



Daimei Chen, Yanping Chen\* and Teng Wang

Department of Applied Mathematics, School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China; daimeich@126.com (D.C.); tengwustb@126.com (T.W.) \* Correspondence: yanpingch@ustb.edu.cn

Abstract: In this paper, we study the two weight commutators theorem of Riesz potential on an arbitrary homogeneous group  $\mathbb{H}$  of dimension N. Moreover, in accordance with the results in the Euclidean space, we acquire the quantitative weighted bound on homogeneous group.

Keywords: commutators; Riesz potential; homogeneous group

## 1. Introduction and Main Results

Suppose  $\mathbb{H}$  is a nilpotent Lie group, which has the multiplication, inverse, expansion and norm configurations  $(x, y) \mapsto xy, x \mapsto x^{-1}, (t, x) \mapsto t \circ x, x \mapsto \rho(x)$  for  $x, y \in \mathbb{H}, t > 0$ , respectively, then we call *H* being a homogeneous group (see [1] or [2]). The multiplication and inverse operations are polynomials and *t*-action is an automorphism of the group structure, where *t* is of the form

$$t \circ (x_1, \ldots, x_n) = (t^{\beta_1} x_1, \ldots, t^{\beta_n} x_n)$$

for some constants  $0 < \beta_1 \le \beta_2 \le \ldots \le \beta_n$ . Besides,  $\rho(x) := \max_{1 \le j \le n} \{|x_j|^{1/\beta_j}\}$  is a norm linked to the expansion configuration. We call the value  $N = \sum_{j=1}^{n} \tilde{\beta_j}^{-}$  the dimensionality of  $\mathbb H$ . In addition to the Euclidean structure,  $\mathbb H$  is equipped with a homogeneous nilpotent Lie group structure, where Lebesgue measure is a bi-invariant Haar measure, the identity is

the origin  $0, x^{-1} = -x$  and multiplication  $xy, x, y \in \mathbb{H}$ , satisfies

(1)  $(ax)(bx) = ax + bx, x \in \mathbb{H}, a, b \in \mathbb{R};$ 

(2)  $t \circ (xy) = (t \circ x)(t \circ y), x, y \in \mathbb{H}, t > 0;$ 

(3) if z = xy, then  $z_k = P_k(x, y)$ , where  $P_1(x, y) = x_1 + y_1$  and  $P_k(x, y) = x_k + y_k + y_k$  $P_k(x, y)$  for  $k \ge 2$  with a polynomial  $P_k(x, y)$  depending only on  $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$ .

Finally, the Heisenberg group on  $\mathbb{R}^3$  is an example of a homogeneous group. If we define the multiplication

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

 $(x, y, u)(x', y', u') \in \mathbb{R}^3$ , the  $\mathbb{R}^3$  with this group law is the Heisenberg group  $\mathbb{H}_1$ ; a dilation is defined by  $t \circ (x, y, u) = (tx, ty, t^2u)$ , that is the parameters  $\beta_1 = 1, \beta_2 = 1, \beta_3 = 2$ .

**Definition 1.** Let w(x) is a function on  $\mathbb{H}$ , which is non-negative locally integrable. For 1 $\infty$ , we call that w is an  $A_p$  weight, denoted by  $w \in A_p$ , if

$$[w]_{A_p}:=\sup_B \Big(\frac{1}{|B|}\int_B w(x)dx\Big)\Big(\frac{1}{|B|}\int_B \Big(\frac{1}{w}\Big)^{\frac{1}{p-1}}dx\Big)^{p-1}<\infty,$$

The supremum here is taken over of all balls  $B \subset \mathbb{H}$ . We call that the quantity  $[w]_{A_p}$  is the  $A_p$ constant of w. For p = 1, if  $M(w)(x) \le cw(x)$  for a.e.  $x \in \mathbb{H}$ , then we say that w is an  $A_1$  weight,



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denoted by  $w \in A_1$ , where M represents the Hardy-Littlewood maximal function. In addition, let  $A_{\infty} := \bigcup_{1 \le p \le \infty} A_p$ , then we have

$$[w]_{A_{\infty}} := \sup_{B} \left( \frac{1}{|B|} \int_{B} w dx \right) \exp\left( \frac{1}{|B|} \int_{B} \log\left( \frac{1}{w} \right) dx \right) < \infty.$$

**Definition 2.** Let  $x \in \mathbb{H}$ , and w(x) be a non-negative locally integrable function. For  $1 , <math>w \in A_{p,q}$  if

$$[w]_{A_{p,q}} := \sup_{B} \Big( \frac{1}{|B|} \int_{B} w^{q} \Big) \Big( \frac{1}{|B|} \int_{B} w^{-p'} \Big)^{\frac{q}{p'}} < \infty,$$

where p' is the conjugate exponent of p, that is  $\frac{1}{p} + \frac{1}{p'} = 1$ .

**Definition 3.** Suppose  $w \in A_{\infty}$ . Let  $b \in L^1_{loc}(\mathbb{H})$ , then  $b(x) \in BMO_w(\mathbb{H})$  if

$$\|b\|_{BMO_w(\mathbb{H})} := \sup_B \frac{1}{w(B)} \int_B |b(x) - b_B| dx < \infty,$$

where  $b_B := \frac{1}{|B|} \int_B b(x) dx$  and the supremum is taken over of all balls  $B \subset \mathbb{H}$ .

We now review the definition of Riesz potential on homogeneous group. For  $0 < \alpha < N$ ,

$$I_{\alpha}f(x) := \int_{\mathbb{H}} \frac{f(y)}{\rho(xy^{-1})^{N-\alpha}} dy,$$

and the corresponding associated maximal function  $M_{\alpha}$  by

$$M_{\alpha}f(x) = \sup_{x \in B} \frac{1}{|B|^{1-\frac{\alpha}{N}}} \int_{B} |f(y)| dy.$$

The reason why we study the weighted estimates for these operators is because they have a wide range of applications in partial differential equations, Sobolev embeddings or quantum mechanics (see [3] or [4]).

Muckenhoupt and Wheeden [5] are the first scholars to study the Riesz potential. When  $\mathbb{H}$  is an isotropic Euclidean space, Muckenhoupt and Wheeden [5] show that  $I_{\alpha}$  is bounded from  $L^p(w^p)$  to  $L^q(w^q)$  for 1 . Moreover, the sharpconstant in this inequality was given in [6]:

$$||I_{\alpha}||_{L^{p}(w^{p})\to L^{q}(w^{q})} \leq C[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max(1,\frac{p'}{q})}.$$

**Definition 4.** Suppose  $b \in L^1_{loc}(\mathbb{H})$ ,  $f \in L^p(\mathbb{H})$ . Let  $[b, I_{\alpha}]$  be the commutator defined by

$$[b, I_{\alpha}]f(x) := b(x)I_{\alpha}(f)(x) - I_{\alpha}(bf)(x).$$

*The iterative commutators*  $(I_{\alpha})_{b}^{m}$ *,*  $m \in \mathbb{N}$ *, are defined naturally by* 

$$(I_{\alpha})_{b}^{m}f(x) := [b, (I_{\alpha})_{b}^{m-1}]f(x), (I_{\alpha})_{b}^{1}f(x) := [b, I_{\alpha}]f(x).$$

In 2016, Holmes, Rahm and Spencer [7] prove that

$$[b, I_{\alpha}]: L^{p}_{w^{p}}(\mathbb{R}^{n}) \to L^{q}_{\lambda^{q}}(\mathbb{R}^{n}) \Leftrightarrow b \in BMO_{\mu}(\mathbb{R}^{n}),$$

where 1 . Later, the quantitative estimates for iterated commutators of fractional integrals was obtained by N. Accomazzo, J. C. Martínez-Perales and I. P. Rivera-Ríos [8].

In 2013, Sato [9] gave the estimates for singular integrals on homogeneous groups. In [10], X. T. Duong, H. Q. Li and J. Li established the Bloom-type two weight estimates for the commutator of Riesz transform on stratified Lie groups. Moreover, Z. Fan and J. Li [11] obtained the quantitative weighted estimates for rough singular integrals on homogeneous groups.

Motivated by the above estimates, we investigate the quantitative weighted estimation for the higher order commutators of fractional integral operators on homogeneous groups.

In this paper, our main result is the follow theorem.

**Theorem 1.** Let  $0 < \alpha < N$  and 1 , <math>q defined by  $\frac{1}{q} + \frac{\alpha}{N} = \frac{1}{p}$ , and m is a positive integer. Assume that  $\mu, \lambda \in A_{p,q}$  and that  $\nu = \frac{\mu}{\lambda}$ . 1. If  $b \in BMO_{\nu^{1/m}}(\mathbb{H})$ , then

$$\|(I_{\alpha})_{b}^{m}f\|_{L^{q}_{\lambda^{q}}(\mathbb{H})} \leq C_{m,N,\alpha,p}\|b\|_{BMO_{\nu^{1/m}}(\mathbb{H})}^{m}\kappa_{m}\|f\|_{L^{p}_{\mu^{p}}(\mathbb{H})},$$
(1)

where

$$\kappa_m = \sum_{k=0}^m \binom{m}{k} \left( [\lambda]_{A_{p,q}}^{\frac{k}{m}} [\mu]_{A_{p,q}}^{\frac{m-k}{m}} \right)^{(1-\frac{\alpha}{N}) \max\{1, \frac{p_j}{q}\}} A(m,k) B(m,k)$$

and

$$A(m,k) \le \left( \left[ \lambda^{q} \right]_{A_{q}}^{\frac{m+k+1}{2}} \left[ \mu^{q} \right]_{A_{q}}^{\frac{m-k-1}{2}} \right)^{\frac{m-k}{m} \max\{1, \frac{1}{q-1}\}}$$
$$B(m,k) \le \left( \left[ \lambda^{p} \right]_{A_{p}}^{\frac{k-1}{2}} \left[ \mu^{p} \right]_{A_{p}}^{m-\frac{k-1}{2}} \right)^{\frac{k}{m} \max\{1, \frac{1}{p-1}\}}.$$

2. For every  $b \in L^1_{loc}(\mathbb{H})$ , if  $(I_{\alpha})^m_b$  is bounded from  $L^p_{\mu^p}(\mathbb{H})$  to  $L^q_{\lambda^q}(\mathbb{H})$ , then  $b \in BMO_{\nu^{1/m}}(\mathbb{H})$  with

$$\|b\|_{BMO_{\nu^{1/m}}(\mathbb{H})}^{m} \lesssim \|(I_{\alpha})_{b}^{m}\|_{L^{p}_{\mu^{p}}(\mathbb{H}) \to L^{q}_{\lambda^{q}}(\mathbb{H})}.$$

### 2. Domination of the Iterated Commutators by Sparse Operators

#### 2.1. A System of Dyadic Cubes

We define a left-unchanged analogous-distance *d* on  $\mathbb{H}$  by  $d(x, y) = \rho(x^{-1}y)$ , which signifies that there has a constant  $A_0 \ge 1$  such that for any  $x, y, z \in \mathbb{H}$ ,

$$d(x,y) \le A_0[d(x,z) + d(z,y)].$$

Next, let  $B(x, r) := \{y \in \mathbb{H} : d(x, y) < r\}$  be the open ball which is centered on  $x \in \mathbb{H}$  and r > 0 is the radius.

Let  $\mathscr{A}_k$  be *k*-th denumerable index set. A denumerable class  $\mathcal{D} := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k, \mathcal{D}_k := \{Q_{\beta}^k : \beta \in \mathscr{A}_k\}$ , of Borel sets  $Q_{\beta}^k \subseteq \mathbb{H}$  is known as a set of dyadic cubes with arguments  $\delta \in (0, 1)$  and  $0 < a_1 \leq A_1 < \infty$  if it has the characteristics below:

(1)  $\mathbb{H} = \bigcup_{\beta \in \mathscr{A}_k} Q_{\beta}^k$  (disjoint union) for all  $k \in \mathbb{Z}$ ;

- (2) If  $\ell \geq k$ , then either  $Q_{\gamma}^{\ell} \subseteq Q_{\beta}^{k}$  or  $Q_{\beta}^{k} \cap Q_{\gamma}^{\ell} = \emptyset$ ;
- (3) For arbitrary  $(k, \beta)$  and for any  $\ell \leq k$ , there is a exclusive  $\gamma$  such that  $Q_{\beta}^{k} \subseteq Q_{\gamma}^{\ell}$ ;

(4) For arbitrary  $(k, \beta)$  there exists no more that M (a settled geometric constant)  $\gamma$  such that  $Q_{\gamma}^{k+1} \subseteq Q_{\beta}^{k}$ , and  $Q_{\beta}^{k} = \bigcup_{Q \in \mathcal{D}_{k+1}, Q \subseteq Q_{\beta}^{k}} Q$ ;

5) 
$$B(x_{\beta}^{k}, a_{1}\delta^{k}) \subseteq Q_{\beta}^{k} \subseteq B(x_{\beta}^{k}, A_{1}\delta^{k}) =: B(Q_{\beta}^{k});$$

(6) If  $\ell \ge k$  and  $Q_{\gamma}^{\ell} \subseteq Q_{\beta}^{k}$ , then  $B(Q_{\gamma}^{\ell}) \subseteq B(Q_{\beta}^{k})$ . The set  $Q_{\beta}^{k}$  is called a dyadic cube of generation k with centre  $x_{\beta}^{k} \in Q_{\beta}^{k}$  and side length  $\ell(Q_{\beta}^{k}) = \delta^{k}$ .

From the natures of the dyadic system above, for any  $Q_{\beta}^{k}$ ,  $Q_{\gamma}^{k+1}$  and  $Q_{\gamma}^{k+1} \subset Q_{\beta}^{k}$ , we get that there is a constant  $\widetilde{A}_{0} > 0$  such that:

$$|Q_{\gamma}^{k+1}| \le |Q_{\beta}^{k}| \le \widetilde{A}_{0}|Q_{\gamma}^{k+1}|.$$

#### 2.2. Adjacent Systems of Dyadic Cubes

Let { $D^t$  : t = 1, 2, ..., T} be a limited set of the dyadic families, then we call that it is a collection of neighbor systems of dyadic cubes with arguments  $\delta \in (0, 1), 0 < a_1 \le A_1 < \infty$  and  $1 \le C_{adj} < \infty$  if it has the following two characteristics:

(1) For any  $t \in \{1, 2, ..., \mathcal{T}\}$ ,  $\mathcal{D}^t$  is a system of dyadic cubes with arguments  $\delta \in (0, 1)$  and  $0 < a_1 \le A_1 < \infty$ ;

(2) For any ball  $B(x,r) \subseteq \mathbb{H}$  with  $\delta^{k+3} < r \le \delta^{k+2}$ ,  $k \in \mathbb{Z}$ , there have  $t \in \{1, 2, ..., \mathcal{T}\}$ and  $Q \in \mathcal{D}^t$  of generation k which is centered on  ${}^t x_{\beta}^k$  such that  $d(x, {}^t x_{\beta}^k) < 2A_0 \delta^k$  and

$$B(x,r) \subseteq Q \subseteq B(x, C_{adj}r).$$
<sup>(2)</sup>

### 2.3. Sparse Operators

We review the concept of sparse family given in [12] on ordinary spaces of homogeneous description in the sense of Coifman and Weiss [13], which is also suitable in the case of homogeneous groups.

**Definition 5.** Let  $0 < \eta < 1$ , for every  $Q \in S$ , we call that the collection  $S \subset D$  of dyadic cubes be a  $\eta$ -sparse, if there exists a measurable subset  $E_Q \subset Q$  such that  $|E_Q| \ge \eta |Q|$  and the sets  $\{E_Q\}_{Q \in S}$  have only limited overlap.

**Definition 6.** Given a sparse family, the sparse operator  $\mathcal{A}_{\mathcal{S}}$  is defined by

$$\mathcal{A}_{\mathcal{S}}(f)(x) = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \chi_Q(x),$$

where  $\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f(x) dx$ .

In this subfraction, the primary target is to reveal the following quantitative edition of Lacey's pointwise domination inequality.

**Proposition 1.** Let  $0 < \alpha < N$ . Let *m* be a nonnegative integer. For every  $f \in C_c^{\infty}(\mathbb{H})$  and  $b \in L_{loc}^m(\mathbb{H})$ , there exits  $\mathcal{T}$  dyadic systems  $\mathcal{D}^t$ ,  $t = 1, 2, ..., \mathcal{T}$  and  $\eta$ -sparse families  $\mathcal{S}_t \subset \mathcal{D}^t$  such that for a.e.  $x \in \mathbb{H}$ ,

$$|(I_{\alpha})_{b}^{m}f| \leq C_{N,m,\alpha} \sum_{t=1}^{\mathcal{T}} \sum_{k=0}^{m} \binom{m}{k} \mathcal{A}_{\alpha,\mathcal{S}_{t}}^{m,k}(b,f)(x), \quad a.e.x \in \mathbb{H},$$
(3)

where for a sparse family S,  $\mathcal{A}_{\alpha,S}^{m,k}(b,\cdot)$  is the sparse operator given by

$$\mathcal{A}^{m,k}_{\alpha,\mathcal{S}}(b,f)(x) = \sum_{Q \in \mathcal{S}} |b(x) - b_Q|^{m-k} |Q|^{\frac{\alpha}{N}} \langle f(b - b_Q)^k \rangle_Q \chi_Q(x).$$

To show the Proposition 1, we need some auxiliary maximal operators. To begin with, let  $\tilde{j}_0$  be the smallest integer such that

$$2^{j_0} > \max\{3A_0, 2A_0C_{adj}\}\tag{4}$$

and let  $C_{\tilde{i}_0} := 2^{\tilde{j}_0 + 2} A_0$ .

Next we define the grand maximal truncated operator  $\mathcal{M}_{I_{\alpha}}$  as follows:

$$\mathcal{M}_{I_{\alpha}}f(x) = \sup_{x\in B} \mathop{\mathrm{ess\,sup}}_{\xi\in B} |I_{\alpha}(f\chi_{\mathbb{H}\setminus C_{\widetilde{j_0}}B})(\xi)|,$$

where the first supremum is taken over of all balls  $B \subset \mathbb{H}$  satisfying  $x \in B$ . We can know that this operator is of vital importance in the following proof, Given a ball  $B_0 \subset \mathbb{H}$ , for  $x \in B_0$  we also define a local edition of  $\mathcal{M}_{I_\alpha}$  by

$$\mathcal{M}_{I_{\alpha},B_{0}}f(x) = \sup_{x \in B \subset B_{0}} \operatorname{ess\,sup}_{\xi \in B} |I_{\alpha}(f\chi_{C_{\widetilde{j_{0}}}B_{0} \setminus C_{\widetilde{j_{0}}}B})(\xi)|.$$

Now, we claim that the following lemma is true.

**Lemma 1.** Let  $0 < \alpha < N$ . The following pointwise estimates holds: 1. For  $a.e.x \in B_0$ ,

$$|I_{\alpha}(f\chi_{C_{\widetilde{j_0}}B_0})(x)| \leq \mathcal{M}_{I_{\alpha},B_0}f(x).$$

2. There exists a constant  $C_{N,\alpha} > 0$  such that for a.e.  $x \in \mathbb{H}$ ,

$$\mathcal{M}_{I_{\alpha}}f(x) \leq C_{N,\alpha}\Big(M_{\alpha}f(x) + I_{\alpha}|f|(x)\Big).$$

Using the results of Lemma 1, we then prove the Proposition 1.

**Proof of Proposition 1.** In order to proof the Proposition 1, we refer to the thinking in [8] for this domination, which is adapted to our situation of homogeneous groups.

Firstly, we suppose that f is supported in a ball  $B_0 := B(x_0, r) \subset \mathbb{H}$ , next we disintegrate  $\mathbb{H}$  which respect to this ball  $B_0$ . We can do it as follows. We start define the annuli  $U_j := 2^{j+1}B_0 \setminus 2^j B_0, j \ge 0$  and select the minimum integer  $j_0$  such that

$$j_0 > j_0$$
 and  $2^{j_0} > 4A_0$  (5)

Next, for any  $U_i$ , we select the balls

$$\{\widetilde{B}_{j,\ell}\}_{\ell=1}^{L_j},\tag{6}$$

centred in  $U_i$  and with radius  $2^{j-j_0}r$  to cover  $U_i$ . From the doubling property [13], we obtain

$$\lim_{j} L_{j} \leq C_{A_{0},\tilde{j_{0}}},$$
(7)

where  $C_{A_0,\tilde{j_0}}$  is an positive constant that only relates on  $A_0$  and  $\tilde{j_0}$ .

s

We now go over the characters of these  $\widetilde{B}_{j,\ell}$ . Denote  $\widetilde{B}_{j,\ell} := B(x_{j,\ell}, 2^{j-\tilde{j_0}}r)$ , where  $\tilde{j_0}$  is defines as in (4). Then we have  $C_{adj}\widetilde{B}_{j,\ell} := B(x_{j,\ell}, C_{adj}2^{j-\tilde{j_0}}r)$ , which was shown in the proof of Theorem 3.7 in [12] that

$$C_{adj}B_{j,\ell} \cap U_{j+j_0} = \emptyset, \quad \forall j \ge 0 \quad \text{and} \quad \forall \ell = 1, 2, \dots, L_j;$$
(8)

and

$$C_{adj}\widetilde{B}_{j,\ell} \cap U_{j-j_0} = \emptyset, \quad \forall j \ge j_0 \quad \text{and} \quad \forall \ell = 1, 2, \dots, L_j.$$
 (9)

Now, because of the Equation (8) and (9), we see that each  $C_{adj}\widetilde{B}_{j,\ell}$ , at most overlap with  $2j_0 + 1$  annuli  $U_j$ 's. Moreover, for every j and  $\ell$ ,  $C_{\tilde{i}_0}\widetilde{B}_{j,\ell}$  covers  $B_0$ .

Next by observing the (2), there is an integer  $t_0 \in \{1, 2, ..., \mathcal{T}\}$  and  $Q_0 \in \mathcal{D}^{t_0}$  such that  $B_0 \subseteq Q_0 \subseteq C_{adj}B_0$ . Additionally, for this  $Q_0$ , as in Section 2.1 the ball that includes  $Q_0$  and has comparable measure to  $Q_0$  is represented by  $B(Q_0)$ . Consequently,  $B_0$  is overwritten by  $B(Q_0)$  and  $|B(Q_0)| \leq |B_0|$ , where the implicit constant relates only to  $C_{adj}$  and  $A_1$ .

Now we claim that there exists a  $\frac{1}{2}$ -sparse family  $\mathcal{F}^{t_0} \subset \mathcal{D}^{t_0}(Q_0)$ , the set of all dyadic cubes in  $t_0$ -th dyadic system that are contained in  $Q_0$ , such that for *a.e.*  $x \in B_0$ ,

$$|(I_{\alpha})_b^m(f\chi_{C_{\widetilde{j}_0}B(Q_0)})(x)| \le C_{N,m,\alpha}\sum_{k=0}^m \binom{m}{k}\mathcal{B}_{\alpha,\mathcal{F}^{t_0}}^{m,k}(b,f)(x),$$
(10)

where

$$\mathcal{B}_{\alpha,\mathcal{F}^{t_0}}^{m,k}(b,f)(x) = \sum_{Q \in \mathcal{F}^{t_0}} |b(x) - b_{R_Q}|^{m-k} |C_{\tilde{j_0}}B(Q)|^{\frac{\alpha}{N}} \langle f(b - b_{R_Q})^k \rangle_{C_{\tilde{j_0}}B(Q)} \chi_Q(x).$$

Here,  $R_Q$  is the dyadic cube in  $\mathscr{D}^t$  for some  $t \in \{1, 2, ..., \mathcal{T}\}$  such that  $C_{\tilde{j}_0}B(Q) \subset R_Q \subset C_{adj} \cdot C_{\tilde{j}_0}B(Q)$ , where B(Q) is defined as in Section 2.1,  $j_0$  defined as in (5) and  $\tilde{j}_0$  defined as in (4).

Assume that we have already proven the assertion (10). Let us take a partition of  $\mathbb{H}$  as follows:

$$\mathbb{H} = \bigcup_{j=0}^{\infty} 2^j B_0.$$

We next consider the annuli  $U_j := 2^{j+1}B_0 \setminus 2^j B_0$  for  $j \ge 0$  and the covering  $\{\widetilde{B}_{j,\ell}\}_{\ell=1}^{L_j}$ of  $U_j$  as in (6). We note that for each  $\widetilde{B}_{j,\ell}$ , there exist  $t_{j,\ell} \in \{1, 2, ..., \mathcal{T}\}$  and  $\widetilde{Q}_{j,\ell} \in \mathcal{D}^{t_{j,\ell}}$  such that  $\widetilde{B}_{j,\ell} \subseteq \widetilde{Q}_{j,\ell} \subseteq C_{adj}\widetilde{B}_{j,\ell}$ . Therefore, we acquire that for each such  $\widetilde{B}_{j,\ell}$ , the enlargement  $C_{\widetilde{i}_0}B(\widetilde{Q}_{j,l})$  covers  $B_0$  since  $C_{\widetilde{i}_0}\widetilde{B}_{j,\ell}$  covers  $B_0$ .

Next, we utilize (10) to each  $\widetilde{B}_{j,\ell}$ , then we acquire a  $\frac{1}{2}$ -sparse family  $\widetilde{\mathcal{F}}_{j,\ell} \subset \mathcal{D}^{t_{j,\ell}}(\widetilde{Q}_{j,\ell})$  such that (10) can be established for *a.e.*  $x \in \widetilde{B}_{j,\ell}$ .

Now, set  $\mathcal{F} := \bigcup_{j,\ell} \widetilde{\mathcal{F}}_{j,\ell}$ . Then we observe that the balls  $C_{adj} \widetilde{B}_{j,\ell}$  are overlapping not more than  $C_{A_0,\widetilde{j_0}}(2j_0 + 1)$  times, where  $C_{A_0,\widetilde{j_0}}$  is the constant in (7). Then, we can obtain that  $\mathcal{F}$  is a  $\frac{1}{2C_{A_0,\widetilde{j_0}}(2j_0+1)}$ -sparse family and for *a.e.*  $c \in \mathbb{H}$ ,

$$|(I_{\alpha})_{b}^{m}(f)(x)| \leq C_{N,m,\alpha} \sum_{k=0}^{m} {m \choose k} \sum_{Q \in \mathcal{F}} \left( |b(x) - b_{R_{Q}}|^{m-k} |C_{\widetilde{j}_{0}}B(Q)|^{\frac{\alpha}{N}} \langle f(b - b_{R_{Q}})^{k} \rangle_{C_{\widetilde{j}_{0}}B(Q)} \right) \chi_{Q}(x).$$

Since  $C_{\tilde{j}_0}B(Q) \subset R_Q$ , and it is clear that  $|R_Q| \leq \overline{C}|C_{\tilde{j}_0}B(Q)|$  ( $\overline{C}$  depends only on  $C_{adj}$ ), we obtain that  $\langle f \rangle_{C_{\tilde{j}_0}B(Q)} \leq \overline{C} \langle f \rangle_{R_Q}$ . Now, we set  $S_t := \{R_Q \in \mathcal{D}^t : Q \in \mathcal{F}\}, t \in \{1, 2, ..., \mathcal{T}\}$ , then since the fact that  $\mathcal{F}$  is  $\frac{1}{2C_{A_0,\tilde{j}_0}(2j_0+1)}$ -sparse, we can acquire that each family  $S_t$  is  $\frac{1}{2C_{A_0,\tilde{j}_0}(2j_0+1)\tilde{c}}$ -sparse.

Now, we let

$$\eta := \frac{1}{2C_{A_0,\widetilde{j_0}}(2j_0+1)\overline{c}},$$

where  $\overline{c}$  is a constant relating only on  $\overline{C}$ ,  $C_{\widetilde{j_0}}$ . Then it follows that (3) holds, which finishes the proof.  $\Box$ 

**Proof of the Assertion (10).** To demonstrate the assertion it suffice to attest the following recursive computation: there exist the cubes  $P_j \in D^{t_0}(Q_0)$  that does not intersect each other such that  $\sum_i |P_i| \leq \frac{1}{2} |Q_0|$  and for *a.e.*  $x \in B_0$ ,

$$\begin{split} &|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)|\chi_{Q_{0}}(x)\\ &\leq C_{N,m,\alpha}\sum_{k=0}^{m}\binom{m}{k}|b(x)-b_{R_{Q_{0}}}|^{m-k}|C_{\tilde{j}_{0}}B(Q_{0})|^{\frac{\alpha}{N}}\langle f(b-b_{R_{Q_{0}}})^{k}\rangle_{C_{\tilde{j}_{0}}B(Q_{0})}\chi_{Q_{0}}(x)\\ &+\sum_{j}|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(P_{j})})(x)|\chi_{P_{j}}(x). \end{split}$$

Iterating this estimate, we acquire (10) with  $\mathcal{F}^{t_0}$  being the union of all the families  $\{P_j^k\}$ , where  $\{P_j^0\} = \{Q_0\}, \{P_j^1\} = \{P_j\}$  as mentioned above, and  $\{P_j^k\}$  are the cubes acquired at the *k*-th stage of the iterative approach. Clearly  $\mathcal{F}^{t_0}$  is a  $\frac{1}{2}$ -sparse family, since let

$$E_{P_j^k} = P_j^k \setminus \cup_j P_j^{k+1}.$$

Now we prove the recursive estimate. For any countable family  $\{P_j\}_j$  of disjoint cubes  $P_j \subset D^{t_0}(Q_0)$ , we have that

$$\begin{split} &|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)\chi_{Q_{0}}(x) \\ &\leq |(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)\chi_{Q_{0}\setminus\cup_{j}P_{j}}(x) + \sum_{j}|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)\chi_{P_{j}}(x) \\ &\leq |(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)\chi_{Q_{0}\setminus\cup_{j}P_{j}}(x) + \sum_{j}|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})\setminus C_{\tilde{j}_{0}}B(P_{j})})(x)\chi_{P_{j}}(x) \\ &+ \sum_{j}|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(P_{j})})(x)\chi_{P_{j}}(x) \end{split}$$

So we just have to reveal that we can opt for a family of pairwise disjoint cubes  $\{P_j\} \subset \mathcal{D}^{t_0}(Q_0)$  such that  $\sum_j |P_j| \leq \frac{1}{2} |Q_0|$  and that for *a.e.*  $x \in B_0$ ,

$$|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)|\chi_{Q_{0}\setminus\cup_{j}P_{j}}(x)+\sum_{j}|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})\setminus C_{\tilde{j}_{0}}B(P_{j})})(x)|\chi_{P_{j}}(x)| \leq C_{N,m,\alpha}\sum_{k=0}^{m}\binom{m}{k}|b(x)-b_{R_{Q_{0}}}|^{m-k}|C_{\tilde{j}_{0}}B(Q_{0})|^{\frac{\alpha}{N}}\langle f(b-b_{R_{Q_{0}}})^{k}\rangle_{C_{\tilde{j}_{0}}B(Q_{0})}\chi_{Q_{0}}(x).$$

Using that  $(I_{\alpha})_{b}^{m} f = (I_{\alpha})_{b-c}^{m} f$  for any  $c \in \mathbb{R}$ , and also that

$$(I_{\alpha})_{b-c}^{m}f = \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} I_{\alpha} ((b-c)^{k}f) (b-c)^{m-k},$$

it follows that

$$\begin{split} &|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)|\chi_{Q_{0}\setminus\cup_{j}P_{j}(x)}+\sum_{j}|(I_{\alpha})_{b}^{m}(f\chi_{C_{\tilde{j}_{0}}B(Q_{0})\setminus C_{\tilde{j}_{0}}B(P_{j})})(x)|\chi_{P_{j}}(x)\\ &\leq \sum_{k=0}^{m}\binom{m}{k}|b(x)-b_{R_{Q_{0}}}|^{m-k}|I_{\alpha}((b-b_{R_{Q_{0}}})^{k}f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)|\chi_{Q_{0}\setminus\cup_{j}P_{j}}(x)\\ &+\sum_{k=0}^{m}\binom{m}{k}|b(x)-b_{R_{Q_{0}}}|^{m-k}\sum_{j}|I_{\alpha}((b-b_{R_{Q_{0}}})^{k}f\chi_{C_{\tilde{j}_{0}}B(Q_{0})\setminus C_{\tilde{j}_{0}}B(P_{j})})(x)|\chi_{P_{j}}(x)\\ &=:W_{1}+W_{2}. \end{split}$$

Now we define the set  $E = \bigcup_{k=0}^{m} E_k$ , where

$$E_{k} = \{x \in B_{0} : \mathcal{M}_{I_{\alpha},B_{0}}((b-b_{R_{Q_{0}}})^{k}f)(x) > C_{N,m,\alpha}|C_{\tilde{j}_{0}}B(Q_{0})|^{\frac{\alpha}{N}}\langle (b-b_{R_{Q_{0}}})^{k}f \rangle_{C_{\tilde{j}_{0}}B(Q_{0})}\},$$

with  $C_{N,m,\alpha}$  being a positive number to be chosen.

From [8], we can choose  $C_{N,m,\alpha}$  big enough (depending on  $C_{\tilde{j}_0}, C_{adj}$ , and  $A_1$ ) such that

$$|E| \leq \frac{1}{4\widetilde{A_0}}|B_0|,$$

where  $\widetilde{A_0}$  is defined in Section 2.1. We now utilize the Calderón-Zygmund decomposition to the function  $\chi_E$  on  $B_0$  at the height  $\lambda := \frac{1}{2\widetilde{A_0}}$ , to acquire pairwise disjoint cubes  $\{P_j\} \subset \mathcal{D}^{t_0}(Q_0)$  such that

$$\frac{1}{2\widetilde{A_0}}|P_j| \le |P_j \cap E| \le \frac{1}{2}|P_j|$$

and  $|E \setminus \bigcup_j P_j| = 0$ . This implies that

$$\sum_j |P_j| \leq rac{1}{2} |B_0|$$
 and  $P_j \cap E^c 
eq arnothing.$ 

Fix some *j*. Since we have  $P_j \cap E^c \neq \emptyset$ , we observe that

$$\mathcal{M}_{I_{\alpha},B_{0}}((b-b_{R_{Q_{0}}})^{k}f)(x) \leq C_{N,m,\alpha}|C_{\tilde{j}_{0}}B(Q_{0})|^{\frac{\alpha}{N}}\langle (b-b_{R_{Q_{0}}})^{k}f\rangle_{C_{\tilde{j}_{0}}B(Q_{0})},$$

which allows us to control the summation in  $W_2$  by considering the cube  $P_j$ .

Now by (i) in Lemma 1, we know that

$$|I_{\alpha}((b-b_{R_{Q_0}})^k f\chi_{C_{\tilde{j}_0}B(Q_0)})(x)| \le \mathcal{M}_{I_{\alpha},B_0}((b-b_{R_{Q_0}})^k f)(x), \quad \text{for } a.e. \ x \in B_0.$$

Since  $|E \setminus \bigcup_j P_j| = 0$ , we have that

$$\mathcal{M}_{I_{\alpha},B_{0}}\left(\left(b-b_{R_{Q_{0}}}\right)^{k}f\right)(x)$$
  
$$\leq C_{N,m,\alpha}|C_{\widetilde{j_{0}}}B(Q_{0})|^{\frac{\alpha}{N}}\left\langle\left(b-b_{R_{Q_{0}}}\right)^{k}f\right\rangle_{C_{\widetilde{j_{0}}}B(Q_{0})}, \quad \text{for } a.e. \ x \in B_{0} \setminus \cup_{j}P_{j}.$$

Consequently,

$$|I_{\alpha}((b-b_{R_{Q_{0}}})^{k}f\chi_{C_{\tilde{j}_{0}}B(Q_{0})})(x)| \leq C_{N,m,\alpha}|C_{\tilde{j}_{0}}B(Q_{0})|^{\frac{\alpha}{N}}\langle (b-b_{R_{Q_{0}}})^{k}f\rangle_{C_{\tilde{j}_{0}}B(Q_{0})}, \quad \text{for } a.e. \ x \in B_{0} \setminus \cup_{j}P_{j}.$$

These estimates allow us to control the remaining terms in  $W_1$ , so we are done.  $\Box$ 

**Proof of Lemma 1.** Now we give the proof process of Lemma 1.

The result in the Euclidean space case can be referred to as [8]. Now, we can adapt the proof in [8] to our setting of homogeneous groups.

(i) Let *r* is close enough to 0 such that  $B(x,r) \subset B_0$ . Then,

$$\begin{split} |I_{\alpha}(f\chi_{C_{\widetilde{j_0}}B_0})(x)| &\leq |I_{\alpha}(f\chi_{C_{\widetilde{j_0}}B(x,r)})(x)| + |I_{\alpha}(f\chi_{C_{\widetilde{j_0}}B_0\setminus C_{\widetilde{j_0}}B(x,r)})(x)| \\ &\leq |I_{\alpha}(f\chi_{C_{\widetilde{j_0}}B(x,r)})(x)| + \mathcal{M}_{I_{\alpha},B_0}f(x), \end{split}$$

the estimate for the first term follows by standard computations involving a dyadic annulitype decomposition of the B(x, r).

$$\begin{split} |I_{\alpha}(f\chi_{C_{\tilde{j}_{0}}B(x,r)})(x)| &= \left| \int_{\mathbb{H}} \frac{f(y)\chi_{C_{\tilde{j}_{0}}B(x,r)}}{d(x,y)^{N-\alpha}}dy \right| \\ &\leq \int_{B(x,C_{\tilde{j}_{0}}r)} \frac{|f(y)|}{d(x,y)^{N-\alpha}}dy \\ &= \sum_{i=-\infty}^{1} \int_{B(x,C_{\tilde{j}_{0}}^{i}r)\setminus B(x,C_{\tilde{j}_{0}}^{i-1}r)} \frac{|f(y)|}{d(x,y)^{N-\alpha}}dy \\ &\leq \sum_{i=-\infty}^{1} (C_{\tilde{j}_{0}}^{i-1}r)^{\alpha-N} \int_{B(x,C_{\tilde{j}_{0}}^{i}r)^{N}} |f(y)|dy \\ &= \sum_{i=-\infty}^{1} (\frac{1}{C_{\tilde{j}_{0}}})^{\alpha-N} (C_{\tilde{j}_{0}}^{i}r)^{\alpha} \frac{1}{(C_{\tilde{j}_{0}}^{i}r)^{N}} \int_{B(x,C_{\tilde{j}_{0}}^{i}r)} |f(y)|dy \\ &\leq C_{N,\alpha,C_{\tilde{j}_{0}}}r^{\alpha}Mf(x). \end{split}$$

Then,

$$|I_{\alpha}(f\chi_{C_{\widetilde{I}\alpha}B_0})(x)| \le C_{N,\alpha,C_{\widetilde{I}\alpha}}r^{\alpha}Mf(x) + \mathcal{M}_{I_{\alpha},B_0}f(x), \tag{11}$$

the estimate in (i) is settled letting  $r \rightarrow 0$  in (11).

(ii) Let  $x, \xi \in B := B(x_0, r)$ . Let  $B_x$  be the closed ball with radius  $4(A_0 + C_{\tilde{j}_0})r$ , which centered at x. Then  $C_{\tilde{j}_0}B \subset B_x$ , and we acquire

$$\begin{split} |I_{\alpha}(f\chi_{\mathbb{H}\backslash C_{\widetilde{j_0}}B})(\xi)| &= |I_{\alpha}(f\chi_{\mathbb{H}\backslash B_x})(\xi) + I_{\alpha}(f\chi_{B_x\backslash C_{\widetilde{j_0}}B})(\xi)| \\ &\leq |I_{\alpha}(f\chi_{\mathbb{H}\backslash B_x})(\xi) - I_{\alpha}(f\chi_{\mathbb{H}\backslash B_x})(x)| \\ &+ |I_{\alpha}(f\chi_{B_x\backslash C_{\widetilde{j_0}}B})(\xi)| + |I_{\alpha}(f\chi_{\mathbb{H}\backslash B_x})(x)| \end{split}$$

For the first term, since  $\rho$  is homogeneous of degree  $\alpha - N$ , and by using the Proposition 1.7 in [1], we get

$$\begin{split} &|I_{\alpha}(f\chi_{\mathbb{H}\backslash B_{x}})(\xi) - I_{\alpha}(f\chi_{\mathbb{H}\backslash B_{x}})(x)| \\ &\leq \int_{\mathbb{H}\backslash B_{x}} |f(y)| \left| \frac{1}{d(y,\xi)^{N-\alpha}} - \frac{1}{d(x,y)^{N-\alpha}} \right| dy \\ &\leq C_{N,\alpha} \int_{\mathbb{H}\backslash B_{x}} \frac{2r}{d(x,y)^{N-\alpha+1}} |f(y)| dy \\ &= C_{N,\alpha} \sum_{i=1}^{\infty} \int_{2^{i}B_{x}\backslash 2^{i-1}B_{x}} \frac{2r}{d(x,y)^{N-\alpha+1}} |f(y)| dy \\ &\leq C_{N,\alpha} \sum_{i=1}^{\infty} \frac{2r}{\left(2^{i-1}|B_{x}|^{\frac{1}{N}}\right)^{N-\alpha+1}} \int_{2^{i}B_{x}} |f(y)| dy \\ &= C_{N,\alpha} \sum_{i=1}^{\infty} \frac{2r}{\left(2^{i-1}2^{2}r(A_{0}+C_{\tilde{j}_{0}})\right)^{N-\alpha+1}} \int_{2^{i}B_{x}} |f(y)| dy \\ &= C_{N,\alpha} \sum_{i=1}^{\infty} \frac{2r}{2^{i+1}r(A_{0}+C_{\tilde{j}_{0}})} \cdot \frac{1}{\left(2^{i+1}r(A_{0}+C_{\tilde{j}_{0}})\right)^{N-\alpha}} \int_{2^{i}B_{x}} |f(y)| dy \\ &\leq C_{N,\alpha} M_{\alpha}f(x). \end{split}$$

Next, for  $\xi \in B, y \in B_x \setminus C_{\tilde{j}_0} B$ , we have  $d(y, \xi) > 2^{\tilde{j}_0} r$ . Then we have

$$\begin{split} |I_{\alpha}(f\chi_{B_{x}\setminus C_{\widetilde{j_{0}}}B})(\xi)| &\leq \int_{B_{x}\setminus C_{\widetilde{j_{0}}}B} \frac{1}{d(y,\xi)^{N-\alpha}} |f(y)| dy \\ &\leq \frac{1}{|2^{\widetilde{j_{0}}}r|^{N-\alpha}} \int_{B_{x}} |f(y)| dy \\ &= C_{N,\alpha} \frac{1}{|4(A_{0}+C_{\widetilde{j_{0}}})r|^{N-\alpha}} \int_{B_{x}} |f(y)| dy \\ &\leq C_{N,\alpha} M_{\alpha}f(x). \end{split}$$

Finally, we observe that

$$\begin{split} |I_{\alpha}(f\chi_{\mathbb{H}\setminus B_{x}})(x)| &= |\int_{\mathbb{H}\setminus B_{x}}\frac{f(y)}{d(x,y)^{N-\alpha}}dx| \\ &\leq \int_{\mathbb{H}}\frac{|f(y)|}{d(x,y)^{N-\alpha}}dx \\ &= I_{\alpha}|f|(x), \end{split}$$

which finishes the proof of (ii).  $\Box$ 

Next, we review that the dyadic weighted *BMO* space associated with the system  $\mathcal{D}^t$  is defined as

$$BMO_{\eta,\mathcal{D}^{t}}(\mathbb{H}) := \{b \in L^{1}_{loc}(\mathbb{H}) : \|b\|_{BMO_{\eta,\mathcal{D}^{t}}} < \infty\},\$$

where  $||b||_{BMO_{\eta,D^t}} = \sup_{Q \in D^t} \frac{1}{\eta(Q)} \int_Q |b(x) - b_Q| dx$ . Then according to the dyadic structure theorem studies in [14], one has

$$BMO_{\eta}(\mathbb{H}) = \bigcap_{t=1}^{\mathcal{T}} BMO_{\eta,\mathcal{D}^{t}}(\mathbb{H}).$$

Now, to verify a function *b* is in  $BMO_{\eta}(\mathbb{H})$ , it suffices to verify it belongs to each weighted dyadic *BMO* space  $BMO_{\eta,\mathcal{D}^t}(\mathbb{H})$ . Given a dyadic cube  $Q \in \mathcal{D}^t$  with  $t = 1, 2, ..., \mathcal{T}$ , and a measurable function *f* on  $\mathbb{H}$ , we define the local mean oscillation of *f* on *Q* by

$$\omega_{\lambda}(f;Q) = \inf_{c \in \mathbb{R}} \left( (f-c)\chi_Q \right)^* (\lambda|Q|), \quad 0 < \lambda < 1,$$

where

$$\left((f-c)\chi_Q\right)^*(\lambda|Q|) = \sup_{E \subset Q, |E|=\lambda|Q|} \inf_{x \in E} |(f-c)(x)|.$$

With these notation and dyadic structure theorem above, following the same proof in [10], we also acquire that for any weight  $\eta \in A_2$ , we have

$$\|b\|_{BMO_{\eta}(\mathbb{H})} \leq C \sum_{t=1}^{\mathcal{T}} \sup_{Q \in \mathcal{D}^{t}} \omega_{\lambda}(b; Q) \frac{|Q|}{\eta(Q)}, \quad 0 < \lambda \leq 2^{N+1},$$
(12)

where *C* depends on  $\eta$ .

**Proposition 2.** Suppose that  $\mathbb{H}$  is a homogeneous group with dimension  $N, b \in L^1_{loc}(\mathbb{H})$ . Then for any cube  $Q \subset \mathbb{H}$ , there exist measurable set  $F_i \subset Q$  with i = 1, 2, such that

$$\omega_{2^{\frac{1}{N+2}}}(b;Q) \le b(x) - b(y), \ \forall (x,y) \in F_1 \times F_2.$$

**Proof.** We take ideas from N. Accomazzo, J. C. Martínez-Perales and I. P. Rivera-Ríos [8]. In [8], for any cube  $Q \in D^t$  with t = 1, 2, ..., T, there exists a subset  $E \subset Q$  with  $|E| = \frac{1}{2^{N+2}}|Q|$  such that for every  $x \in E$ ,

$$\omega_{2^{\frac{1}{N+2}}}(b;Q) \le |b(x) - m_b(Q)|,$$

where  $m_b(Q)$  is a not necessarily unique number that satisfies

$$\max\left\{|\{x \in Q : b(x) > m_b(Q)\}|, |\{x \in Q : b(x) < m_b(Q)\}|\right\} \le \frac{|Q|}{2}.$$

Let  $E_1 \subset Q$  with  $|E| = \frac{1}{2}|Q|$  and such that  $b(x) \ge m_b(Q)$  for every  $x \in E_1$ . Further let  $E_2 = Q \setminus E_1$ , then  $|E_2| = \frac{1}{2}|Q|$  and for every  $x \in E_2$ ,  $b(x) \le m_b(Q)$ .

We obtain that at least half of the set *E* is contained either in  $E_1$  or in  $E_2$  since *Q* is the disjoint union of  $E_1$  and  $E_2$ . Without loss of generality, we assume that half of *E* is in  $E_1$ , then we let  $F_1 = E \cap E_1, F_2 = E_2 \cap (E \cap E_1)^c$ , we have

$$|F_1| = |E| - |E \cap (E \cap E_1)^C| \ge |E| - \frac{|E|}{2} = \frac{|Q|}{2^{N+3}},$$

and

$$|F_2| = |E_2| - |E_2 \cap (E \cap E_1)| \ge \frac{1}{2}|Q| - \frac{1}{2^{N+3}}|Q| = (\frac{1}{2} - \frac{1}{2^{N+3}})|Q|.$$

Then if  $x \in F_1$  and  $y \in F_2$ , we have that

$$\omega_{2^{\frac{1}{N+2}}}(b;Q) \leq b(x) - m_b(Q) \leq b(x) - b(y),$$

which shows that Proposition 2 holds.  $\Box$ 

Given a dyadic grid  $\mathcal{D}$ , define the dyadic Riesz potential operator

$$I_{\alpha}^{\mathcal{D}}f(x) = \sum_{Q \in \mathcal{D}} \frac{1}{|Q|^{1-\frac{\alpha}{N}}} \int_{Q} |f(y)| dy \chi_{Q}(x).$$

**Proposition 3.** *Given*  $0 < \alpha < N$ *, then for any dyadic grid* D*,* 

$$I^{\mathcal{D}}_{\alpha}f(x) \lesssim I_{\alpha}f(x). \tag{13}$$

**Proof.** The result in the Euclidean setting is from the Proposition 2.1 in [15]. Here, we can adapt the proof in [15] to our setting of spaces of homogeneous type.  $\Box$ 

# 3. Proof of Theorem 1

To proof (i), we are following the ideas in [16] or [8].

Let  $\mathcal{D}$  be a dyadic system in  $\mathbb{H}$  and let  $\mathcal{S}$  be a sparse family from  $\mathcal{D}$ . We know

$$\mathcal{A}_{\alpha,\mathcal{S}}^{m,k}(b,f)(x) = \sum_{Q \in \mathcal{S}} |b(x) - b_Q|^{m-k} |Q|^{\frac{\alpha}{N}} \langle (b - b_Q)^k f \rangle_Q \chi_Q(x),$$

by duality, we have that

$$\begin{split} \|\mathcal{A}_{\alpha,\mathcal{S}}^{m,k}(b,f)(x)\|_{L^{q}_{\lambda^{q}}(\mathbb{H})} &\leq \sup_{g:\|g\|_{L^{q'}_{\lambda^{q'}}(\mathbb{H})} = 1} \sum_{Q \in \mathcal{S}} \left( \int_{Q} |g(x)\lambda^{q}| |b(x) - b_{Q}|^{m-k} dx \right) |Q|^{\frac{\alpha}{N}} \\ &\times \left( \frac{1}{|Q|} \int_{Q} |b(x) - b_{Q}|^{k} |f(x)| dx \right). \end{split}$$

By Lemma 3.5 in [12], there exists a sparse family  $\tilde{S} \subset D$  such that  $S \subset \tilde{S}$  and for every cube  $Q \in \tilde{S}$ , for *a.e.*  $x \in Q$ ,

$$|b(x) - b_Q| \le C_N \sum_{P \in \widetilde{\mathcal{S}}, P \subset Q} \Omega(b, P) \chi_P(x),$$

where  $\Omega(b, P) = \frac{1}{|P|} \int_{P} |b(x) - b_P| dx$ 

Assume that  $b \in BMO_{\eta}(\mathbb{H})$  with  $\eta$  to be chosen, then we have for *a.e.*  $x \in Q$ ,

$$\begin{split} |b(x) - b_Q| &\leq C_N \sum_{P \in \widetilde{\mathcal{S}}, P \subset Q} \frac{1}{\eta(P)} \int_P |b(x) - b_P| dx \cdot \frac{\eta(P)}{|P|} \chi_P(x) \\ &\leq C_N \|b\|_{BMO_\eta}(\mathbb{H}) \sum_{P \in \widetilde{\mathcal{S}}, P \subset Q} \frac{\eta(P)}{|P|} \chi_P(x). \end{split}$$

Then, we further have

$$\begin{split} &\|\mathcal{A}_{\alpha,\mathcal{S}}^{m,k}(b,f)(x)\|_{L^{q}_{\lambda^{q}}(\mathbb{H})} \\ &\leq C_{N}\|b\|_{BMO_{\eta}(\mathbb{H})}^{m} \sup_{g:\|g\|_{L^{q'}_{\lambda^{q'}}(\mathbb{H})} = 1} \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_{Q} |g(x)\lambda^{q}| \Big(\sum_{P \in \widetilde{\mathcal{S}}, P \subset Q} \frac{\eta(P)}{|P|} \chi_{P}(x)\Big)^{m-k} dx\right) \\ &\times \left(\frac{1}{|Q|} \int_{Q} \Big(\sum_{P \in \widetilde{\mathcal{S}}, P \subset Q} \frac{\eta(P)}{|P|} \chi_{P}(x)\Big)^{k} |f(x)| dx\right) \cdot |Q| \cdot |Q|^{\frac{\kappa}{N}}. \end{split}$$

Next, note that for each  $\ell \in \mathbb{N}$ , from [12], for an arbitrary function *h*, we have

$$\begin{split} &\int_{Q} |h(x)| \Big(\sum_{Q \in \widetilde{\mathcal{S}}, P \subset Q} \frac{\eta(P)}{|P|} \chi_{P}(x)\Big)^{\ell} dx \\ &\lesssim \int_{Q} \mathcal{A}_{\widetilde{\mathcal{S}}, \eta}^{\ell}(|h|)(x) dx, \end{split}$$

where  $\mathcal{A}_{\widetilde{S},\eta}(|h|)(x) := \mathcal{A}_{\widetilde{S}}(|h|)\eta$ ,  $\mathcal{A}_{\widetilde{S}}(h) := \sum_{Q \in \widetilde{S}} h_Q \chi_Q$  and  $\mathcal{A}_{\widetilde{S},\eta}^{\ell} f$  stands for the  $\ell$ -th iteration of  $\mathcal{A}_{\widetilde{S},\eta}$ .

Then we have

$$\begin{split} &\|\mathcal{A}_{\alpha,\mathcal{S}}^{m,k}(b,f)(x)\|_{L^{q}_{\lambda^{q}}(\mathbb{H})} \\ &\leq C_{N}\|b\|_{BMO_{\eta}(\mathbb{H})}^{m} \sup_{g:\|g\|_{L^{q'}_{\lambda^{q'}}(\mathbb{H})} = 1} \sum_{Q\in\mathcal{S}} \left(\int_{Q} \mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{m-k}(|g|\lambda^{q})\right) \cdot \frac{1}{|Q|^{1-\frac{\alpha}{N}}} \left(\int_{Q} \mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\right) \\ &\leq C_{N}\|b\|_{BMO_{\eta}(\mathbb{H})}^{m} \sup_{g:\|g\|_{L^{q'}_{\lambda^{q'}}(\mathbb{H})} = 1} \int_{\mathbb{H}} \sum_{Q\in\mathcal{S}} \frac{1}{|Q|^{1-\frac{\alpha}{N}}} \left(\int_{Q} \mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\chi_{Q}(x)\right) \cdot \mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{m-k}(|g|\lambda^{q}) \\ &= C_{N}\|b\|_{BMO_{\eta}(\mathbb{H})}^{m} \sup_{g:\|g\|_{L^{q'}_{\lambda^{q'}}(\mathbb{H})} = 1} \int_{\mathbb{H}} I_{\mathcal{S}}^{\alpha} \left(\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\right)(x) \left(\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{m-k}(|g|\lambda^{q})\right)(x) dx, \end{split}$$

where  $I_{\mathcal{S},\eta}^{\alpha}f := I_{\mathcal{S}}^{\alpha}(f)\eta$  and  $I_{\mathcal{S}}^{\alpha}f(x) = \sum_{Q \in \mathcal{S}} \frac{1}{|Q|^{1-\frac{\alpha}{N}}} \int_{Q} |f|\chi_{Q}(x).$ 

From (13) and the boundedness of  $I_{\alpha}f$ , if p, q,  $\alpha$  are as in the hypothesis of Theorem 1.1 and  $w \in A_{p,q}$ ,  $S \subset D$ , then

$$\|I_{\mathcal{S}}^{\alpha}\|_{L^{p}_{w^{p}}(\mathbb{H}) \to L^{q}_{w^{q}}(\mathbb{H})} \leq C_{N,p,q,\alpha}[w]_{A_{p,q}}^{(1-\frac{\alpha}{N})\max\{1,\frac{p_{r}}{q}\}}.$$
(14)

Observe that  $\mathcal{A}_{\widetilde{\mathcal{S}}}$  is self-adjoint, then

$$\int_{\mathbb{H}} I^{\alpha}_{\mathcal{S}} \Big( \mathcal{A}^{k}_{\widetilde{\mathcal{S}}, \eta}(|f|) \Big) \Big( \mathcal{A}^{m-k}_{\widetilde{\mathcal{S}}, \eta}(|g|\lambda^{q}) \Big) = \int_{\mathbb{H}} \mathcal{A}_{\widetilde{\mathcal{S}}} \mathcal{A}^{m-k-1}_{\widetilde{\mathcal{S}}, \eta} \Big[ I^{\alpha}_{\mathcal{S}, \eta} \Big( \mathcal{A}^{K}_{\widetilde{\mathcal{S}}, \eta}(|f|) \Big) \Big] |g|\lambda^{q}.$$

By Hölder inequality, we have that

$$\|\mathcal{A}_{\alpha,\mathcal{S}}^{m,k}(b,f)(x)\|_{L^{q}_{\lambda^{q}}(\mathbb{H})} \leq C_{N}\|b\|_{BMO_{\eta}(\mathbb{H})}^{m}\|\mathcal{A}_{\widetilde{\mathcal{S}}}\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{m-k-1}I^{\alpha}_{\mathcal{S},\eta}\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\|_{L^{q}_{\lambda^{q}}(\mathbb{H})}.$$

Applying that  $\|\mathcal{A}_{\widetilde{\mathcal{S}}}\|_{L^p_w(\mathbb{H})} \leq C_{N,p}[w]^{\max\{1,\frac{1}{p-1}\}}_{A_p}$  (see, e.g., [17] ),

$$\begin{split} &\|\mathcal{A}_{\widetilde{\mathcal{S}}}\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{m-k-1}I_{\mathcal{S},\eta}^{\alpha}\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\|_{L^{q}_{\lambda^{q}}(\mathbb{H})} \\ &\leq C_{N,p}[\lambda^{q}]_{A_{q}}^{\max\{1,\frac{1}{q-1}\}}\|\mathcal{A}_{\widetilde{\mathcal{S}}}\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{m-k-2}I_{\mathcal{S},\eta}^{\alpha}\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\|_{L^{q}_{\lambda^{q}\eta^{q}}(\mathbb{H})} \\ &\leq C_{N,p}[\lambda^{q}]_{A_{q}}^{\max\{1,\frac{1}{q-1}\}}[\lambda^{q}\eta^{q}]_{A_{q}}^{\max\{1,\frac{1}{q-1}\}}\|\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{m-k-2}I_{\mathcal{S},\eta}^{\alpha}\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\|_{L^{q}_{\lambda^{q}\eta^{q}}(\mathbb{H})} \\ &\leq C_{N,p}\Big(\prod_{i=0}^{m-k-1}[\lambda^{q}\eta^{iq}]_{A_{q}}\Big)^{\max\{1,\frac{1}{q-1}\}}\|I_{\mathcal{S},\eta}^{\alpha}\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\|_{L^{q}_{\lambda^{q}\eta(m-k-1)q}(\mathbb{H})}. \end{split}$$

Using (14), we have that

$$\begin{split} \|I^{\alpha}_{\mathcal{S},\eta}\mathcal{A}^{k}_{\widetilde{\mathcal{S}},\eta}(|f|)\|_{L^{q}_{\lambda^{q}\eta^{(m-k-1)q}}(\mathbb{H})} &= \|I^{\alpha}_{\mathcal{S}}\mathcal{A}^{k}_{\widetilde{\mathcal{S}},\eta}(|f|)\|_{L^{q}_{\lambda^{q}\eta^{(m-k)q}}(\mathbb{H})} \\ &\leq C_{N,p,\alpha}[\lambda\eta^{m-k}]^{(1-\frac{\alpha}{N})\max\{1,\frac{p'}{q}\}}_{A_{p,q}}\|\mathcal{A}^{k}_{\widetilde{\mathcal{S}},\eta}(|f|)\|_{L^{p}_{\lambda^{p}\eta^{(m-k)p}}(\mathbb{H})}, \end{split}$$

and applying again  $\|\mathcal{A}_{\widetilde{\mathcal{S}}}\|_{L^p_w(\mathbb{H})} \leq C_{N,p}[w]^{\max\{1,\frac{1}{p-1}\}}_{A_p}$ ,

$$\|\mathcal{A}_{\widetilde{\mathcal{S}},\eta}^{k}(|f|)\|_{L^{p}_{\lambda^{p}\eta^{(m-k)p}}(\mathbb{H})} \leq C_{N,p} \Big(\prod_{i=m-k+1}^{m} [\lambda^{p}\eta^{ip}]_{A_{p}}\Big)^{\max\{1,\frac{1}{p-1}\}} \|f\|_{L^{p}_{\lambda^{p}\eta^{mp}}(\mathbb{H})},$$

which, along with the previous estimate, yields

$$\begin{aligned} &\|\mathcal{A}_{\alpha,\mathcal{S}}^{m,\kappa}(b,f)(x)\|_{L^{q}_{\lambda^{q}}(\mathbb{H})} \\ &\leq C_{N,p,\alpha}\|b\|_{BMO_{\eta}(\mathbb{H})}^{m}A(m,k)B(m,k)[\lambda\eta^{m-k}]^{(1-\frac{\alpha}{N})\max\{1,\frac{p'}{q}\}}\|f\|_{L^{p}_{\lambda^{p}\eta^{mp}}(\mathbb{H})}. \end{aligned}$$

where

$$A(m,k) = \left(\prod_{i=0}^{m-k-1} [\lambda^{q} \eta^{iq}]_{A_{q}}\right)^{\max\{1,\frac{1}{q-1}\}},$$

and

$$B(m,k) = \left(\prod_{i=m-k+1}^{\infty} [\lambda^p \eta^{ip}]_{A_p}\right)^{\max\{1,\frac{1}{p-1}\}}$$

Hence, setting  $\eta = \nu^{1/m}$ , where  $\nu = (\frac{\mu}{\lambda})^{1/p}$ , it reading follows from Hölder's inequality

$$[\lambda^s \nu^{s\frac{i}{m}}]_{A_s} \le [\lambda^s]_{A_s}^{\frac{m-i}{m}} [\mu^s]_{A_s}^{\frac{i}{m}}, \quad s = p, q.$$

Thus, we acquire that

$$A(m,k) \leq \left(\prod_{i=0}^{m-k-1} [\lambda^{q}]_{A_{q}}^{\frac{m-i}{m}} [\mu^{q}]_{A_{q}}^{\frac{i}{m}}\right)^{\max\{1,\frac{1}{q-1}\}} \leq \left([\lambda^{q}]_{A_{q}}^{\frac{m+k+1}{2}} [\mu^{q}]_{A_{q}}^{\frac{m-k-1}{2}}\right)^{\frac{m-k}{m}\max\{1,\frac{1}{q-1}\}},$$

and

$$B(m,k) \leq \Big(\prod_{i=m-k+1}^{m} [\lambda^p]_{A_p}^{\frac{m-i}{m}} [\mu^p]_{A_p}^{\frac{i}{m}}\Big)^{\max\{1,\frac{1}{p-1}\}} \leq \Big([\lambda^p]_{A_p}^{\frac{k-1}{2}} [\mu^p]_{A_p}^{m-\frac{k-1}{2}}\Big)^{\frac{k}{m}\max\{1,\frac{1}{p-1}\}}.$$

Combining all the preceding estimates obtains (i).

To proof (ii), we are going to follow ideas in [10]. Based on (12), it suffices to show that there exists a positive constant *C* such that for all dyadic cubes  $Q \in D^t$ ,

$$\omega_{2^{\frac{1}{N+2}}}(b;Q)^{m} \le C \left(\frac{\nu^{1/m}(Q)}{|Q|}\right)^{m} \|(I_{\alpha})_{b}^{m}\|_{L^{p}_{\mu^{p}}(\mathbb{H}) \to L^{q}_{\lambda^{q}}(\mathbb{H})}$$
(15)

Using Proposition 2 and Hölder inequality implies that

$$\begin{split} \omega_{2^{\frac{1}{N+2}}}(b;Q)^{m}|F_{1}||F_{2}| &\leq \int_{F_{1}} \int_{F_{2}} \left(b(x) - b(y)\right)^{m} dx dy \\ &\leq dima(Q)^{N-\alpha} \int_{F_{1}} \int_{F_{2}} \frac{\left(b(x) - b(y)\right)^{m}}{d(x,y)^{N-\alpha}} dx dy \\ &= dima(Q)^{N-\alpha} \int_{F_{1}} (I_{\alpha})^{m}_{b}(\chi_{F_{2}})(x) dx \\ &\leq C|Q|^{1-\frac{\alpha}{N}} \left(\int_{Q} \lambda^{-q'}\right)^{\frac{1}{q'}} \cdot \left(\int_{\mathbb{H}} [(I_{\alpha})^{m}_{b}(\chi_{F_{2}})]^{q} \lambda^{q} dx\right)^{\frac{1}{q}} \\ &\leq C|Q|^{1-\frac{\alpha}{N}} \left(\int_{Q} \lambda^{-q'}\right)^{\frac{1}{q'}} \cdot \left(\int_{Q} \mu^{p}\right)^{\frac{1}{p}} \|(I_{\alpha})^{m}_{b}\|_{L^{p}_{\mu^{p}}(\mathbb{H}) \to L^{q}_{\lambda^{q}}(\mathbb{H})} \\ &= C|Q|^{2} \left(\frac{1}{|Q|} \int_{Q} \lambda^{-q'}\right)^{\frac{1}{q'}} \cdot \left(\frac{1}{|Q|} \int_{Q} \mu^{p}\right)^{\frac{1}{p}} \|(I_{\alpha})^{m}_{b}\|_{L^{p}_{\mu^{p}}(\mathbb{H}) \to L^{q}_{\lambda^{q}}(\mathbb{H})}, \end{split}$$

where we used that  $\frac{1}{q} + \frac{\alpha}{N} = \frac{1}{p}$ . Further, this yields

 $\omega_{2^{\frac{1}{N+2}}}(b;Q)^{m} \leq C \left(\frac{1}{|Q|} \int_{Q} \lambda^{-q'}\right)^{\frac{1}{q'}} \cdot \left(\frac{1}{|Q|} \int_{Q} \mu^{p}\right)^{\frac{1}{p}} \|(I_{\alpha})^{m}_{b}\|_{L^{p}_{\mu^{p}}(\mathbb{H}) \to L^{q}_{\lambda^{q}}(\mathbb{H})}.$ 

Then from [8], we have

$$\left(\frac{1}{|Q|}\int_{Q}\mu^{p}\right)^{\frac{1}{p}} \leq C\left(\frac{1}{|Q|}\int_{Q}\nu^{1/m}\right)^{m}\left(\frac{1}{|Q|}\int_{Q}\lambda^{p}\right)^{\frac{1}{p}}$$

so the

$$\omega_{2^{\frac{1}{N+2}}}(b;Q)^{m} \leq C\left(\frac{1}{|Q|}\int_{Q}\nu^{1/m}\right)^{m}\left(\frac{1}{|Q|}\int_{Q}\lambda^{-q'}\right)^{\frac{1}{q'}}\left(\frac{1}{|Q|}\int_{Q}\lambda^{p}\right)^{\frac{1}{p}}\|(I_{\alpha})^{m}_{b}\|_{L^{p}_{\mu^{p}}(\mathbb{H})\to L^{q}_{\lambda^{q}}(\mathbb{H})}.$$

Now we observe that since q > p then by Hölder inequality,

$$\left(\frac{1}{|Q|}\int_{Q}\lambda^{p}\right)^{\frac{1}{p}} \leq \left(\frac{1}{|Q|}\int_{Q}\lambda^{q}\right)^{\frac{1}{q}} \text{ and } \left(\frac{1}{|Q|}\int_{Q}\lambda^{-q'}\right)^{\frac{1}{q'}} \leq \left(\frac{1}{|Q|}\int_{Q}\lambda^{-p'}\right)^{\frac{1}{p'}},$$

then

$$\left(\frac{1}{|Q|}\int_{Q}\lambda^{-q'}\right)^{\frac{1}{q'}}\left(\frac{1}{|Q|}\int_{Q}\lambda^{p}\right)^{\frac{1}{p}} \leq \left[\left(\frac{1}{|Q|}\int_{Q}\lambda^{q}\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_{Q}\lambda^{-p'}\right)^{\frac{q}{p'}}\right]^{\frac{1}{q}}.$$

Consequently, since  $\lambda \in A_{p,q}$ , we finally get

$$\omega_{2^{\frac{1}{N+2}}}(b;Q)^m \leq C\left(\frac{1}{|Q|}\int_Q \nu^{1/m}\right)^m \|(I_\alpha)_b^m\|_{L^p_{\mu^p}(\mathbb{H}) \to L^q_{\lambda^q}(\mathbb{H})}.$$

Thus, (15) holds and hence, the proof of (ii) is complete. Therefore, we complete the proof of Theorem 1. **Author Contributions:** Writing—original draft preparation, D.C.; methodology, Y.C.; check, T.W. All authors have read and agreed to the published version of the manuscript.

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#### Abbreviation

BMO Bounded Mean Oscillation

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