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Contracting and Involutive Negations of Probability Distributions

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Abstract: A dozen papers have considered the concept of negation of probability distributions (pd) introduced by Yager. Usually, such negations are generated point-by-point by functions defined on a set of probability values and called here negators. Recently the class of pd-independent linear negators has been introduced and characterized using Yager's negator. The open problem was how to introduce involutive negators generating involutive negations of pd. To solve this problem, we extend the concepts of contracting and involutive negations studied in fuzzy logic on probability distributions. First, we prove that the sequence of multiple negations of pd generated by a linear negator converges to the uniform distribution with maximal entropy. Then, we show that any pd-independent negator is non-involutive, and any non-trivial linear negator is strictly contracting. Finally, we introduce an involutive negator in the class of pd-dependent negators. It generates an involutive negation of probability distributions.

Keywords: probability distribution; negation of probability distribution; contracting negation; involutive negation; linear negator; involutive negator; entropy

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1. Introduction

The concept of negation of a probability distribution (pd) was introduced by Yager [1]. He is concerned with the representation of the knowledge contained in the negation of a probability distribution. He considered an example of a rule-based system consisting of rules of the form: If V is Tall, then U is b, and If V is Not Tall, then U is d. If Tall is represented as a fuzzy set, then the process of obtaining Not Tall is well known. If Tall is represented by a probability distribution, then to determine Not Tall, we need to define the negation of a probability distribution.

Yager [1] defined the negation $\bar{P}$ of a finite probability distribution $P = (p_1, \dots, p_n)$ by: $\bar{P} = (\bar{p}_1, \dots, \bar{p}_n)$, where $\bar{p}_i$ is defined by: $\bar{p}_i = \frac{1-p_i}{n-1}$. He noted that other negations of probability distributions (pd) could exist. Further, negations of probability distributions were considered in many works [2–12]. The properties of Yager's negation of a probability distribution are studied in [2]. The authors of [3] studied uncertainty related to Yager's negation. The authors of [4,5] studied the convergence of the sequence of multiple Yager's negations of pd to the uniform distribution. Yager's negation is used in a multi-criteria decision-making procedure in [6]. The authors of [7] introduced another negation of probability distributions based on Tsallis entropy. A negation of basic probability assignment in Dempster-Shafer theory is considered in [8]. The properties of the negation of basic probability assignment based on a total uncertainty measure are studied in [9]. A definition of negation in basic belief assignment in the Dempster-Shafer theory using matrix operators is given in [10]. This matrix negation was also considered in [11].

The authors of [12] studied functions called negators defined on the set of probability values and point-by-point transforming pd into its negation. Two types of negators are considered: pd-independent and pd-dependent. In the class of pd-independent negators it was introduced the class of linear negators [12]. It was shown that a negator is linear if and only if it is a convex combination of Yager's and uniform negators. Hence the Yager's

negator plays a crucial role in the definition of pd-independent linear negators. The current paper studies new properties of negators introduced in the paper [12].

The non-solved problem was how to introduce involutive negators generating involutive negations of pd. Involutive and non-involutive negations are studied in detail in fuzzy logic and for lexicographic valuations of plausibility [13–19]. It was shown that any fuzzy negation at any point of $[0,1]$ is contracting, expanding, or involutive. This paper aims to extend the concepts of contracting, expanding, and involutive negations from fuzzy logic on the set of probability values in probability distributions and to introduce involutive negators and negations of pd.

The principal contributions of the paper are the following: we prove that the sequence of multiple negations of pd generated by a linear negator converges to the uniform distribution with maximal entropy. We show that any pd-independent negator is non-involutive, and any non-trivial linear negator is strictly contracting. Finally, we introduce an involutive negator in the class of pd-dependent negators that generates an involutive negation of probability distributions.

The paper has the following structure. Section 2 considers basic definitions of negators and negations of pd generated by negators and describes pd-independent and linear negators' properties from [12] used in the following sections. Section 3 presents the results of the paper. Section 3.1 introduces a general form of multiple linear negators and finds the limit of the sequence of such multiple negators. Section 3.2 shows non-involutivity of pd-independent negators, and shows that non-trivial linear negators are strictly contracting. In Section 3.3, we introduce an involutive pd-dependent negator that defines the involutive negation of probability distribution. Section 4 contains a discussion.

2. Materials and Methods

2.1. Negations of Discrete Probability Distributions

A set $P = \{p_1, \dots, p_n\}$, of n real values p_i , where $n \geq 2$, is referred to as a (finite discrete) *probability distribution (pd) of the length n* , if it satisfies for all $i = 1, \dots, n$, the following properties:

$$0 \leq p_i \leq 1, \sum_{i=1}^n p_i = 1. \quad (1)$$

One can consider p_i as a probability of an outcome x_i in some experiment X with outcomes $\{x_1, \dots, x_n\}$, $n \geq 2$. Let $\mathcal{P}_n$ be the set of all possible probability distributions of the length n defined on X . For simplicity of the interpretation, we will fix the ordering of outcomes and the ordering of corresponding probability values according to their indexing $i = 1, \dots, n$, and represent the probability distribution $P = \{p_1, \dots, p_n\}$ as n -tuple $P = (p_1, \dots, p_n)$.

The probability distribution $P_{(i)} = (p_1, \dots, p_n)$ satisfying the property: $p_i = 1$ for some i in $\{1, \dots, n\}$, and $p_j = 0$ for all $j \neq i$, will be referred to as a *degenerate or point distribution* [20]. For example, for $i = 1$ and $i = n$ we have the following point distributions: $P_{(1)} = (1, 0, \dots, 0)$, and $P_{(n)} = (0, \dots, 0, 1)$.

The simplest example of pd is the *uniform distribution*: $P_U = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$.

A transformation $neg(P)$ of probability distributions $P = (p_1, \dots, p_n)$ from $\mathcal{P}_n$ into probability distributions $neg(P) = Q = (q_1, \dots, q_n)$ in $\mathcal{P}_n$ is called a *negation of probability distributions* if, for all $i, j = 1, \dots, n$, the following properties are satisfied [12]:

$$0 \leq q_i \leq 1, \sum_{i=1}^n q_i = 1, \quad (2)$$

$$\text{if } p_i \leq p_j, \text{ then } q_i \geq q_j, \quad (3)$$

From (3) it follows for all $i, j = 1, \dots, n$:

$$\text{if } p_i = p_j, \text{ then } q_i = q_j. \quad (4)$$

A negator N is a function of probability values p_i point-by-point transforming probability distributions $P = (p_1, \dots, p_n)$ into probability distributions: $neg(P) = (N(p_1), \dots, N(p_n))$, hence, for all $i = 1, \dots, n$, the following properties are satisfied:

$$0 \leq N(p_i) \leq 1, \sum_{i=1}^n N(p_i) = 1, \quad (5)$$

$$\text{if } p_i \leq p_j, \text{ then } N(p_i) \geq N(p_j). \quad (6)$$

We will say that a negator N generates (point-by-point) a negation $neg(P) = (N(p_1), \dots, N(p_n))$ of probability distribution P .

A negator N is called *pd-independent* [12] if for any pd $P = (p_1, \dots, p_n)$ in $\mathcal{P}_n$ the negator $N(p_i)$ depends only on the value p_i but not on other values p_j in P . A negator that is not pd-independent will be referred to as *pd-dependent*. A negation $neg_N(P) = (N(p_1), \dots, N(p_n))$ of a probability distribution $P = (p_1, \dots, p_n)$ will be called a *pd-independent negation* of pd if it is generated by pd-independent negator N .

Yager's negator [1]:

$$N_Y(p) = \frac{1-p}{n-1}, \text{ for all } p \text{ in } [0, 1], \quad (7)$$

is a pd-independent negator. For any pd $P = (p_1, \dots, p_n)$ in $\mathcal{P}_n$ it defines negation of P :

$$neg_Y(P) = (N_Y(p_1), \dots, N_Y(p_n)) = \left(\frac{1-p_1}{n-1}, \dots, \frac{1-p_n}{n-1} \right).$$

The uniform negator [12]:

$$N_U(p) = \frac{1}{n}, \text{ for all } p \text{ in } [0, 1], \quad (8)$$

is another example of pd-independent negator. For any pd $P = (p_1, \dots, p_n)$ in $\mathcal{P}_n$ negator N_U defines its negation: $neg_U(P) = (N_U(p_1), \dots, N_U(p_n)) = \left(\frac{1}{n}, \dots, \frac{1}{n} \right) = P_U$, where $P_U = \left(\frac{1}{n}, \dots, \frac{1}{n} \right)$ is the uniform distribution.

The following negator, introduced by Zhang et al. [7]:

$$N_T(p_i) = \frac{1-p_i^k}{n - \sum_{j=1}^n p_j^k}, \quad k \neq 0.$$

is an example of pd-dependent negator.

In the following sections, we will show that all pd-independent negators are non-involutive. We will introduce an involutive negator in the class of pd-dependent negators. This involutive negator generates an involutive negation of probability distributions satisfying the property: $neg(neg(P)) = P$.

2.2. Properties of Pd-Independent and Linear Negators

The authors of [12] showed that Yager's negator plays a crucial role in the construction of pd-independent linear negators: any linear negator is a convex combination of Yager's and uniform negators; hence it is a function of Yager's negator. Let us consider some properties of pd-independent and linear negators that will be used further in this paper.

An element p in $[0, 1]$ is called a *fixed point of a negator* N if $N(p) = p$. A probability distribution P is called a *fixed point of a negation* neg if $neg(P) = P$.

Proposition 1 ([12]). Any negator N has a fixed point $p = \frac{1}{n}$, i.e., $N\left(\frac{1}{n}\right) = \frac{1}{n}$, and the negation of probability distribution $neg(P) = (N(p_1), \dots, N(p_n))$ generated by N have the fixed point:

$$neg(P_U) = P_U. \quad (9)$$

Theorem 1 ([12]). Any pd -independent negator N has the unique fixed point $p = \frac{1}{n}$ and any pd -independent negation of probability distributions neg_N has a unique fixed point P_U .

Corollary 1. Any pd -independent negator N satisfies the following property:

$$N(x) = x, \text{ if and only if, } x = \frac{1}{n}. \quad (10)$$

Proposition 2 ([12]). Any pd -independent negator N satisfies the property:

$$N(0) = \frac{1 - N(1)}{n - 1}. \quad (11)$$

From (11), we obtain the following:

$$N(1) = 1 - (n - 1)N(0). \quad (12)$$

Proposition 3 ([12]). Any pd -independent negator N satisfies the following properties:

$$N(1) \in \left[0, \frac{1}{n}\right], \quad (13)$$

$$N(0) \in \left[\frac{1}{n}, \frac{1}{n-1}\right]. \quad (14)$$

Theorem 2 ([12]). Any pd -independent negator N satisfies the following properties:

$$N(p) \in \left[0, \frac{1}{n}\right] \quad \text{if } p \geq \frac{1}{n}, \quad (15)$$

$$N(p) \in \left[\frac{1}{n}, \frac{1}{n-1}\right] \quad \text{if } p \leq \frac{1}{n}. \quad (16)$$

A pd -independent negator N is referred to as a *linear negator* [12] if $N(p)$ is a linear function of $p \in [0, 1]$. The negation $neg_N(P) = (N(p_1), \dots, N(p_n))$ of a probability distribution $P = (p_1, \dots, p_n)$ is called a *linear negation of pd* if $N(p)$ is a linear negator.

Theorem 3 ([12]). A function $N(p)$ is a linear negator if and only if it is a convex combination of negators N_U and N_Y , i.e., for some $\alpha \in [0, 1]$ for all p in $[0, 1]$ the following property is satisfied:

$$N(p) = \alpha N_U(p) + (1 - \alpha) N_Y(p) = \alpha \frac{1}{n} + (1 - \alpha) \frac{1 - p}{n - 1}, \quad (17)$$

where $\alpha \in [0, 1]$ is a parameter of the convex combination.

From (13), we have $nN(1) \in [0, 1]$, hence in (17), we can use $\alpha = nN(1)$ and represent (17) as a function of Yager's negator N_Y [12]:

$$N(p) = N(1) + (1 - nN(1)) \frac{1 - p}{n - 1} = N(1) + (1 - nN(1)) N_Y(p), \quad (18)$$

where $N(1)$ is a parameter, $N(1) \in \left[0, \frac{1}{n}\right]$, defining the value of $N(p)$ for $p = 1$.

The authors of [12] formulate an Open Problem: *Prove or disprove a hypothesis that any pd -independent negator is linear.* We suppose that any pd -independent negator is linear.

In such a case, the properties established in [12] and in this paper for pd-independent negators will be fulfilled for linear negators and vice versa.

Section 3 presents the new property of linear negations: the sequence of multiple linear negations of a pd converges to the uniform distribution with the maximal entropy.

3. Results

3.1. Multiple Linear Negators and Negations

Let N be a pd-independent negator. For all $k = 1, 2, \dots$, denote $N^k(p) = N(N^{k-1}(p))$, where $N^0(p) = p$ and $N^1(p) = N(p)$. We have: $N^{k+2}(p) = N(N^{k+1}(p)) = N(N(N^k(p))) = N^2(N^k(p))$.

For any p in $[0,1]$ denote $d = p - \frac{1}{n}$, then $p = \frac{1}{n} + d$.

Applying (12): $N(1) = 1 - (n-1)N(0)$, in linear negator (18): $N(p) = N(1) + (1 - nN(1))\frac{1-p}{n-1}$, after equivalent transformations, we obtain another representation of linear negator:

$$N(p) = N(0) + (1 - nN(0))p. \quad (19)$$

Substituting in (19): $p = \frac{1}{n} + d$ and $A = 1 - nN(0)$ we obtain the following:

$$N(p) = N(0) + (1 - nN(0))\left(\frac{1}{n} + d\right) = N(0) + \frac{1}{n} - nN(0)\frac{1}{n} + Ad = \frac{1}{n} + Ad. \quad (20)$$

Proposition 4. For linear negator N , the following formula holds for any p in $[0,1]$ and for all $k = 0, 1, 2, \dots$:

$$N^k(p) = \frac{1}{n} + A^k d, \text{ where } A = 1 - nN(0), \text{ and } d = p - \frac{1}{n}. \quad (21)$$

Proof. The formula (21) holds for $k = 0$: $N^0(p) = p = \frac{1}{n} + A^0 d = \frac{1}{n} + p - \frac{1}{n} = p$, and (21) holds also for $k = 1$ in (20). Suppose that (21) holds for $k \geq 1$. Using (19) and (20) show that (21) holds for $k + 1$: $N^{k+1}(p) = N(N^k(p)) = N(0) + (1 - nN(0))N^k(p) = N(0) + (1 - nN(0))\left(\frac{1}{n} + A^k d\right) = N(0) + \frac{1}{n} - nN(0)\frac{1}{n} + (1 - nN(0))A^k d = \frac{1}{n} + A^{k+1}d$. $\square$

Theorem 4. For linear negator N for any p in $[0,1]$, it holds:

$$\lim_{k \rightarrow \infty} (N^k(p)) = \frac{1}{n}. \quad (22)$$

Proof. From (14), we have $\frac{1}{n} \leq N(0) \leq \frac{1}{n-1}$; hence for $A = 1 - nN(0)$ we obtain: $1 - n\frac{1}{n-1} \leq A \leq 1 - n\frac{1}{n}$, i.e.,

$$-\frac{1}{n-1} \leq A \leq 0, \text{ and } |A| < 1. \quad (23)$$

For $d = p - \frac{1}{n}$ and $p \in [0,1]$ we have: $-\frac{1}{n} \leq d \leq \frac{n-1}{n}$. Taking into account these possible values of A and d , we obtain from (21): $\lim_{k \rightarrow \infty} (N^k(p)) = \lim_{k \rightarrow \infty} \left(\frac{1}{n} + A^k d\right) = \frac{1}{n} + \lim_{k \rightarrow \infty} (A^k d) = \frac{1}{n} + d \lim_{k \rightarrow \infty} (A^k) = \frac{1}{n} + d \cdot 0 = \frac{1}{n}$. $\square$

The definition of linear negation says that the negation $neg_N(P) = (N(p_1), \dots, N(p_n))$ of a probability distribution $P = (p_1, \dots, p_n)$ is a linear negation of pd, if $N(p)$ is a linear

negator. Define: $neg_N^k(P) = (N^k(p_1), \dots, N^k(p_n))$. From the construction of $N^k(p)$ it is clear that $neg_N^k(P)$ is a probability distribution, i.e., $N^k(p_i)$ satisfy the properties (5). Let us check, for example, that $\sum_{i=1}^n N^k(p_i) = 1$. Indeed, from (21), we have: $\sum_{i=1}^n N^k(p_i) = \sum_{i=1}^n \left(\frac{1}{n} + A^k \left(p_i - \frac{1}{n} \right) \right) = \sum_{i=1}^n \frac{1}{n} + A^k \sum_{i=1}^n \left(p_i - \frac{1}{n} \right) = 1 + A^k \left(\sum_{i=1}^n p_i - \sum_{i=1}^n \frac{1}{n} \right) = 1 + A^k(1 - 1) = 1$.

From Theorem 4, we have the following.

Theorem 5. If N is a linear negator, then for the corresponding linear negation of pd $neg_N(P) = (N(p_1), \dots, N(p_n))$ for any pd P in $\mathcal{P}_n$ it holds:

$$\lim_{k \rightarrow \infty} (neg_N^k(P)) = neg_U(P) = P_U.$$

Proof. $\lim_{k \rightarrow \infty} (neg_N^k(P)) = \lim_{k \rightarrow \infty} (N^k(p_1), \dots, N^k(p_n)) = \left(\lim_{k \rightarrow \infty} N^k(p_1), \dots, \lim_{k \rightarrow \infty} N^k(p_n) \right) = \left(\frac{1}{n}, \dots, \frac{1}{n} \right) = P_U = neg_U(P)$. $\square$

As it follows from Theorem 5, the multiple negations of probability distributions have as the limit the uniform distribution with the maximal entropy value [1]:

$$H(P) = \sum_{i=1}^n (1 - p_i) p_i.$$

Results obtained for linear negators also fulfilled for Yager's negator, because it is a linear negator. Hence, for Yager's negator we have from (21): $A = 1 - nN_Y(0) = 1 - n \frac{1-0}{n-1} = -\frac{1}{n-1}$, and for all $k = 0, 1, 2, \dots$ it is fulfilled:

$$N_Y^k(p) = \frac{1}{n} + (-1)^k \frac{1}{(n-1)^k} \left(p - \frac{1}{n} \right). \quad (24)$$

3.2. Contracting Negators

The concepts of contracting and expanding negations have been introduced and studied in [14–17] on the sets of lexicographic valuations, multisets, and membership values [18,19]. Here we extend these concepts on pd -independent negators.

Definition 1. Let p be a probability value from $[0,1]$. A negator N is called contracting in p if

$$\min\{p, N(p)\} \leq N(N(p)) \leq \max\{p, N(p)\}, \quad (25)$$

expanding in p if

$$\min\{N(p), N(N(p))\} \leq p \leq \max\{N(p), N(N(p))\}, \quad (26)$$

and involutive in p if

$$N(N(p)) = p. \quad (27)$$

A negator N is called contracting, expanding, or involutive (on $[0,1]$) if it satisfies the corresponding property for all p in $[0,1]$. A negator N is called non-involutive if it is not involutive. A negator N is called strictly contracting if for any $p \neq \frac{1}{n}$ all inequalities in (25) are strict:

$$\min\{p, N(p)\} < N(N(p)) < \max\{p, N(p)\}. \quad (28)$$

A negation $neg_N(P) = (N(p_1), \dots, N(p_n))$ of probability distributions $P = (p_1, \dots, p_n)$ will be called contracting if it is generated by contracting negator N .

Theorem 6. Any negator N for any p in $[0,1]$ satisfies (25) or (26), hence it is contracting or expanding in p . N satisfies both properties (25) and (26) if and only if N is involutive in p .

Proof. Suppose $p \leq N(p)$, then from (6), we obtain: $N(N(p)) \leq N(p)$, that gives $p \leq N(N(p)) \leq N(p)$ or $N(N(p)) \leq p \leq N(p)$, and hence (25) or (26), respectively, fulfilled.

Dually, if $N(p) \leq p$, then from (6) we obtain $N(p) \leq N(N(p))$, that gives $N(p) \leq p \leq N(N(p))$ or $N(p) \leq N(N(p)) \leq p$, and hence (26) or (25), respectively, fulfilled.

If N is involutive in p , then (25) coincides with (26), and both hold. Suppose both (25) and (26) hold together. If $p \leq N(p)$ then from (6) we obtain $N(N(p)) \leq N(p)$, and from (25) and (26) we obtain $p \leq N(N(p))$ and $N(N(p)) \leq p$ hence (27) is fulfilled. Similarly, we obtain (27) from $N(p) \leq p$, (6), (25), and (26). $\square$

Theorem 7. Any pd -independent negator N is non-involutive.

Proof. From (13) and (16) we have: $N(1) \in [0, \frac{1}{n}]$, and $N(N(1)) \in [\frac{1}{n}, \frac{1}{n-1}]$, hence $N(N(1)) < 1$, and N is non-involutive. $\square$

From this theorem, it follows that involutive negators we need to look for between pd -dependent negators. Such negator we introduce in the following section.

A linear negator N such that $N \neq N_U$ will be referred to as a *non-trivial linear negator*.

Theorem 8. Any non-trivial linear negator is strictly contracting.

Proof. From $N \neq N_U$, Corollary 1 and (14) it follows $N(p) \neq \frac{1}{n}$ for all $p \neq \frac{1}{n}$, hence $N(0) \neq \frac{1}{n}$, and $\frac{1}{n} < N(0) \leq \frac{1}{n-1}$, and in (21) for $A = 1 - nN(0)$ we have: $-\frac{1}{n-1} \leq A < 0$, i.e., A is negative, and $|A| < 1$. Using representation (21): $N^k(p) = \frac{1}{n} + A^k d$, and $p = \frac{1}{n} + d$, we need to prove that (28) is satisfied for all $p \neq \frac{1}{n}$ in $[0,1]$, i.e., the following inequalities are fulfilled:

$$\min\{p, N(p)\} = \min\left\{\frac{1}{n} + d, \frac{1}{n} + Ad\right\} < N(N(p)) = \frac{1}{n} + A^2 d < \max\{p, N(p)\} = \max\left\{\frac{1}{n} + d, \frac{1}{n} + Ad\right\}. \quad (29)$$

If $d > 0$ we have: $\min\left\{\frac{1}{n} + d, \frac{1}{n} + Ad\right\} = \frac{1}{n} + Ad < \frac{1}{n} + A^2 d < \frac{1}{n} + d = \max\left\{\frac{1}{n} + d, \frac{1}{n} + Ad\right\}$, i.e., (29) and hence (28) are satisfied.

If $d < 0$ we have: $\min\left\{\frac{1}{n} + d, \frac{1}{n} + Ad\right\} = \frac{1}{n} + d < \frac{1}{n} + A^2 d < \frac{1}{n} + Ad = \max\left\{\frac{1}{n} + d, \frac{1}{n} + Ad\right\}$, i.e., (29) and hence (28) are satisfied.

Hence linear negator is strictly contracting. $\square$

Strictly contracting linear negator can be represented by a contracting spiral in Figure 1, which depicts a sequence of linear negator values $p = N^0(p)$, $N^1(p)$, $N^2(p)$, $N^3(p)$, $\dots$, for $p > \frac{1}{n}$ in the form of a spiral, contracting around the fixed point $\frac{1}{n}$. Figure 2 depicts the sequence $N^k(p)$ from Theorem 4 with the limit $\frac{1}{n}$. Similar figures can be depicted for $p < \frac{1}{n}$.

Figure 1. Contracting negator as a spiral contracting around the fixed point $\frac{1}{n}$.

Figure 2. Contracting negator as a sequence of $N^k(p)$ with the limit $\frac{1}{n}$.

3.3. Involutive Negators and Involutive Negations

Let $P = (p_1, \dots, p_n)$ be a probability distribution from $\mathcal{P}_n$, and $\text{neg}(P)$ be a negation of pd P . It will be called an *involutive negation* if the following property is satisfied:

$$\text{neg}(\text{neg}(P)) = P, \text{ for all } P \text{ in } \mathcal{P}_n. \quad (30)$$

As in the previous sections, consider negations $\text{neg}(P)$ acting element-by-element on pd P using some negator N :

$$\text{neg}(P) = (N(p_1), \dots, N(p_n)), \quad (31)$$

but in this case, to obtain an involutive negation $\text{neg}(P)$ satisfying (30) a negator N should be involutive, such that for any pd $P = (p_1, \dots, p_n)$ it is fulfilled:

$$N(N(p_i)) = p_i, \text{ for all } i = 1, \dots, n. \quad (32)$$

As it follows from Theorem 7, an involutive negator cannot be pd-independent. It means that $N(p_i)$ depends not only on the value of p_i but possibly on other values of probability distribution $P = (p_1, \dots, p_n)$. Hence $\text{neg}(P)$ can be considered as an *operator* acting on pd P and transforming it element-by-element into a new pd Q :

$$Q = \text{neg}(P) = (N_P(p_1), \dots, N_P(p_n)) = (q_1, \dots, q_n),$$

where $N_P(p_i)$ is a pd-dependent negator depending on P . In such notation, the involutivity (30) of a negation neg will have the form:

$$\text{neg}(\text{neg}(P)) = \text{neg}(Q) = (N_Q(q_1), \dots, N_Q(q_n)) = (N_Q(N_P(p_1)), \dots, N_Q(N_P(p_n))) = P = (p_1, \dots, p_n),$$

which, instead of (32), gives $N_Q(N_P(p_i)) = p_i$, for all $i = 1, \dots, n$. The indexes Q and P in N can be used for not forgetting that N is pd-dependent and to what pd negation neg was applied; hence it can be $N_Q(p) \neq N_P(p)$ for some p in $[0, 1]$. But when it is clear

to what pd negation neg is applied, instead of N_Q and N_P we will write simply N . In the previous sections, we considered pd-independent negators $N(p)$ defined on $[0, 1]$ as functions generating element-by-element a negation $neg_N(P)$ of probability distributions P . Now we say that negation $neg(P)$ applies element-by-element transformation $N(p_i)$ of the components of pd $P = (p_1, \dots, p_n)$ that considered as a pd-dependent negator N .

Let $P = (p_1, \dots, p_n)$ be a probability distribution. Denote $\max(P) = \max_i \{p_i\} = \max\{p_1, \dots, p_n\}$, $\min(P) = \min_i \{p_i\} = \min\{p_1, \dots, p_n\}$ and $MP = \max(P) + \min(P)$.

Theorem 9. Let $P = (p_1, \dots, p_n)$ be a probability distribution. Then the function:

$$N(p_i) = \frac{\max(P) + \min(P) - p_i}{n(\max(P) + \min(P)) - 1} = \frac{MP - p_i}{nMP - 1}, \text{ for all } i = 1, \dots, n, \quad (33)$$

is an involutive negator, i.e., the function (31) is an involutive negation, satisfying (30).

Proof. Let us check the fulfillment for (33) the properties (5), (6), and (32).

From (33) and (1) we obtain: $\sum_{i=1}^n N(p_i) = \sum_{i=1}^n \frac{MP - p_i}{nMP - 1} = \frac{\sum_{i=1}^n MP - \sum_{i=1}^n p_i}{nMP - 1} = \frac{nMP - 1}{nMP - 1} = 1$. From (33) we obtain: $N(p_i) \geq 0$, and from $\sum_{i=1}^n N(p_i) = 1$ it follows $N(p_i) \leq 1$ for all $i = 1, \dots, n$, hence (5) is fulfilled.

$N(p_i)$ defined by (33), is a decreasing function of p_i ; hence (6) is fulfilled.

Let us prove (32). Denote $Q = neg(P) = (N(p_1), \dots, N(p_n)) = (q_1, \dots, q_n)$, where $q_i = N(p_i)$ for all $i = 1, \dots, n$. From (33), we have: $q_i = N(p_i) = \frac{MP - p_i}{nMP - 1}$, and $\max(Q) = \max_i \{N(p_i)\} = \max_i \left\{ \frac{MP - p_i}{nMP - 1} \right\} = \frac{MP - \min_i \{p_i\}}{nMP - 1} = \frac{\max(P) + \min(P) - \min(P)}{nMP - 1} = \frac{\max(P)}{nMP - 1}$.

Dually obtain: $\min(Q) = \frac{\min(P)}{nMP - 1}$, and:

$$MQ = \max(Q) + \min(Q) = \frac{\max(P)}{nMP - 1} + \frac{\min(P)}{nMP - 1} = \frac{\max(P) + \min(P)}{nMP - 1} = \frac{MP}{nMP - 1}. \quad (34)$$

From (33) and (34) we obtain the following:

$$N(N(p_i)) = N(q_i) = \frac{MQ - q_i}{nMQ - 1} = \frac{\frac{MP}{nMP - 1} - \frac{MP - p_i}{nMP - 1}}{\frac{nMP}{nMP - 1} - 1} = \frac{\frac{p_i}{nMP - 1}}{\frac{nMP - nMP + 1}{nMP - 1}} = p_i;$$

hence (32) is fulfilled, N is involutive negator, and the negation (31) is involutive. $\square$

We show that the probability value $p \in [0, 1]$ is a fixed point of negator (33) if, for any probability distribution P , it is fulfilled:

$$N(p) = p.$$

Proposition 6. The involutive negator N defined by (33) has the unique fixed point $p = \frac{1}{n}$, and the uniform distribution $P_U = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$ is the unique fixed point of the involutive negation neg with the negator (33).

Proof. We have $N\left(\frac{1}{n}\right) = \frac{MP - \frac{1}{n}}{nMP - 1} = \frac{\frac{1}{n}(nMP - 1)}{nMP - 1} = \frac{1}{n}$, i.e., $p = \frac{1}{n}$ is a fixed point of N . If p is a fixed point of the negator (33), that is: $N(p) = \frac{MP - p}{nMP - 1} = p$, then we obtain sequentially: $MP - p = p(nMP - 1) = pnMP - p$; $MP = pnMP$; $1 = pn$; and finally: $p = \frac{1}{n}$. Hence, $p = \frac{1}{n}$ is the unique fixed point of negator N defined by (33), and:

$$N\left(\frac{1}{n}\right) = \frac{1}{n}. \quad (35)$$

From (35), it follows that the negation of probability distributions $neg(P) = (N(p_1), \dots, N(p_n))$ has a fixed point $P_U = (\frac{1}{n}, \dots, \frac{1}{n})$:

$$neg(P_U) = \left(N\left(\frac{1}{n}\right), \dots, N\left(\frac{1}{n}\right)\right) = \left(\frac{1}{n}, \dots, \frac{1}{n}\right) = P_U.$$

Suppose $P = (p_1, \dots, p_n)$ is a fixed point of negation neg with negator N defined by (33). Then $neg(P) = (N(p_1), \dots, N(p_n)) = (p_1, \dots, p_n)$, hence $N(p_i) = p_i$, for all $i = 1, \dots, n$, i.e., p_i are fixed points of N , hence $p_i = \frac{1}{n}$, and $P = (\frac{1}{n}, \dots, \frac{1}{n}) = P_U$. $\square$

Since the involutive negator (33) is strictly decreasing function, from $N(\frac{1}{n}) = \frac{1}{n}$ it follows for any probability distribution $P = (p_1, \dots, p_n)$ and any $i = 1, \dots, n$:

$$\begin{cases} \text{if } p_i < \frac{1}{n} \text{ then } N(p_i) > \frac{1}{n} \\ N(\frac{1}{n}) = \frac{1}{n} \\ \text{if } p_i > \frac{1}{n} \text{ then } N(p_i) < \frac{1}{n} \end{cases} \quad (36)$$

If $P = (p_1, \dots, p_n) \neq (\frac{1}{n}, \dots, \frac{1}{n}) = P_U$, i.e., not all p_i equal to $\frac{1}{n}$, then from $\sum_{i=1}^n p_i = 1$ it follows that $\min(P) < \frac{1}{n} < \max(P)$, and from (36) and strict monotonicity of N it follows that: $N(\min(P)) > \frac{1}{n} > N(\max(P))$, and for all p_i , $i = 1, \dots, n$, from (33) it is fulfilled: $\max_i \{N(p_i)\} = N(\min(P)) = \frac{\max(P)}{nMP-1} \geq N(p_i) \geq N(\max(P)) = \frac{\min(P)}{nMP-1} = \min_i \{N(p_i)\}$.

To summarize the last considerations, we can say that for any probability distribution $P = (p_1, \dots, p_n)$ the values of negations $N(p_i)$, $i = 1, \dots, n$, are located on the decreasing line connecting two points in 2-dimensional space with coordinates: $(\min(P), \frac{\max(P)}{nMP-1})$ and $(\max(P), \frac{\min(P)}{nMP-1})$, and passing through the fixed point $(\frac{1}{n}, \frac{1}{n})$.

When $MP = 1$, the involutive negator (33) coincides with Yager's negator. This situation appears when $n = 2$ or for a point distribution $P_{(i)} = (p_1, \dots, p_n)$ satisfying the property: $p_i = 1$ for some $i = 1, \dots, n$, and $p_j = 0$ for all $j \neq i$. For example, for $P_{(1)} = (1, 0, \dots, 0)$, we have: $neg(P_{(1)}) = (N(1), N(0), \dots, N(0)) = (0, \frac{1}{n-1}, \dots, \frac{1}{n-1})$.

Example 1. Consider probability distribution $P = (0.1, 0.2, 0.15, 0.3, 0.25)$. We have $n = 5$, fixed point: $\frac{1}{n} = \frac{1}{5} = 0.2$, and $N(0.2) = 0.2$, $\max(P) = 0.3$, $\min(P) = 0.1$, $MP = 0.4$, $nMP - 1 = 5(0.4) - 1 = 1$, $N(p_i) = \frac{MP-p_i}{nMP-1} = \frac{0.4-p_i}{1} = 0.4 - p_i$. $neg(P) = (N(0.1), N(0.2), N(0.15), N(0.3), N(0.25)) = (0.3, 0.2, 0.25, 0.1, 0.15)$. After similar calculations, we obtain: $neg(neg(P)) = P$.

4. Discussion

The paper studied negators generating element-by-element negations of probability distributions (pd). We showed that the sequence of multiple negations of pd generated by a pd-independent linear negator converges to the uniform distribution with maximal entropy. We showed that all pd-independent negators are non-involutive, and non-trivial linear negators are strictly contracting; hence, we need to look for an involutive negator in the class of pd-dependent negators. Finally, we introduced an involutive negator in the class of pd-dependent negators. It generates involutive negation of probability distributions. Such involutive negation can formalize a probability distribution NOT(P), where P is some linguistic concept like HIGH defined on a set of probability distributions. The involutivity of negation, like NOT(NOT(P)) = P, is a common property for many logical systems and can be used in reasoning systems operating with terms represented by probability distributions.

We plan to apply the proposed involutive negation in Dempster-Shaffer theory, as it was proposed in the original work of Yager [1], and also in data analysis.

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References

1. Yager, R.R. On the maximum entropy negation of a probability distribution. *IEEE Transact. Fuzzy Syst.* **2015**, *23*, 1899–1902. [CrossRef]
2. Srivastava, A.; Maheshwari, S. Some new properties of negation of a probability distribution. *Int. J. Intell. Syst.* **2018**, *33*, 1133–1145. [CrossRef]
3. Srivastava, A.; Kaur, L. Uncertainty and negation—Information theoretic applications. *Int. J. Intell. Syst.* **2019**, *34*, 1248–1260. [CrossRef]
4. Yin, L.; Deng, X.; Deng, Y. The negation of a basic probability assignment. *IEEE Transact. Fuzzy Syst.* **2018**, *27*, 135–143. [CrossRef]
5. Xie, D.; Xiao, F. Negation of basic probability assignment: Trends of dissimilarity and dispersion. *IEEE Access* **2019**, *7*, 111315–111323. [CrossRef]
6. Sun, C.; Li, S.; Deng, Y. Determining weights in multi-criteria decision making based on negation of probability distribution under uncertain environment. *Mathematics* **2020**, *8*, 191. [CrossRef]
7. Zhang, J.; Liu, R.; Zhang, J.; Kang, B. Extension of Yager’s negation of a probability distribution based on Tsallis entropy. *Int. J. Intell. Syst.* **2020**, *35*, 72–84. [CrossRef]
8. Gao, X.; Deng, Y. The negation of basic probability assignment. *IEEE Access* **2019**, *7*, 107006–107014. [CrossRef]
9. Xie, K.; Xiao, F. Negation of belief function based on the total uncertainty measure. *Entropy* **2019**, *21*, 73. [CrossRef] [PubMed]
10. Luo, Z.; Deng, Y. A matrix method of basic belief assignment’s negation in Dempster-Shafer theory. *IEEE Transact. Fuzzy Syst.* **2019**, *28*, 2270–2276. [CrossRef]
11. Li, S.; Xiao, F.; Abawajy, J.H. Conflict management of evidence theory based on belief entropy and negation. *IEEE Access* **2020**, *8*, 37766–37774. [CrossRef]
12. Batyrshin, I.; Villa-Vargas, L.A.; Ramírez-Salinas, M.A.; Salinas-Rosales, M.; Kubysheva, N. Generating Negations of Probability Distributions. *Soft Comput.* **2021**. Available online: <https://arxiv.org/abs/2103.14986> (accessed on 27 March 2021). [CrossRef]
13. Trillas, E. Sobre funciones de negacion en la teoria de conjuntos difusos. *Stochastica* **1979**, *3*, 47–59.
14. Batyrshin, I.Z. Lexicographic valuations of plausibility with universal bounds. II. Negation operations. *Izv. Ross. Akad. Nauk. Teor. Sist. Upr.* **1995**, *5*, 133–151. (In Russian)
15. Batyrshin, I.; Wagenknecht, M. Contracting and expanding negations on $[0,1]$. *J. Fuzzy Math.* **1998**, *6*, 133–140.
16. Batyrshin, I.Z. *Basic Operations of Fuzzy Logic and Their Generalizations*; Otechestvo Publisher: Kazan, Russia, 2001.
17. Batyrshin, I. On the structure of involute, contracting and expanding negations. *Fuzzy Sets Syst.* **2003**, *139*, 661–672. [CrossRef]
18. Sheremetov, L.; Batyrshin, I.; Filatov, D.; Martinez, J.; Rodriguez, H. Fuzzy expert system for solving lost circulation problem. *Appl. Soft Comput.* **2008**, *8*, 14–29. [CrossRef]
19. Batyrshin, I. Uncertainties with memory in construction of strict monotonic t-norms and t-conorms for finite ordinal scales: Basic definitions and applications. *Appl. Comput. Math.* **2011**, *10*, 498–513.
20. Evans, M.J.; Rosenthal, J.S. *Probability and Statistics: The Science of Uncertainty*, 2nd ed.; W. H. Freeman and Company: New York, NY, USA, 2010.