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Assessing the Non-Linear Dynamics of a Hopf–Langford Type System

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Abstract: In this paper, the non-linear dynamical behavior of a 3D autonomous dissipative system of Hopf–Langford type is investigated. Through the help of a mode transformation (as the system's energy is included) it is shown that the 3D nonlinear system can be separated of two coupled subsystems in the master (drive)-slave (response) synchronization type. After that, based on the computing first and second Lyapunov values for master system, we have attempted to give a general framework (from bifurcation theory point of view) for understanding the structural stability and bifurcation behavior of original system. Moreover, a family of exact solutions of the master system is obtained and discussed. The effect of synchronization on the dynamic behavior of original system is also studied by numerical simulations.

Keywords: analysis; synchronization; nonlinear dynamics; Hopf–Langford system

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1. Introduction

Non-linear chaotic behavior has been observed in systems of different nature as this motion is based on homoclinic (heteroclinic) structures which instability accompanied by local divergence and global contraction [1–3]. It is well-known that autonomous nonlinear differential system of the form

$$\dot{x} = \frac{dx}{dt} = f(x, \lambda)\,,\; x \in R^n\,,$$

where $n \geq 3$, λ is the vector of parameters and $f : R^n \to R^n$ is a smooth vector function (i.e., continuously differentiable) in some domain Ω, can display a rich diversity of periodic, multiple periodic, chaotic and hyperchaotic flows dependent upon the specific values of one or more bifurcation (control) parameters [4,5].

The investigation of dynamical processes in coupled nonlinear systems is an interesting problem from both theoretical (mathematical) and applied (engineering) points of view. Phenomena such as stability in interacting subsystems can be observed in nature and science. Usually, that phenomenon is called synchronization [6,7]. There are four basic types of synchronization known: complete, generalized, phase and lag synchronization [8]. Phase synchronization is the phenomenon of the onset of balance between the phases of the subsystems state variables oscillations, which is caused by an onset of the energy balance.

A principal problem toward complete understanding of nonlinear interactions is to identify where in its phase space one dynamical system is structurally stable. For example, in a small neighborhood of a structurally stable Poincaré homoclinic orbit lie only saddle-type periodic orbits. On the contrary, both structurally unstable and attractive periodic orbits, in addition to saddle ones, may exist near a structurally unstable homoclinic orbit [9]. Note that after Smale's works [10,11] these systems are said to be Morse–Smale systems.

The structural stability (roughness) investigation of steady state and of limit cycles or other types of trajectories is a main problem in bifurcation theory. It is well-known that there is critical dependence of the stability conditions of limit cycles on the stability conditions of its steady states. Based on classical works [12–14], it was defined that by knowing the sign of Lyapunov values, $L_k(\lambda_i)$, (also called focus values, Lyapunov quantities (coefficients)) we can efficiently study the structure of complicated nonlinear system trajectories. In other words, the type of: (1) stability loss of equilibrium and (2) winding/unwinding of system trajectories in small neighborhoods of equilibrium depend on the sign of Lyapunov value [15,16]. Furthermore, the (sequential) number of the last non zero Lyapunov value determines the number of possible limit cycles (stable/unstable).

In the present paper, we focus our study on the following system

$$\begin{aligned} \dot{x}_1 &= (\mu-\alpha)x_1 - \beta x_2 + x_1x_3 + l x_1(1-x_3^2), \\ \dot{x}_2 &= \beta x_1 + (\mu-\alpha)x_2 + x_2x_3 + l x_2(1-x_3^2), \\ \dot{x}_3 &= \mu x_3 - \gamma(x_1^2+x_2^2+x_3^2), \end{aligned} \tag{1}$$

where μ, α, β, γ and l are positive control parameters. The infinite structure from ordinary differential equations of system (1) was originally introduced by Hopf [17] in order to describe a fluid's turbulence dynamics. Later, firstly in a private communication and after that in a paper [18], Langford introduced (1) for $\alpha=\beta=\gamma=1$ and $l=0$. In our next considerations, when $\alpha\neq 1, \beta\neq 1$, $\gamma=1$ and $l\neq 0$, we will call system (1)—a generalized Hopf–Langford system (GHL). After the works of Hassard et al. [19] and Nikolov et al. [20], even today, the non-linear dynamics of GHL system has been intensively investigated with the help of both theoretical and numerical approaches [21–23].

Here we investigate the previously unexplored parameter regions of the generalized Hopf–Langford system (GHL). A new qualitative picture of behavior near bifurcation points can be obtained when the GHL system (1) has the following modified form

$$\begin{aligned} \dot{x}_1 &= (\mu-\alpha)x_1 - \beta x_2 + x_1x_3 + l x_1(1-x_3^2), \\ \dot{x}_2 &= \beta x_1 + (\mu-\alpha)x_2 + x_2x_3 + l x_2(1-x_3^2), \\ \dot{x}_3 &= \mu x_3 - 2E, \\ \dot{E} &= 2(\mu+l-\alpha)E - (l-\alpha)x_3^2 - 2l x_3^2 E - x_3^3 + l x_3^4, \end{aligned} \tag{2}$$

where $E=\frac{1}{2}(x_1^2+x_2^2+x_3^2)$ is the energy. Note that in Appendix B, we show how $\dot{E}$ in (2) is obtained. Moreover, we qualitatively and numerically investigate the structural stability of (2) (which is equivalent to the original system (1)), whereas for this goal a specific version of bifurcation theory, based on the computing of Lyapunov values (not exponents), $L_k(\lambda_i)$, was used.

The equilibrium (steady state) points (FPs) of system (2) are found by equating the right-hand sides of (2) to zero (see Appendix D). Thus, we obtain that equilibrium points of the system are

$$\mathbf{O_1}:\ \overline{x}_1=\overline{x}_2=\overline{x}_3=\overline{E}=0\,,\ \text{first EP}, \tag{3}$$

$$\mathbf{O_{2,3,4}}:\ \begin{array}{l} \overline{x}_1=\overline{x}_2=0,\ \overline{E}=\frac{\mu}{2}\overline{x}_3, \\ l\,\overline{x}_3^3-(\mu l+1)\overline{x}_3^2+(\alpha-l)\overline{x}_3+\mu(\mu+l-\alpha)=0\,. \end{array}\ \text{second, third and fourth EPs}. \tag{4}$$

Here we note that the polynomial function for $\overline{x}_3$ in (4) (according to Cardan's method [24], p. 23, the substitution $\overline{x}_3=z-\frac{m_1}{3}$ is valid, i.e., $z^3+K_1z+K_2=0$) has three different real roots if

$$Q=\left(\frac{K_1}{3}\right)^3+\left(\frac{K_2}{2}\right)^2<0\,, \tag{5}$$

where $K_1=-\frac{m_1^2}{3}+m_2\,, K_2=2\left(\frac{m_1}{3}\right)^3-\frac{m_1m_2}{3}+m_3,\ m_1=-\frac{\mu l+1}{l}\,, m_2=\frac{\alpha-l}{l}$ and $m_3=\frac{\mu(\mu+l-\alpha)}{l}$. Moreover, according to [24], if: (i) $Q>0$, then the equation for z (the third equation in (4), respectively) has one real root and a pair of imaginary roots and (ii) $Q=0$, then for $K_1 = K_2 = 0$ all the roots of the same equation are zero and for $K_1 \neq 0, K_2 \neq 0$ two real roots.

While dealing with modified system (2), the main simplification is that it can be separated of two coupled subsystems in the master (drive)–slave (response) synchronization type. As we can see, the master system has the following form

$$\begin{aligned} &\dot{x}_3 = \mu x_3 - 2E, \\ &\dot{E} = 2(\mu + l - \alpha)E - (l - \alpha)x_3^2 - 2l x_3^2 E - x_3^3 + l x_3^4 . \end{aligned} \tag{6}$$

Since it is obvious that the corresponding dynamic behavior of the original system (2) principally depends on the behavior of the master system [7,25], below we investigate only the bifurcation dynamic of this system. It is seen that master system describes the evolution in time of variable x_3 and energy E.

The paper consists of three sections apart from the Introduction. In Section 2 we present analytical results concerning the system (6). Section 3 gives an overview of numerical simulations, and finally in Section 4 we summarize and conclude our results.

2. Analytical Results

2.1. Qualitative Analysis

Our objective here is to consider the local (near equilibrium points) behavior of system (6) which presents an autonomous nonlinear two-dimensional dynamical model. According to the general theory of ordinary differential equations [26], the equilibrium (steady state) values of system (6) are as those in (3) and (4)—the part for x_3 and E. In order to determine the type of these equilibrium points, we make the following substitutions into (6)

$$x_3 = \overline{x}_3 + x, \; E = \overline{E} + y . \tag{7}$$

Hence, after accomplishing some transformations, system (6) (in local coordinates) is given through

$$\begin{aligned} &\dot{x} = \mu\, x - 2y\, , \\ &\dot{y} = c_1 x + c_2 y + c_3 x^2 - c_4 xy + c_5 x^3 - c_6 x^2 y + l x^4\, , \end{aligned} \tag{8}$$

where

$$\begin{aligned} &c_1 = \overline{x}_3 \left[4l\, \overline{x}_3^2 - 3\overline{x}_3 - 4l\overline{E} - 2(l - \alpha)\right], c_2 = 2\left(\mu + l - \alpha - l\, \overline{x}_3^2\right); \\ &c_3 = \left(6l\, \overline{x}_3^2 - 3\overline{x}_3 - 2l\overline{E} - l + \alpha\right), c_4 = 4l\, \overline{x}_3\, , c_5 = 4l\, \overline{x}_3 - 1\, , c_6 = 2l\, . \end{aligned} \tag{9}$$

The stability of equilibrium points (3) and (4) is defined by the following Routh–Hurwitz conditions

$$R \equiv p = -(\mu + c_2) = -3\mu + 2\left(l\, \overline{x}_3^2 + \alpha - l\right) > 0, \tag{10}$$

$$q = \mu c_2 + 2c_1 = 2\left\{\mu\left(\mu + l - \alpha - l\, \overline{x}_3^2\right) + \overline{x}_3\left[4l\, \overline{x}_3^2 - (3 + 2l\mu)\overline{x}_3 - 2(l - \alpha)\right]\right\} > 0\, . \tag{11}$$

We remark that the notations p, q and R in (10) and (11) are taken from [14], as they are the coefficients of the characteristic equation $\chi^2 + p\chi + q = 0$. It is seen that first equilibrium point O_1(Equation (3)) is always stable if

$$\left|\begin{aligned} &\alpha < \mu + l\, , \\ &\alpha > \tfrac{2l+3\mu}{2} \end{aligned}\right. . \tag{12}$$

If (12) (i.e., the Routh–Hurwitz conditions) is not valid, then O_1 is: (i) always unstable or (ii) is degenerate (critical case). Note that in case (ii) (i.e., $q = 0$) the implicit function theorem may no longer be applied, and the existence of O_1 in a neighboring system is not necessarily guaranteed. For investigation of this critical case, the method of reduction to the center manifold and the method of normal forms can be used [3]. By definition, when condition (10) is not valid, the steady states (3) and (4) become unstable, as in this case according to [3,14] "soft" (reversible) or "hard" (un-reversible) stability loss takes place. To define the type of stability loss of steady states (3) and (4) it is necessary to calculate the so-called first Lyapunov value ($L_1(\lambda_0)$) on the boundary of stability $R = 0$. For instance, in [3], Sections 9.2 and 9.3 and [27] the analytical results (in the form of Theorems) for stability and stability loss regarding limit cycle are shown. In the case of two first order differential equations, this value can by determined analytically by the formula in [14] (for details see Appendix C). For system (8) we have

$$\begin{aligned} &a_{11} = a_{02} = a_{20} = a_{30} = a_{21} = a_{12} = b_{02} = b_{03} = b_{12} = 0\,, a = \mu\,, \\ &b = -2\,, c = c_1\,, d = c_2\,, b_{20} = c_3\,, b_{11} = -c_4\,, b_{30} = c_5\,, b_{21} = -c_6\,. \end{aligned} \tag{13}$$

Thus, in this case, $L_1(\lambda_0)$ has the form:

$$L_1(\lambda_0) = \frac{\pi l}{2q\sqrt{q}}\left[16l\,\overline{x}_3^3 - 2(3+4\mu l)\overline{x}_3^2 + \mu^2\right] \tag{14}$$

It can be seen that: (i) if $l = 0$ then $L_1 = 0$; (ii) if $l \neq 0$ and $\overline{x}_3 = 0$ then $L_1 = \frac{\pi l \mu^2}{2q\sqrt{q}} > 0$ and (iii) if $l \neq 0, \overline{x}_3 \neq 0$ then L_1 can be positive, negative or equal to zero. In accordance with the Lyapunov–Andronov theory, if $L_1 = 0$, then the second one (L_2) must be calculated. Notice that sign of L_2 determines the stability/instability of the external limit cycle. For $L_2 > 0$, the external limit cycle is unstable. On the other hand, for $L_2 < 0$, the external limit cycle is stable—for a detailed discussion see Appendix A or [3,12–14,28].

Let us now calculate L_2, when $L_1 = R = 0$. We do the calculation in the above case (iii) where we can use formula (A3) in the Appendix A. The reader can easily complete the details or find them in the following literature—[3,12,14]. In this case, we can introduce $b_{04} = l$ and (13) for system (8), and the expression of second Lyapunov value becomes:

$$L_2 = \frac{\pi}{24}\left[c_4\left(c_3c_4^2 - c_4c_6 - 8c_3c_5 - 5l\right) + 3c_5c_6\right] \tag{15}$$

If we consider (15), it follows that in dependence of the system's parameters L_2 can have a different sign or to be zero.

In order to simplify our numerical calculations for L_1 and L_2, some of the parameters are kept constant. The parameter values are: $\alpha = 1$ and $\beta = 0.9$. The first Lyapunov value obtained for system (6) as a function of bifurcation parameters $\mu \in [0.53, 0.533]$ and $l \in [0.208, 0.24]$ is plotted in Figure 1. It is seen that for smaller values of μ and larger values of l, that L_1 has negative values, i.e., a soft stability loss takes place. On the contrary, the first Lyapunov value is positive (a hard stability loss occurs) for larger values of μ and smaller values of l. Indeed, using this fact, we conclude that for some combinations of parameter's values L_1 can be zero. Thus, when $L_1 = 0$, we are interested in knowing the sign of the second Lyapunov value.

In Figure 2, L_2 (calculated at $R = L_1 = 0$ in (15)) is shown for different values of the bifurcation parameters μ and l, i.e., $\mu \in [0.53, 0.533]$ and $l \in [0.208, 0.24]$. It can be seen that L_2 is always negative. Therefore, following the theory introduced in [12,14] we have soft stability loss for the external limit cycle. Hence, one of the situations shown in Figures A3 and A4 (see Appendix A) is valid.

It will be shown in Section 3 that the situation in Figure A4 is valid for system (6).

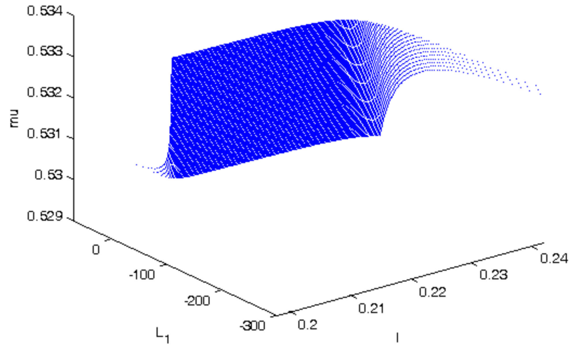

Figure 1. The graph of the first Lyapunov value, L_1, as a function of parameters μ and l, when $\alpha = 1$ and $\beta = 0.9$.

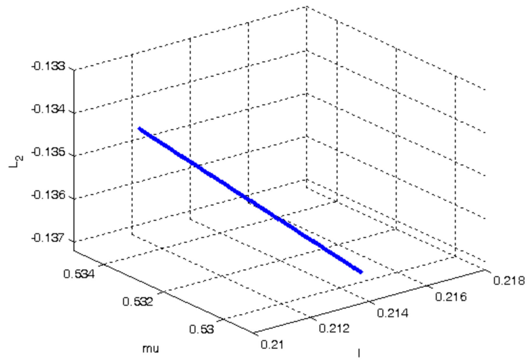

Figure 2. The graph of the second Lyapunov value, L_2, as a function of parameters μ and l, when $\alpha = 1$ and $\beta = 0.9$. For more details see text.

2.2. A Family of Exact Solutions

Let us return to system (6) in which x_3 and E are denoted by u and v, respectively, that is

$$\begin{aligned} \dot{u} &= \mu u - 2v\,, \\ \dot{v} &= 2(\mu + l - \alpha)v - (l - \alpha)u^2 - 2lu^2v - u^3 + lu^4. \end{aligned} \tag{16}$$

Solving the first one of the above equations with respect to v one obtains

$$v = \frac{1}{2}(\mu u - \dot{u}),\ \dot{v} = \frac{1}{2}(\mu\dot{u} - \ddot{u})\,. \tag{17}$$

Then, substituting expressions (17) into the second equation of Equation (16) one arrives at the following second-order nonlinear ordinary differential equation for the function $u(t)$, namely,

$$\ddot{u} + g(u)\dot{u} + h(u) = 0, \tag{18}$$

where

$$g(u) = 2\alpha - 3\mu - 2l + 2lu^2,\ h(u) = 2u(\mu - u)\left(l + \mu - \alpha + u - lu^2\right). \tag{19}$$

Equation (18) is of Liénard type [29] (see also, e.g., [30,31] and the references therein). Equations of this family describe the behavior of a large number of physical systems. For instance, the famous classical and generalized force-free Duffing and van del Pol oscillator equations, the so-called modified Painlevé–Ince equations and many other important equations belong to this family (see, e.g., [32–34]).

Fortunately, it turned out that Equation (18) in which the functions $g(u)$ and $h(u)$ are given by expressions (19) can be factorized, without any restriction on the involved parameters α, μ and l, in the following explicit form

$$\left[\frac{d}{dt} - \phi_2(u)\right]\left[\frac{d}{dt} - \phi_1(u)\right]u(t) = 0, \tag{20}$$

where

$$\phi_1(u) = \mu - u, \tag{21}$$

$$\phi_2(u) = 2\left(1 + \mu - \alpha + u - lu^2\right). \tag{22}$$

This result, which can be easily verified by direct computation, opens a way of constructing a family of particular solutions for the regarded Equation (18) since each solution of the simpler equation

$$\left[\frac{d}{dt} - \phi_1(u)\right]u(t) = 0 \tag{23}$$

is, evidently, a particular solution of Equation (18) as well. On account of the expression (21) for the function $\phi_1(u)$, Equation (23) above takes the form of the following generalized Riccati equation

$$\dot{u} - \mu u + u^2 = 0 \tag{24}$$

(cf. [35], Sec. 2.15) whose general solution reads

$$u(t) = \frac{\mu e^{(t+t_0)\mu}}{\lambda + e^{(t+t_0)\mu}}, \tag{25}$$

where t_0 and λ are arbitrary constants.

In this way, substituting Equation (25) into the first equation of Equation (17), we obtain a two-parameter family of exact solutions of the regarded system (16), namely,

$$u(t) = \frac{\mu e^{(t+t_0)\mu}}{\lambda + e^{(t+t_0)\mu}},\ v(t) = \frac{1}{2}u^2(t) = \frac{1}{2}\left[\frac{\mu e^{(t+t_0)\mu}}{\lambda + e^{[t+t_0]\mu}}\right]^2 \tag{26}$$

Actually, in the (u, v)-plane each such solution lies on the parabola $v = \frac{1}{2}u^2$, see Figure 3. The right-hand sides of formulas (26) simply show how this parabola is traversed over time. They imply: (i) when $\lambda > 0$ the function u is positive for each $t \in (-\infty, +\infty)$ and tends to zero or μ when the time tends to $-\infty$ or $+\infty$, respectively; (ii) if $\lambda < 0$ is such that $\lambda + e^{(t_1+t_0)\mu} = 0$, then the function u is negative when $t \in (-\infty, t_1)$ and tends to zero or $-\infty$ when the time tends to $-\infty$ or t_1 (from below), respectively, while, when $t \in (t_1, +\infty)$ the function u is positive then we have the stationary solution $u = \mu, v = \frac{1}{2}\mu^2$.

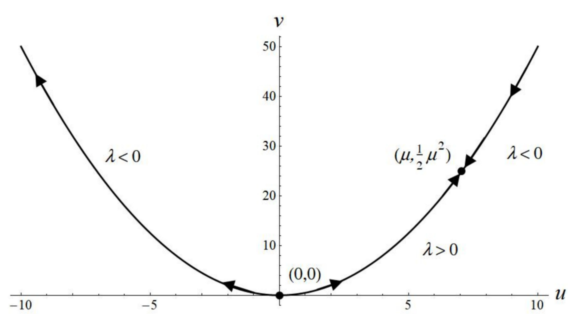

Figure 3. The parabolic trajectory $v = \frac{1}{2}u^2$ corresponding to the solutions of system (16) of form (26). Here, (0, 0) and $\left(\mu, \frac{1}{2}\mu^2\right)$ are equilibrium points of this system.

3. Numerical Analysis

For all numerical calculations, we restricted ourselves to the variation of μ and l. The other parameters remain constant at the values $\alpha = 1$ and $\beta = 0.9$. The solutions for $(x_1(t), x_2(t), x_3(t), E(t))$ were calculated with the fixed initial conditions (0.1, 0.1, 0.07, 0.03). Moreover, the differential equations in Equation (2) were solved numerically using MATLAB (The MathWorks, Inc., Natick, MA, USA)-ODE45 solver (which is based on an explicit Runge–Kutta (4,5) formula, the Dormand–Prince pair).

In Figures 4 and 5, we numerically demonstrate that when master subsystem (6) is stable (i.e., the energy E of the system tends to a constant) or has a stable limit cycle (i.e., $L_1 < 0$) then the slave subsystem (x_1, x_2) into (2) has periodic solutions with period one, or quasi-periodic solutions (oscillations) with period different to one. Note here that those data are in accordance with the results obtained in our analytical study.

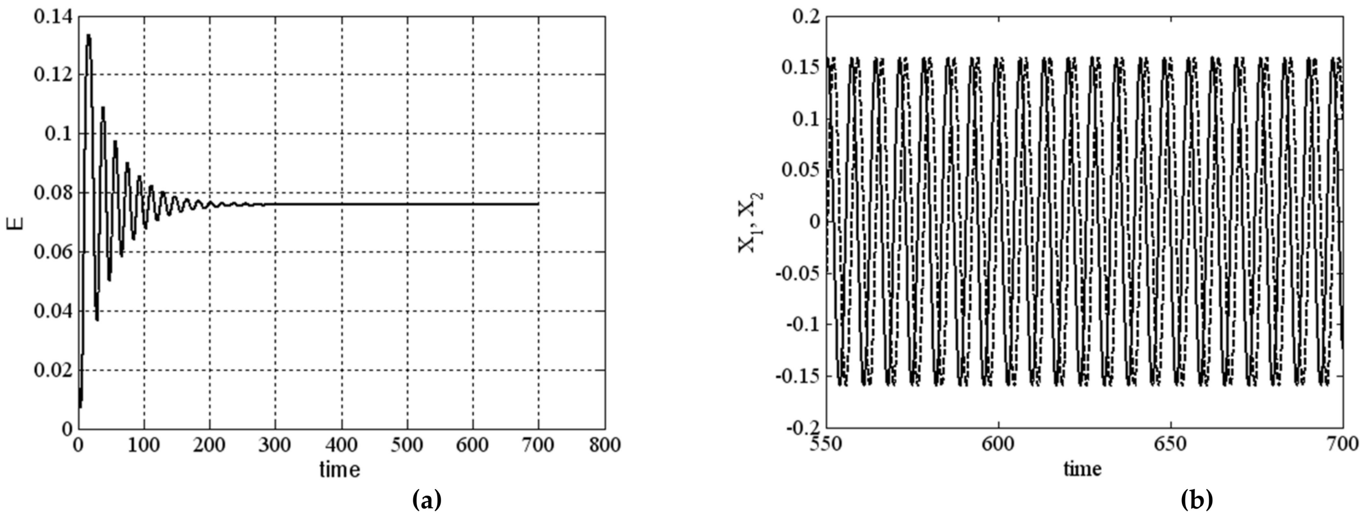

Figure 4. Periodic solutions (**b**) for the slave system, when energy E (**a**) tends to a constant. The parameters are: $\alpha = 1$, $\beta = 0.9$, $\mu = 0.53$ and $l = 0.2$.

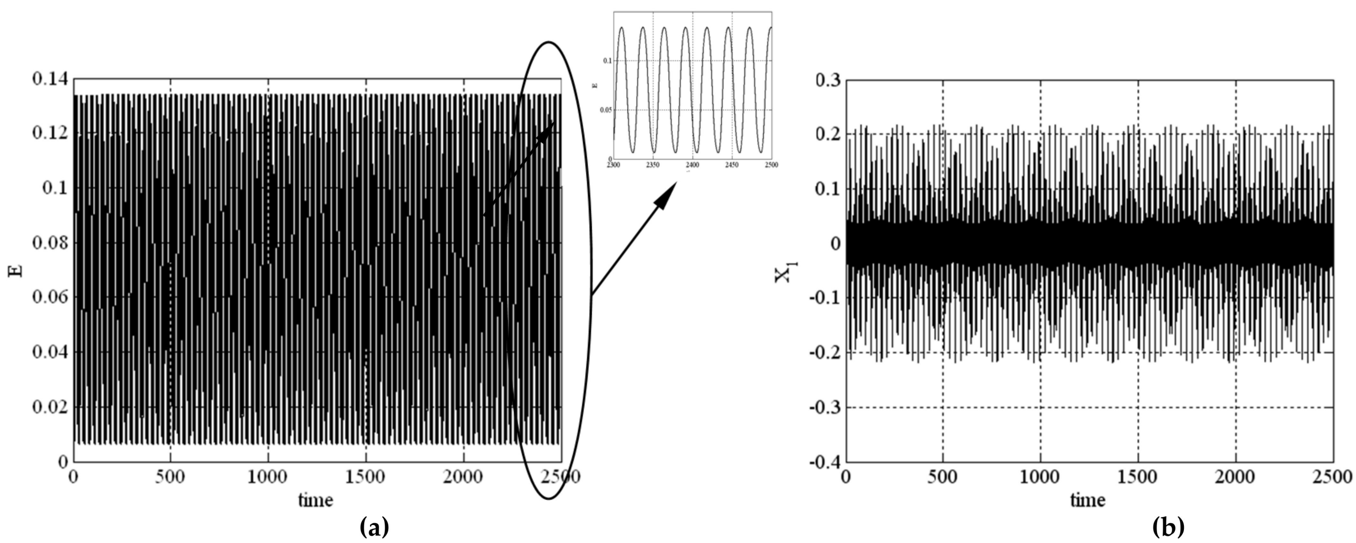

Figure 5. Quasi-periodic solutions (**b**) of the slave system, when energy E (**a**) oscillates—stable limit cycle. The parameters are: $\alpha = 1$, $\beta = 0.9$, $\mu = 0.531$ and $l = 0.2216$.

At $\mu = 0.533$ and $l = 0.21907$, another qualitative behavior of master (x_3, E) subsystem and slave (x_1, x_2) subsystem can be observed—see Figure 6. Here, subsystem (x_3, E) is (structurally) unstable, whereas subsystem (x_1, x_2) has chaotic solutions. Remarkably, this is valid only for $x_3 > 0$ and energy E has a still smaller amplitude than one borderline case-limit value. It should be remarked here that, in a strict sense, the energy of original system (2) is a major regulator of its qualitative behavior.

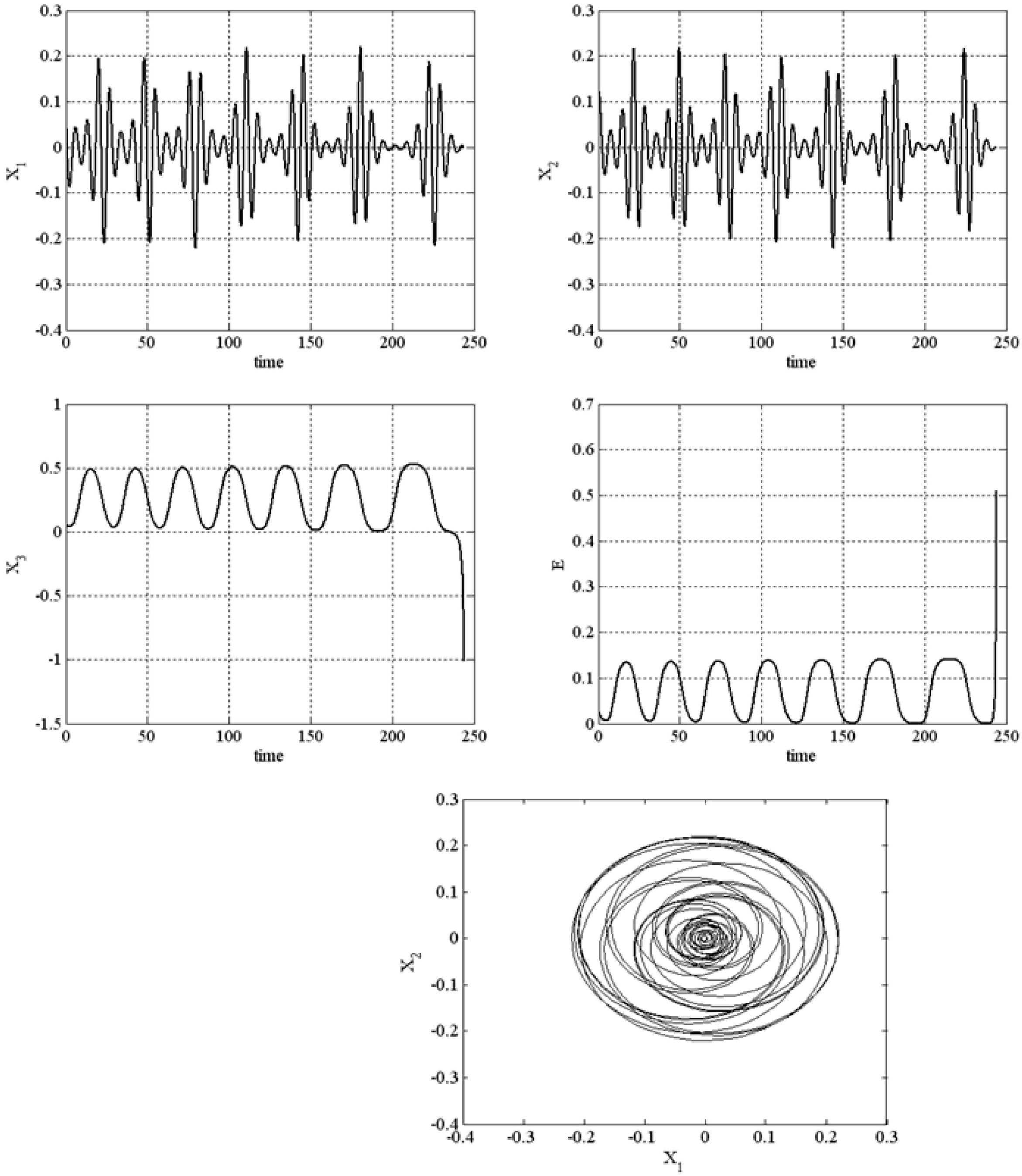

Figure 6. Chaotic like solutions of slave system (x_1, x_2), when master system (x_3, E) is (structurally) unstable. The parameters are: $\alpha = 1$, $\beta = 0.9$, $\mu = 0.533$ and $l = 0.21907$.

For clarity, we briefly discuss the dynamics of master subsystem (6) for different values of either L_1, L_2, or both. If $L_1 < 0$, then reversible (soft) stability loss takes place, and (6) is structurally stable—see Figures 4 and 5. In the case of $L_1 > 0$, the behavior of the system is expected to be structurally unstable—see Figure 6. Finally, if $L_1 = 0$, then for $L_2 < 0$, system (6) is structurally stable similar to the cases in Figures 4 and 5—not shown here.

4. Conclusions

Assume a three-dimensional autonomous nonlinear dissipative system of Hopf–Langford type. The introduction of energy as a new variable in the original system then leads to an equivalent four-dimensional system which can be separated into two coupled subsystems in the master (drive) (x_3, E)– slave (response) (x_1, x_2) synchronization type. Here we qualitatively and numerically investigated the nonlinear behavior (such as effect of synchronization) of an equivalent 4D dissipative system. As our study has shown, if the master system (energy) solutions: (1) tend to a constant, then the slave system (x_1, x_2) has periodic solutions; (2) oscillate, then the slave system has quasi-periodic behavior and (3) are (structurally) unstable, then the slave system has chaotic solutions. Our results for the first Lyapunov value, L_1 and second Lyapunov value, L_2, (when $R = L_1 = 0$) presented in Section 2 suggest that system (2) is structurally stable for some intervals of its parameters, and therefore soft stability loss takes place. It is also evident from the analytical calculations that master system (6) can be presented as the famous Liénard type. Hence, a family of exact solutions is obtained.

Many different aspects of this paper motivate further investigation. For example, the four-dimensional system has been suspected of showing hyperchaos (two positive Lyapunov's exponents) for some values of the control parameters μ, α, β, γ and l. The knowledge that the master subsystem is a primary regulator can really simplify the analysis of the original 3D system.

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Appendix A

Analytical Calculation of Second Lyapunov Value—L_2 (λ_0)

We consider a two-dimensional autonomous system

$$\left| \begin{array}{l} \dot{x} = ax + by + X(x, y)\,, \\ \dot{y} = cx + dy + Y(x, y)\,, \end{array} \right. \tag{A1}$$

where a, b, c and d are the parameters, and X and Y are polynomials in the form:

$$\begin{array}{l} X(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \ldots\ h.o.t., \\ Y(x, y) = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + \ldots\ h.o.t. \end{array} \tag{A2}$$

If $p = R = -(a + d) = 0, q = r = ad - bc > 0$ and $L_1 = 0$, then according to [14], the second Lypaunov value $L_2(\lambda_0)$ for system (Equation (A1)) can be determined analytically by the formula:

$$
\begin{aligned}
L_2 = \frac{\pi}{24}&[a_{02}b_{20}\left(5a_{02}b_{11} + 10a_{02}a_{20} + 4b_{11}^2 + 11a_{20}b_{11} + 6a_{20}^2 - 5a_{11}b_{20} - 10b_{20}b_{02} - 4a_{11}^2 - 11a_{11}b_{02} - 6b_{02}^2\right) + \\
&+a_{20}b_{02}\left(6b_{02}^2 - 5a_{11}b_{02} + 10b_{02}b_{20} - 2a_{11}^2 - 5a_{11}b_{20} + 5a_{20}b_{11} - 6a_{20}^2 - 10a_{20}a_{02} + 2b_{11}^2 + 5a_{02}b_{11}\right) + \\
&+a_{02}b_{02}\left(5b_{11}^2 - a_{11}^2 - 6a_{11}b_{02}\right) - a_{20}b_{20}\left(5a_{11}^2 - b_{11}^2 - 6a_{20}b_{11}\right) + a_{11}^3(a_{20} + a_{02}) - b_{11}^3(b_{02} + b_{20}) - \\
&-5b_{20}^2(a_{12} + 3b_{03}) + b_{02}^2(3b_{21} - 6a_{12} - 5a_{30}) + a_{11}^2(a_{12} + a_{30}) + b_{20}b_{02}(5b_{21} - 5a_{12} - 9b_{03} + 5a_{30}) - \\
&-b_{20}a_{11}(4a_{12} + 9b_{03} + 5a_{30}) + b_{02}a_{11}(3b_{21} - a_{12} + 4a_{30}) - 5a_{02}^2(b_{21} + 3a_{30}) + a_{20}^2(3a_{12} - 6b_{21} - 5b_{03}) + \\
&+b_{11}^2(b_{21} + b_{03}) + a_{20}a_{02}(5a_{12} - 5b_{21} - 9a_{30} + 5b_{03}) - a_{02}b_{11}(4b_{21} + 9a_{30} + 5b_{03}) + \\
&+a_{20}b_{11}(3a_{12} - b_{21} + 4b_{03}) + 4b_{20}b_{11}(2b_{30} + b_{12}) + b_{02}b_{11}(7b_{30} - a_{21} + 5b_{12} + a_{03}) + 2a_{11}b_{11}(a_{03} + b_{30}) + \\
&+2a_{20}b_{20}(8b_{30} - 5a_{21} - b_{12}) + 2a_{20}b_{02}(4b_{30} - 5a_{21} - 5b_{12} + 4a_{03}) - 2a_{02}b_{20}(a_{21} + b_{12}) + \\
&+a_{20}a_{11}(b_{30} + 5a_{21} - b_{12} + 7a_{03}) + 2a_{02}b_{02}(8a_{03} - 5b_{12} - a_{21}) + 4a_{02}a_{11}(2a_{03} + a_{21}) + \\
&+b_{11}(5b_{04} - b_{22} + 2a_{13} - 3b_{40}) + a_{02}(2b_{22} + 20b_{04} + 5a_{13} + 3a_{31}) + 3b_{41} + 3b_{23} + 15b_{05} + \\
&+a_{20}(4b_{22} + 22b_{04} + 7a_{13} - 6b_{40} + 9a_{31}) - b_{20}(2a_{22} + 20a_{40} + 5b_{31} + 3b_{13}) + 15a_{50} + 3a_{32} + 3a_{14} - \\
&-b_{20}(2a_{22} + 20a_{40} + 5b_{31} + 3b_{13}) - a_{11}(5a_{40} - a_{22} + 2b_{31} - 3a_{04}) + 3a_{21}(2a_{30} + b_{03} + a_{12}) - \\
&-3b_{12}(2b_{03} + a_{30} + b_{21}) + 3a_{03}(a_{12} + 3b_{03}) - 3b_{30}(b_{21} + 3a_{30}) - \\
&-b_{02}(4a_{22} + 22a_{40} + 7b_{31} - 6a_{04} + 9b_{13})].
\end{aligned} \tag{A3}
$$

Note that the maximal power n (into Equation (A2)) must be in accordance with the number of the calculated Lyapunov value, i.e., the Lyapunov value L_k depends from the terms with power $n = 2k + 1$. It follows from works [3,12,13] that the role of second Lyapunov value is similar with those of first Lyapunov value, i.e., if the external (second) limit cycle is stable/unstable.

In Figures A1–A4, the possible types of qualitative behavior of system (Equation (A1)) are shown, when $L_2(\lambda_0) \neq 0$. If $L_2(\lambda_0) > 0$ (at $R = L_1 = 0$), then in the neighborhood of the fixed point the following structures are valid: (i) an unstable limit cycle exists, and the fixed point is stable focus—see Figure A2a (ii) an external unstable limit cycle and a stable (with small amplitude) limit cycle occur, when the safe boundary of stability is crossed. In this case, if the initial perturbations are smaller from the amplitude of the external cycle, then the system will be stay in the neighborhood of the fixed point/stable limit cycle—see Figure A2b; and (iii) irreversible behavior, as after very small perturbation the system leave forever vicinity of the fixed point—see Figure A2c.

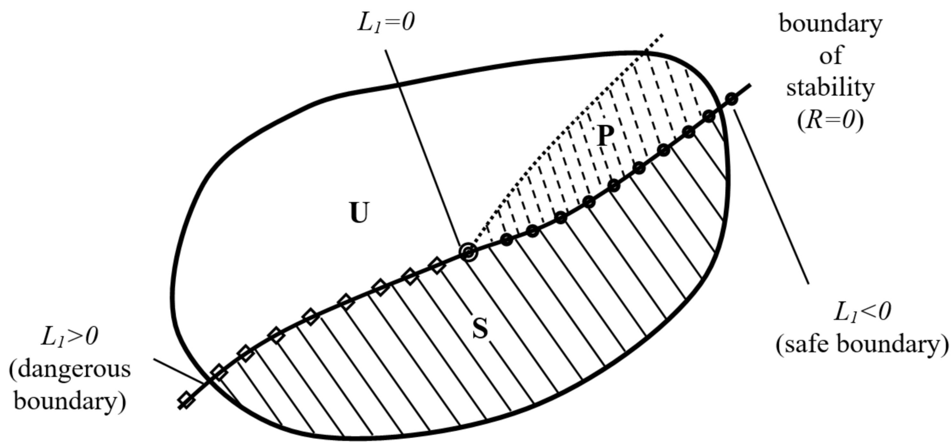

Figure A1. Vicinity of a point from boundary of stability $R = 0$, in which first Lyapunov value L_1 is zero, and second Lyapunov value L_2 is positive. Legend: **U**—unstable zone ($R < 0$); **S**—stable zone ($R > 0$); **P**—particular zone.

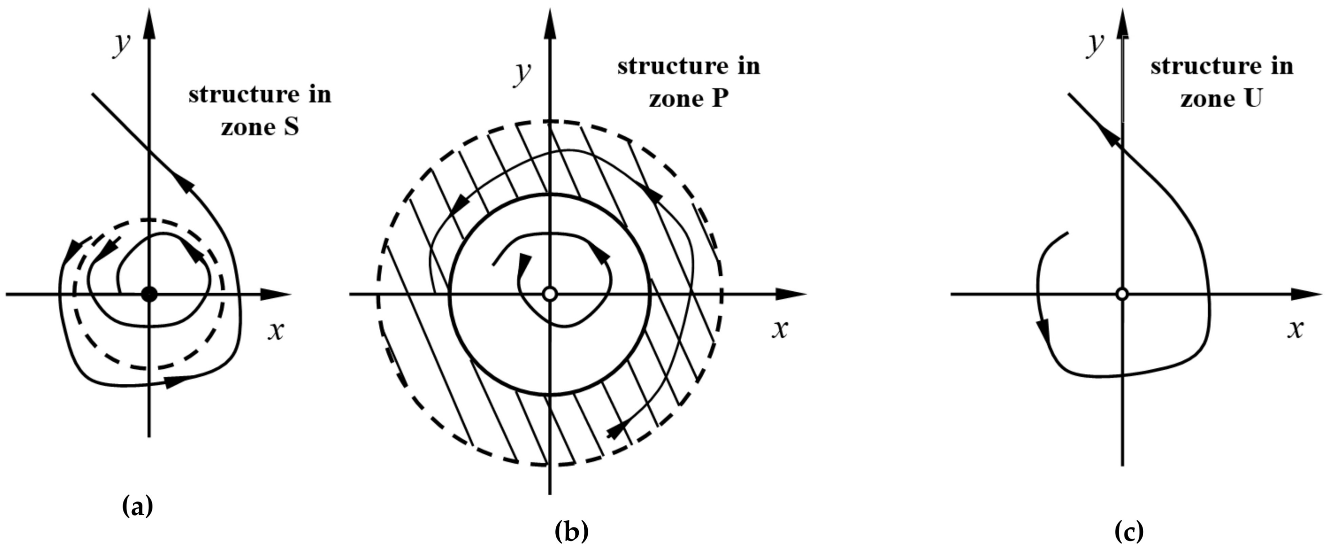

Figure A2. Qualitative structures in the vicinity of a fixed point, when second Lyapunov value L_2 is positive. Note that different zones U, S and P are those in Figure A1. Legend: solid circle line—stable limit cycle; dashed circle line—unstable limit cycle. (**a**) an unstable limit cycle exists, and the fixed point is stable focus; (**b**) an external unstable limit cycle and a stable limit cycle occur; (**c**) irreversible behavior, as after very small perturbation the system leave forever vicinity of the fixed point.

In Figure A3, the qualitative behavior of system (Equation (A1)) is shown, when $L_2(\lambda_0) < 0$ (at $R = L_1 = 0$). For small variation of the system's parameters, in vicinity of a fixed point three different structures are possible—see Figure A4. In this case, the system (Equation (A1)) will stay in the neighborhood of fixed point independently from the perturbation size, and the external cycle is stable.

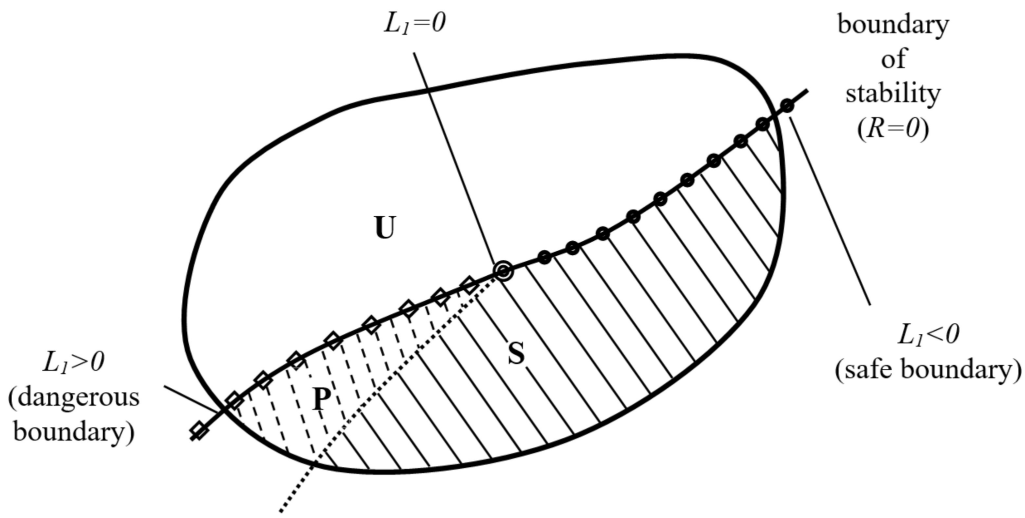

Figure A3. Vicinity of a point from boundary of stability $R = 0$, in which first Lyapunov value L_1 is zero, and second Lyapunov value L_2 is negative. Legend: **U**—unstable zone ($R < 0$); **S**—stable zone ($R > 0$); **P**—particular zone.

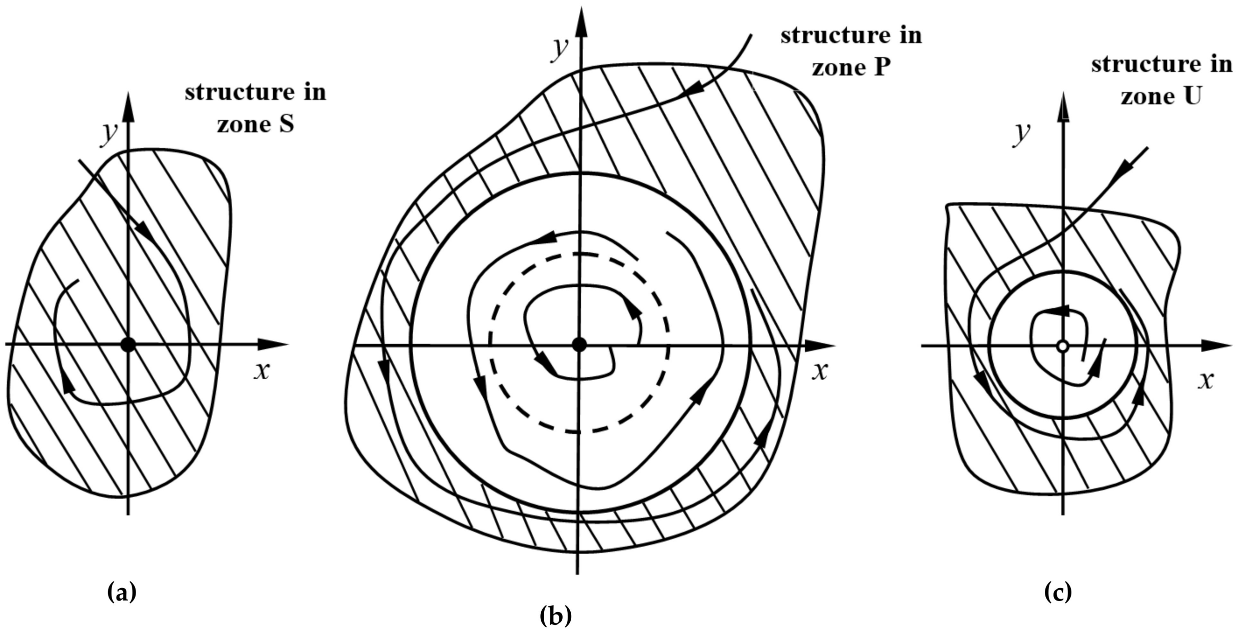

Figure A4. Qualitative structures in the vicinity of a fixed point, when second Lyapunov value L_2 is negative. Note that different zones U, S and P are those in Figure A3. Legend: solid circle line—stable limit cycle; dashed circle line—unstable limit cycle. (**a**) structure in zone S; (**b**) structure in zone P; (**c**) structure in zone U.

Appendix B

Derivation of $\dot{E}$

To find the fourth equation in Equation (2), we differentiate $E = \frac{1}{2}(x^2 + y^2 + z^2)$, obtaining

$$\dot{E} = x\dot{x} + y\dot{y} + z\dot{z}, \tag{A4}$$

where $\dot{E} = dE/dt$. Substituting $\dot{x}$, $\dot{y}$ and $\dot{z}$ (from Equation (2)) into Equation (A4), we derive

$$\dot{E} = (\mu - a)\left(x_1^2 + x_2^2\right) + l\left(1 - x_3^2\right)\left(x_1^2 + x_2^2\right) + x_3\left(x_1^2 + x_2^2\right) + \mu x_3^2 - 2Ex_3 \tag{A5}$$

or

$$\dot{E} = (\mu - a)\left(2E - x_3^2\right) + l\left(1 - x_3^2\right)\left(2E - x_3^2\right) + x_3\left(2E - x_3^2\right) + \mu x_3^2 - 2Ex_3 \tag{A6}$$

where $x_1^2 + x_2^2 = 2E - x_3^2$. Accomplishing some algebraic operations, finally for $\dot{E}$, we have

$$\dot{E} = 2(\mu + l - \alpha)E - (l - \alpha)x_3^2 - 2lx_3^2E - x_3^3 + lx_3^4. \tag{A7}$$

Appendix C

First Lyapunov Value—L_1

According to [14], in the case of two first-order nonlinear differential equations (in the form Equation (A1)), the analytical formula for first Lyapunov value L_1 is:

$$\begin{aligned} L_1 = -\tfrac{\pi}{4bq\sqrt{q}}\{&[ac\left(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}\right) + ab\left(b_{11}^2 + a_{20}b_{11} + a_{11}b_{20}\right) + c^2(a_{11}a_{02} + 2a_{02}b_{02}) - \\ &-2ac\left(b_{02}^2 - a_{20}a_{02}\right) - 2ab\left(a_{20}^2 - b_{20}b_{02}\right) - b^2(2a_{20}b_{20} + b_{11}b_{20}) + \left(bc - 2a^2\right)(b_{11}b_{02} - a_{11}a_{20})] - \\ &\left(a^2 + bc\right)[3(cb_{03} - ba_{30}) + 2a(a_{21} + b_{12}) + (ca_{12} - bb_{21})]\}. \end{aligned} \tag{A8}$$

It is seen that corresponding coefficients for system (2) are:

$$\begin{array}{l} a_{11} = a_{02} = a_{20} = a_{30} = a_{21} = a_{12} = b_{02} = b_{03} = b_{12} = 0\,, a = \mu\,, \\ b = -2\,, c = c_1\,, d = c_2\,, b_{20} = c_3\,, b_{11} = -c_4\,, b_{30} = c_5\,, b_{21} = -c_6\,. \end{array} \tag{A9}$$

Appendix D

Derivation of Equilibrium Points

To examine the qualitative behavior of the system (2), we begin to look at any equilibrium points. These are solutions, which arise at $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{E} = 0$. Thus, we have

$$\overline{x}_1 = \overline{x}_2 = 0, \overline{E} = \frac{\mu}{2}\overline{x}_3 \text{ and } \overline{x}_3\left[l\,\overline{x}_3^3 - (\mu l + 1)\,\overline{x}_3^2 + (\alpha - l)\,\overline{x}_3 + \mu(\mu + l - \alpha)\right] = 0\,. \tag{A10}$$

Hence, for the equilibrium points of the system (2), we obtain

$$\mathbf{O_1} : \overline{x}_1 = \overline{x}_2 = \overline{x}_3 = \overline{E} = 0, \textit{ first EP}, \tag{A11}$$

$$\mathbf{O_{2,3,4}} : \begin{array}{l} \overline{x}_1 = \overline{x}_2 = 0\,,\ \overline{E} = \frac{\mu}{2}\overline{x}_3\,, \\ l\,\overline{x}_3^3 - (\mu l + 1)\overline{x}_3^2 + (\alpha - l)\overline{x}_3 + \mu(\mu + l - \alpha) = 0\,. \end{array} \textit{ second, third and fourth EPs.} \tag{A12}$$

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