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n-th Order Functional Problems with Resonance of Dimension One

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Abstract: We consider the nonlinear n-th order boundary value problem $Lu = u^{(n)} = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) = Nu$ given arbitrary bounded linear functional conditions $B_i(u) = 0$, $i = 1, \ldots, n$ and develop a method that allows us to study all such resonance problems of order one, as well as implementing a more general constructive method for deriving existence criteria in the framework of the coincidence degree method of Mawhin. We demonstrate applicability of the formalism by giving an example for $n = 4$.

Keywords: Carathéodory conditions; coincidence degree theory; functional condition; resonance

MSC: 34B10; 34B15

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1. Introduction

We consider the differential equation

$$u^{(n)} = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)), \quad 0 < t < 1, \tag{1}$$

together with the functional conditions

$$B_i(u) = 0, \quad i = 1, .., n. \tag{2}$$

The topic of existence results for resonance problems in the view of topological degree due to Mawhin has a long history [1–11]. Many results span not just ordinary differential equations, but even fractional differential equations [12,13]. Among the degree based methods, the coincidence degree theory continues to play an important role [2,4–7,14] just to name a few. There are two particular matters of import, that being the structure of the projectors and that of the order of resonance. A common theme emerges from recent research when it comes to finding suitable projectors: fix the boundary conditions away from a more general form in order to construct suitable projectors. We highlight a few examples.

In [9], the author study the problem

$$x''(t) = q(t)f(t, x(t), x'(t)), \quad t \in (0, \infty),$$

$$x(0) = 0, \quad x'(+\infty) = \int_0^\infty x'(s)\, dg(s),$$

where $g : [0, \infty) \to [0, \infty)$ is nondecreasing and such that $g(0) = 0$. The resonance condition $g(\infty) = 1$. The existence results were obtained using the coincidence degree theorem of Mawhin stated below, which is the method of our paper, as well.

In [11], which is a generalization of [9],the authors analyzed an n-th order boundary value problem.

$$u^{(n)}(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t)), \quad t \in (0, \infty),$$
$$u^{(i)}(0) = 0,\, i = 1, \dots, n-3, \quad u^{(n-1)}(\infty) = 0,$$
$$\Gamma_1(u) = \Gamma_2(u) = 0,$$

where $n \geq 3, \Gamma_1, \Gamma_2 : C^{(n-1)}[0, \infty) \to \mathbb{R}$. In this work, the resonance occurs if $\Gamma_1(1)\Gamma_2(t^{n-2}) = \Gamma_1(t^{n-2})\Gamma_2(1)$.

The above shows a similar trend: they offer a particular type of bounded linear boundary conditions. Using said conditions they obtain projectors P and Q and conclude that indeed solutions exist because of coincidence degree theory due to Mawhin. The problem here is that this obscures the methodology of obtaining projectors necessary for the theory in general, since the choice of boundary conditions are too specific. We provide a method for these of the form of (1) of resonance one in which we assume very little about the boundary conditions (2), in order to illuminate this process of constructing projectors. This is a further extension of [7] in which they apply a similar methodology to solve all similar problems to that of (1) but $n = 2$. Here n is arbitrary. This is beneficial as there exist problems of higher order which are of interest, such as fourth order problems dealing with thermo-elasticity, mechanics, and flow as seen in papers such as [15]. We will use [7] in several cases to confirm that our results do indeed generalize their methods.

We assume the following:

(B_1) The linear functionals $B_i : X \to \mathbb{R},\ i = 1, \dots, n$, are such that the matrix

$$K = [\sigma_{ij}]_{i,j=1,\dots,n}, \quad \sigma_{ij} = \frac{1}{(n-j)!} B_i(t^{n-j}), \tag{3}$$

has rank $n - 1$;

(B_2) The functionals $B_i : X \to \mathbb{R}$ are continuous with the respective norms β_i, where $X = C^{(n-1)}[0, 1]$ with the norm $||u||_X = \max_{i=1,\dots,n} ||u^{(i-1)}||_0$, where $|| \cdot ||_0$ is the max norm.

Set

$$\operatorname{dom} L = \{u \in X : u^{(n)} \in Z \text{ and } \hat{B}u = 0\}$$

where $\hat{B}(u) = [B_i(u)]^T_{i=1,\dots,n}$ and $Z = L_1[0, 1]$ with the usual norm $|| \cdot ||_Z$. Define the mapping $L : \operatorname{dom} L \subset X \to Z$ by

$$Lu = u^{(n)}. \tag{4}$$

We define the mapping $N : X \to Z$ by

$$Nu(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t)),$$

where f is Carathéodory. Then (1) and (2) is equivalent to the coincidence equation $Lu = Nu$. The following results come from [8].

Definition 1. *Let X and Z be real normed spaces. A linear mapping $L : \operatorname{dom} L \subset X \to Z$ is called a Fredholm mapping if* $\ker L$ *has finite dimension and* $\operatorname{Im} L$ *is closed and has a finite co-dimension. If L is a Fredholm mapping, its (Fredholm) index is the integer $Ind\ L = \dim \ker L - codim\ \operatorname{Im} L$.*

In this instance we are concerned with a Fredholm mapping of index zero. Thus we construct continuous projectors $P : X \to X$, $Q : Z \to Z$ so that

$$\operatorname{Im} P = \ker L, \ker Q = \operatorname{Im} L, X = \ker L \oplus \ker P, Z = \operatorname{Im} L \oplus \operatorname{Im} Q. \tag{5}$$

The map

$$L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

has the inverse denoted by $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Z \to \operatorname{dom} L \cap \ker P$ is defined by $K_P(I - Q)$. If L is a Fredholm mapping of index zero, then, for every isomorphism $J : \operatorname{Im} Q \to \ker L$, the mapping $JQ + K_{P,Q} : Z \to \operatorname{dom} L$ is an isomorphism and

$$(JQ + K_{P,Q})^{-1} u = (L + J^{-1}P)u, u \in \operatorname{dom} L.$$

Definition 2. *Let $L : \operatorname{dom} L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N : E \to Z$ be a mapping. We say that N is L-compact on E if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are continuous and compact on E. In addition, we say that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.*

For our methods to apply, we need that our given N is L-compact. Since f is Carathéodory, however, this follows from the dominated convergence theorem and the Kolmogorov-Riesz criterion:

Theorem 1. *For $1 \leq p \leq \infty$, $E \subset L_p[0,1]$ is compact if*

- *E is bounded;*
- *the limit*

$$\lim_{\epsilon \to 0} \int_0^1 |g(s+\epsilon) - g(s)|^p \, ds = 0$$

is uniform in E.

To show the existence of a solution to (1) and (2), expressed as $Lu = Nu$, we apply the following theorem from [8]:

Theorem 2. *Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following are satisfied:*

1. *$Lu \neq Nu$ for every $(u, \lambda) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times (0,1)$;*
2. *$Nu \notin \operatorname{Im} L$ for every $u \in \ker L \cap \partial\Omega$;*
3. *$deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, with $Q : Z \to Z$ a continuous projector such that $\ker Q = \operatorname{Im} L$, and $J : \operatorname{Im} Q \to \ker L$ is an isomorphism.*

Then the equation $Lu = Nu$ has at least one solution in $\operatorname{dom} L \subset \overline{\Omega}$.

2. Technical Lemmas

The set

$$\begin{aligned} T_n &= \left\{ \sum_{i=1}^{n} \frac{\beta_i t^{n-i}}{(n-i)!} : \hat{\beta} = [\beta_i]_{i=1,\ldots,n}^T \in \mathbb{R}^n \right\} \\ &= \left\{ u \in AC^{n-1}[0,1] : u(t) = \sum_{i=1}^{n} \frac{u^{(n-i)}(0) t^{n-i}}{(n-i)!} \right\} \end{aligned}$$

is the solution space of the homogeneous Equation (1). Let (3) represent a linear map $\mathcal{K} : \mathbb{R}^n \to \mathbb{R}^n$. By (2) and (B_1), $u \in \ker L \subseteq T_n$ if and only if

$$B_i\left(\sum_{j=1}^{n} \frac{\beta_j t^{n-j}}{(n-j)!} \right) = \sum_{j=1}^{n} \frac{\beta_j}{(n-j)!} B_i(t^{n-j}) = 0, \, i = 1, \ldots, n.$$

Then $\ker \mathcal{K} = \left\{ \hat{\beta} \in \mathbb{R}^n : K\hat{\mathcal{L}}u = \hat{0} \right\}$, $\dim \ker \mathcal{K} = 1$, where

$$\hat{\mathcal{L}}u = [u^{(n-1)}(0), u^{(n-2)}(0), \ldots, u(0)]^T, \tag{6}$$

if and only if (B_1) is fulfilled. We define a map $\hat{G} : Z \to \mathbb{R}^n$ by

$$\hat{G}(g) = [B_i G(g)]^T_{i=1,\dots,n}, \quad G(g)(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} g(s)\,ds. \tag{7}$$

Note that $u \in \operatorname{dom} L$ implies $Lu = g(t)$ for some $g \in Z$. Hence

$$u(t) = \sum_{i=1}^{n} \frac{u^{(n-i)}(0)t^{n-j}}{(n-j)!} + G(g)(t),$$

which, together with (2), yields

$$\hat{G}(g) + K\hat{\mathcal{L}}u = \hat{0}.$$

We state our next lemma in terms of (3) and (7).

Lemma 1. *The functional differential problem (1) and (2) is at resonance of dimension 1 if and only if the condition (B_1) is satisfied. Moreover,*

$$\ker L = \left\{u \in T_n : K\hat{\mathcal{L}}u = \hat{0}\right\} \text{ and } \operatorname{Im} L = \left\{g \in Z : \hat{G}(g) \in \operatorname{Im}\mathcal{K}\right\}.$$

Let $\mathcal{K}' : \mathbb{R}^n \to \mathbb{R}^n$ denote the adjoint of $\mathcal{K}$. Then

$$\ker\mathcal{K}' \oplus \operatorname{Im}\mathcal{K} = \ker\mathcal{K} \oplus \operatorname{Im}\mathcal{K}' = \mathbb{R}^n.$$

Let $\{\hat{\kappa}_i : i = 1,\dots,n\}$ and $\{\hat{\rho}_i, i = 1,\dots,n\}$ be such that

$$\operatorname{span}\{\hat{\kappa}_1\} = \ker\mathcal{K} \text{ and } \operatorname{span}\{\hat{\kappa}_2,\dots,\hat{\kappa}_n\} = \operatorname{Im}\mathcal{K}', \tag{8}$$

$$\operatorname{span}\{\hat{\rho}_1\} = \ker\mathcal{K}' \text{ and } \operatorname{span}\{\hat{\rho}_2,\dots,\hat{\rho}_n\} = \operatorname{Im}\mathcal{K} \tag{9}$$

and, in addition,

(P_1) Let $\{\hat{\kappa}_i : i = 1,\dots,n\}$ be a basis of unit vectors in $\mathbb{R}^n$ such that $\langle\hat{\kappa}_1, \hat{\kappa}_j\rangle = 0$ for all $j = 2,\dots,n$,

(P_2) Let $\{\hat{\rho}_i : i = 1,\dots,n\}$ be a basis of unit vectors in $\mathbb{R}^n$ such that $\langle\hat{\rho}_1, \hat{\rho}_j\rangle = 0$ for all $j = 2,\dots,n$.

Remark 1. *Notice that the original problem outlined in [7] offers*

$$K = \begin{bmatrix} \alpha b & \alpha a \\ b & a \end{bmatrix}.$$

From this we can tell that $\ker\mathcal{K} = \{c[a,-b]^T : c \in \mathbb{R}\}$*; compare it to* $\ker L = \{c(at-b) : c \in \mathbb{R}\}$ *in [7].*

3. Projectors in X and Z

We present a strategy for "isolating" an arbitrary subspace for T_n by a continuous linear projector in X. This task is essentially algebraic and the continuity follows from properties of the Wronskian. The construction to follow is independent of the functional problem in question.

We consider $M = [\hat{\kappa}_1|\dots|\hat{\kappa}_n] = [\kappa_{ij}]_{i,j=1,\dots,n}$ and define

$$\psi_{\hat{\kappa}_i}(t) = \sum_{j=1}^{n} \frac{\kappa_{ji}}{(n-j)!} t^{n-j}, i = 1,\dots,n$$

so that $\hat{\mathcal{L}}\psi_{\hat{\kappa}_i} = \hat{\kappa}_i$.

Theorem 3. *Assume that* (P_1) *holds. Then there exist linear projectors* $P, \tilde{P} : X \to X$ *such that* $X = \ker P \oplus \operatorname{Im} P$*, with* $T_n = \operatorname{Im} \tilde{P} \oplus \operatorname{Im} P$*, where* $\operatorname{Im} P = span\{\psi_{\hat{\kappa}_1}\}$ *and* $\ker P = \{u \in X : \hat{\mathcal{L}}u \in span\{\hat{\kappa}_i : i = 2, \ldots, n\}\}$*.*

Proof. Let $u \in X$ and define $P : X \to X$ by

$$Pu(t) = \langle \hat{\kappa}_1, \hat{\mathcal{L}}u \rangle \psi_{\hat{\kappa}_1}(t). \tag{10}$$

Since $P\left(\frac{t^{n-i}}{(n-i)!}\right)(t) = \kappa_{i1}\psi_{\hat{\kappa}_1}$, $i = 0, \ldots, n-1$, then $P\psi_{\hat{\kappa}_1}(t) = \psi_{\hat{\kappa}_1}(t)$ and so $P^2 = P$, $X = \ker P \oplus \operatorname{Im} P$ with $\operatorname{Im} P = \operatorname{span}\{\psi_{\hat{\kappa}_1}\}$ and $\ker P = \{u \in X : \hat{\mathcal{L}}u \in \operatorname{span}\{\hat{\kappa}_i : i = 2, \ldots, n\}\}$. □

Now, let

$$\tilde{P}u(t) = \langle M^{-1}\hat{\mathcal{L}}u, \hat{\psi}_0 \rangle \tag{11}$$

where $\hat{\psi}_0 = [0, \psi_{\hat{\kappa}_2}(t), \ldots, \psi_{\hat{\kappa}_n}(t)]^T$. With $M^{-1}\hat{\mathcal{L}}u = [\omega_i]_{i=1,\ldots,n}$, this can be written as

$$\tilde{P}u(t) = \sum_{i=2}^{n} \omega_i \psi_{\hat{\kappa}_i}(t).$$

This reveals that

$$\hat{\mathcal{L}}\tilde{P}u = \sum_{i=2}^{n} \omega_i \hat{\kappa}_i = MUM^{-1}\hat{\mathcal{L}}u$$

where U is the identity matrix with the first column replaced with the zero column and so

$$\tilde{P}^2u(t) = \langle M^{-1}\hat{\mathcal{L}}\tilde{P}u, \hat{\psi}_0 \rangle = \langle UM^{-1}\hat{\mathcal{L}}u, \hat{\psi}_0 \rangle = \tilde{P}u(t).$$

We can see that $\tilde{P}\psi_{\hat{\kappa}_i}(t) = \psi_i(t)$ for $i = 2, \ldots, n$ because of the properties of $\hat{\mathcal{L}}$ and the M^{-1} factor and so $\operatorname{Im} \tilde{P} = \operatorname{span}\{\psi_{\hat{\kappa}_i} : i = 2, \ldots, n\}$. But then $\ker P|_{T_n} = \operatorname{Im} \tilde{P}|_{T_n}$ because of the properties of $\hat{\mathcal{L}}$.

Remark 2. *Note that one could rewrite* $Pu(t)$ *as*

$$Pu(t) = \langle M^{-1}\hat{\mathcal{L}}u, \hat{\psi}_1 \rangle$$

where $\hat{\psi}_1 = [\psi_{\hat{\kappa}_1}, 0, \ldots, 0]^T$ *and so*

$$Pu(t) + \tilde{P}u(t) = \langle M^{-1}\hat{\mathcal{L}}u, \hat{\psi} \rangle$$

where $\hat{\psi} = \hat{\psi}_0 + \hat{\psi}_1$*. Define*

$$\hat{\alpha} = \left[\frac{1}{(n-1)!}t^{n-1}, \frac{1}{(n-2)!}t^{n-2}, \ldots, t, 1\right]^T.$$

Then its clear that $\hat{\psi} = M^T\hat{\alpha}$*. This means that*

$$\begin{aligned} Pu(t) + \tilde{P}u(t) &= \langle M^{-1}\hat{\mathcal{L}}u, \hat{\psi} \rangle = \langle \hat{\mathcal{L}}u, (M^T)^{-1}\hat{\psi} \rangle \\ &= \langle \hat{\mathcal{L}}u, (M^T)^{-1}M^T\hat{\alpha} \rangle = \langle \hat{\mathcal{L}}u, \hat{\alpha} \rangle \\ &= \sum_{i=1}^{n} \frac{1}{(n-i)!} t^{n-i} u^{(n-i)}(0) = P_0u(t). \end{aligned}$$

The same technique seen in the previous theorem can be used to show that $P_0^2 = P_0$*,* $\ker P_0 = \{u \in X : \hat{\mathcal{L}}u = \hat{0}\}$*,* $\operatorname{Im} P_0 = T_n$*, and* $X = \ker P_0 \oplus \operatorname{Im} P_0$*.*

Remark 3. *In the problem outline in [7], we would have $\kappa_1 = [a, -b]^T$ with $\psi_{\hat{\kappa}_1} = at - b$ meaning one has the projector*

$$Pu(t) = \frac{1}{a^2 + b^2}(au'(0) - bu(0))(at - b),$$

which is the exactly the projector P identified in the paper.

This leaves our projector Q.

Lemma 2. *Let $\hat{\rho}_i \in \mathbb{R}^n$, $i = 1, \dots, n$, satisfy (P_2). Then there exists a linear projector $Q : Z \to Z$ such that $Z = \ker Q \oplus \operatorname{Im} Q$, where $\operatorname{Im} Q = span\{h\}$ for some $h \in Z$ and $\ker Q = \{g \in Z : \hat{G}(g) \in span\{\hat{\rho}_i : i = 2, \dots, n\}\}$.*

Proof. Although an abuse of notation, consider the operator

$$B = \langle \hat{\rho}_1, \hat{B} \rangle.$$

We show that there exists an $h \in Z$ such that $B(Gh) \neq 0$. Note that

$$B\left(\frac{t^{n-i}}{(n-i)!}\right) = \left\langle \hat{\rho}_1, \hat{B}\left(\frac{t^{n-i}}{(n-i)!}\right)\right\rangle = 0$$

because $\hat{\rho}_1$ is the vector that spans $\ker \mathcal{K}'$ and $\frac{\hat{B}(t^{n-i})}{(n-i)!}$ is just the ith row of K^T which is the associated matrix for $\mathcal{K}'$. But at the same time, each B_i is linearly independent, meaning there exists a $u_0 \in X \backslash T_n$ such that $B(u_0) \neq 0$. By the continuity of B there exists a polynomial p such that $||p - u_0||_X < \epsilon$ and $Lp \neq 0$ and so set $h = Lp$. Then

$$BG(h) = BG(Lp) = B\left(p - \sum_{i=1}^{n} \frac{u^{(n-i)}(0)t^{n-i}}{(n-i)!}\right) = Bp \neq 0.$$

Since $B \circ G$ is linear, we can choose h such that $BG(h) = 1$. Define $Q : Z \to Z$ by

$$Qg(t) = BG(g)h(t) = \left(\sum_{i=1}^{n} \rho_{i1} B_i\right) G(g)h(t). \tag{12}$$

Since $Qh(t) = h(t)$, we have that $Q^2 g = Qg$, $g \in Z$. Obviously, Q is a continuous map and $Z = \ker Q \oplus \operatorname{Im} Q$, $\operatorname{Im} Q = \operatorname{span}\{h\}$, $\ker Q = \{g \in Z : \hat{G}(g) \in \operatorname{span}\{\hat{\rho}_i : i = 2, \dots, n\}\}$. $\square$

Remark 4. *In [7], one had that*

$$Qg(t) = (B_1 - \alpha B_2)\left(\int_0^t (t-s)g(s)\, ds\right)h(t).$$

Note that $\ker \mathcal{K}' = span\{[1, -\alpha]^T\}$ in the case of $n = 2$ from their paper and so constitutes a quick example of the above.

4. Main Results

The formulas for P and Q in the previous section are not presumed to depend on the geometry of a particular problem. Now this connection is made and we obtain suitable decompositions of the spaces X and Z by an exact pair (P, Q) of projectors. This is done in the next lemma.

Lemma 3. *Let (B_1) and (B_2) hold and $\hat{G}$ be given by (7). Then the mapping $L : \operatorname{dom} L \subset X \to Z$ is a Fredholm mapping of index zero.*

Proof. We utilize the projectors P and Q from Lemma 3, Lemma 2 respectively.

By Lemma 2 and that $\operatorname{Im}\mathcal{K} = \operatorname{span}\{\hat{\rho}_i : i = 2, \dots, n\}$,

$$\ker Q = \{g \in Z : \hat{G}(g) \in \operatorname{Im}\mathcal{K}\} = \operatorname{Im} L \subset X$$

for the projector (12). In particular, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$ and $\operatorname{codim} \operatorname{Im} L = 1$.

By (3) and that $\operatorname{Im} P = \operatorname{span}\{\psi_{\hat{\kappa}_1}\} = \{u \in T_n : K\hat{\mathcal{L}}u = \hat{0}\}$ since with $\hat{\mathcal{L}}(\psi_{\hat{\kappa}_1})$ one would obtain the vector $\hat{\kappa}_1$ by construction,

$$\operatorname{Im} P = \{u \in T_n : K\hat{\mathcal{L}}u = \hat{0}\} = \ker L$$

for the projector (10). In particular, $X = \ker L \oplus \ker P$ and $\dim \ker L = 1$. Thus L is a Fredholm map of index zero. $\square$

Now, consider K and recall that K is of rank $n-1$. Let

$$K_b = [\sigma'_{lk}]_{l,k=1,\dots,n-1}, \text{ where } \sigma'_{lk} = \sigma_{i_l j_k},\ i_l < i_{l+1},\ j_k < j_{k+1}$$

so that $\det K_b$ is a basis minor. Let $g \in \operatorname{Im} L$ and choose $\hat{\kappa}_0 = \hat{\kappa}_0(g) \in \mathbb{R}^n$ satisfy the equation $K\hat{\kappa} = -\hat{G}(g)$ with $\hat{G}$ given by (7). In particular, with the matrix K_b, $\hat{\kappa}_0 = [\kappa_{0,i}]^T_{i=1,\dots,n-1}$ can be determined by

$$\kappa_{0,i_k} = \begin{cases} -(K_b^{-1}\hat{G}_b(g))_k, & \text{if } k = 1, \dots, n-1 \\ 0, & k = n \end{cases} \tag{13}$$

Note that

$$K\hat{\kappa}_0 = KMM^{-1}\hat{\kappa}_0 = \sum_{i=2}^{n} (M^{-1}\hat{\kappa}_0)_i K\hat{\kappa}_i = \sum_{i=2}^{n} (M^{-1}\hat{\kappa}_0)_i K\hat{\kappa}_i;$$

this suggests defining

$$\phi(g) = \sum_{i=2}^{n} (M^{-1}\hat{\kappa}_0)_i \psi_{\hat{\kappa}_i}. \tag{14}$$

Subsequently, we define $K_P : \operatorname{Im} L \to X$ by

$$K_P g = \phi(g) + G(g). \tag{15}$$

The map K_P is well-defined. Indeed, if $\hat{\omega}_1, \hat{\omega}_2 \in \mathbb{R}^n$ satisfy the equation $K\hat{\kappa} = -\hat{G}(g)$, then $\hat{\omega}_1 - \hat{\omega}_2 \in \ker \mathcal{K}$, meaning $\hat{\omega}_1 = \hat{\omega}_2 + \hat{\zeta}$, $\hat{\zeta} \in \ker \mathcal{K}$. This means that

$$\phi(g)_{\hat{\omega}_1} = \sum_{i=2}^{n} (M^{-1}\hat{\omega}_1)_i \psi_{\hat{\kappa}_i} = \sum_{i=2}^{n} (M^{-1}\hat{\omega}_2)_i \psi_{\hat{\kappa}_i} = \phi(g)_{\hat{\omega}_2}.$$

We have

$$LK_P g = L\phi(g) + LG(g) = g$$

by definition of G and since $\phi(g) \in T_n$. Also

$$\hat{B}\phi(g) = \sum_{i=2}^{n} (M^{-1}\hat{\kappa}_0)_i \hat{B}\psi_{\hat{\kappa}_i} = \sum_{i=2}^{n} (M^{-1}\hat{\kappa}_0)_i K\hat{\kappa}_i$$

and

$$\hat{B}G(g) = \hat{G}(g) = -K\hat{\kappa}_0.$$

Hence

$$\hat{B}K_P g = \hat{B}\phi(g) + \hat{B}G(g) = K\left(\sum_{i=2}^{n}(M^{-1}\hat{\kappa}_0)_i\hat{\kappa}_i - \hat{\kappa}_0\right)$$
$$= -K\left(M^{-1}\hat{\kappa}_0)_1\hat{\kappa}_1\right)$$
$$= \hat{0}.$$

Note that $\hat{\mathcal{L}}G(g) = \hat{0}$ and $\phi(g) \in \operatorname{Im}\tilde{P}$. Hence $K_P g \in \ker P$ since $PK_P g = P\phi(g) + PG(g) = 0$. Thus $K_P g \in \operatorname{dom} L \cap \ker P$.

Now, let $u \in \operatorname{dom} L \cap \ker P$ and $g = Lu = u^{(n)}$. By Lemmas 1 and 3, $K\hat{\mathcal{L}}u = \hat{G}(u^{(n)})$ and

$$\hat{\mathcal{L}}u = \sum_{i=2}^{n}(M^{-1}\hat{\mathcal{L}}u)_i\hat{\kappa}_i \in \operatorname{Im}\mathcal{K}'.$$

Thus

$$\phi(u^{(n)}) = \sum_{i=2}^{n}(M^{-1}\hat{\mathcal{L}}u)_i\psi_{\hat{\kappa}_i} = \tilde{P}u.$$

Therefore

$$K_P Lu = \phi(Lu) + G(Lu) = \tilde{P}u + \left(u - \sum_{i=1}^{n} u^{(n-i)}(0)\frac{t^{n-i}}{(n-i)!}\right)$$
$$= \tilde{P}u + u - P_0 u$$
$$= u - Pu$$
$$= u$$

since $u \in \ker P$.

We can summarize the above as a lemma.

Lemma 4. *For L defined by (4), $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ defined by (15) satisfies*

$$K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}.$$

Remark 5. *Again, considering [7], we see that we may remove the first row and column of K; this results in $K_b = [a]$ and so $\hat{\kappa}_0 = [0, -\frac{1}{a}B_2G(g)]^T$. We find that $[b,a]^T$ is a vector orthogonal to $[a,-b]^T$ and so $\psi_{\hat{\kappa}_2} = bt + a$ and*

$$M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, M^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and so

$$K_p g = -\frac{bt+a}{a^2+b^2}B_2G(g) + G(g)$$

which is exactly the generalized inverse found in said paper.

Assume the following conditions on the function $f(t, x_1, \ldots, x_n)$ are satisfied:

(H_1) there exists a constant $K_0 > 0$ such that, for each $u \in \operatorname{dom} L \backslash \ker L$ with $\sum_{i=0}^{n-1}|u^{(i)}(t)| > K_0$, $t \in [0,1]$, we have $QNu \neq 0$.

(H_2) there exist functions $\delta_i \in L^1[0,1]$, $i = 0, \ldots, n$ such that, for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and a.e. $t \in [0,1]$,

$$|f(t, x_1, \ldots, x_n)| \leq \delta_0 + \sum_{i=1}^{n}\delta_i(t)|x_i|.$$

(H_3) there exists a constant $K_1 > 0$ such that if $|c| > K_1$ then $cB(GNu_c) > 0$ where $u_c = c\psi_{\hat{\kappa}_1}$.

This leads us into the position to prove the following existence theorem.

Theorem 4. *If (B_1), (B_2), (H_1)–(H_3) hold, then the functional problem (1), (2) has at least one solution provided*

$$\sum_{i=1}^{n} ||\delta_i||_Z < 1.$$

Proof. Let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L : Lu = \lambda Nu, \lambda \in (0,1)\}$. If $u \in \Omega_1$, it follows from (H_1) that there exists a $t_0 \in [0,1]$ such that $|u^{(i)}(t_0)| \leq K_0$, $i = 0, \ldots, n-1$. Of course, one has

$$u^{(n-1)}(t) = u^{(n-1)}(t_0) + \int_{t_0}^{t} u^{(n)}(s)\, ds,$$

and, in conjunction with above, the inequality

$$|u^{(n-1)}(t)| \leq |u^{(n-1)}(t_0)| + \int_0^1 |u^{(n)}(s)|\, ds < K_0 + ||Lu||_Z.$$

Suppose that one has

$$|u^{(n-i)}(t)| < iK_0 + ||Lu||_Z.$$

Then

$$|u^{(n-i-1)}(t)| < K_0 + \int_0^1 |u^{(i)}(s)|\, ds < K_0 + iK_0 + ||Lu||_Z = (i+1)K_0 + ||Lu||_Z.$$

This results in

$$||u||_X \leq ||Lu||_Z + nK_0 < ||Nu||_Z + nK_0 \leq ||h_0||_Z + \sum_{i=1}^{n} ||h_i|_Z||u||_X + nK_0$$

and so

$$||u||_X \leq \frac{||h_0||_Z + nK_0}{1 - \sum_{i=1}^{n} ||h_i||_Z}$$

meaning Ω_1 is bounded. □

Define $\Omega_2 = \{u \in \ker L : Nu \in \operatorname{Im} L\}$. Then $u = c\psi_{\hat{\kappa}_1}$ for some $c \in \mathbb{R}$. Since $Nu \in \operatorname{Im} L = \ker Q$, $B(GNu) = 0$. By (H_3), $|c| \leq K_1$; that is, Ω_2 is bounded.

Define $J : \operatorname{Im} Q \to \ker L$ by

$$Jg(t) = BG(g)\psi_{\hat{\kappa}_1}.$$

Then $J(ch)(t) = cBG(h)\psi_{\hat{\kappa}_1} = c\psi_{\hat{\kappa}_1}$, meaning we have an isomorphism.

Let $\Omega_3 = \{u \in \ker L : \lambda u + (1-\lambda)JQNu = 0, \lambda \in [0,1]\}$. Let u be denoted by $u_c = c\psi_{\hat{\kappa}_1}$. Then $\lambda u + (1-\lambda)JQNu = 0$ implies $\lambda u_c + (1-\lambda)JQNu_c = 0$. If $\lambda = 0$ then $JQNu_c = 0$; that is, $u \in \Omega_2$, which is bounded. If $\lambda = 1$, then $c = 0$. If $\lambda \in (0,1)$ then, by (H_3),

$$0 < \lambda c^2 = -(1-\lambda)cB(GNu_c) < 0,$$

which is a contradiction. Thus Ω_3 is bounded.

Let Ω be open and bounded such that $\cup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$. Then the first two assumptions of Theorem 2 are fulfilled. Lemma 3 states that L is Fredholm of index zero. We are left with only determining the third assumption of Theorem 2.

We apply the degree property of invariance under a homotopy to

$$H(u,\lambda) = \lambda Iu + (1-\lambda)JQNu, \quad (u,\lambda) \in X \times [0,1].$$

If $u \in \ker L \cap \partial\Omega$, then

$$\begin{aligned}\deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) &= \deg(H(\cdot, 0), \Omega \cap \ker L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \ker L, 0) \\ &= \deg(I, \Omega \cap \ker L, 0) \\ &\neq 0.\end{aligned}$$

Thus a solution exists on $\operatorname{dom} L \cap \overline{\Omega}$.

Remark 6. *Considering $H(u, \lambda) = -\lambda I u + (1-\lambda) JQNu$ and $\Omega_3 = \{u \in \ker L : -\lambda u + (1-\lambda) JQNu = 0, \lambda \in [0,1]\}$, the proof holds similarly for the case where one swaps out (H_3) with (H_4) there exists a constant $K_1 > 0$ such that if $|c| > K_1$ then $cB(GNu_c) < 0$ where $u_c = c\psi_{\hat{\kappa}_1}$.*

The method seen in Theorem 4 cannot be used if ($H1$) goes unfulfilled, since a reduction of order technique cannot be utilized. In order to attempt a different approach, we will need the following norm estimates.

Lemma 5. *The map $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ satisfies*

$$||K_P g||_X \leq C||g||_Z$$

where

$$C = \frac{1}{(n-1)!} + ||M^{-1}|| \sum_{i=2}^{n} ||\psi_{\hat{\kappa}_i}||_X ||K_b^{-1}|| \left(\sum_{k=1}^{n-1} ||B_{l_k}||^2 \right)^{1/2}.$$

In particular,

$$||(K_P g)^{(j)}||_0 \leq A_j ||g||_Z,$$

for $j = 0, \ldots, n-1$ where

$$A_j = \frac{1}{(n-j)!} + ||M^{-1}|| \sum_{i=2}^{n} ||\psi_{\hat{\kappa}_i}^{(i)}||_0 ||K_b^{-1}|| \left(\sum_{k=1}^{n-1} ||B_{l_k}||^2 \right)^{1/2}.$$

By (15) we obtain

$$||K_P g||_X \leq ||\phi(g)||_X + ||G(g)||_X \leq ||\phi(g)||_X + \frac{1}{(n-1)!}||g||_Z.$$

Recall that $\phi(g) = \sum_{i=2}^{n} (M^{-1}\hat{\kappa}_0)_i \psi_{\hat{\kappa}_i}$, and note that $||\hat{\kappa}_0|| \leq ||K_b^{-1}|| ||\hat{G}_b(g)||$, where $||\cdot||$ stands both for the Euclidean norm on $\mathbb{R}^{n-1}$ or, without loss of clarity, the compatible matrix norm. Hence, recalling $\hat{G}_b(g) = [B_{l_k} G(g)]^T_{k=1,\ldots,n-1}$, we obtain

$$\begin{aligned}||\hat{\kappa}_0|| \leq ||K_b^{-1}|| ||\hat{G}(g)|| &\leq ||K_b^{-1}|| \left(\sum_{k=1}^{n-1} ||B_{l_k}||^2 \right)^{1/2} ||G(g)||_X \\ &\leq ||K_b^{-1}|| \left(\sum_{k=1}^{n-1} ||B_{l_k}||^2 \right)^{1/2} ||g||_Z.\end{aligned}$$

Hence,

$$\begin{aligned}||\phi(g)||_X &\leq \sum_{i=2}^{n} |(M^{-1}\hat{\kappa}_0)_i| ||\psi_{\hat{\kappa}_i}||_X \\ &\leq ||M^{-1}|| \sum_{i=2}^{n} ||\psi_{\hat{\kappa}_i}||_X ||K_b^{-1}|| \left(\sum_{k=1}^{n-1} ||B_{l_k}||^2 \right)^{1/2} ||g||_Z.\end{aligned}$$

The combination nets us C. For A_j, simply note that we would do the above but for a specific derivative under the max norm, and not move to $||\cdot||_X$.

Now, we change one of the leading assumptions to the main result, (H_1), where here $j \in \{0, \dots, n-1\}$:
(H_5) There exists a constant $K_0 > 0$ such that $u \in \operatorname{dom} L \setminus \ker L$ with $|u^{(j)}(t)| > K_0$ implies $QNu \neq 0$ in $[0,1]$.

Theorem 5. *For $j \in \{0, \dots, n-1\}$, if (B_1), (B_2), (H_2), (H_3) (or (H_4)), and (H_5) hold, then the functional problem (1), (2) has at least one solution provided $\psi^{(j)}_{\hat{\kappa}_1} \neq 0$ on $[0,1]$ and*

$$D^* \left(\sum_{i=1}^{n} ||\delta_i||_Z \right) < 1$$

where

$$D^* = \frac{A_j ||\psi_{\hat{\kappa}_1}||_X}{\min_{t\in[0,1]|\psi^{(j)}_{\hat{\kappa}_1}|}} + C.$$

Proof. Consider $u \in \Omega_1$ as outlined in Theorem 4, with $u = u_1 + u_2$, $u_1 = Pu$, $u_2 = (I-P)u = K_P Lu = \lambda K_P Nu$. We have, by Lemma 5,

$$||u_2^{(j)}||_0 \leq A_j ||Nu||_Z, ||u_2||_X \leq C||Nu||_Z.$$

Now, $u_1 = u - u_2$, so that $|(Pu)^{(j)}(t_0)| = |u_1^{(j)}(t_0)| \leq |u(t_0)| + |u_2^{(j)}(t_0)| < K_0 + A_j||Nu||_Z$. We have

$$|u_1^{(j)}(t_0)| = |\langle \hat{\rho}_1, \hat{\mathcal{L}}u \rangle| |\psi^{(j)}_{\hat{\kappa}_1}| < K_0 + A_j ||Nu||_Z.$$

In particular,

$$|\langle \hat{\rho}_1, \hat{\mathcal{L}}u \rangle| \leq \frac{A_j ||Nu||_Z}{\min_{t\in[0,1]|\psi^{(j)}_{\hat{\kappa}_1}|}},$$

meaning

$$||u_1||_X = ||Pu||_X \leq (K_0 + A_j ||Nu||_Z) \frac{||\psi_{\hat{\kappa}_1}||_X}{\min_{t\in[0,1]} |\psi^{(j)}_{\hat{\kappa}_1}|}$$

and so

$$\begin{aligned}
||u||_X &\leq ||u_1||_X + ||u_2||_X \\
&< C_1 + \left(\frac{A_j ||\psi_{\hat{\kappa}_1}||_X}{\min_{t\in[0,1]} |\psi^{(j)}_{\hat{\kappa}_1}|} + C \right) ||Nu||_Z \\
&< C_2 + \left(\frac{A_j ||\psi_{\hat{\kappa}_1}||_X}{\min_{t\in[0,1]} |\psi^{(j)}_{\hat{\kappa}_1}|} + C \right) \left(\sum_{i=1}^{n} ||\delta_i||_Z \right) ||u||_X \\
&< C_2 + D^* \left(\sum_{i=1}^{n} ||\delta_i||_Z \right) ||u||_X
\end{aligned}$$

and therefore Ω_1 is bounded. The rest of the proof replicates that of Theorem 4. □

5. Example

Consider

$$Lu(t) = u^{(4)}(t) = A(1 + u(t) + 2\sin(u''(t) + u'''(t))$$

under the conditions

$$
\begin{aligned}
B_1(u) &= \frac{5}{3}u'''(1) + \frac{5}{3}u''(1/3) - u'(2/3) = 0,\\
B_2(u) &= 2u'''(0) - \frac{1}{6}u''(1/3) + u'(2/3) - u(1) = 0,\\
B_3(u) &= -\frac{1}{3}u'''(0) + \frac{1}{6}u''(1/3) + 2u'(2/3) - u(1) = 0,\\
B_4(u) &= \frac{10}{3}u'''(0) + \frac{5}{2}u''(1/3) - u(1) = 0.
\end{aligned}
$$

One obtains

$$
K = \begin{bmatrix} 2 & 1 & -1 & 0\\ 2 & 0 & 0 & -1\\ 0 & 1 & 1 & -1\\ 4 & 2 & -1 & -1 \end{bmatrix},
$$

$$
\ker\mathcal{K} = \operatorname{span}\left\{\frac{1}{3}\begin{bmatrix}1\\0\\2\\2\end{bmatrix}\right\}, \ker\mathcal{K}' = \operatorname{span}\left\{\frac{\sqrt{15}}{15}\begin{bmatrix}3\\1\\1\\-2\end{bmatrix}\right\},
$$

$$
\operatorname{Im}\mathcal{K} = \operatorname{span}\left\{\frac{\sqrt{10}}{30}\begin{bmatrix}4\\3\\-7\\4\end{bmatrix}, \frac{\sqrt{21}}{21}\begin{bmatrix}-2\\2\\-2\\-3\end{bmatrix}, \frac{\sqrt{14}}{42}\begin{bmatrix}-2\\9\\5\\2\end{bmatrix}\right\},
$$

$$
\operatorname{Im}\mathcal{K}' = \operatorname{span}\left\{\frac{\sqrt{5}}{15}\begin{bmatrix}2\\0\\-5\\4\end{bmatrix}, \frac{\sqrt{5}}{5}\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\0\end{bmatrix}\right\},
$$

with the fundamental solution set being of course $\left\{\frac{t^3}{6}, \frac{t^2}{2}, t, 1\right\}$ and so

$$
\begin{aligned}
\psi_{\hat{\kappa}_1}(t) &= \frac{1}{18}\left(t^3 + 12t + 12\right),\\
\psi_{\hat{\kappa}_2}(t) &= \frac{\sqrt{5}}{45}\left(t^3 - 15t + 12\right),\\
\psi_{\hat{\kappa}_3}(t) &= \frac{\sqrt{5}}{15}\left(3 - t^3\right),\\
\psi_{\hat{\kappa}_4}(t) &= -\frac{1}{2}t^2.
\end{aligned}
$$

We see that $h(t) = \frac{\sqrt{15}}{5}$ suffices for $B(Gh) = 1$ and so we have

$$
Pu(t) = \frac{1}{54}\left(u'''(0) + 2u'(0) + 2u(0)\right)\left(12 + 12t + t^3\right),
$$

$$
\begin{aligned}
Qg(t) &= \frac{1}{5}(3B_1(G(g)) + B_2(G(g)) + B_3(G(g)) - 2B_4(G(g)))\\
&= \int_0^1 g(s)\,ds
\end{aligned}
$$

are suitable projectors.

Because of $\ker \mathcal{K}'$, we have that $3B_1 + B_2 + B_3 - 2B_4 = 0$, which we will use shortly. For the generalized inverse, we have

$$K_b = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \ K_b^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & -2 & 0 \end{bmatrix},$$

$$\hat{\kappa}_0 = \begin{bmatrix} 0 \\ -\frac{1}{2}(B_1(G(g)) - B_2(G(g))) + B_3(G(g))) \\ \frac{1}{2}(B_2(G(g)) + B_3(G(g)) - B_4(G(g))) \\ B_3(G(g)) \end{bmatrix},$$

$$M^{-1}\hat{\kappa}_0 = \begin{bmatrix} \frac{1}{3}(B_1(G(g)) + 3B_2(G(g)) - B_3(G(g))) \\ -\frac{\sqrt{5}}{30}(5B_1(G(g)) - 3B_2(G(g)) - 5B_3(G(g))) \\ \frac{\sqrt{5}}{5}B_2(G(g)) \\ \frac{1}{2}(B_1(G(g)) - B_2(G(g)) + B_3(G(g))) \end{bmatrix}$$

so that

$$\begin{aligned} K_P(g)(t) = &- \frac{1}{270}(5B_1(G(g)) - 3B_2(G(g)) - 5B_3(G(g)))(t^3 - 15t + 12) \\ &+ \frac{1}{15}B_2(G(g))(3 - t^3) - \frac{1}{4}(B_1(G(g)) - B_2(G(g)) + B_3(G(g))t^2 + G(g). \end{aligned}$$

We know that the proposed $\hat{\kappa}_0$ is a solution to the problem $K\hat{\kappa}_0 = -\hat{G}(g)$ by a simple calculation and noting that $-B_4(G(g)) = -\frac{1}{2}(3B_1(G(g)) + B_2(G(g)) + B_3(G(g)))$. Now, clearly $LK_P g = g$, and so we wish to show the reverse. Utilizing

$$\begin{aligned} B_1(G(u^{(4)})) &= u'(0) - u''(0) - 2u'''(0), \\ B_2(G(u^{(4)})) &= u(0) - 2u'''(0), \\ B_3(G(u^{(4)})) &= u(0) - u''(0) - u'(0), \end{aligned}$$

one has

$$\begin{aligned} K_P(u^{(4)})(t) = &\bigg[-\frac{1}{135}(5u'(0) - 4u(0) - 2u'''(0)))(t^3 - 15t + 12) \\ &+ \frac{1}{15}(u(0) - 2u'''(0))(3 - t^3) \\ &+ \frac{1}{2}u''(0)t^2\bigg] + \bigg[u(t) - \frac{1}{6}u'''(0)t^3 - \frac{1}{2}u''(0)t^2 - u'(0)t - u(0)\bigg]. \end{aligned}$$

Note then that the above becomes, after expanding amongst the basic polynomials, $u(t) - Pu(t)$ and thus with $u \in \operatorname{dom} L \cap \ker P$,

$$K_P(Lu)(t) = u(t)$$

and so indeed $K_P g$ is the generalized inverse. The result above establishes an example of the proof of Lemma 4, notably in how $K_P(g)$ is well defined by utilizing the B_i functionals and $Lu = u^{(4)}$.

We attempt to secure a possible solution in view of Theorem 5. If $u(t) > 4$ then $Nu > 1$ and if $u(t) < -4$ then $Nu < -1$; we see that $K_0 = 4$ is appropriate for (H_5) with $j = 0$ as the polynomials noted within the integrals for Q are all strictly positive. We also see that

$$\begin{aligned} cB(GNu_c) &= \frac{A}{18}\int_0^1 \left[18c + c^2(s^3 + 12s + 12) + 36c\sin\left(\frac{c}{3}(s+1)\right)\right] ds \\ &= \frac{A}{18}\left[\frac{73c^2}{4} + 18c + 108\left(\cos\left(\frac{c}{3}\right) - \cos\left(\frac{2c}{3}\right)\right)\right] \end{aligned}$$

Note the the last term is bounded in c and for large K in terms of magnitude, the above will be strictly positive for all values of c for which $|c| > K$; namely, $K = \frac{15{,}552}{73}$ could suffice. In addition, with Theorem 5 in mind, we know that $\psi_{\hat{\kappa}_1}(t) \neq 0$ on $[0,1]$, and so we could determine a small enough A so that the theorem is fulfilled. A solution exists.

Remark 7. *It should be noted that for the given example, we utilized $j = 0$. If we had used $j = 1$ (which is initially allowable since $\psi'_{\hat{\kappa}_1}(t) \neq 0$ on $[0,1]$) we would run into a problem as a bound on u' would not constitute a bound on Nu since its placed inside a $\sin(x)$ term. Now, if we considered some alternative problem such as with*

$$N^*u = A(1 + u'(t) + \sin(u''(t) + u'''(t))$$

then we could have used $j = 1$ instead. This would change some of the argumentation for $cB(GN^*u_c)$ but not much; we would still obtain a quadratic form and so as long as $|c|$ is large enough there is no problem. This would not be true for $j = 2$ however as we would obtain $\psi''_{\hat{\kappa}_1}(0) = 0$.

6. Conclusions

We considered the nonlinear n-th order boundary value problem at resonance subject to abstract linear functional conditions and developed a method that allows us to study all resonance scenarios of order one. In particular, we implemented a general constructive method for deriving existence criteria in the framework of the coincidence degree approach. The method is linear-algebraic and thus has applications to similar problems of fractional order.

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