New Bounds for the Sine Function and Tangent Function

Ling Zhu

Department of Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China; zhuling@zjgsu.edu.cn

Abstract: Using the power series expansion technique, this paper established two new inequalities for the sine function and tangent function bounded by the functions $x^2(\sin(\lambda x)/(\lambda x))^{\alpha}$ and $x^2(\tan(\mu x)/(\mu x))^{\beta}$. These results are better than the ones in the previous literature.

Keywords: Mitrinović–Adamović-type inequality; Becker–Stark-type inequality; circular functions

1. Introduction

Because of the fact the functions $\cos x$ and $(\sin x)/x$ are less than 1 for $x \in (0, \pi/2)$, in order to determine this relationship $(\sin x)/x$ and the weighted geometric mean of $\cos x$ and 1, we examine the Taylor expansion of the following function:

$$\frac{\sin x}{x} - (\cos x)^{\beta} = \left(\frac{1}{2}\beta - \frac{1}{6}\right)x^2 + \left(-\frac{1}{8}\beta^2 + \frac{1}{12}\beta + \frac{1}{120}\right)x^4 + \left(\frac{1}{48}\beta^3 - \frac{1}{24}\beta^2 + \frac{1}{45}\beta - \frac{1}{5040}\right)x^6 + O(x^8).$$

When choosing $\beta = 1/3$ in above formula we can obtain the following fact

$$\frac{\sin x}{x} - (\cos x)^{1/3} = \frac{1}{45}x^4 + \frac{19}{5670}x^6 + O(x^8),$$

which will motivate us to prove the following inequality

$$\frac{\sin x}{x} > (\cos x)^{1/3} \quad (1)$$

holds for $0 < x < \pi/2$. The above inequality was confirmed by Mitrinović and Adamović in [1], so we call it *Mitrinović–Adamović inequality*. On the other hand, the relationship between $(\sin x)/x$ and the weighted arithmetic mean of $\cos x$ and 1 has been discussed in Zhu [2] just published, described as the following inequality similarly:

$$\frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3}\cos x \quad (2)$$

or

$$\frac{3\sin x}{2 + \cos x} < x. \quad (3)$$

In 1451, using a geometrical method Nicolaus De Cusa (1401–1464) discovered (3), and in 1664 when considering the estimation of π Christian Huygens (1629–1695) confirmed (2). In view of the above historical facts (see [3–10]), we call the inequality (2) *Cusa–Huygens inequality*.

In 2018, Zhu [11] shown two improvements to (3) as follows: the inequalities

$$\frac{1}{180}x^5 < x - \frac{3\sin x}{2 + \cos x} \quad (4)$$

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and

$$\frac{1}{2100}x^7 < x - \frac{3 \sin x}{2 + \cos x} \left[1 + \frac{(1 - \cos x)^2}{9(3 + 2 \cos x)} \right] \quad (5)$$

hold for all $x \in (0, \pi]$, where $1/180$ and $1/2100$ are the best constants in previous inequalities, respectively. Two results of previous proposition are corrections of Theorem 3.4.20 from monograph Mitrinović [9]. Malešević et al. [12] made a bilateral supplement to the above two inequalities. Chen and Cheung [13] obtained the bounds for $(\sin x)/x$ in term of $((2 + \cos x)/3)^\delta$ as follows

$$\left(\frac{2 + \cos x}{3} \right)^{\theta_0} < \frac{\sin x}{x} < \left(\frac{2 + \cos x}{3} \right)^{\vartheta_0} \quad (6)$$

holds for all $0 < x < \pi/2$, where $\vartheta_0 = 1$ and $\theta_0 = (\ln \pi - \ln 2)/(\ln 3 - \ln 2)$ are the best possible constants in (6). The double inequality (6) was proved by Bagul [14] and Zhu [15] in different ways. In Zhu [15] some new improvements to inequality (2) can be found:

$$\left(1 - \frac{x^3}{\pi^3} \right) \frac{2 + \cos x}{3} < \frac{\sin x}{x} < \left(1 - \frac{x^4}{180} \right) \frac{\cos x + 2}{3}, \quad (7)$$

$$\left[1 + \frac{8(\pi - 3)}{\pi^3} x^2 \right] \frac{2 + \cos x}{3} - \frac{8(\pi - 3)}{\pi^3} x^2 < \frac{\sin x}{x} < \left(1 + \frac{1}{30} x^2 \right) \frac{2 + \cos x}{3} - \frac{1}{30} x^2, \quad (8)$$

and

$$\begin{aligned} & \left[1 + \frac{1}{30} x^2 + \frac{2(240\pi - \pi^3 - 720)}{15\pi^5} x^4 \right] \frac{2 + \cos x}{3} - \left[\frac{1}{30} x^2 + \frac{2(240\pi - \pi^3 - 720)}{15\pi^5} x^4 \right] \\ & < \frac{\sin x}{x} \\ & < \left(1 + \frac{1}{30} x^2 + \frac{1}{840} x^4 \right) \frac{2 + \cos x}{3} - \left(\frac{1}{30} x^2 + \frac{1}{840} x^4 \right) \end{aligned} \quad (9)$$

hold for $0 < x < \pi/2$.

Bercu [16] used the truncations of Fourier cosine series to the inequality (2) and obtained an enhanced form of (2):

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\frac{1}{45} (1 - \cos x)^2, \quad 0 < x < \frac{\pi}{2}. \quad (10)$$

Bagul et al. [17] draw two conclusions about the improvement of inequality (2):

$$-\left(\frac{2}{3} - \frac{2}{\pi} \right) \frac{1}{(\pi/2 - 1)} (x - \sin x) < \frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left(\frac{2}{3} - \frac{2}{\pi} \right) \frac{1}{(\pi/2 - 1)^2} (x - \sin x)^2 \quad (11)$$

and

$$-\left(\frac{2}{3} - \frac{2}{\pi} \right) (\sin x - x \cos x) < \frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left(\frac{2}{3} - \frac{2}{\pi} \right) (\sin x - x \cos x)^2 \quad (12)$$

hold for $0 < x < \pi/2$.

Recently, Zhu [2] improved the famous inequality (2) using two different technology paths and draw two results as follows: for $0 < x < \pi/2$, the two inequalities

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\frac{1}{180} x^4 \left(\frac{\sin x}{x} \right)^{2/7} \quad (13)$$

and

$$\frac{\sin x}{x} - \frac{2 + \cos x}{3} < -\left(\frac{2}{3} - \frac{2}{\pi}\right)(\sin x - x \cos x)^{(\pi^2-12)/(3\pi^2-\pi^3)} \quad (14)$$

hold with the best constant $-1/180$ and $(2/3 - 2/\pi)$ respectively.

Inequalities on two functions $(\sin x)/x$ and $(\tan x)/x$ arouse great enthusiasm of researchers. Interested readers can refer to these literatures [18–61] and monograph [62] which was edited by Rassias and Andrica.

This paper focuses on some new bounds for the functions $\sin(x)/x$ and $\tan(x)/x$ and wants to improve the following inequalities:

$$1 - \frac{\sin x}{x} > 0 \text{ and } \frac{\tan x}{x} - 1 > 0. \quad (15)$$

Recently, Wu and Bercu [63] thought of Fourier series technology to approximate these two functions. They considered the power series expansion of the following function

$$\begin{aligned} & 1 - \frac{\sin x}{x} - (a + b \cos x + c \cos 2x + d \cos 3x) \\ &= -(a + b + c + d) + x^2 \left(\frac{1}{2}b + 2c + \frac{9}{2}d + \frac{1}{6} \right) - x^4 \left(\frac{1}{24}b + \frac{2}{3}c + \frac{27}{8}d + \frac{1}{120} \right) \\ &+ x^6 \left(\frac{1}{720}b + \frac{4}{45}c + \frac{81}{80}d + \frac{1}{5040} \right) + O(x^8). \end{aligned}$$

To obtain a slightly higher precision approximation, they let

$$\begin{cases} a + b + c + d &= 0 \\ \frac{1}{2}b + 2c + \frac{9}{2}d + \frac{1}{6} &= 0 \\ \frac{1}{24}b + \frac{2}{3}c + \frac{27}{8}d + \frac{1}{120} &= 0 \\ \frac{1}{720}b + \frac{4}{45}c + \frac{81}{80}d + \frac{1}{5040} &= 0 \end{cases}$$

to obtain these constants

$$a = \frac{359}{945}, b = -\frac{167}{420}, c = \frac{2}{105}, d = -\frac{1}{756}$$

and find

$$1 - \frac{\sin x}{x} - \left(\frac{359}{945} - \frac{167}{420} \cos x + \frac{2}{105} \cos 2x - \frac{1}{756} \cos 3x \right) = \frac{23}{226800} x^8 + O(x^{10}),$$

which leads them to prove

$$\begin{aligned} 1 - \frac{\sin x}{x} &> \frac{359}{945} - \frac{167}{420} \cos x + \frac{2}{105} \cos 2x - \frac{1}{756} \cos 3x \\ &= \frac{(1 - \cos x)(-31 \cos x + 5 \cos^2 x + 341)}{945}. \end{aligned}$$

This technique can be used to deal with the approximation problem of another function $\tan(x)/x - 1$, and then they obtain the following results.

Proposition 1 ([63]). *The following inequalities*

$$1 - \frac{\sin x}{x} > \frac{(1 - \cos x)(-31 \cos x + 5 \cos^2 x + 341)}{945}, \quad (16)$$

$$\frac{\tan x}{x} - 1 > \frac{(1 - \cos x)(604 \cos^2 x - 1817 \cos x + 1843)}{945}, \quad (17)$$

$$\begin{aligned} 1 - \frac{\sin x}{x} &> 1 - \left(\frac{2 + \cos x}{3} - \frac{1}{180}x^4 + \frac{1}{3780}x^6 \right) \\ &> 1 - \left(1 + \frac{(1 - \cos x)(-5 \cos^2 x + 31 \cos x - 341)}{945} \right) \\ &> 1 - \frac{2 + \cos x}{3} \end{aligned} \quad (18)$$

hold for all $x \in (0, \pi/2)$.

In this paper, we want to obtain an approximation with appropriate accuracy about these two functions. We examine the power series expansion of function in the following form

$$\begin{aligned} &1 - \frac{\sin x}{x} - ax^2 \left(\frac{\sin bx}{bx} \right)^c \\ &= -x^2 \left(a - \frac{1}{6} \right) + x^4 \left(\frac{1}{6}ab^2c - \frac{1}{120} \right) + x^6 \left[a \left(\frac{1}{180}b^4c - \frac{1}{72}b^4c^2 \right) + \frac{1}{5040} \right] \\ &+ x^8 \left[\frac{1}{45360}ab^6c(35c^2 - 42c + 16) - \frac{1}{362880} \right] + O(x^{10}), \end{aligned}$$

and let

$$\begin{cases} a - \frac{1}{6} = 0 \\ \frac{1}{6}ab^2c - \frac{1}{120} = 0 \\ a \left(\frac{1}{180}b^4c - \frac{1}{72}b^4c^2 \right) + \frac{1}{5040} = 0 \end{cases}$$

to determine

$$a = \frac{1}{6}, b = \pm \frac{\sqrt{7}}{14}, c = \frac{42}{5}.$$

We can obtain that

$$1 - \frac{\sin x}{x} - \frac{1}{6}x^2 \left(\frac{\sin \frac{\sqrt{7}}{14}x}{\frac{\sqrt{7}}{14}x} \right)^{\frac{42}{5}} = \frac{1}{4116000}x^8 + O(x^{10}).$$

In the same way, we obtain

$$\frac{\tan x}{x} - 1 - \frac{1}{3}x^2 \left(\frac{\tan \frac{\sqrt{43}}{7}x}{\frac{\sqrt{43}}{7}x} \right)^{\frac{294}{215}} = \frac{12416}{18907875}x^8 + O(x^{10}).$$

With the above foreshadowing, we can now announce the main work of this paper which established two inequalities of exponential type for the functions $1 - (\sin x)/x$ and $(\tan x)/x - 1$ bounded by the function $x^2(\sin(\lambda x)/(\lambda x))^a$ and $x^2(\tan(\mu x)/(\mu x))^b$ as follows.

Theorem 1. Let $0 < |x| < \pi/2$, $\phi = 1$ and

$$\varphi = \left[\frac{24^5 \pi^{27} (\pi - 2)^5}{4^{42} 7^{21} \left(\sin \frac{\sqrt{7}}{28} \pi \right)^{42}} \right]^{1/5} > 1.$$

Then the double inequality

$$\phi \left[\frac{1}{6} x^2 \left(\frac{\sin \frac{1}{2\sqrt{7}} x}{\frac{1}{2\sqrt{7}} x} \right)^{\frac{42}{5}} \right] < 1 - \frac{\sin x}{x} < \varphi \left[\frac{1}{6} x^2 \left(\frac{\sin \frac{1}{2\sqrt{7}} x}{\frac{1}{2\sqrt{7}} x} \right)^{\frac{42}{5}} \right] \quad (19)$$

holds with the best constants ϕ and φ .

Theorem 2. Let $0 < |x| < \pi/2$. Then

$$\frac{\tan x}{x} - 1 > \frac{1}{3} x^2 \left(\frac{\tan \frac{\sqrt{43}}{7} x}{\frac{\sqrt{43}}{7} x} \right)^{\frac{294}{215}} \quad (20)$$

holds with the best constant $1/3$.

2. Lemmas

The proof of the main conclusions (Theorems 1 and 2) of this paper needs the following lemmas as the basis.

Lemma 1. Let $n \geq 3$, $n \in \mathbb{N}$,

$$\begin{aligned} \sigma_1 &= \frac{\sqrt{7} + 14}{14} \approx 1.1890, \\ \sigma_2 &= \frac{14 - \sqrt{7}}{14} \approx 0.81102, \\ \sigma_3 &= \frac{\sqrt{7}}{14} \approx 0.18898, \end{aligned} \quad (21)$$

and for $k \geq 4$,

$$\begin{aligned} a_k &= -\frac{27}{2} \sqrt{7} \frac{\sigma_1^{2k+2}}{(2k+2)!} + \frac{27}{2} \sqrt{7} \frac{\sigma_2^{2k+2}}{(2k+2)!} - 21 \frac{\sigma_3^{2k}}{(2k)!} \\ &\quad + 32 \sqrt{7} \frac{\sigma_3^{2k+1}}{(2k+1)!} + \frac{5\sqrt{7} + 21}{2} \frac{\sigma_2^{2k+1}}{(2k+1)!} + \frac{21 - 5\sqrt{7}}{2} \frac{\sigma_1^{2k+1}}{(2k+1)!}. \end{aligned} \quad (22)$$

Then $2a_{2n} - 5a_{2n+1} > 0$.

Proof. Since

$$\begin{aligned} a_{2n} &= -\frac{27}{2} \sqrt{7} \frac{\sigma_1^{4n+2}}{(4n+2)!} + \frac{27}{2} \sqrt{7} \frac{\sigma_2^{4n+2}}{(4n+2)!} - 21 \frac{\sigma_3^{4n}}{(4n)!} \\ &\quad + 32 \sqrt{7} \frac{\sigma_3^{4n+1}}{(4n+1)!} + \frac{5\sqrt{7} + 21}{2} \frac{\sigma_2^{4n+1}}{(4n+1)!} + \frac{21 - 5\sqrt{7}}{2} \frac{\sigma_1^{4n+1}}{(4n+1)!}, \\ a_{2n+1} &= -\frac{27}{2} \sqrt{7} \frac{\sigma_1^{4n+4}}{(4n+4)!} + \frac{27}{2} \sqrt{7} \frac{\sigma_2^{4n+4}}{(4n+4)!} - 21 \frac{\sigma_3^{4n+2}}{(4n+2)!} \\ &\quad + 32 \sqrt{7} \frac{\sigma_3^{4n+3}}{(4n+3)!} + \frac{5\sqrt{7} + 21}{2} \frac{\sigma_2^{4n+3}}{(4n+3)!} + \frac{21 - 5\sqrt{7}}{2} \frac{\sigma_1^{4n+3}}{(4n+3)!}, \end{aligned}$$

we compute to obtain

$$(4n+4)!(2a_{2n} - 5a_{2n+1}) = \left[\frac{5}{2}u(n)\sigma_1^{4n+1} - 512v(n)\sigma_3^{4n} \right] + \frac{5}{2}w(n)\sigma_2^{4n+1},$$

where

$$\begin{aligned} u(n) &= \left(\frac{2688}{5} - 128\sqrt{7} \right) n^3 - \left(\frac{2304}{5}\sqrt{7} - \frac{5616}{5} \right) n^2 - \left(\frac{17559}{35}\sqrt{7} - \frac{3277}{5} \right) n \\ &\quad - \left(\frac{19459}{140}\sqrt{7} - \frac{31019}{280} \right), \\ v(n) &= (n+1) \left(21n^3 + \frac{55}{2}n^2 + \frac{1193}{128}n + \frac{1445}{3584} \right), \\ w(n) &= \left(128\sqrt{7} + \frac{2688}{5} \right) n^3 + \left(\frac{2304}{5}\sqrt{7} + \frac{5616}{5} \right) n^2 + \left(\frac{17559}{35}\sqrt{7} + \frac{3277}{5} \right) n \\ &\quad + \left(\frac{14436973}{109760}\sqrt{7} + \frac{75065}{784} \right) \end{aligned}$$

are positive for $n \geq 3$. In order to prove Lemma 1, it suffices to prove that for $n \geq 3$,

$$\frac{5}{2}u(n)\sigma_1^{4n+1} - 512v(n)\sigma_3^{4n} > 0 \iff \left(\frac{\sigma_1}{\sigma_3} \right)^{4n} > \frac{1024}{5\sigma_1} \frac{v(n)}{u(n)}.$$

To note the fact

$$\frac{\sigma_1}{\sigma_3} = 1 + 2\sqrt{7} \approx 6.2915 > 6,$$

we only need to prove

$$6^{4n} > \frac{1024}{5\sigma_1} \frac{v(n)}{u(n)}. \quad (23)$$

By mathematical induction, we can prove the inequality (23). First, the inequality (23) is obviously true for $n = 3$. Assume that (23) holds for $n = m \geq 3$, that is,

$$6^{4m} > \frac{1024}{5\sigma_1} \frac{v(m)}{u(m)}$$

holds. In the following, we shall prove that (23) holds for $n = m + 1$. Since

$$6^{4(m+1)} = 6^4 \cdot 6^{4m} > 1296 \left[\frac{1024}{5\sigma_1} \frac{v(m)}{u(m)} \right],$$

we can complete the proof of (23) when showing that

$$1296 \left[\frac{1024}{5\sigma_1} \frac{v(m)}{u(m)} \right] > \frac{1024}{5\sigma_1} \frac{v(m+1)}{u(m+1)},$$

or

$$\frac{A}{B} := \frac{1296v(m)}{u(m)} > \frac{v(m+1)}{u(m+1)} := \frac{C}{D}$$

In fact,

$$\begin{aligned}
 AD - BC &= 1296v(m)u(m+1) - v(m+1)u(m) \\
 &= \left(\frac{48\,280\,304\,638\,987\,025}{200\,704} - \frac{7757\,241\,843\,454\,045}{100\,352} \sqrt{7} \right) \\
 &\quad + \left(\frac{12\,227\,207\,591\,977\,687}{28\,672} - \frac{1898\,338\,908\,715\,519}{14\,336} \sqrt{7} \right) (m-3) \\
 &\quad + \left(\frac{11\,559\,549\,671\,546\,923}{35\,840} - \frac{10\,809\,910\,365\,147}{112} \sqrt{7} \right) (m-3)^2 \\
 &\quad + \left(\frac{17\,268\,465\,300\,163}{128} - \frac{24\,831\,245\,768\,359}{640} \sqrt{7} \right) (m-3)^3 \\
 &\quad + \left(\frac{674\,131\,077\,463}{20} - \frac{18\,578\,001\,121}{2} \sqrt{7} \right) (m-3)^4 \\
 &\quad + \left(5030\,226\,642 - \frac{6621\,922\,062}{5} \sqrt{7} \right) (m-3)^5 \\
 &\quad + \left(\frac{2075\,901\,072}{5} - 104\,111\,168 \sqrt{7} \right) (m-3)^6 \\
 &\quad + \left(14\,620\,032 - 3480\,960 \sqrt{7} \right) (m-3)^7 \\
 &> 0
 \end{aligned}$$

holds for all $m \geq 3$. This completes the proof of Lemma 1. $\square$

It is not difficult to prove the following conclusion in the similar way.

Lemma 2. Let $n \geq 2, n \in \mathbb{N}$,

$$\begin{aligned}
 \rho_1 &= \frac{2}{7}\sqrt{43} + 2 \approx 3.873\,6, \\
 \rho_2 &= \frac{2}{7}\sqrt{43} \approx 1.873\,6, \\
 \rho_3 &= 2 - \frac{2}{7}\sqrt{43} \approx 0.126\,45,
 \end{aligned} \tag{24}$$

and for $k \geq 4$,

$$\begin{aligned}
 b_k &= \frac{17\sqrt{43}}{903 \cdot 2} \frac{\rho_1^{2k+1}}{(2k+1)!} - \frac{117\sqrt{43}}{4816} \frac{\rho_1^{2k+2}}{(2k+2)!} \\
 &\quad + \frac{1}{2} \frac{2^{2k}}{(2k)!} - \frac{1}{2} \frac{2^{2k+1}}{(2k+1)!} \\
 &\quad + \frac{283\sqrt{43}}{3612} \frac{\rho_2^{2k+1}}{(2k+1)!} - \frac{17\sqrt{43}}{903 \cdot 2} \frac{\rho_3^{2k+1}}{(2k+1)!} + \frac{117\sqrt{43}}{4816} \frac{\rho_3^{2k+2}}{(2k+2)!}.
 \end{aligned} \tag{25}$$

Then $2b_{2n} - 5b_{2n+2} > 0$.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let

$$F(x) = \ln\left(1 - \frac{\sin x}{x}\right) - \ln\left[\frac{1}{6}x^2 \left(\frac{\sin \frac{1}{2\sqrt{7}}x}{\frac{1}{2\sqrt{7}}x}\right)^{\frac{42}{5}}\right],$$

where $0 < |x| < \pi/2$. Since this function $F(x)$ is even, let's consider the problem on interval $(0, \pi/2)$. We compute to obtain

$$F'(x) = \frac{2\sqrt{7}}{35} \frac{f(x)}{x(\cos \sigma_1 x - \cos \sigma_2 x + 2x \sin \sigma_3 x)},$$

where

$$\begin{aligned} f(x) &= \frac{27}{2}\sqrt{7}\cos\sigma_1 x - \frac{27}{2}\sqrt{7}\cos\sigma_2 x - 21x^2\cos\sigma_3 x + 32\sqrt{7}x\sin\sigma_3 x \\ &+ \left(\frac{5}{2}\sqrt{7} + \frac{21}{2}\right)x(\sin\sigma_2 x) - \left(\frac{5}{2}\sqrt{7} - \frac{21}{2}\right)x(\sin\sigma_1 x). \end{aligned} \quad (26)$$

and σ_i ($i = 1, 2, 3$) are defined as (21). Substituting

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad (27)$$

into (26), we can obtain that

$$\begin{aligned} f(x) &:= \sum_{k=4}^{\infty} (-1)^k a_k x^{2k+2} = \sum_{n=2}^{\infty} (a_{2n} x^{4n+2} - a_{2n+1} x^{4n+4}) \\ &= \left(\frac{1}{274400} x^{10} - \frac{163}{1521273600} x^{12} \right) + \sum_{n=3}^{\infty} (a_{2n} x^{4n+2} - a_{2n+1} x^{4n+4}), \end{aligned}$$

where a_k is defined by (22). Since

$$\frac{1}{274400} x^{10} - \frac{163}{1521273600} x^{12} > 0$$

for all $x \in (0, \pi/2)$, we can determine the positive definiteness of the function $f(x)$ on $(0, \pi/2)$ when proving

$$a_{2n} x^{4n+2} - a_{2n+1} x^{4n+4} > 0 \iff x^2 < \frac{a_{2n}}{a_{2n+1}} \quad (28)$$

for $n \geq 3$. Since

$$x^2 < \left(\frac{\pi}{2}\right)^2 \approx 2.4674 < \frac{5}{2},$$

we can prove (28) when proving that for $n \geq 3$,

$$\frac{5}{2} < \frac{a_{2n}}{a_{2n+1}},$$

or

$$2a_{2n} - 5a_{2n+1} > 0,$$

which comes from Lemma 1.

So $f(x) > 0$ and $F(x)$ is increasing on $(0, \pi/2)$. In view of

$$F(0^+) = 0, \quad F(1^-) = \frac{1}{5} \ln \left(\frac{24^5 \pi^{27} (\pi - 2)^5}{4^{42} 7^{21} \left(\sin \frac{\sqrt{7}}{28} \pi \right)^{42}} \right) > 0,$$

the proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. Let

$$G(x) = \ln\left(\frac{\tan x}{x} - 1\right) - \ln\left[\frac{1}{3}x^2\left(\frac{\tan \frac{\sqrt{43}}{7}x}{\frac{\sqrt{43}}{7}x}\right)^{\frac{294}{215}}\right], \quad 0 < x < \frac{\pi}{2}.$$

Then

$$G'(x) = \frac{1806\sqrt{43}}{9245} \frac{g(x)}{x\left(\tan \frac{\sqrt{43}}{7}x\right)(\tan x - x)\cos^2 \frac{\sqrt{43}}{7}x \cos^2 x},$$

where

$$\begin{aligned} g(x) = & \frac{117\sqrt{43}}{4816} \cos \rho_1 x - \frac{117\sqrt{43}}{4816} \cos \rho_3 x \\ & - \frac{1}{2}x \sin 2x + \frac{1}{2}x^2 \cos 2x + \frac{1}{2}x^2 \\ & + \frac{283\sqrt{43}}{3612}x \sin \rho_2 x + \frac{17\sqrt{43}}{903 \cdot 2}x \sin \rho_1 x - \frac{17\sqrt{43}}{903 \cdot 2}x \sin \rho_3 x, \end{aligned} \quad (29)$$

where ρ_i ($i = 1, 2, 3$) are defined as (24). Substituting (27) into (29), we can obtain that

$$g(x) =: \sum_{k=4}^{\infty} (-1)^k b_k x^{2k+2},$$

where b_k is defined by (25). We can rewrite $g(x)$ as

$$g(x) = \sum_{n=2}^{\infty} (b_{2n} x^{4n+2} - b_{2n+1} x^{4n+4}).$$

Then, we determine the positive definiteness of the function $g(x)$ on $(0, \pi/2)$ when proving

$$b_{2n} x^{4n+2} - b_{2n+1} x^{4n+4} > 0 \iff x^2 < \frac{b_{2n}}{b_{2n+1}} \quad (30)$$

for $n \geq 3$. Since

$$x^2 < \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4} \approx 2.4674 < \frac{5}{2},$$

we can prove (30) when proving that for $n \geq 3$,

$$\frac{5}{2} < \frac{b_{2n}}{b_{2n+1}},$$

which is the result of Lemma 2. So $g(x) > 0$ and $G(x)$ is strictly increasing on $(0, \pi/2)$. Therefore $G(x) > G(0^+) = 0$. Considering the reason

$$\lim_{x \rightarrow 0^+} \frac{\frac{\tan x}{x} - 1}{x^2 \left(\frac{\tan \frac{\sqrt{43}}{7}x}{\frac{\sqrt{43}}{7}x}\right)^{\frac{294}{215}}} = \frac{1}{3},$$

the proof of Theorem 2 is completed. $\square$

4. Comparison of New and Old Results

When letting $\phi = 1$ in (19) we can obtain

$$1 - \frac{\sin x}{x} > \frac{1}{6}x^2 \left(\frac{\sin \frac{1}{2\sqrt{7}}x}{\frac{1}{2\sqrt{7}}x}\right)^{\frac{42}{5}}. \quad (31)$$

By using the similar proof method of Theorem 1 it is not difficult to prove the following results:

$$\frac{1}{6}x^2 \left(\frac{\sin \frac{1}{2\sqrt{7}}x}{\frac{1}{2\sqrt{7}}x} \right)^{\frac{42}{5}} > 1 - \left(\frac{2 + \cos x}{3} - \frac{1}{180}x^4 + \frac{1}{3780}x^6 \right), \quad (32)$$

$$\frac{1}{3}x^2 \left(\frac{\tan \frac{\sqrt{43}}{7}x}{\frac{\sqrt{43}}{7}x} \right)^{\frac{294}{215}} > \frac{(1 - \cos x)(604 \cos^2 x - 1817 \cos x + 1843)}{945} \quad (33)$$

hold for all $x \in (0, \pi/2)$. So new results (19) and (20) are better than the old ones (16) and (17), respectively. In addition, there are deeper conclusions:

$$\begin{aligned} 1 - \frac{\sin x}{x} &> \frac{1}{6}x^2 \left(\frac{\sin \frac{1}{2\sqrt{7}}x}{\frac{1}{2\sqrt{7}}x} \right)^{\frac{42}{5}} \\ &> 1 - \left(\frac{2 + \cos x}{3} - \frac{1}{180}x^4 + \frac{1}{3780}x^6 \right) \\ &> \frac{(1 - \cos x)(5 \cos^2 x - 31 \cos x + 341)}{945} > 1 - \frac{2 + \cos x}{3}. \end{aligned} \quad (34)$$

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