Article

Periodic Points of Modular Firmly Mappings in the Variable Exponent Sequence Spaces $\ell_{p(\cdot)}$

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Abstract: In this work, we investigate the existence of periodic points of mappings defined on nonconvex domains within the variable exponent sequence spaces $\ell_{p(\cdot)}$. In particular, we consider the case of modular firmly nonexpansive and modular firmly asymptotically nonexpansive mappings. These kinds of results have never been obtained before.

Keywords: electrorheological fluids; fixed point; modular firmly nonexpansive mapping; modular vector spaces; periodic point; strictly convex; uniformly convex

MSC: primary 47H09; 47H10

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1. Introduction

A central concept in nonlinear analysis and optimization is that of a firmly nonexpansive mapping, mainly due to its relation with maximally monotone operators, as evidenced in [1–5] and the references therein. In 1962, Minty [6,7] implicitly employed the idea of firmly nonexpansive mappings to study monotone mappings. The definition of firmly nonexpansive mappings on Banach spaces was introduced by Bruck [8] in 1973. The fixed point results for such mappings obtained by Kaczor [9] stand out among others of their kind. In fact, Kaczor's results may be seen as a wonderful extension to Smarzewski's work [10].

In principle, the concept of firm nonexpansiveness can be carried over to the case of modular spaces. The first attempt in this direction is due to Khan [11].

This work delves deeper into firm nonexpansiveness in the modular setting; specifically, we recover most of the results by [9] for mappings defined on the variable exponent sequence spaces $\ell_{p(\cdot)}$.

These spaces originated from the consideration of certain problems related to Fourier series in the work by Orlicz [12]. The study of these spaces was furthered in [13–16]; they played a motivational role in the introduction of the concept of modular [17]. Moreover, the above mentioned sequence spaces are the discrete version of the Lebesgue variable exponent spaces $L^{p(\cdot)}$. In recent years, these spaces took an important role in the study of many problems for which the classical L^p spaces are not appropriate to be used. Due to their importance in applications, the variable exponent Lebesgue spaces became a major area of mathematical work [18]. In particular, it is worth noticing the importance of these spaces in the treatment of the hydrodynamical behavior of non-Newtonian fluids [19,20] and its applications to several areas of Engineering and Medicine [21–24].

In this work, we investigate the existence of periodic points for firmly asymptotically nonexpansive mappings defined within the spaces $\ell_{p(\cdot)}$. In particular, we extend Kaczor's results [9] in the modular sense. As was noted in [25], the main major issue with the results of [11] is the assumption made by the author about the domain being convex.

Our work requires the tools from the field of fixed point theory, for which the reader is referred to the books [26–28].

2. Basic Definitions & Results

We start this section by summarizing some simple facts about the linear spaces $\ell_{p(\cdot)}$.

Definition 1 ([12]). *the linear spaces $\ell_{p(\cdot)}$ are defined as*

$$\ell_{p(\cdot)} = \left\{ (x_n) \subset \mathbb{R}^{\mathbb{N}}; \sum_{n=0}^{\infty} \frac{1}{p(n)} \left| \frac{x_n}{\beta} \right|^{p(n)} < +\infty, \text{ for some } \beta > 0 \right\},$$

where $p : \mathbb{N} \rightarrow [1, \infty)$.

In [12], Orlicz introduced these spaces using a slightly different terminology and notation. These spaces inspired Nakano who developed a more general theory known as modular vector spaces [14,17,29].

Proposition 1 ([13,15,17]). *On $\ell_{p(\cdot)}$, we introduce the functional $\varrho : \ell_{p(\cdot)} \rightarrow [0, \infty]$ by*

$$\varrho(u) = \varrho((u_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)} |u_n|^{p(n)}.$$

The function ϱ has the following properties:

- (i) $\varrho(u) = 0$ if and only if $u = 0$;
- (ii) $\varrho(\pm u) = \varrho(u)$;
- (iii) $\varrho(\alpha u + (1 - \alpha)v) \leq \alpha \varrho(u) + (1 - \alpha) \varrho(v)$, for all $\alpha \in [0, 1]$,
for any $u, v \in \ell_{p(\cdot)}$.

In addition, ϱ is left-continuous, that is, one has $\lim_{r \rightarrow 1^-} \varrho(rw) = \varrho(w)$, for any $w \in \ell_{p(\cdot)}$. Next we adapt to the modular case a few concepts that are commonplace in the metric setting.

Definition 2 ([25,28]).

1. A sequence $\{u_n\} \subset \ell_{p(\cdot)}$ is ϱ -convergent to $u \in \ell_{p(\cdot)}$ if and only if $\varrho(u_n - u) \rightarrow 0$. It is obvious that, if ϱ -limit exists, then it must be unique;
2. A sequence $\{u_n\} \subset \ell_{p(\cdot)}$ is ϱ -Cauchy if $\varrho(u_n - u_m) \rightarrow 0$ as $n, m \rightarrow \infty$;
3. A set $C \subset \ell_{p(\cdot)}$ is said to be ϱ -closed if for any sequence $\{u_n\} \subset C$ that ϱ -converges to u , it follows that $u \in C$;
4. A set $C \subset \ell_{p(\cdot)}$ is ϱ -bounded if $\delta_{\varrho}(C) = \sup\{\varrho(u - v); u, v \in C\} < \infty$;
5. Let $u \in \ell_{p(\cdot)}$ and $R \geq 0$. We define the ϱ -ball centered at u with radius R as

$$B_{\varrho}(u, R) = \{v \in \ell_{p(\cdot)}; \varrho(u - v) \leq R\}.$$

Moreover, ϱ satisfies the Fatou property, that is to say, if a sequence $\{v_n\} \subseteq \ell_{p(\cdot)}$ ϱ -converges to v , then, for any $u \in \ell_{p(\cdot)}$, it holds

$$\varrho(u - v) \leq \liminf_{n \rightarrow \infty} \varrho(u - v_n).$$

It is easy to show that the Fatou's property is equivalent to the ϱ -closedness of the ϱ -balls. We pass to a fundamental property when it comes to modular vector spaces, namely, the Δ_2 -condition.

Definition 3. The functional ϱ is said to satisfy the Δ_2 -condition if there exists $C \geq 0$ such that

$$\varrho(2w) \leq C \varrho(w),$$

for all $w \in \ell_{p(\cdot)}$.

We leave up to the reader the verification of the fact that the modular ϱ satisfies the Δ_2 -condition if and only if $p^+ = \sup_{n \in \mathbb{N}} p(n) < \infty$ [13,15,17].

The Luxemburg norm associated to the modular ϱ on $\ell_{p(\cdot)}$ is given by:

$$\|x\|_{\varrho} = \inf \left\{ \lambda > 0; \varrho \left(\frac{1}{\lambda} x \right) \leq 1 \right\}.$$

Endowed with the Luxemburg norm, $(\ell_{p(\cdot)}, \|\cdot\|_{\varrho})$ becomes a Banach space. Most of the geometric properties enjoyed by Banach spaces will hold when $p^- = \inf_{n \in \mathbb{N}} p(n)$ and $p^+ = \sup_{n \in \mathbb{N}} p(n)$ are away from 1 and $+\infty$. In particular, the uniform convexity holds in $(\ell_{p(\cdot)}, \|\cdot\|_{\varrho})$ if and only if $1 < p^- \leq p^+ < +\infty$ [15].

In the next definition, we introduce the class of mappings for which we will investigate the existence of periodic points.

Definition 4. Let D be a nonempty subset of $\ell_{p(\cdot)}$. Fix $\lambda \in (0, 1)$. A map $T : D \rightarrow \ell_{p(\cdot)}$ is said to be

1. Ref. [30] ϱ -nonexpansive, if

$$\varrho(T(x) - T(y)) \leq \varrho(x - y),$$

for all $x, y \in D$.

2. Ref. [28] ϱ -asymptotically nonexpansive, if there exists a sequence of positive numbers $\{k_p\}$ such that $\lim_{p \rightarrow \infty} k_p = 1$ and

$$\varrho(T^p(x) - T^p(y)) \leq k_p \varrho(x - y),$$

for all $p \geq 1$ and all $x, y \in D$.

3. Ref. [11,25] λ - ϱ -firmly nonexpansive, if

$$\varrho(T(x) - T(y)) \leq \varrho((1 - \lambda)(x - y) + \lambda(T(x) - T(y))),$$

for all $x, y \in D$.

4. λ - ϱ -firmly asymptotically nonexpansive if there exists a sequence of positive numbers $\{k_p\}$ such that $\lim_{p \rightarrow \infty} k_p = 1$ and

$$\varrho(T^p(x) - T^p(y)) \leq k_p \varrho((1 - \lambda)(x - y) + \lambda(T^p(x) - T^p(y))),$$

for all $p \geq 1$ and all $x, y \in D$.

A fixed point of T is any $x \in D$ such that $T(x) = x$. The set of fixed points of T will be denoted by $\text{Fix}(T)$. The fixed points of T^n are known as periodic points of T , for $n \geq 1$.

The concept of λ - ϱ -firmly asymptotically nonexpansive mappings is new and was not studied before. The authors of [25] looked at the case of a non convex domain and obtained some interesting results about firmly nonexpansive mappings in the modular sense in $\ell_{p(\cdot)}$. In the next section, we extend their main results to this new class of mappings.

3. Main Results

Periodic points are a special case of the fixed point property. Most of the interesting results in this area use some kind of geometric properties. This is the reason many authors worked on establishing the modular version of these geometric properties, such as strict convexity, uniform convexity and so forth (see, for example, refs. [28,31,32]).

The following definition was introduced in [25] and proved to be useful.

Definition 5. Let D be a nonempty subset of $\ell_{p(\cdot)}$. We will say that D is q -connected if and only if D is not a finite union of disjoint q -closed nonempty subsets.

Note that q -closed convex subsets of $\ell_{p(\cdot)}$ are q -connected. Next, recall that a subset K of $\ell_{p(\cdot)}$ is said to have the fixed point property for q -nonexpansive self-mappings if any q -nonexpansive mapping $S : K \rightarrow K$ has a fixed point. Convex subsets of $\ell_{p(\cdot)}$ having the fixed point property for q -nonexpansive self-mappings were extensively studied [28]. Note that in [32], it is proved that if $\{n \in \mathbb{N}; p(n) = 1\}$ has at most one element, then any nonempty q -bounded convex q -closed subset K of $\ell_{p(\cdot)}$ has the fixed point property for q -nonexpansive mappings provided K enjoys the property (R). This property is very similar to the reflexivity or weak-compactness in Banach spaces. Recall that K enjoys the property (R) if and only if for any decreasing sequence $\{K_n\}_{n \geq 1}$ of q -closed, convex nonempty subsets of K , has a nonempty intersection.

Theorem 1. Let K be a union of nonempty, pairwise disjoint, q -closed and q -connected subsets $K_1, K_2, \dots, K_n$ of $\ell_{p(\cdot)}$, with $n \geq 2$. Assume that one of the subsets $K_1, K_2, \dots, K_n$ has the fixed point property for q -nonexpansive self-mappings. Then any q -nonexpansive mapping $S : K \rightarrow K$ has a periodic point.

Proof. Since S is q -nonexpansive, then $S^{-1}(K_i)$ is q -closed, for any $i = 1, \dots, n$. Clearly, we have $K_j = \bigcup_{i=1}^n S^{-1}(K_i) \cap K_j$. Since $\{S^{-1}(K_i)\}$ are q -closed disjoint subsets and each K_j is q -connected, we deduce that, for any $i \in \{1, \dots, n\}$, there exists $k \in \{1, \dots, n\}$ such that $S(K_i) \subset K_k$. Without loss of any generality, we assume that K_1 has the fixed point property for q -nonexpansive self-mappings. Therefore, there exists $p \in \{1, \dots, n\}$ such that $S^p(K_1) \subset K_1$. Since S^p is q -nonexpansive, we conclude that S^p has a fixed point in K_1 , that is, T has a periodic point in K_1 . $\square$

In general, there is no reason for the periodic point to be a fixed point of the mapping. If we assume that the modular functional q satisfies some geometric properties, then this will hold. This is the main driving argument behind the main results of [25].

Theorem 2 ([25]). Assume $\{n \in \mathbb{N}; p(n) = 1\}$ has at most one element. Let $C = \bigcup_{i=1}^n C_i$, with $n \geq 2$, be a union of nonempty, pairwise disjoint, q -closed and q -connected subsets C_i of $\ell_{p(\cdot)}$.

- (1) Assume that C_1 has the fixed point property for q -nonexpansive self-mappings. If for some $\lambda \in (0, 1)$, $T : C \rightarrow C$ is λ - q -firmly nonexpansive, then T has a fixed point in C .
- (2) Assume that C_1 is q -bounded, convex and has the property (R). If for some $\lambda \in (0, 1)$, $T : C \rightarrow C$ is λ - q -firmly nonexpansive, then T has a fixed point in C .

As we said earlier, the main results of our work are the extension of the above conclusions to λ - q -firmly asymptotically nonexpansive mappings. The following technical simple fact will be useful.

Lemma 1. Let C be a nonempty subset of the vector space $\ell_{p(\cdot)}$. If for some $\lambda \in (0, 1)$, $T : C \rightarrow C$ is λ - q -firmly asymptotically nonexpansive, then there exists $p_0 \geq 1$, such that T^p is q -asymptotically nonexpansive, for any $p \geq p_0$.

Proof. Since T is λ -firmly asymptotically nonexpansive, there exists a sequence of positive numbers $\{k_p\}$ such that $\lim_{p \rightarrow \infty} k_p = 1$ and

$$\varrho\left(T^p(x) - T^p(y)\right) \leq k_p \varrho\left((1 - \lambda)(x - y) + \lambda(T^p(x) - T^p(y))\right),$$

for any $x, y \in C$ and $p \in \mathbb{N}$. Since $\lim_{p \rightarrow \infty} \lambda k_p = \lambda < 1$, there exists $p_0 \geq 1$ such that $\lambda k_p < 1$, for any $p \geq p_0$. Using the fact that ϱ is convex, we get

$$\varrho\left(T^p(x) - T^p(y)\right) \leq k_p (1 - \lambda) \varrho(x - y) + k_p \lambda \varrho(T^p(x) - T^p(y)),$$

which implies

$$\varrho\left(T^p(x) - T^p(y)\right) \leq \frac{k_p(1 - \lambda)}{1 - \lambda k_p} \varrho(x - y),$$

for $p \geq p_0$ and $x, y \in C$. Since

$$\lim_{p \rightarrow \infty} \frac{k_p(1 - \lambda)}{1 - \lambda k_p} = 1,$$

we conclude that T^p is ϱ -asymptotically nonexpansive, for any $p \geq p_0$. $\square$

Before we state our next result on ϱ -asymptotically nonexpansive mappings, we recall that if $p^- = \inf_{n \in \mathbb{N}} p(n) > 1$, then ϱ enjoys a modular uniform convexity [31], which will secure that any nonempty ϱ -bounded convex ϱ -closed subset C of $\ell_{p(\cdot)}$ has the fixed point property for ϱ -asymptotically nonexpansive self-mappings [28].

Theorem 3. Assume $p^- > 1$. Let $K = \bigcup_{i=1}^n K_i$, with $n \geq 2$, are a union of nonempty, pairwise disjoint, ϱ -closed and ϱ -connected subsets K_i of $\ell_{p(\cdot)}$. Assume that K_1 is a nonempty ϱ -bounded convex ϱ -closed subset C of $\ell_{p(\cdot)}$. If, for some $\lambda \in (0, 1)$, $T : K \rightarrow K$ is λ - ϱ -firmly asymptotically nonexpansive, then T has a periodic point in K .

Proof. Lemma 1 implies the existence of $p_0 \geq 1$ such that T^p is ϱ -asymptotically nonexpansive, for any $p \geq p_0$. Fix $p > p_0$. Set $S = T^p$. Using the same argument in the proof of Theorem 1, there exists $n \geq 1$ such that $S^n(K_1) \subset K_1$, i.e., $T^{np}(K_1) \subset K_1$. Since K_1 has the fixed point property for ϱ -asymptotically nonexpansive self-mappings, we conclude that T^{np} has a fixed point in K_1 , which is a periodic point of T . $\square$

A similar result to Theorem 2 for λ - ϱ -firmly asymptotically nonexpansive mappings is the following:

Theorem 4. Assume $p^- > 1$. Let $K = \bigcup_{i=1}^n K_i$, with $n \geq 2$, be a union of nonempty, pairwise disjoint, ϱ -closed and ϱ -connected subsets K_i of $\ell_{p(\cdot)}$. Assume that K_1 is a nonempty ϱ -bounded convex ϱ -closed subset K of $\ell_{p(\cdot)}$. If for some $\lambda \in (0, 1)$, $T : K \rightarrow K$ is λ - ϱ -firmly asymptotically nonexpansive, then T has a fixed point in K .

Proof. Using Theorem 3, there exists a periodic point $a_0 \in K$ of T , that is, $T^q(a_0) = a_0$, for some $q \geq 1$. Without loss of generality, we may assume $q \geq 2$. Since T is λ - ϱ -firmly asymptotically nonexpansive, there exists a sequence of positive numbers $\{k_p\}$ such that $\lim_{p \rightarrow \infty} k_p = 1$ and

$$\varrho\left(T^p(a) - T^p(b)\right) \leq k_p \varrho\left((1 - \lambda)(a - b) + \lambda(T^p(a) - T^p(b))\right),$$

for any $a_i \in K$ and $p \in \mathbb{N}$. Set $a_i = T^i(a_0)$, for $i \geq 1$. We have $a_{i+q} = a_i$, for $i \in \mathbb{N}$, which implies $T^{pq}(a_i) = a_{i+pq} = a_i$, for any $i, p \in \mathbb{N}$. Hence for any $i, j \geq 1$ and $p \in \mathbb{N}$, we have

$$\begin{aligned} \varrho(T(a_i) - T(a_j)) &= \varrho(T^{pq+1}(a_i) - T^{pq+1}(a_j)) \\ &\leq k_{pq+1} \varrho((1-\lambda)(a_i - a_j) + \lambda(T^{pq+1}(a_i) - T^{pq+1}(a_j))) \\ &\leq k_{pq+1} \varrho((1-\lambda)(a_i - a_j) + \lambda(T(a_i) - T(a_j))). \end{aligned}$$

If we let $p \rightarrow \infty$ and using $\lim_{n \rightarrow \infty} k_n = 1$, we get

$$\varrho(T(a_i) - T(a_j)) \leq \varrho((1-\lambda)(a_i - a_j) + \lambda(T(a_i) - T(a_j))),$$

which implies $\varrho(T(a_i) - T(a_j)) \leq \varrho(a_i - a_j)$, for any $i, j \geq 1$. Moreover, we have

$$\varrho(a_i - a_j) = \varrho(a_{i+q} - a_{j+q}) \leq \varrho(a_{i+1} - a_{j+1}) = \varrho(T(a_i) - T(a_j)),$$

since $q \geq 2$, which implies

$$\varrho(a_{i+1} - a_{j+1}) = \varrho(T(a_i) - T(a_j)) = \varrho(a_i - a_j),$$

for any $i, j \geq 1$. In other words, the restriction of T to the set $\{a_0, a_1, \dots, a_{q-1}\}$ is a ϱ -isometry. Define $T_\lambda = (1-\lambda)I + \lambda T$, where I is the identity map. Using the inequality (FN), we get

$$\varrho(a_i - a_j) = \varrho(T(a_i) - T(a_j)) \leq \varrho(T_\lambda(a_i) - T_\lambda(a_j)) \leq \varrho(a_i - a_j),$$

for any $i, j \geq 1$. Hence, the restriction of T_λ to the set $\{a_0, a_1, \dots, a_{q-1}\}$ is also a ϱ -isometry. Fix $i \in [1, q-1]$ and set $u = a_i - a_{i-1}$ and $v = T(a_i) - T(a_{i-1}) = a_{i+1} - a_i$. Note that $\varrho(u) = \varrho(v)$. Moreover, we have:

$$\begin{aligned} \varrho((1-\lambda)u + \lambda v) &= \varrho(T_\lambda(a_i) - T_\lambda(a_{i-1})) \\ &= \varrho(a_i - a_{i-1}) \\ &= (1-\lambda)\varrho(u) + \lambda\varrho(v). \end{aligned}$$

Our assumption on $p(\cdot)$ implies ϱ is (SC) [32], which implies $u = v$, that is, $a_i - a_{i-1} = a_{i+1} - a_i$. Since i was taken arbitrarily, we get

$$a_1 - a_0 = a_2 - a_1 = \dots = a_{p-1} - a_{p-2} = a_q - a_{q-1},$$

which implies

$$q(a_1 - a_0) = \sum_{i=0}^{q-1} a_{i+1} - a_i = a_q - a_0 = a_0 - a_0 = 0.$$

In other words, we have $T(a_0) = a_1 = a_0$, that is, a_0 is a fixed point of T . $\square$

It is not known under what general conditions the periodic point discovered in Theorem 3 is a fixed point. However, if ϱ satisfies the triangle inequality, that is,

$$\varrho(x+y) \leq \varrho(x) + \varrho(y),$$

for any $x, y \in \ell_{p(\cdot)}$, then we can show that the periodic point discovered in Theorem 3 is in fact a fixed point. It is not hard to see that q satisfies the triangle inequality if and only if $p(n) = 1$, for any $n \in \mathbb{N}$.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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