(t^*(\xi,\eta,\varepsilon)) = y_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon) \end{aligned}$$

has a solution $(x_+^2(t,\xi,\eta,\varepsilon), y_+^2(t,\xi,\eta,\varepsilon))$ such that

$$\begin{aligned} &\sup_{t^*-1\le t\le \bar{T}+1} |x_+^2(t,\xi,\eta,\varepsilon) - u_+(t,\xi,y_+^2(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon))| = O(\varepsilon) \\ &\sup_{t^*-1\le t\le \bar{T}+1} |y_+^2(t,\xi,\eta,\varepsilon) - y_-(t^*(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon)| = O(\varepsilon) \end{aligned}$$

from which we also get, using (17)

$$\begin{aligned} &\sup_{t^*-1\le t\le \bar{T}+1} |x_+^2(t,\xi,\eta,\varepsilon) - u_+(t,\xi,\eta)| = O(\varepsilon) \\ &\sup_{t^*-1\le t\le \bar{T}+1} |y_+^2(t,\xi,\eta,\varepsilon) - \eta| = O(\varepsilon). \end{aligned} \tag{21}$$

Moreover, there exists $T(\xi,\eta,\varepsilon)$ such that

$$\begin{aligned} &|T(\xi,\eta,\varepsilon) - T(\xi,\eta)| = O(\varepsilon) \\ &x_+(T(\xi,\eta,\varepsilon),\xi,\eta,\varepsilon) \in L. \end{aligned} \tag{22}$$

We set

$$x(t,\xi,\eta,\varepsilon) = \begin{cases} x_+^1(t,\xi,\eta,\varepsilon) & \text{if } 0 \le t \le t_*(\xi,\eta,\varepsilon) \\ x_-(t,\xi,\eta,\varepsilon) & \text{if } t_*(\xi,\eta,\varepsilon) \le t \le t^*(\xi,\eta,\varepsilon) \\ x_+^2(t,\xi,\eta,\varepsilon) & \text{if } t^*(\xi,\eta,\varepsilon) \le t \le T(\xi,\eta,\varepsilon) \end{cases}$$

and similarly

$$y(t,\xi,\eta,\varepsilon)=\begin{cases} y_+^1(t,\xi,\eta,\varepsilon) & \text{if } 0\le t\le t_*(\xi,\eta,\varepsilon)\\ y_-(t,\xi,\eta,\varepsilon) & \text{if } t_*(\xi,\eta,\varepsilon)\le t\le t^*(\xi,\eta,\varepsilon)\\ y_+^2(t,\xi,\eta,\varepsilon) & \text{if } t^*(\xi,\eta,\varepsilon)\le t\le T(\xi,\eta,\varepsilon)\end{cases}$$

From Equations (17), (19) and (21) we see that (i)–(vi) hold. In particular

$$\begin{array}{l}\sup_{0\le t\le T(\xi,\eta,\varepsilon)}|x(t,\xi,\eta,\varepsilon)-u(t,\xi,\eta)|=O(\varepsilon)\\ \sup_{0\le t\le T(\xi,\eta,\varepsilon)}|y(t,\xi,\eta,\varepsilon)-\eta|=O(\varepsilon).\end{array} \tag{23}$$

So, for any ε sufficiently small, say $|\varepsilon|\le\bar\varepsilon$, we have $y(t,\xi,\eta,\varepsilon)\in B(y_0,r_2)$. □

3. Periodic Solutions

In this section we prove a theorem concerning the existence of a periodic solution $(x(t,\varepsilon),y(t,\varepsilon))$ of system (1) such that

$$\sup_{0\le t\le T}|x(t,\varepsilon)-u(t,x_0,y_0)|+|y(t,\varepsilon)-y_0|\to 0 \quad \text{as } \varepsilon\to 0.$$

For $(\xi,\eta)\in L\times\mathbb{R}^m$ let $T(\xi,\eta,\varepsilon)$, $t_*(\xi,\eta,\varepsilon)$, $t^*(\xi,\eta,\varepsilon)$ be the C^r functions whose existence is stated in Theorem 1. We set

$$\begin{aligned} J_{11}=&\int_0^T f_x(u(t,x_0,y_0),y_0)u_\xi(t,x_0,y_0)dt+f(x_0,y_0)T_\xi(x_0,y_0)\\ &+[f_+(u(t_*,x_0,y_0),y_0)-f_-(u(t_*,x_0,y_0),y_0))]\tfrac{\partial t_*}{\partial\xi}(x_0,y_0)\\ &+[f_-(u(t^*,x_0,y_0),y_0)-f_+(u(t^*,x_0,y_0),y_0)]\tfrac{\partial t^*}{\partial\xi}(x_0,y_0))\end{aligned}$$

$$\begin{aligned} J_{12}=&\int_0^T f_x(u(t,x_0,y_0),y_0)u_\eta(t,x_0,y_0)+f_y(u(t,x_0,y_0),y_0)dt+f(x_0,y_0)T_\eta(x_0,y_0)\\ &+[f_+(u(t_*,x_0,y_0),y_0)-f_-(u(t_*,x_0,y_0),y_0))]\tfrac{\partial t_*}{\partial\eta}(x_0,y_0)\\ &+[f_-(u(t^*,x_0,y_0),y_0)-f_+(u(t^*,x_0,y_0),y_0)]\tfrac{\partial t^*}{\partial\eta}(x_0,y_0))\end{aligned}$$

$$\begin{aligned} J_{21}=&\int_0^T g_x(u(t,x_0,y_0),y_0,0)u_\xi(t,x_0,y_0)dt+g(x_0,y_0,0)T_\xi(x_0,y_0)\\ J_{22}=&\int_0^T g_x(u(t,x_0,y_0),y_0,0)u_\eta(t,x_0,y_0)+g_y(u(t,x_0,y_0),y_0,0)dt\\ &+g(x_0,y_0,0)T_\eta(x_0,y_0)\end{aligned}$$

Note that the derivatives in the previous formulae are the derivatives of the restrictions of the various functions to $(\xi,\eta)\in L\times\mathbb{R}^m$. For example, $T_\xi(\xi,\eta)$ denotes the derivative of $T:L\times\mathbb{R}^m\to\mathbb{R}$ and similarly for the other derivatives with respect to ξ.

We prove the following

Theorem 2. *Suppose that (A_1)–(A_3) hold and that*

$$\int_0^T g(u(t,x_0,y_0),y_0,0)dt=0. \tag{24}$$

Suppose, further, that the linear map $J:T_{x_0}L\times\mathbb{R}^m\to\mathbb{R}^n\times\mathbb{R}^m$:

$$J:\begin{pmatrix}\xi\\ \eta\end{pmatrix}\mapsto\begin{pmatrix}J_{11}\xi+J_{12}\eta\\ J_{21}\xi+J_{22}\eta\end{pmatrix},\quad \xi\in T_{x_0}L,\ \eta\in\mathbb{R}^m,$$

has maximum rank $(= n + m - 1)$*. Then there exists* $\varepsilon_0 > 0$ *such that for* $|\varepsilon| < \varepsilon_0$ *system* (1) *has a unique periodic solution* $(x(t, \varepsilon), y(t, \varepsilon))$ *of period* $T(\varepsilon)$ *such that*

$$\lim_{\varepsilon \to 0} T(\varepsilon) = T \tag{25}$$

and such that

$$\sup_{0 \le t \le T} |x(t, \varepsilon) - u(t, x_0, y_0)| + |y(t, \varepsilon) - y_0| \to 0 \quad \text{as } \varepsilon \to 0. \tag{26}$$

Moreover, the map $\varepsilon \mapsto (x(t, \varepsilon), y(t, \varepsilon))$ *into the space of bounded functions is* C^{r-1}*.*

Remark 2. *(i)* $J : T_{x_0}L \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ *defines a* $(n + m) \times (n + m - 1)$ *matrix. However, it will be seen during the proof of Theorem* 2 *that*

$$J : T_{x_0}L \times \mathbb{R}^m \to T_{x_0}L \times \mathbb{R}^m. \tag{27}$$

although this does not result immediately. This is why we made the assumption on the rank. By the way, because of (27) *the assumption is equivalent to the fact that* $J : T_{x_0}L \times \mathbb{R}^m \to T_{x_0}L \times \mathbb{R}^m$ *is an isomorphism. Note, also, that* $T_{x_0}L = \{v_0\}^\perp$*.*

(ii) Condition (24) *is a* 0*-average condition for* $g(u(t, \xi, \eta), \eta, 0)$ *at* (x_0, y_0) *and implies that* $y_\varepsilon(t, x_0, y_0, 0)$ *is a* T*-periodic solution of*

$$\dot{v} = g(u(t, x_0, y_0), y_0, 0).$$

Note that (24) *corresponds to* $M_3^{p/q}(I_0, \theta_0, z_0) = 0$ *with* $p = q$ *in ([1], Theorem 3.1) where the authors search for subharmonic periodic solutions. Here, we do not have to take into account the extra parameter* θ *because of the autonomous character of Equation* (1)*. Note, also, that, differentiating* $\dot{y}_\varepsilon(t, x_0, y_0) = g(u(t, x_0, y_0), y_0, 0)$ *with respect to* t*, we get*

$$\ddot{y}_\varepsilon(t, x_0, y_0) = g_x(u(t, x_0, y_0), y_0, 0)\dot{u}(t, x_0, y_0)$$

and then

$$\int_0^T g_x(u(t, x_0, y_0), y_0, 0) f(u(t, x_0, y_0), y_0) dt = \dot{y}_\varepsilon(T, x_0, y_0) - \dot{y}_\varepsilon(0, x_0, y_0) = 0.$$

Proof. Let $B(x_0, r_1)$, $B(y_0, r_2)$ be as in Theorem 1. To obtain a periodic solution of Equation (1) we solve the system

$$\begin{aligned} x(T(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon) - \xi = 0 \\ y(T(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon) - \eta = 0 \end{aligned} \tag{28}$$

for $(\xi, \eta) \in (B_1 \cap L) \times B_2$ where $B_1 \times B_2 \subset B(x_0, r_1) \times B(y_0, r_2)$ is a small neighborhood of (x_0, y_0). When $\varepsilon = 0$ the second equation in (28) reads $\eta = \eta$ and is satisfied for any $\eta \in B(y_0, r_2)$. So we replace (28) with

$$\begin{aligned} x(T(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon) - \xi = 0 \\ \varepsilon^{-1}[y(T(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon) - \eta] = 0. \end{aligned} \tag{29}$$

Since $y(t, \xi, \eta, 0) = \eta$, the function

$$\begin{cases} \varepsilon^{-1}[y(T(\xi, \eta, \varepsilon), \xi, \eta, \varepsilon) - \eta] & \text{if } \varepsilon \neq 0 \\ y_\varepsilon(T(\xi, \eta), \xi, \eta, 0) & \text{if } \varepsilon = 0 \end{cases}$$

is C^{r-1} in $B(x_0,\rho_1) \times B(x_0,\rho_2) \times]-]\bar{\varepsilon},\bar{\varepsilon}[$. Then, when $\varepsilon = 0$ (29) reads:

$$\begin{array}{l} u(T(\xi,\eta),\xi,\eta) - \xi = 0 \\ y_\varepsilon(T(\xi,\eta),\xi,\eta,0) = 0 \end{array} \tag{30}$$

where $(\xi,\eta) \in L \times \mathbb{R}^m$ and $T(\xi,\eta) = T(\xi,\eta,0)$, because $x(t,\xi,\eta,0) = u(t,\xi,\eta)$. From (10) it follows that equation $u(T(\xi,\eta),\xi,\eta) - \xi = 0$ has the solution $(\xi,\eta) = (x_0,y_0)$. Next, since

$$\dot{y}_\varepsilon(t,\xi,\eta,0) = g(u(t,\xi,\eta),\eta,0), \quad y_\varepsilon(0,\xi,\eta,0) = 0$$

we get

$$y_\varepsilon(T(\xi,\eta),\xi,\eta,0) = \int_0^{T(\xi,\eta)} g(u(t,\xi,\eta),\eta,0)dt. \tag{31}$$

From (24) we conclude that Equation (30) has the solution $(\xi,\eta) = (x_0,y_0)$. Recall that $T = T(x_0,y_0)$.

We now compute the Jacobian matrix $J(x_0,y_0)$ of the left-hand side of Equation (30) at (x_0,y_0). We know that $u(t,\xi,\eta)$ satisfies the integral equation

$$u(t,\xi,\eta) = \xi + \int_0^t f(u(t,\xi,\eta),\eta)dt. \tag{32}$$

and that $u(T(\xi,\eta),\xi,\eta) - \xi \in \{f(x_0,y_0)\}^\perp$. More explicitly, taking into account the definition of $f(x,y)$:

$$\begin{array}{l} u(T(\xi,\eta),\xi,\eta) - \xi = \displaystyle\int_0^{t_*(\xi,\eta)} f_+(u(t,\xi,\eta),\eta)dt + \\ \displaystyle\int_{t_*(\xi,\eta)}^{t^*(\xi,\eta)} f_-(u(t,\xi,\eta),\eta)dt + \int_{t^*(\xi,\eta)}^{T(\xi,\eta)} f_+(u(t,\xi,\eta),\eta)dt \in \{f(x_0,y_0\}^\perp. \end{array}$$

Differentiating, we get

$$\begin{array}{l} \dfrac{\partial}{\partial\xi}[u(T(\xi,\eta),\xi,\eta) - \xi] = \\ f_+(u(t_*(\xi,\eta),\xi,\eta),\eta)\dfrac{\partial t_*}{\partial\xi}(\xi,\eta) + \displaystyle\int_0^{t_*(\xi,\eta)} f_{+,x}(u(t,\xi,\eta),\eta)u_\xi(t,\xi,\eta)dt \\ +f_-(u(t^*(\xi,\eta),\xi,\eta),\eta)\dfrac{\partial t^*}{\partial\xi}(\xi,\eta) - f_-(u(t_*(\xi,\eta),\xi,\eta),\eta)\dfrac{\partial t_*}{\partial\xi}(\xi,\eta) \\ + \displaystyle\int_{t_*(\xi,\eta)}^{t_*(\xi,\eta)} f_{-,x}(u(t,\xi,\eta),\eta)u_\xi(t,\xi,\eta)dt \\ +f_+(u(T(\xi,\eta),\xi,\eta),\eta))T_\xi(\xi,\eta) - f_+(u(t^*(\xi,\eta),\xi,\eta),\eta))\dfrac{\partial t^*}{\partial\xi}(\xi,\eta) \\ + \displaystyle\int_{t^*(\xi,\eta)}^{T(\xi,\eta)} f_{+,x}(u(t,\xi,\eta),\eta)u_\xi(t,\xi,\eta)dt = \\ f_+(u(T(\xi,\eta),\xi,\eta),\eta)T_\xi(\xi,\eta) + \displaystyle\int_0^{T(\xi,\eta)} f_x(u(t,\xi,\eta),\eta)u_\xi(t,\xi,\eta)dt \\ +[f_+(u(t_*(\xi,\eta),\xi,\eta),\eta) - f_-(u(t_*(\xi,\eta),\xi,\eta),\eta)]\dfrac{\partial t_*}{\partial\xi}(\xi,\eta) \\ +[f_-(u(t^*(\xi,\eta),\xi,\eta),\eta) - f_+(u(t^*(\xi,\eta),\xi,\eta),\eta)]\dfrac{\partial t^*}{\partial\xi}(\xi,\eta). \end{array}$$

where

$$f_x(\xi,\eta) = \begin{cases} f_{+,x}(\xi,\eta) & \text{if } (\xi,\eta) \in \Omega_+ \\ f_{-,x}(\xi,\eta) & \text{if } (\xi,\eta) \in \Omega_-. \end{cases} \tag{33}$$

So, using $u(T,x_0,y_0) = x_0$, $f_+(x_0,y_0) = f(x_0,y_0)$:

$$\frac{\partial}{\partial\xi}[u(T(\xi,\eta),\xi,\eta) - \xi]_{\xi=x_0,\eta=y_0} = J_{11}.$$

Similarly we have:

$$\begin{aligned}
&\frac{\partial}{\partial \eta}[u(T(\xi,\eta),\xi,\eta)-\xi] = f_+(u(T(\xi,\eta),\xi,\eta),\eta)T_\eta(\xi,\eta) \\
&+\int_0^{T(\xi,\eta)} f_x(u(t,\xi,\eta),\eta)u_\eta(t,\xi,\eta)+f_y(u(t,\xi,\eta),\eta)dt \\
&+[f_+(u(t_*(\xi,\eta),\xi,\eta),\eta)-f_-(u(t_*(\xi,\eta),\xi,\eta),\eta)]\tfrac{\partial t_*}{\partial \eta}(\xi,\eta) \\
&+[f_-(u(t^*(\xi,\eta),\xi,\eta),\eta)-f_+(u(t^*(\xi,\eta),\xi,\eta),\eta)]\tfrac{\partial t^*}{\partial \eta}(\xi,\eta).
\end{aligned}$$

and hence

$$\frac{\partial}{\partial \eta}[u(T(\xi,\eta),\xi,\eta)-\xi]_{\xi=x_0,\eta=y_0} = J_{12}.$$

Next, using (31) we get

$$\begin{aligned}
&\tfrac{\partial}{\partial \xi}y_\varepsilon(T(\xi,\eta),\xi,\eta,0) = g(u(T(\xi,\eta),\xi,\eta),\eta,0)T_\xi(\xi,\eta) \\
&\quad +\int_0^{T(\xi,\eta)} g_x(u(t,\xi,\eta),\eta,0)u_\xi(t,\xi,\eta)dt
\end{aligned}$$

$$\begin{aligned}
&\tfrac{\partial}{\partial \eta}y_\varepsilon(T(\xi,\eta),\xi,\eta,0) = g(u(T(\xi,\eta),\xi,\eta),\eta,0)T_\eta(\xi,\eta) \\
&\quad +\int_0^{T(\xi,\eta)} g_x(u(t,\xi,\eta),\eta,0)u_\eta(t,\xi,\eta)+g_y(u(t,\xi,\eta),\eta,0)dt
\end{aligned}$$

hence

$$J(x_0,y_0) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.$$

Since $u(T(\xi,\eta),\xi,\eta)-\xi \in \{f(x_0,y_0)\}^\perp$ we see that

$$J_{11}\xi + J_{12}\eta \in \{f(x_0,y_0)\}^\perp$$

for any $(\xi,\eta) \in \{f(x_0,y_0)\}^\perp \times \mathbb{R}^m$. Hence the assumption of the rank of $J(x_0,y_0)$ is equivalent to the fact that $J(x_0,y_0) : \{f(x_0,y_0)\}^\perp \times \mathbb{R}^m \to \{f(x_0,y_0)\}^\perp \times \mathbb{R}^m$ is an isomorphism. From the Implicit Function Theorem it follows then the existence of $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ and $\varepsilon \neq 0$, there exist $\xi = \xi(\varepsilon)$, $\eta = \eta(\varepsilon)$ such that

$$\begin{aligned}
x(T(\xi(\varepsilon),\eta(\varepsilon),\varepsilon),\xi(\varepsilon),\eta(\varepsilon),\varepsilon)-\xi(\varepsilon)=0 \\
y(T(\xi(\varepsilon),\eta(\varepsilon),\varepsilon),\xi(\varepsilon),\eta(\varepsilon),\varepsilon)-\eta(\varepsilon)=0
\end{aligned}$$

Setting $T(\varepsilon) = T(\xi(\varepsilon),\eta(\varepsilon),\varepsilon)$, $x(t,\varepsilon) = x(t,\xi(\varepsilon),\eta(\varepsilon),\varepsilon)$, $y(t,\varepsilon) = y(t,\xi(\varepsilon),\eta(\varepsilon),\varepsilon)$ and recalling that $x(0,\xi(\varepsilon),\eta(\varepsilon),\varepsilon) = \xi(\varepsilon) = 0$ and $y(0\xi(\varepsilon),\eta(\varepsilon),\varepsilon) = \eta(\varepsilon)$ we see that $x(t,\varepsilon),y(t,\varepsilon))$ is a $T(\varepsilon)$ periodic solution of Equation (1). (25) and (26) follow from (23), (13), and (i) of Lemma 1. □

Remark 3. *(i) Note that, since $u(T(\xi,\eta),\xi,\eta) \in L = \{x_0\} + \{v_0\}^\perp$, we have*

$$v_0^t[u(T(\xi,\eta),\xi,\eta)-x_0] = 0.$$

Hence differentiating with respect to ξ and η:

$$\begin{aligned}
v_0^t[\dot u(T(\xi,\eta),\xi,\eta)T_\xi(\xi,\eta)+u_\xi(T(\xi,\eta),\xi,\eta)]=0 \\
v_0^t[\dot u(T(\xi,\eta),\xi,\eta)T_\eta(\xi,\eta)+u_\eta(T(\xi,\eta),\xi,\eta)]=0
\end{aligned} \tag{34}$$

and then

$$\begin{pmatrix} T_\xi(x_0,y_0) \\ T_\eta(x_0,y_0) \end{pmatrix} = -\frac{1}{v_0^t f(x_0,y_0)} \begin{pmatrix} v_0^t u_\xi(T,x_0,y_0) \\ v_0^t u_\eta(T,x_0,y_0) \end{pmatrix}. \tag{35}$$

(ii) Suppose the following condition holds.

(A) *The linear maps $J_{11} : \{v_0\}^\perp \to \{v_0\}^\perp$ and $J : \{v_0\}^\perp \times \mathbb{R}^m \to \{v_0\}^\perp \times \mathbb{R}^m$ are both invertible.*

For $\xi \in L \cap B(x_0, r_1)$, $\eta \in B(y_0, r_2)$, consider the function $\Phi(\xi, \eta) = u(T(\xi, \eta), \xi, \eta) - \xi$. From (10) we get $\Phi(x_0, y_0) = 0$; moreover,

$$\Phi_\xi(x_0, y_0) = J_{11}.$$

Hence there exists $r_1, r_2 > 0$ and a unique function $\bar{u} : B(y_0, r_2) \to B(x_0, r_1) \cap L$ such that $\bar{u}(y_0) = x_0$ and

$$\Phi(\bar{u}(y), y) = u(T(\bar{u}(y), y), \bar{u}(y), y) - \bar{u}(y) = 0.$$

For any $y \in B(y_0, r_2)$ the function $u(t, \bar{u}(y), y)$ is then a $T(\bar{u}(y), y)$-periodic solution of the discontinuous equation $\dot{x} = f(x, y)$. Next, suppose also that (24) holds, that is the equation

$$\Psi(y) := \int_0^{T(\bar{u}(y),y)} g(u(t, \bar{u}(y), y), y, 0)dt = 0 \tag{36}$$

has the solution $y = y_0$. We have

$$\Psi'(y_0) = J_{21}\bar{u}'(y_0) + J_{22}.$$

We prove that $\Psi'(y_0)$ is invertible. Indeed, suppose that $y \neq 0$ exists such that $\Psi'(y_0)y = 0$. Then

$$\begin{aligned} J\begin{pmatrix} \bar{u}'(y_0)y \\ y \end{pmatrix} &= \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}\begin{pmatrix} \bar{u}'(y_0)y \\ y \end{pmatrix} = \begin{pmatrix} [J_{11}\bar{u}'(y_0) + J_{12}]y \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} [\Phi_\xi(x_0, y_0)\bar{u}'(y_0) + \Phi_\eta(x_0, y_0)]y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

contradicting the fact that J is invertible. Note that, since $\bar{u} : \mathbb{R}^m \to L$, we get $\bar{u}'(y_0) : \mathbb{R}^m \to \{v_0\}^\perp$. So, if (A) holds, besides (A_1)–(A_3), we conclude that Equation (30) has the unique solution (x_0, y_0) and the Jacobian matrix at this point is invertible. Thus the conclusion of Theorem 2 holds. In this case the invertibility of J_{11} implies the existence of a family of periodic solution to the unperturbed equation $\dot{x} = f(x, y)$; however, the invertibility of J implies that only one of these solutions persists for the perturbed equation.

An Example

In this subsection we give an example of application of Theorem 2. The system we consider is

$$\begin{cases} \dot{x}_1 = \left[x_1 + \frac{1}{2}(\operatorname{sgn}(x_2) - 1)a\right]y + x_2 \\ \dot{x}_2 = -\left[x_1 + \frac{1}{2}(\operatorname{sgn}(x_2) - 1)a\right] + yx_2 \\ \dot{y} = \varepsilon g(x, y, \varepsilon) \end{cases} \tag{37}$$

or, in matrix form:

$$\begin{cases} \dot{x} = \begin{cases} A(y)x & \text{if } x_2 > 0 \\ A(y)\left(x - \begin{pmatrix} a \\ 0 \end{pmatrix}\right) & \text{if } x_2 < 0 \end{cases} \\ \dot{y} = \varepsilon g(x, y, \varepsilon) \end{cases} \tag{38}$$

where $a > 0$ and

$$A(y) = \begin{pmatrix} y & 1 \\ -1 & y \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad y, \varepsilon \in \mathbb{R}.$$

Note that

$$\Omega_\pm = \{(x_1,\ x_2) | \pm x_2 > 0\}$$

that is $h(x, y) = x_2$. Let $a > 0$. We prove the following result

Proposition 1. *For any $\eta \in \mathbb{R}$, $\eta > 0$ there exists a unique 2π-periodic solution of*

$$\dot{x} = \begin{cases} A(\eta)x & \text{if } x_2 > 0 \\ A(\eta)\left(x - \begin{pmatrix} a \\ 0 \end{pmatrix}\right) & \text{if } x_2 < 0 \end{cases} \tag{39}$$

given by

$$\hat{u}(t,\eta) = \begin{cases} \xi_0(\eta)e^{\eta t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} & \text{for } 0 \le t \le \frac{\pi}{2} \\ \begin{pmatrix} a \\ 0 \end{pmatrix} + \xi_0(\eta)e^{\eta(t-\pi)}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} & \text{for } \frac{\pi}{2} \le t \le \frac{3}{2}\pi \\ \xi_0(\eta)\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}e^{\eta(t-2\pi)} & \text{for } \frac{3}{2}\pi \le t \le 2\pi \end{cases} \tag{40}$$

where

$$\xi_0(\eta) = \frac{a}{2\sinh(\frac{\pi}{2}\eta)}.$$

Moreover, suppose that $y_0 > 0$ exists such that the function

$$G(\eta) := \int_0^{2\pi} g(\hat{u}(t,\eta),\eta,0)dt$$

has a simple zero at $\eta = y_0$. Then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ there exist $T(\varepsilon)$ such that $\lim_{\varepsilon\to 0}|T(\varepsilon) - 2\pi| = 0$ and Equation (38) has a unique, piecewise smooth, $T(\varepsilon)$-periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$, intersecting transversally the discontinuity line $x_2 = 0$ and such that

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le 2\pi}\{|x(t,\varepsilon) - u(t,\xi_0(y_0),y_0)| + |y(t,\varepsilon) - y_0|\} = 0.$$

Proof. Note that the assumption on $G(\eta)$ means that

$$\int_0^{2\pi} g(\hat{u}(t,y_0),y_0,0)dt = 0 \tag{41}$$

and

$$\int_0^{2\pi} g_x(\hat{u}(t,y_0),y_0,0)\hat{u}_\eta(t,y_0) + g_y(\hat{u}(t,y_0),y_0,0)dt \ne 0. \tag{42}$$

For any $\xi \in \mathbb{R}$, $\xi > 0$, we consider the point $(0,\xi) \in \mathbb{R}^2$ and set $L = \text{span}\{e_2\}$, where $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Note that, for any $\eta \in \mathbb{R}$, L is a transverse hyperplane in $\mathbb{R}^2$ to

$$f_+((0,\xi),\eta) = \xi\begin{pmatrix} 1 \\ \eta \end{pmatrix}.$$

We prove that, for $|\eta - y_0|$ sufficiently small, assumptions (A_1)–(A_3) are satisfied at the point $(\xi_0(\eta),\eta)$.

To this end we first describe the solutions $u(t,\xi,\eta) = (u_1(t,\xi,\eta), u_2(t,\xi,\eta))$ of the unperturbed Equation (39) when $\eta \in I_a$, $|\eta - y_0| \le \sigma y_0$, $0 < \sigma < 1$, and $(0,\xi) \in L \cap \Omega_+$ such that $|\xi - \xi_0(\eta)|$ is sufficiently small. We have

$$u(t,\xi,\eta) = e^{A(\eta)t}\begin{pmatrix} 0 \\ \xi \end{pmatrix} = \xi e^{\eta t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \tag{43}$$

for all $t \geq 0$ as long as $\cos t > 0$, that is for all $0 \leq t \leq \frac{\pi}{2}$. As $h(x,y) = x_2$ we get:

$$h_x(x,y)f_+(x,y) = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, A(y)x \rangle = yx_2 - x_1$$
$$h_x(x,y)f_-(x,y) = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, A(y)\begin{pmatrix} x_1 - a \\ x_2 \end{pmatrix} \rangle = yx_2 + a - x_1$$

So

$$\begin{aligned} h_x(x,y)f_+(x,y) < 0 \Leftrightarrow yx_2 - x_1 < 0 \\ h_x(x,y)f_-(x,y) < 0 \Leftrightarrow yx_2 + a - x_1 < 0 \end{aligned} \tag{44}$$

Being $a > 0$ both conditions are satisfied if $a < x_1 - yx_2$. Since $u(\frac{\pi}{2}, \xi, \eta) = \xi e^{\frac{\pi}{2}\eta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we see that

$$h_x\Big(u\Big(\frac{\pi}{2}, \xi, \eta\Big), \eta\Big) f_\pm\Big(u\Big(\frac{\pi}{2}, \xi, \eta\Big), \eta\Big) < 0 \Leftrightarrow \xi > ae^{-\frac{\pi}{2}\eta}$$

Since $\xi_0(\eta) > ae^{-\frac{\pi}{2}\eta}$, (44) is satisfied provided $|\xi - \xi_0(\eta)|$ is sufficiently small. Indeed let $|\xi - \xi_0(\eta)| < \delta$. Then $\xi - ae^{-\frac{\pi}{2}\eta} > \xi_0(\eta) - ae^{-\frac{\pi}{2}\eta} - \delta = \frac{ae^{-\pi\eta}}{2\sinh \frac{\pi}{2}\eta} - \delta$. So $\xi - ae^{-\frac{\pi}{2}\eta} > 0$ if $0 < \delta < \delta_0(\eta) := \frac{ae^{-\pi\eta}}{2\sinh \frac{\pi}{2}\eta}$. Note

$$\frac{ae^{-\pi(\eta)(1+\sigma)y_0}}{2\sinh \frac{\pi}{2}(\eta)(1+\sigma)y_0} \leq \delta_0(\eta) \leq \frac{ae^{-\pi(\eta)(1-\sigma)y_0}}{2\sinh \frac{\pi}{2}(\eta)(1-\sigma)y_0}$$

for $|\eta - y_0| \leq \sigma y_0$. Next, for $t \geq \frac{\pi}{2}$, $u(t,\xi,\eta) = \begin{pmatrix} u_1(t,\xi,\eta) \\ u_2(t,\xi,\eta) \end{pmatrix}$ solves the equation:

$$\dot{x} = A(\eta)\Big(x - \begin{pmatrix} a \\ 0 \end{pmatrix}\Big)$$
$$x(\tfrac{\pi}{2}) = \xi e^{\frac{\pi}{2}\eta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

until $u_2(t,\xi,\eta) = 0$. Hence:

$$\begin{aligned} u(t,\xi,\eta) &= \begin{pmatrix} a \\ 0 \end{pmatrix} + e^{\eta(t-\frac{\pi}{2})}\begin{pmatrix} \cos(t-\frac{\pi}{2}) & \sin(t-\frac{\pi}{2}) \\ -\sin(t-\frac{\pi}{2}) & \cos(t-\frac{\pi}{2}) \end{pmatrix}\begin{pmatrix} \xi e^{\frac{\pi}{2}\eta} - a \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ 0 \end{pmatrix} + e^{\eta t}\Big(\xi - ae^{-\frac{\pi}{2}\eta}\Big)\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \end{aligned} \tag{45}$$

Note that $u_2(t,\xi,\eta) = (\xi - ae^{-\frac{\pi}{2}\eta})e^{\eta t}\cos t$ and then, since $\xi > ae^{-\frac{\pi}{2}\eta}$, (45) holds for $\frac{\pi}{2} \leq t \leq \frac{3}{2}\pi$. Moreover,

$$u(\tfrac{3}{2}\pi, \xi, \eta) = \begin{pmatrix} a \\ 0 \end{pmatrix} - e^{\frac{3}{2}\pi\eta}\begin{pmatrix} \xi - ae^{-\frac{\pi}{2}\eta} \\ 0 \end{pmatrix} = \begin{pmatrix} a(1+e^{\pi\eta}) - \xi e^{\frac{3}{2}\pi\eta} \\ 0 \end{pmatrix}.$$

Arguing as before, we see that $h_x(x,y)f_\pm(x,y) > 0$ holds if and only if $yx_2 - x_1 > 0$ and when $x = u(\frac{3}{2}\pi, \xi, \eta)$, $y = \eta$ this last condition is equivalent to

$$a(1+e^{\eta\pi}) - \xi e^{\frac{3}{2}\pi\eta} < 0$$

i.e.,

$$\xi > ae^{-\frac{\pi}{2}\eta}(1+e^{-\pi\eta}). \tag{46}$$

It is easily seen that

$$\xi_0(\eta) - ae^{-\frac{\pi}{2}\eta}(1+e^{-\pi\eta}) = \frac{ae^{-2\pi\eta}}{2\sinh \frac{\pi}{2}\eta} > 0.$$

Hence, if $|\xi - \xi_0(\eta)| < \delta$ we get

$$\xi - ae^{-\frac{\pi}{2}\eta}(1 + e^{-\pi\eta}) > \frac{ae^{-2\pi\eta}}{2\sinh\frac{\pi}{2}\eta} - \delta > 0$$

for $\delta < \delta_1(\eta) := \frac{ae^{-2\pi\eta}}{2\sinh\frac{\pi}{2}\eta}$. Thus

$$h_x\left(u\left(\frac{3}{2}\pi, \xi, \eta\right), \eta\right) f_\pm\left(u\left(\frac{3}{2}\pi, \xi, \eta\right), \eta\right) > 0$$

provided $|\xi - \xi_0(\eta)|$ is sufficiently small. Note that condition (46) implies $\xi > ae^{-\frac{\pi}{2}\eta}$ and hence (ii) and (iii) of Lemma 1 hold. So for $|\xi - \xi_0(\eta)| < \frac{ae^{-2\pi\eta}}{2\sinh\frac{\pi}{2}\eta}$, and $|\eta - y_0| \le \sigma y_0$, (46) holds and then $u(t, \xi, \eta)$, $\xi \in L \cap \Omega_+$, intersect transversally the negative x_1 axis at the point

$$\begin{pmatrix} a(1 + e^{\eta\pi}) - \xi e^{\frac{3}{2}\pi\eta} \\ 0 \end{pmatrix}.$$

Next, for $t \ge \frac{3}{2}\pi$, $u(t, \xi, \eta)$ solves the equation:

$$\begin{aligned} &\dot{x} = A(y)x \\ &x(\tfrac{3}{2}\pi) = \begin{pmatrix} a(1 + e^{\eta\pi}) - \xi e^{\frac{3}{2}\pi\eta} \\ 0 \end{pmatrix} \end{aligned}$$

for all $t > \frac{3}{2}\pi$ such that $u(t, \xi, \eta) \in \Omega_+$. Hence

$$\begin{aligned} u(t, \xi, \eta) &= e^{(t - \frac{3}{2}\pi)\eta} \begin{pmatrix} \cos(t - \frac{3}{2}\pi) & \sin(t - \frac{3}{2}\pi) \\ -\sin(t - \frac{3}{2}\pi) & \cos(t - \frac{3}{2}\pi) \end{pmatrix} \begin{pmatrix} a(1 + e^{\eta\pi}) - \xi e^{\frac{3}{2}\pi\eta} \\ 0 \end{pmatrix} \\ &= [a(1 + e^{\eta\pi}) - \xi e^{\frac{3}{2}\pi\eta}] e^{(t - \frac{3}{2}\pi)\eta} \begin{pmatrix} -\sin\ t \\ -\cos t \end{pmatrix} \\ &= [\xi - a(1 + e^{-\eta\pi}) e^{-\frac{\pi}{2}\eta}] e^t \begin{pmatrix} \sin\ t \\ \cos t \end{pmatrix} \end{aligned}$$

So:

$$u_2(t, \xi, \eta) = [\xi - a(1 + e^{-\eta\pi}) e^{-\frac{\pi}{2}\eta}] e^t \cos t > 0$$

for $\frac{3}{2}\pi \le t \le \frac{5}{2}\pi$, since $\xi > ae^{-\frac{\pi}{2}\eta}(1 + e^{-\pi\eta})$. Collecting all together, we see that, for $|\eta - y_0| \le \sigma y_0$ and $|\xi - \xi_0(\eta)|$ sufficiently small

$$u(t, \xi, \eta) = \begin{cases} \xi e^{\eta t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} & \text{for } 0 \le t \le \frac{\pi}{2} \\ \begin{pmatrix} a \\ 0 \end{pmatrix} + e^{\eta t}\left(\xi - ae^{-\frac{\pi}{2}\eta}\right) \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} & \text{for } \frac{\pi}{2} \le t \le \frac{3}{2}\pi \\ \left[\xi - a(1 + e^{-\eta\pi}) e^{-\frac{\pi}{2}\eta}\right] e^{\eta t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} & \text{for } \frac{3}{2}\pi \le t \le \frac{5}{2}\pi. \end{cases} \tag{47}$$

Note that

$$u(2\pi, \xi, \eta) = \left[\xi - a(1 + e^{-\pi\eta}) e^{-\frac{\pi}{2}\eta}\right] e^{2\pi\eta} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{48}$$

hence $T(\xi, \eta) = 2\pi$ for any $(\xi, \eta) \in L \times \mathbb{R}$ with $|\xi - \xi_0(\eta)|$ sufficiently small and $|\eta - y_0| \le \sigma y_0$. We obtain a 2π-periodic solution of Equation (39) if and only if $\xi = u(2\pi, \xi, \eta)$ that is if and only if

$$\xi = [\xi - a(1 + e^{-\pi\eta}) e^{-\eta t_*}] e^{2\pi\eta}$$

and this holds if and only if $(e^{2\pi\eta}-1)\xi = ae^{2\pi\eta}(1+e^{-\eta\pi})e^{-\frac{\pi}{2}\eta}$ or

$$\xi = \frac{ae^{-\frac{\pi}{2}\eta}}{1-e^{-\pi\eta}} = \frac{a}{2\sinh(\frac{\pi}{2}\eta)} = \xi_0(\eta).$$

Note that

$$\xi > \frac{ae^{-\frac{\pi}{2}\eta}}{1-e^{-\pi\eta}} \Leftrightarrow |u(2\pi,\xi,\eta)| > \xi$$

and

$$\xi < \frac{ae^{-\frac{\pi}{2}\eta}}{1-e^{-\pi\eta}} \Leftrightarrow |u(2\pi,\xi,\eta)| < \xi.$$

Hence, for $|\eta - y_0| \le \sigma y_0$, Equation (38) has the unique (up to time translation) unstable 2π-periodic solution:

$$u(t,\xi_0(\eta),\eta) = \begin{cases} \xi_0(\eta)e^{\eta t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} & \text{for } 0 \le t \le \frac{\pi}{2} \\ \begin{pmatrix}a\\ 0\end{pmatrix} + \xi_0(\eta)e^{\eta(t-\pi)}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} & \text{for } \frac{\pi}{2} \le t \le \frac{3}{2}\pi \\ \xi_0(\eta)\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}e^{\eta(t-2\pi)} & \text{for } \frac{3}{2}\pi \le t \le 2\pi. \end{cases}$$

For any $\eta > 0$, $|\eta - y_0| \le \sigma y_0$ we have then a unique (unstable) 2π-periodic solution of Equation (37), (or (38)) and we have seen that (A_1)–(A_3) are satisfied. Note that

$$u(t,\xi_0(\eta),\eta) = \hat{u}(t,\eta) \tag{49}$$

hence

$$\int_0^{2\pi} g(u(t,x_0,y_0),y_0,0)dt = 0,$$

where $x_0 = \xi_0(y_0)$, because of (41). Hence (24) in Theorem 2 is satisfied.

Next we compute the matrix $J(x_0,y_0)$. Recall that $L = \{e_1\}^\perp$ where $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$. With reference to Lemma 1 we also have

$$t_*(\xi,\eta) = \frac{\pi}{2}, \quad t^*(\xi,\eta) = \frac{3}{2}\pi, \quad T(\xi,\eta) = 2\pi$$

so

$$\frac{\partial t_*}{\partial \xi}(\xi,\eta) = \frac{\partial t_*}{\partial \eta}(\xi,\eta) = \frac{\partial t^*}{\partial \xi}(\xi,\eta) = \frac{\partial t^*}{\partial \eta}(\xi,\eta) = \frac{\partial T}{\partial \xi}(\xi,\eta) = \frac{\partial T}{\partial \eta}(\xi,\eta) = 0$$

and

$$\begin{aligned} J_{11}(x_0,y_0) &= \int_0^{2\pi} A(y_0)u_\xi(t,x_0,y_0)dt \\ J_{12}(x_0,y_0) &= \int_0^{2\pi} A(y_0)u_\eta(t,x_0,y_0) + f_y(u(t,x_0,y_0),y_0)dt \\ J_{21}(x_0,y_0) &= \int_0^{2\pi} g_x(u(t,x_0,y_0),y_0,0)u_\xi(t,x_0,y_0)dt \\ J_{22}(x_0,y_0) &= \int_0^{2\pi} g_x(u(t,x_0,y_0),y_0,0)u_\eta(t,x_0,y_0) + g_y(u(t,x_0,y_0),y_0,0)dt. \end{aligned} \tag{50}$$

Now differentiating (47) with respect to ξ we get

$$u_\xi(t,x_0,y_0) = e^{\eta t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}. \tag{51}$$

Similarly:

$$u_\eta(t,x_0,y_0)=\begin{cases} t x_0 e^{y_0 t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} & \text{for } 0\le t\le \frac{\pi}{2}\\ x_0 e^{y_0(t-\pi)}\left[t+\frac{\pi}{2}(e^{\pi y_0}-1)\right]\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} & \text{for } \frac{\pi}{2}\le t\le \frac{3}{2}\pi\\ x_0 e^{y_0(t-2\pi)}\left[t+\frac{\pi}{2}(e^{\pi y_0}-1)(e^{\pi y_0}+3)\right]\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} & \\ \qquad \text{for } \frac{3}{2}\pi\le t\le 2\pi & \end{cases} \tag{52}$$

So we get, after some algebra:

$$J_{11}=\int_0^{2\pi} A(y_0)u_\xi(t,x_0,y_0)dt=\int_0^{2\pi} e^{y_0 t}\begin{pmatrix}y_0\sin t+\cos t\\ y_0\cos t-\sin t\end{pmatrix}dt=(e^{2\pi y_0}-1)\begin{pmatrix}0\\1\end{pmatrix}$$

and similarly

$$\begin{aligned}
&\int_0^{2\pi} u_\eta(t,x_0,y_0)dt=\int_0^{\frac{\pi}{2}} t x_0 e^{y_0 t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt\\
&\quad+\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} x_0 e^{y_0(t-\pi)}\left[t+\frac{\pi}{2}(e^{\pi y_0}-1)\right]\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt\\
&\quad+\int_{\frac{3}{2}\pi}^{2\pi} x_0 e^{y_0(t-2\pi)}\left[t+\frac{\pi}{2}(e^{\pi y_0}-1)(e^{\pi y_0}+3)\right]\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt\\
&=\int_0^{\frac{\pi}{2}} t x_0 e^{y_0 t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x_0 e^{y_0 t}\left[t+\frac{\pi}{2}(e^{\pi y_0}+1)\right]\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt\\
&\quad+\int_{-\frac{\pi}{2}}^{0} x_0 e^{y_0 t}\left[t+\frac{\pi}{2}(e^{\pi y_0}+1)^2\right]\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt\\
&=\frac{\pi}{2}x_0(e^{\pi y_0}+1)^2\int_{-\frac{\pi}{2}}^{0} e^{y_0 t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt-\frac{\pi}{2}x_0(e^{\pi y_0}+1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{y_0 t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt\\
&=\frac{\pi}{2}x_0(e^{\pi y_0}+1)\left[(e^{\pi y_0}+1)\int_{-\frac{\pi}{2}}^{0} e^{y_0 t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{y_0 t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}dt\right]\\
&=\frac{\pi}{2}x_0(e^{\pi y_0}+1)\left[\frac{e^{\pi y_0}+1}{y_0^2+1}\begin{pmatrix}y_0 e^{-\frac{\pi}{2}y_0}-1\\ e^{-\frac{\pi}{2}y_0}+y_0\end{pmatrix}-\frac{1}{y_0^2+1}\begin{pmatrix}y_0 e^{\frac{\pi}{2}y_0}+y_0 e^{-\frac{\pi}{2}y_0}\\ e^{\frac{\pi}{2}y_0}+e^{-\frac{\pi}{2}y_0}\end{pmatrix}\right]\\
&=\frac{\pi}{2}x_0\frac{e^{\pi y_0}+1}{y_0^2+1}\begin{pmatrix}-(e^{\pi y_0}+1)\\ y_0(e^{\pi y_0}+1)\end{pmatrix}=\frac{\pi}{2}x_0\frac{(e^{\pi y_0}+1)^2}{y_0^2+1}\begin{pmatrix}-1\\ y_0\end{pmatrix}
\end{aligned}$$

Moreover, it is easy to check that

$$f_y(x,y)=\begin{cases} x & \text{if } x_2>0\\ x-\begin{pmatrix}a\\0\end{pmatrix} & \text{if } x_2<0\end{cases}$$

from which it easily follows that:

$$\int_0^{2\pi} f_y(u(t,x_0,y_0),y_0)dt=\begin{pmatrix}0\\0\end{pmatrix}.$$

So

$$J_{12}=\int_0^{2\pi} A(y_0)u_\eta(t,x_0,y_0)dt=\frac{\pi}{2}x_0(e^{\pi y_0}+1)^2\begin{pmatrix}0\\1\end{pmatrix}.$$

Note that, according to Remark 2, both $J_{11}(x_0, y_0)$ and $J_{12}(x_0, y_0)$ belong to $T_{x_0}L$. Next

$$J_{21}(x_0, y_0) = \int_0^{2\pi} e^{y_0 t} g_x(u(t, x_0, y_0), y_0, 0) \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt$$

$$J_{22}(x_0, y_0) = \int_0^{2\pi} g_x(u(t, x_0, y_0), y_0, 0) u_\eta(t, x_0, y_0) + g_y(u(t, x_0, y_0), y_0, 0) dt$$

We can simplify the expression for $J_{22}(x_0, y_0)$. Differentiating (49) with respect to η at $\eta = y_0$ we get

$$u_\xi(t, x_0, y_0)\xi_0'(y_0) + u_\eta(t, x_0, y_0) = \hat{u}_\eta(t, y_0)$$

so

$$\begin{aligned} &J_{22}(x_0, y_0) \\ &\quad = \int_0^{2\pi} g_x(\hat{u}(t, y_0), y_0, 0)[\hat{u}_\eta(t, y_0) - u_\xi(t, x_0, y_0)\xi_0'(y_0)] + g_y(\hat{u}(t, y_0), y_0, 0) dt \\ &\quad = \int_0^{2\pi} g_x(\hat{u}(t, y_0), y_0, 0)\hat{u}_\eta(t, y_0) + g_y(\hat{u}(t, y_0), y_0, 0) dt - J_{21}(x_0, y_0)\xi_0'(y_0). \end{aligned}$$

Hence we see that the conditions of Theorem 2 are satisfied if and only if the matrix

$$\begin{pmatrix} e^{\pi y_0} - 1 & \frac{\pi}{2} x_0 (e^{\pi y_0} + 1) \\ J_{21}(x_0, y_0) & \int_0^{2\pi} g_x(\hat{u}(t, y_0), y_0, 0)\hat{u}_\eta(t, y_0) + g_y(\hat{u}(t, y_0), y_0, 0) dt - J_{21}(x_0, y_0)\xi_0'(y_0) \end{pmatrix} \tag{53}$$

is invertible. Noting that

$$\frac{J_{12}}{J_{11}} = \frac{\pi}{2} x_0 \frac{e^{\pi y_0} + 1}{e^{\pi y_0} - 1} = \frac{\pi}{2} \frac{a}{2 \sinh \frac{\pi}{2} y_0} \frac{\cosh \frac{\pi}{2}}{\sinh \frac{\pi}{2} y_0} = \frac{a\pi}{4} \frac{\cosh \frac{\pi}{2}}{\sinh^2 \frac{\pi}{2} y_0}$$

and

$$\xi_0'(y_0) = -\frac{a\pi}{4} \frac{\cosh \frac{\pi}{2} y_0}{\sinh^2 \frac{\pi}{2} y_0}$$

we see that

$$\det \begin{pmatrix} e^{\pi y_0} - 1 & \frac{\pi}{2} x_0 (e^{\pi y_0} + 1) \\ J_{21}(x_0, y_0) & -J_{21}(x_0, y_0)\xi_0'(y_0) \end{pmatrix} = 0$$

and then the matrix in (53) is invertible if and only if

$$\begin{pmatrix} e^{\pi y_0} - 1 & 0 \\ \int_0^{2\pi} e^{y_0 t} g_x(\hat{u}(t, y_0), y_0, 0) \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt & \int_0^{2\pi} g_x(\hat{u}(t, y_0), y_0, 0)\hat{u}_\eta(t, y_0) + g_y(\hat{u}(t, y_0), y_0, 0) dt \end{pmatrix}$$

that is if and only if (42) holds. The conclusion follows from Theorem 2. $\square$

As a concrete example we consider $g(x, y, \varepsilon) = \ell(y)^t x$, where $\ell(y) = \ell_1(y)e_1 + \ell_2(y)e_2$. We have

$$\begin{aligned} &\int_0^{2\pi} g_x(\hat{u}(t, y_0), y_0, 0)\hat{u}_\eta(t, y_0) + g_y(\hat{u}(t, y_0), y_0, 0) dt \\ &= \int_0^{2\pi} \ell(y_0)^t \hat{u}_\eta(t, y_0) + \ell'(y_0)^t \hat{u}(t, y_0) dt. \end{aligned}$$

Now
$$
\begin{aligned}
\int_0^{2\pi} \hat{u}_\eta(t,y_0) = & \int_0^{\frac{\pi}{2}} [\xi_0'(y_0) + t\xi_0(y_0)] e^{y_0 t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt \\
& + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} [\xi_0'(y_0) + (t-\pi)\xi_0(y_0)] e^{y_0(t-\pi)} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt \\
& + \int_{\frac{3}{2}\pi}^{2\pi} [\xi_0'(y_0) + (t-2\pi)\xi_0(y_0)] e^{y_0(t-2\pi)} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt \\
= & \int_0^{\frac{\pi}{2}} [\xi_0'(y_0) + t\xi_0(y_0)] e^{y_0 t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt \\
& - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\xi_0'(y_0) + t\xi_0(y_0)] e^{y_0 t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt \\
& + \int_{-\frac{\pi}{2}}^{0} [\xi_0'(y_0) + t\xi_0(y_0)] e^{y_0 t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt = 0
\end{aligned}
$$
and
$$
\begin{aligned}
\int_0^{2\pi} \hat{u}(t,y_0) = & \int_0^{\frac{\pi}{2}} \xi_0(y_0) e^{y_0 t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt + \begin{pmatrix} a\pi \\ 0 \end{pmatrix} + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \xi_0(y_0) e^{y_0(t-\pi)} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt \\
& + \int_{\frac{3}{2}\pi}^{2\pi} \xi_0(y_0) e^{y_0(t-2\pi)} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt = \begin{pmatrix} a\pi \\ 0 \end{pmatrix}
\end{aligned}
$$

Hence, when $g(x,y,\varepsilon) = \ell^t(y)x$, the conclusion of Proposition 1 holds if the function $\langle \ell(y), e_1 \rangle$ has a simple zero at $y = y_0$ that is
$$
\ell_1(y_0) = 0, \quad \ell_1'(y_0) \neq 0. \tag{54}
$$

Now we take a concrete form of (37)
$$
\left\{ \begin{array}{l}
\dot{x}_1 = \left[x_1 + \frac{1}{2}(\mathrm{sgn}(x_2) - 1)\right] y + x_2 \\
\dot{x}_2 = -\left[x_1 + \frac{1}{2}(\mathrm{sgn}(x_2) - 1)\right] + y x_2 \\
\dot{y} = \varepsilon((y-1)x_1 + y x_2)
\end{array} \right. \tag{55}
$$
so $a = 1$ and $g(x,y,\varepsilon) = (y-1)x_1 + yx_2$. Since $\ell_1(y) = y - 1$, (54) holds for $y_0 = 1$. The unperturbed system of (55) has a form
$$
\left\{ \begin{array}{l}
\dot{x}_1 = \left[x_1 + \frac{1}{2}(\mathrm{sgn}(x_2) - 1)\right] + x_2 \\
\dot{x}_2 = -\left[x_1 + \frac{1}{2}(\mathrm{sgn}(x_2) - 1)\right] + x_2
\end{array} \right. \tag{56}
$$
with periodic solution (40) for $\eta = 1$, $\xi_0(1) = \frac{1}{2\sinh(\frac{\pi}{2})} \cong 0.217269$ and with vector plot on Figures 1 and 2.

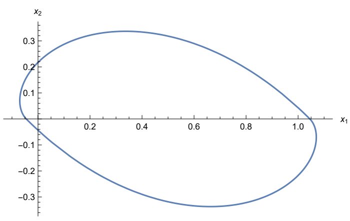

Figure 1. Periodic solution of (56).

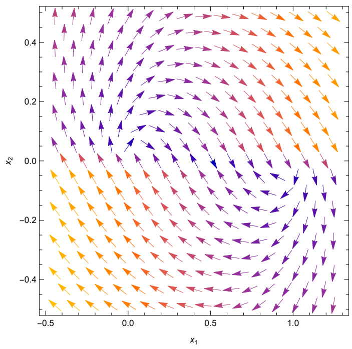

Figure 2. Vector plot of (56).

The periodic solution of (55) with $\varepsilon = 0.01$ is presented in Figures 3–6.

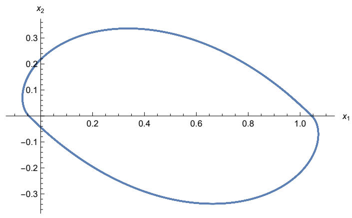

Figure 3. $(x_1(t), x_2(t))$ component of periodic solution of (55) with $\varepsilon = 0.01$.

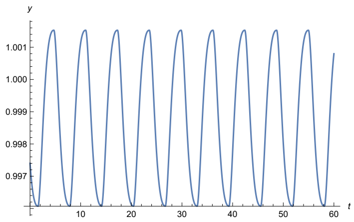

Figure 4. $y(t)$ component of periodic solution of (55) with $\varepsilon = 0.01$.

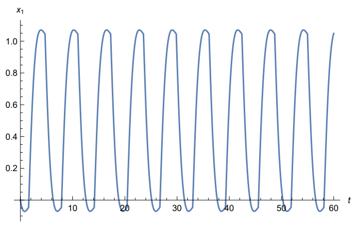

Figure 5. $x_1(t)$ component of periodic solution of (55) with $\varepsilon = 0.01$.

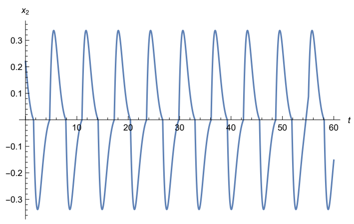

Figure 6. $x_2(t)$ component of periodic solution of (55) with $\varepsilon = 0.01$.

4. Discussion

In this paper we study a persistence of periodic solutions of perturbed slowly varying discontinuous differential equations for a non degenerate case where the unperturbed discontinuous system (3) has a periodic solution for $y = y_0$ and certain non degenerateness conditions are satisfied. We construct a Jacobian matrix and show that, if it is invertible then the perturbed system has a unique periodic solution near the periodic solution of the unperturbed system. We plan to consider a more degenerate case in a forthcoming paper when (3) has a smooth family of periodic solutions.

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