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Investigation of the Fractional Strongly Singular Thermostat Model via Fixed Point Techniques

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Abstract: Our main purpose in this paper is to prove the existence of solutions for the fractional strongly singular thermostat model under some generalized boundary conditions. In this way, we use some recent nonlinear fixed-point techniques involving α-ψ-contractions and α-admissible maps. Further, we establish the similar results for the hybrid version of the given fractional strongly singular thermostat control model. Some examples are studied to illustrate the consistency of our results.

Keywords: boundary conditions; hybrid differential equation; fractional thermostat model; strong singularity; the Caputo derivative; α-ψ-contraction

1. Introduction

Fractional calculus is one of the most important branches of mathematics that derives and studies many different properties of integration and derivation operators of non-integer orders via singular and nonsingular kernels. These operators are called fractional integrals and derivatives [1–4]. Because of the importance, potential, high accuracy and flexibility of the mentioned fractional operators, the attention of engineers and applied researchers has been drawn in this direction. One can find several published works regarding applications of this field in mathematical models. For instance, Baleanu et al. [5] designed a novel model of FBVP on glucose graph, or Mohammadi et al. [6] studied a fractional mathematical model of Mumps virus in the context of the Caputo–Fabrizio operators. In [7], Boutiara et al. used the Caputo–Hadamard fractional operators to study the solutions of a three-point BVP—see [8–30].

In 2005, the first mathematical model based on thermostat control was designed in the following structure by Webb [31] as

$$\begin{cases} v''(\mathfrak{t}) + g(\mathfrak{t})H(\mathfrak{t}, v(\mathfrak{t})) = 0, \\ v'(0) = 0, bv'(1) + v(\xi) = 0, \end{cases}$$

for $\mathfrak{t} \in [0,1]$ and $b > 0$. Then, in [32], Shen, Zhou and Yang considered the thermostat differential equation in the non-integer format and with the same boundary conditions as

$$\begin{cases} D^p v(\mathfrak{t}) + \lambda H(\mathfrak{t}, v(\mathfrak{t})) = 0, \\ v'(0) = 0, \ \ bD^{p-1}v(\mathfrak{t}) + v(\xi) = 0, \end{cases}$$

where $\mathfrak{t} \in [0,1]$, $b > 0$ and $p \in (1,2]$, $\xi \in (0,1)$, $b\Gamma(p) > (1-\xi)^{p-1}$, $\lambda > 0$ and $H : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. In the subsequent years, other researchers investigated different structures of the fractional model of thermostat. In [33], Baleanu et al. designed the hybrid fractional model of thermostat control for the first time and by utilizing the Dhage's method, established their desired purposes on the existence of solution, which takes such a format

$${}^c\mathcal{D}^p\Big[\frac{v(\mathfrak{t})}{h(\mathfrak{t}, v(\mathfrak{t}))}\Big] + H(\mathfrak{t}, v(\mathfrak{t})) = 0,$$

via the hybrid boundary conditions

$${}^c\mathcal{D}^1\Big[\frac{v(\mathfrak{t})}{h(\mathfrak{t}, v(\mathfrak{t}))}\Big]\Big|_{\mathfrak{t}=0} = 0, \quad b\,{}^c\mathcal{D}^{p-1}\Big[\frac{v(\mathfrak{t})}{h(\mathfrak{t}, v(\mathfrak{t}))}\Big]\Big|_{\mathfrak{t}=1} + \Big[\frac{v(\mathfrak{t})}{h(\mathfrak{t}, v(\mathfrak{t}))}\Big]\Big|_{\mathfrak{t}=\xi} = 0,$$

in which $p \in (1,2]$, $\xi \in (0,1)$, $b > 0$, ${}^c\mathcal{D}^1 = \dfrac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}$, ${}^c\mathcal{D}^q$ stands for the Caputo derivative for given order $q \in \{p, p-1\}$ and $H, h \in C([0,1] \times \mathbb{R}, \mathbb{R})$ with $h \neq 0$.

Recently in 2021, Thaiprayoon et al. [34] devoted to investigating a class of ϕ-Hilfer nonlinear implicit fractional model describing thermostat control as

$$\begin{cases} {}^H D^{p,\rho;\phi} v(\mathfrak{t}) = H(\mathfrak{t}, v(\beta\mathfrak{t}), I^{q;\phi}v(\theta\mathfrak{t})), \quad \mathfrak{t} \in [0,T], \\ \sum_{i=1}^{m} a_i\,{}^H D^{\gamma_i,\rho;\phi} v(\zeta_i) = A_1, \qquad \sum_{j=1}^{n} b_j\,{}^H D^{\mu_j,\rho;\phi} v(\delta_j) + \sum_{k=1}^{r} c_k v(\xi_k) = A_2, \end{cases}$$

in which ${}^H D^{\alpha,\rho;\phi}$ stands for the ϕ-Hilfer derivative of order $\alpha = \{p, \gamma_i, \mu_j\}$, $p \in (1,2]$, γ_i, $\mu_j \in (0,1]$, $A_1, A_2, a_i, b_j, c_k \in \mathbb{R}$, ζ_i, δ_j, $\xi_k \in (0,T)$, $\rho \in [0,1]$, $I^{q;\phi}$ is the ϕ-RL-integral for given order $q > 0$, $\beta, \theta \in (0,1]$ and $H \in C([0,T] \times \mathbb{R}^2, \mathbb{R})$.

Naturally, in many real-world mathematical models, in some points of the existing domain, there is the singularity and this implies that the computation and finding possible solutions of the given fractional system becomes a complicated process. Due to such a difficulty, some researchers are interested in the investigation of singular fractional BVPs. For instance, see [35,36].

The importance and existing complexity in studying fractional structures having singular points motivate us to investigate the existence and uniqueness of solutions for some real mathematical models in engineering in which solutions possess singular points; therefore, by using main ideas of above mathematical models, in this manuscript, we find a theoretical method to investigate the existence of solutions for the strongly singular fractional model of thermostat control given as

$${}^c D^{\omega} x(\mathfrak{t}) + g(\mathfrak{t})\mathfrak{f}(x(\mathfrak{t})) = 0, \qquad (\omega \geq 2, \ \ \omega \in (n-1, n]), \tag{1}$$

with initial conditions $x^{(j)}(0) = 0$ for $j \in \{0, 1, \ldots, n-1\}$ with $j \neq k$ coupled with the boundary condition

$$\big(p(\mathfrak{t})x(\mathfrak{t})\big)'\big|_{\mathfrak{t}=1} + ax(\eta) = 0,$$

where $\eta \in (0,1)$, $a > 0$, $(\omega - k - 1)p(1) > a\eta^k$, $g : [0,1] \to \mathbb{R}$ is singular or strongly singular at some points of $[0,1]$, $p : [0,1] \to [0,\infty)$ is differentiable in $\mathfrak{t} = 1$, $\mathfrak{f} \in C(\mathbb{R},\mathbb{R})$ and ${}^cD^{\omega}$ displays the Caputo derivative for given order ω.

Note that the above boundary problem caused by the singular fractional model of thermostat control is new and it has not been studied in any other paper so far and this guarantees the novelty of the present paper. Further, the used technique to confirm the existence of solutions for such a new singular system is based on the special subclass of operators called α-ψ-contractions and α-admissible maps.

The construction of the paper is organized as: Section 2 is devoted to recalling some basic notions. Section 3 is devoted to deriving a corresponding integral equation for the given singular model of thermostat control (1) and proving the existence of solution by making use of α-ψ-contractions. In Section 4, the hybrid version of the aforementioned strongly singular model of thermostat control is proved by means of the same technique. Two illustrative examples for both cases are simulated in Section 5 to confirm the correctness of the findings. At last, the conclusion remarks are stated in Section 6.

2. Basic Notions

Before recalling some basic notions, note that in this article, we apply $\|.\|_1$ for the norm of $L^1[0,1]$ and $\|.\|$ as the sup-norm for the space $X = C([0,1],\mathbb{R})$.

Definition 1 ([3])**.** *The p-th Riemann–Liouville integral of $h \in C([0,+\infty),\mathbb{R})$ for given order $p > 0$ is formulated by*

$$I^p h(\mathfrak{t}) = \int_0^{\mathfrak{t}} \frac{(\mathfrak{t} - r)^{p-1}}{\Gamma(p)} h(r)\,\mathrm{d}r$$

if it is finite-valued.

Definition 2 ([3])**.** *Let $n = [p] + 1$. For $h \in C(\mathbb{R}^{\geq 0},\mathbb{R})$, the p-th Riemann–Liouville derivative is given as*

$$D^p h(\mathfrak{t}) = \left(\frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\right)^n \int_0^{\mathfrak{t}} \frac{(\mathfrak{t} - r)^{n-p-1}}{\Gamma(n-p)} h(r)\,\mathrm{d}r$$

if it is finite-valued.

Definition 3 ([3])**.** *Let $n = [p] + 1$. For $h \in AC^{(n)}(\mathbb{R}^{\geq 0},\mathbb{R})$, the p-th Caputo derivative is presented by*

$${}^cD^p h(\mathfrak{t}) = \int_0^{\mathfrak{t}} \frac{(\mathfrak{t} - r)^{n-p-1}}{\Gamma(n-p)} h^{(n)}(r)\,\mathrm{d}r$$

if it is finite-valued.

Proposition 1 ([1])**.** *Let $n - 1 < p < n$. Then $\forall\, h \in C^{n-1}([0,+\infty))$,*

$$I^p\left({}^cD^p h\right)(\mathfrak{t}) = h(\mathfrak{t}) + \sum_{i=0}^{n-1} c_i \mathfrak{t}^i = c_0 + c_1\mathfrak{t} + c_2\mathfrak{t}^2 + \cdots + c_{n-1}\mathfrak{t}^{n-1}$$

for some $c_i \in \mathbb{R}$.

In 2012, Samet et al. [37] turned to introduction of a new subclass of special functions, which will be applied in our existence method here.

We introduce via Ψ, the subclass of non-decreasing mappings such as $\psi : [0,\infty) \to [0,\infty)$ with

$$\sum_{n=1}^{\infty} \psi^n(\mathfrak{t}) < \infty$$

for all $\mathfrak{t} > 0$. Further, $\forall\, \mathfrak{t} > 0$, $\psi(\mathfrak{t}) < \mathfrak{t}$ [37]. In the sequel, regard X as a complete metric space.

Definition 4 ([37])**.** *Consider $h : X \to X$ and $\alpha : X \times X \to [0, \infty)$ as two mappings. We name h to be an α-admissible whenever $\alpha(hx, hy) \geq 1$ if $\alpha(x, y) \geq 1$.*

Definition 5 ([37])**.** *Consider $\psi \in \Psi$ and $\alpha : A \to [0, \infty)$ with $A = X \times X$. A self-mapping h on X is named as an α-ψ-contraction if*

$$\alpha(x, y) d(hx, hy) \leq \psi(d(x, y)), \quad \forall\, x, y \in X.$$

In this study, the next theorem will be useful for establishing the fundamental theorems.

Theorem 1 ([37])**.** *Consider $\psi \in \Psi$, $\alpha : A \to [0, \infty)$ with $A = X \times X$ and $h : X \to X$ as a continuous α-admissible α-ψ-contraction. If $\exists\, x_0 \in X$ so that $\alpha(x_0, hx_0) \geq 1$, then h admits a fixed-point in X.*

3. Main Results

Here, the existence of solution for the aforesaid strongly singular fractional model of thermostat control (1) is discussed. At first, we provide a key lemma.

Lemma 1. *Let $\omega \geq 2$, $\omega \in [n-1, n)$, $\eta \in (0, 1)$, $a > 0$, $k \in \{0, 1, \ldots, n-2\}$, $\mathfrak{f} \in L^1$ and $p : [0, 1] \to \mathbb{R}$ be differentiable at $\mathfrak{t} = 1$ with $(\omega - k - 1)p(1) > a\eta^k > 0$. Then v as a solution of the linear differential equation ${}^cD^{\omega}(x(\mathfrak{t})) + \mathfrak{f}(\mathfrak{t}) = 0$ via given BCs*

$$\begin{cases} x^{(j)}(0) = 0, \quad \forall\, j \in \{0, 1, \ldots, n-1\}, \; j \neq k, \\ (p(\mathfrak{t})x(\mathfrak{t}))'\big|_{\mathfrak{t}=1} + ax(\eta) = 0, \end{cases} \tag{2}$$

is given as

$$x(\mathfrak{t}) = \int_0^1 G(\mathfrak{t}, \mathfrak{s}) \mathfrak{f}(\mathfrak{s}) d\mathfrak{s}, \tag{3}$$

where

$$G(\mathfrak{t}, \mathfrak{s}) = \frac{1}{\Gamma(\omega)} \left[-(\mathfrak{t} - \mathfrak{s})^{\omega-1} + \left(p'(1)(1-\mathfrak{s})^{\omega-1} + \omega p(1)(1-\mathfrak{s})^{\omega-2} + a(\eta - \mathfrak{s})^{\omega-1} \right) \frac{\mathfrak{t}^k}{\Delta} \right]$$

whenever $0 \leq \mathfrak{s} \leq \mathfrak{t} \leq 1$ and $\mathfrak{s} \leq \eta$,

$$G(\mathfrak{t}, \mathfrak{s}) = \frac{1}{\Gamma(\omega)} \left(p'(1)(1-\mathfrak{s})^{\omega-1} + \omega p(1)(1-\mathfrak{s})^{\omega-2} + a(\eta - \mathfrak{s})^{\omega-1} \right) \frac{\mathfrak{t}^k}{\Delta}$$

whenever $0 \leq \mathfrak{t} \leq \mathfrak{s} \leq \eta \leq 1$,

$$G(\mathfrak{t}, \mathfrak{s}) = \frac{1}{\Gamma(\omega)} \left[-(\mathfrak{t} - \mathfrak{s})^{\omega-1} + \left(p'(1)(1-\mathfrak{s})^{\omega-1} + \omega p(1)(1-\mathfrak{s})^{\omega-2} \right) \frac{\mathfrak{t}^k}{\Delta} \right]$$

whenever $0 \leq \eta \leq \mathfrak{s} \leq \mathfrak{t} \leq 1$,

$$G(t, \mathfrak{s}) = \frac{1}{\Gamma(\omega)} \left(p'(1)(1-\mathfrak{s})^{\omega-1} + \omega p(1)(1-\mathfrak{s})^{\omega-2} \right) \frac{\mathfrak{t}^k}{\Delta}$$

whenever $0 \leq \mathfrak{t} \leq \mathfrak{s} \leq 1$ and $\mathfrak{s} \geq \eta$ and also $\Delta = p'(1) + kp(1) + a\eta^k$.

Proof. Let v be the solution of the given linear BVP and satisfies (2). By using Proposition 1, some real constants $c_0, \ldots, c_{n-1}$ exist provided that

$$v(\mathfrak{t}) = -I^{\omega}(\mathfrak{f}(\mathfrak{t})) + c_0 + c_1\mathfrak{t} + \ldots + c_{n-1}\mathfrak{t}^{n-1}.$$

Since $v^{(j)}(0) = 0$ for all $j \in \{0, 1, \ldots, n-1\}$ with $j \neq k$, we obtain $c_j = 0$ for $j \neq k$ and so

$$v(\mathfrak{t}) = -\frac{1}{\Gamma(\omega)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + c_k \mathfrak{t}^k. \tag{4}$$

Hence $p(\mathfrak{t})v(\mathfrak{t}) = \dfrac{-p(\mathfrak{t})}{\Gamma(\omega)} \displaystyle\int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + c_k p(\mathfrak{t})\mathfrak{t}^k$. Thus,

$$\big(p(\mathfrak{t})v(\mathfrak{t})\big)' = \frac{-p'(\mathfrak{t})}{\Gamma(\omega)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} - \frac{p(\mathfrak{t})}{\Gamma(\omega-1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\omega-2} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + k c_k p(\mathfrak{t})\mathfrak{t}^{k-1} + c_k p'(\mathfrak{t})\mathfrak{t}^k.$$

Now for each $k \geq 1$, we have

$$\begin{aligned}
\big(p(\mathfrak{t})v(\mathfrak{t})\big)'\big|_{\mathfrak{t}=1} &= \frac{-p'(1)}{\Gamma(\omega)} \int_0^1 (1 - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} \\
&\quad - \frac{p(1)}{\Gamma(\omega-1)} \int_0^1 (1 - \mathfrak{s})^{\omega-2} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + c_k p'(1) + k c_k p(1) \\
&= -\int_0^1 \Big(\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)} + \frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)}\Big) \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + (p'(1) + kp(1))c_k,
\end{aligned}$$

and for $k = 0$, we have

$$\begin{aligned}
\big(p(\mathfrak{t})v(\mathfrak{t})\big)' &= \left(\frac{-p(\mathfrak{t})}{\Gamma(\omega)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + c_0 p(\mathfrak{t})\right)' \\
&= \frac{-p'(\mathfrak{t})}{\Gamma(\omega)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} - \frac{p(\mathfrak{t})}{\Gamma(\omega-1)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\omega-2} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + c_0 p'(\mathfrak{t}).
\end{aligned}$$

This implies that

$$\begin{aligned}
\big(p(\mathfrak{t})v(\mathfrak{t})\big)'\big|_{\mathfrak{t}=1} &= \frac{-p'(1)}{\Gamma(\omega)} \int_0^1 (1 - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} - \frac{p(1)}{\Gamma(\omega-1)} \int_0^1 (1 - \mathfrak{s})^{\omega-2} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + c_0 p'(1) \\
&= -\int_0^1 \Big(\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)} + \frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)}\Big) \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + (p'(1) + kp(1))c_k.
\end{aligned}$$

On the other hand, $av(\eta) = -\dfrac{a}{\Gamma(\omega)} \displaystyle\int_0^{\eta} (\eta - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} + c_k a \eta^k$. Since

$$\big(p(t)v(t)\big)'\big|_{t=1} + au(\eta) = 0,$$

we obtain

$$\begin{aligned}
c_k(p'(1) + kp(1) + a\eta^k) &= \int_0^1 \Big(\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)} + \frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)}\Big) \mathfrak{f}(\mathfrak{s}) d\mathfrak{s} \\
&\quad + \frac{a}{\Gamma(\omega)} \int_0^{\eta} (\eta - \mathfrak{s})^{\omega-1} \mathfrak{f}(\mathfrak{s}) d\mathfrak{s}
\end{aligned}$$

and so

$$c_k = \frac{1}{\Delta}\int_0^1 (\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)} + \frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)})\mathfrak{f}(\mathfrak{s})d\mathfrak{s}$$

$$+ \frac{a}{\Delta\Gamma(\omega)}\int_0^\eta (\eta-\mathfrak{s})^{\omega-1}\mathfrak{f}(\mathfrak{s})d\mathfrak{s},$$

where $\Delta = p'(1) + kp(1) + a\eta^k$. Hence, by inserting c_k into (4),

$$v(\mathfrak{t}) = -\frac{1}{\Gamma(\omega)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{\omega-1}\mathfrak{f}(\mathfrak{s})d\mathfrak{s}$$

$$+ \frac{\mathfrak{t}^k}{\Delta}\int_0^1 (\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)} + \frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)})\mathfrak{f}(\mathfrak{s})d\mathfrak{s}$$

$$+ \frac{a\mathfrak{t}^k}{\Delta\Gamma(\omega)}\int_0^\eta (\eta-\mathfrak{s})^{\omega-1}\mathfrak{f}(\mathfrak{s})d\mathfrak{s},$$

and so we obtain

$$v(\mathfrak{t}) = \int_0^1 G(\mathfrak{t},\mathfrak{s})\mathfrak{f}(\mathfrak{s})d\mathfrak{s},$$

in which $G(\mathfrak{t},\mathfrak{s})$ is given by (3) and the argument is completed. □

Remark 1. *In the special case $k = 0$, Green function is reduced to:*

$$G(\mathfrak{t},\mathfrak{s}) = \frac{1}{\Gamma(\omega)}\left[-(\mathfrak{t}-\mathfrak{s})^{\omega-1} + \left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2} + a(\eta-\mathfrak{s})^{\omega-1}\right)\frac{1}{\Delta}\right]$$

when $0 \le \mathfrak{s} \le \mathfrak{t} \le 1$ and $\mathfrak{s} \le \eta$,

$$G(\mathfrak{t},\mathfrak{s}) = \frac{1}{\Gamma(\omega)}\left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2} + a(\eta-\mathfrak{s})^{\omega-1}\right)\frac{1}{\Delta}$$

when $0 \le \mathfrak{t} \le \mathfrak{s} \le \eta \le 1$,

$$G(\mathfrak{t},\mathfrak{s}) = \frac{1}{\Gamma(\omega)}\left[-(\mathfrak{t}-\mathfrak{s})^{\omega-1} + \left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2}\right)\frac{1}{\Delta}\right]$$

when $0 \le \eta \le \mathfrak{s} \le \mathfrak{t} \le 1$ and

$$G(\mathfrak{t},\mathfrak{s}) = \frac{1}{\Gamma(\omega)}\left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2}\right)\frac{1}{\Delta}.$$

Remark 2. *Note that for each $\mathfrak{t},\mathfrak{s} \in [0,1]$, we have*

$$G(\mathfrak{t},\mathfrak{s}) \ge \frac{1}{\Gamma(\omega)}\left[-(\mathfrak{t}-\mathfrak{s})^{\omega-1} + \left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2}\right)\frac{\mathfrak{t}^k}{\Delta}\right].$$

Since $k \le \omega - 1$, we have

$$\mathfrak{t}^k(1-\mathfrak{s})^{\omega-1} \ge t^{\omega-1}(1-\mathfrak{s})^{\omega-1} = (\mathfrak{t}-\mathfrak{t}\mathfrak{s})^{\omega-1} \ge (\mathfrak{t}-\mathfrak{s})^{\omega-1}$$

and so

$$-(\mathfrak{t}-\mathfrak{s})^{\omega-1} + \left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2}\right)\frac{\mathfrak{t}^k}{\Delta}$$

$$\geq -(\mathfrak{t}-\mathfrak{s})^{\omega-1} + \left(p'(1)(\mathfrak{t}-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(\mathfrak{t}-\mathfrak{s})^{\omega-1}\right)\frac{1}{\Delta}$$

$$\geq (\mathfrak{t}-\mathfrak{s})^{\omega-1}\left(-kp(1) + (\omega-1)p(1) - a\eta^k\right)\frac{1}{\Delta}$$

$$= (\mathfrak{t}-\mathfrak{s})^{\omega-1}\left((\omega-k-1)p(1) - a\eta^k\right)\frac{1}{\Delta} \geq 0.$$

Hence,

$$G(\mathfrak{t},\mathfrak{s}) \geq \frac{(\mathfrak{t}-\mathfrak{s})^{\omega-1}}{\Delta\Gamma(\omega)}\left((\omega-k-1)p(1) - a\eta^k\right) \geq 0.$$

Remark 3. *Further, $G(0,\mathfrak{s}) = 0$ and the maximum of $G(\mathfrak{t},\mathfrak{s})$ is obtained if $0 \leq \mathfrak{t} \leq \mathfrak{s} \leq \eta \leq 1$ and accordingly,*

$$G(\mathfrak{t},\mathfrak{s}) \leq \frac{1}{\Gamma(\omega)}\left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2} + a(\eta-\mathfrak{s})^{\alpha-1}\right)\frac{\mathfrak{t}^k}{\Delta}$$

$$\leq \frac{1}{\Delta\Gamma(\omega)}\left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2} + a(\eta-\mathfrak{s})^{\omega-1}\right)\mathfrak{s}^k.$$

One can simply see that G is continuous by terms of $\mathfrak{t}$. Moreover, for $k \geq 1$ we have

$$\frac{\partial G}{\partial \mathfrak{t}}(t,\mathfrak{s}) = \frac{-(\mathfrak{t}-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)} + \frac{k\mathfrak{t}^{k-1}}{\Delta\Gamma(\omega)}\left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\alpha-2} + a(\eta-\mathfrak{s})^{\omega-1}\right)$$

whenever $0 \leq \mathfrak{s} \leq \mathfrak{t} \leq 1$ and $\mathfrak{s} \leq \eta$,

$$\frac{\partial G}{\partial \mathfrak{t}}(\mathfrak{t},\mathfrak{s}) = \frac{k\mathfrak{t}^{k-1}}{\Delta\Gamma(\omega)}\left(p'(1)(1-\mathfrak{s})^{\alpha-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2} + a(\eta-\mathfrak{s})^{\omega-1}\right)$$

whenever $0 \leq \mathfrak{t} \leq \mathfrak{s} \leq \eta \leq 1$,

$$\frac{\partial G}{\partial \mathfrak{t}}(\mathfrak{t},\mathfrak{s}) = \frac{-(\mathfrak{t}-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)} + \frac{k\mathfrak{t}^{k-1}}{\Delta\Gamma(\omega)}\left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2}\right)$$

whenever $0 \leq \eta \leq \mathfrak{s} \leq \mathfrak{t} \leq 1$,

$$\frac{\partial G}{\partial \mathfrak{t}}(\mathfrak{t},\mathfrak{s}) = \frac{k\mathfrak{t}^{k-1}}{\Delta\Gamma(\omega)}\left(p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2}\right)$$

and for case $k = 0$, $\frac{\partial G}{\partial \mathfrak{t}}(\mathfrak{t},\mathfrak{s}) = \frac{-(\mathfrak{t}-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)}$ when $0 \leq \mathfrak{s} \leq \mathfrak{t} \leq 1$ and $\frac{\partial G}{\partial \mathfrak{t}}(\mathfrak{t},\mathfrak{s}) = 0$ when $0 \leq \mathfrak{t} \leq \mathfrak{s} \leq 1$. Thus, $\frac{\partial G}{\partial \mathfrak{t}}$ will be continuous w.r.t the variable $\mathfrak{t}$.

We assume that $X = C[0,1]$ is furnished with $\|x\| = \sup\{|x(\mathfrak{t})| : \mathfrak{t} \in [0,1]\}$, which will be a Banach space and $H : X \to X$ is given by

$$Hx(\mathfrak{t}) = \int_0^1 G(\mathfrak{t},\mathfrak{s})g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s}$$

$$= -\frac{1}{\Gamma(\omega)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\omega-1}g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s}$$

$$+\frac{\mathfrak{t}^k}{\Delta}\int_0^1\Big(\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)}+\frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)}\Big)g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s}$$

$$+\frac{a\mathfrak{t}^k}{\Delta\Gamma(\omega)}\int_0^\eta(\eta-\mathfrak{s})^{\omega-1}g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s} \tag{5}$$

for all $\mathfrak{t}\in[0,1]$. In this case, $x_0\in X$ is a solution for the singular fractional model of thermostat control (1) iff x_0 is a fixed point of H.

In the next theorem, we suppose that the map $g:[0,1]\to\mathbb{R}$ may be singular at some points $\{\mathfrak{t}_i\}_{i=0}^r$ subject to $0=\mathfrak{t}_0<\mathfrak{t}_1<\ldots<\mathfrak{t}_{r-1}<\mathfrak{t}_r=1$. Put

$$n_0=\left[\frac{2}{\min_{0\le i\le r}(\mathfrak{t}_i-\mathfrak{t}_{i-1})}\right]+1.$$

Note that, $\mathfrak{t}_i+\frac{1}{n}<\mathfrak{t}_{i+1}-\frac{1}{n}$ for $n\ge n_0$. Now, for $n\ge n_0$, define $H_n:X\to X$ by

$$H_nx(\mathfrak{t})=\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}}G(\mathfrak{t},\mathfrak{s})g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s}$$

$$=-\frac{1}{\Gamma(\omega)}\sum_{i=0}^{r-1}\int_{[0,\mathfrak{t}]\cap[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]}(\mathfrak{t}-\mathfrak{s})^{\omega-1}g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s}$$

$$+\frac{\mathfrak{t}^k}{\Delta}\sum_{i=0}^{r-1}\int_{[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]}\Big(\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)}+\frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)}\Big)g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s}$$

$$+\frac{a\mathfrak{t}^k}{\Delta\Gamma(\omega)}\sum_{i=0}^{r-1}\int_{[0,\eta]\cap[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]}(\eta-\mathfrak{s})^{\omega-1}g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s}. \tag{6}$$

Theorem 2. *Assume that:*

(i) $p:[0,1]\to\mathbb{R}$ *is differentiable at* $\mathfrak{t}=1$ *and* $p(1)>a\eta^k>0$ *for* $k\in\{0,1,\ldots,n-1\}$.

(ii) *two non-decreasing maps* $M,N:\mathbb{R}^+\to\mathbb{R}^+$ *exist such that*

$$\lim_{z\to0^+}\frac{M(z)}{Q(z)}=q\in[0,\infty)\quad\text{and}\quad\lim_{z\to0^+}\frac{N(z)}{z}=\theta\in[0,\infty),$$

in which $Q:\mathbb{R}\to\mathbb{R}$ *is increasing and* $\lim_{z\to0^+}Q(z)=0$.

(iii) $\mathfrak{f}:\mathbb{R}\to\mathbb{R}$ *fulfills* $|\mathfrak{f}(x)-\mathfrak{f}(y)|\le M(|x-y|)$ *and* $|\mathfrak{f}(z)|\le N(z)$ *for all* $x,y,z\in\mathbb{R}$ *and also* $\|\tilde{g}\|=\int_0^1(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|d\mathfrak{s}<\infty$.

Then the singular fractional model of thermostat control (1) admits a solution if

$$\frac{\max\{\theta,q\}}{\Delta\Gamma(\omega)}(p'(1)+(\omega-1)p(1)+a)\sum_{i=0}^{r-1}\|\tilde{g}_i\|<1,$$

where $\Delta=p'(1)+kp(1)+a\eta^k$ *and* $\|\tilde{g}_i\|=\int_{\mathfrak{t}_i}^{\mathfrak{t}_{i+1}}(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|d\mathfrak{s}$.

Proof. At first, we check the continuity of H_n given by (6). Let $\epsilon>0$ be given. Since $Q(z)\to0$ as $z\to0^+$, $\delta_1>0$ exists so that $z\in(0,\delta_1]$ implies $Q(z)<\epsilon$. Since $\frac{M(z)}{Q(z)}$ tends to $q \in [0, \infty)$ as $z \to 0^+$, thus $\delta_2 > 0$ exists such that $z \in (0, \delta_2]$ yields $\dfrac{M(z)}{Q(z)} \le q + \epsilon$. Hence, $M(z) \le (q+\epsilon)Q(z)$ for all $z \in (0, \delta']$, where $\delta' \le \delta_2$. If

$$\delta := \min\{\epsilon, \delta_1, \delta_2\} \quad \text{and} \quad z := \|x - y\| < \epsilon,$$

then

$$M(\|x-y\|) \le (q+\epsilon)Q(\|x-y\|) < (q+\epsilon)\epsilon,$$

and so $\forall\, x, y \in X$ with $\|x - y\| < \delta$, $n \ge n_0$ and $t \in [0,1]$, we obtain

$$\begin{aligned}
|H_n x(\mathfrak{t}) - H_n y(\mathfrak{t})| &\le \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} G(\mathfrak{t}, \mathfrak{s}) |g(\mathfrak{s})| |\mathfrak{f}(x(\mathfrak{s})) - \mathfrak{f}(y(\mathfrak{s}))| d\mathfrak{s} \\
&\le \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} G(\mathfrak{t}, \mathfrak{s}) |g(\mathfrak{s})| M(|x(\mathfrak{s}) - y(\mathfrak{s})|) d\mathfrak{s} \\
&\le \frac{1}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} (p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2} \\
&+ a(\eta - \mathfrak{s})^{\omega-1})\mathfrak{s}^k |g(\mathfrak{s})| M(\|x-y\|) d\mathfrak{s} \\
&\le \frac{(q+\epsilon)\epsilon}{\Delta\Gamma(\omega)} \left[\sum_{i=0}^{r-1} \int_{t_i + \frac{1}{n}}^{t_{i+1} - \frac{1}{n}} p'(1)(1-\mathfrak{s})^{\omega-1} \mathfrak{s}^k |g(\mathfrak{s})| d\mathfrak{s} \right. \\
&\left. + \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2} \mathfrak{s}^k |g(\mathfrak{s})| d\mathfrak{s} + \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} a(\eta - \mathfrak{s})^{\omega-1} \mathfrak{s}^k |g(\mathfrak{s})| d\mathfrak{s} \right] \\
&\le \frac{(q+\epsilon)\epsilon}{\Delta\Gamma(\omega)} \left[\sum_{i=0}^{r-1} p'(1) \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} (1-\mathfrak{s})^{\omega-2} \mathfrak{s}^k |g(\mathfrak{s})| d\mathfrak{s} \right. \\
&\left. + (\omega-1)p(1) \int_{t_i + \frac{1}{n}}^{t_{i+1} - \frac{1}{n}} (1-\mathfrak{s})^{\omega-2} \mathfrak{s}^k |g(\mathfrak{s})| d\mathfrak{s} + a \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} (1-\mathfrak{s})^{\omega-2} \mathfrak{s}^k |g(\mathfrak{s})| d\mathfrak{s} \right] \\
&= \frac{(q+\epsilon)\epsilon}{\Delta\Gamma(\omega)} (p'(1) + (\omega-1)p(1) + a) \sum_{i=0}^{r-1} \|\tilde{g}_{i,n}\|.
\end{aligned}$$

Hence,

$$\|H_n x - H_n y\| \le \frac{(q+\epsilon)\epsilon}{\Delta\Gamma(\omega)} (p'(1) + (\omega-1)p(1) + a) \sum_{i=0}^{r-1} \|\tilde{g}_{i,n}\|,$$

and so $H_n x \to H_n y$ in X as $x \to y$. Thus, H_n is continuous for each $n \ge n_0$. On the other side, since $\lim_{z \to 0^+} \dfrac{M(z)}{H(z)} = q$, so $\delta_1(\epsilon) > 0$ exists such that $z \in (0, \delta_1'(\epsilon)]$ gives

$$M(z) < (q+\epsilon)Q(z) \tag{7}$$

for all $\delta_1'(\epsilon) \le \delta_1(\epsilon)$. Moreover, since

$$\frac{q}{\Delta\Gamma(\omega)} (p'(1) + (\omega-1)p(1) + a) \sum_{i=0}^{r-1} \|\tilde{g}_{i,n}\| < 1,$$

there exists $\epsilon_0 > 0$ such that

$$\frac{q+\epsilon_0}{\Delta\Gamma(\omega)}(p'(1)+(\omega-1)p(1)+a)\sum_{i=0}^{r-1}\|\tilde{g}_{i,n}\| < 1.$$

Since $\lim_{z\to 0^+}\dfrac{N(z)}{z} = \theta$, there is $\delta = \delta(\epsilon) > 0$ so that $N(z) \leq (\theta+\epsilon)z$ for all $0 < z \leq \delta$. As

$$\frac{\theta}{\Delta\Gamma(\omega)}(p'(1)+(\omega-1)p(1)+a)\sum_{i=0}^{r-1}\|\tilde{g}_{i,n}\| < 1,$$

so $\epsilon_1 > 0$ exists such that

$$\frac{\theta+\epsilon_1}{\Delta\Gamma(\omega)}(p'(1)+(\omega-1)p(1)+a)\sum_{i=0}^{r-1}\|\tilde{g}_{i,n}\| < 1.$$

Put $\delta_1 = \delta(\epsilon_1)$ and $\tilde{r} = \min\{\dfrac{\delta_0(\epsilon_0)}{2}, \delta_2(\epsilon_1)\}$. For $z = \tilde{r}$,

$$N(\tilde{r}) \leq (\theta+\epsilon_1)\tilde{r}.$$

Let $C = \{x \in X : \|x\| \leq \tilde{r}\}$. Define $\alpha : X^2 \to [0,\infty)$ by $\alpha(x,y) = 1$ if $x, y \in C$ and $\alpha(x,y) = 0$ otherwise. If $\alpha(x,y) \geq 1$, then

$$\|x\| \leq \tilde{r} \ \text{ and } \ \|y\| \leq \tilde{r}.$$

We verify in this case, that $\alpha(H_n x, H_n y) \geq 1$. To do this, for each $\mathfrak{t} \in [0,1]$ and $n \geq n_0$, we may write

$$\begin{aligned}
|H_n x(\mathfrak{t})| &\leq \sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} G(\mathfrak{t},\mathfrak{s})|g(\mathfrak{s})||\mathfrak{f}(x(\mathfrak{s}))|d\mathfrak{s} \\
&\leq \frac{1}{\Delta\Gamma(\omega)}\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} (p'(1)(1-\mathfrak{s})^{\omega-1}+(\omega-1)p(1)(1-\mathfrak{s})^{\omega-2}+a(\eta-\mathfrak{s})^{\omega-1}) \\
&\times \mathfrak{s}^k|g(\mathfrak{s})|N(\|x\|)d\mathfrak{s} \\
&\leq \frac{N(\tilde{r})}{\Delta\Gamma(\omega)}(p'(1)+(\omega-1)p(1)+a)\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}}(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|d\mathfrak{s} \\
&\leq \frac{(\theta+\epsilon_1)}{\Delta\Gamma(\omega)}(p'(1)+(\omega-1)p(1)+a)\sum_{i=0}^{r-1}\|\tilde{g}_{i,n}\| \leq \tilde{r}.
\end{aligned}$$

So $\|H_n x\| \leq \tilde{r}$. By the same reason $\|H_n y\| \leq \tilde{r}$, consequently $\alpha(H_n x, H_n y) \geq 1$. Further, if $x \in C$, then $H_n x \in C$ and since $C \neq \phi$, thus $x_0 \in C$ exists so that $\alpha(x_0, H_n x_0) \geq 1$. Now let $\alpha(x,y) = 1$. Then $\|x\| \leq \tilde{r}$ and $\|y\| \leq \tilde{r}$, $\|x-y\| \leq \|x\| + \|y\| \leq 2\tilde{r} \leq \delta_0(\epsilon_0)$ and so by using (7), we obtain

$$M(\|x-y\|) < (q+\epsilon_0)H(\|x-y\|). \tag{8}$$

Therefore, in this case, we have

$$|H_n x(\mathfrak{t}) - H_n y(\mathfrak{t})| \leq \sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} G(\mathfrak{t},\mathfrak{s})|g(\mathfrak{s})||\mathfrak{f}(x(\mathfrak{s}))-\mathfrak{f}(y(\mathfrak{s}))|d\mathfrak{s}$$

$$\leq \frac{1}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} (p'(1)(1-\mathfrak{s})^{\omega-1} + (\omega-1)p(1)(1-\mathfrak{s})^{\omega-2} + a(\eta-\mathfrak{s})^{\omega-1})$$

$$\times \mathfrak{s}^k |g(\mathfrak{s})| M(\|x-y\|) d\mathfrak{s}$$

$$\leq \frac{(q+\epsilon_0)Q(\|x-y\|)}{\Delta\Gamma(\omega)} (p'(1) + (\omega-1)p(1) + a) \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} (1-\mathfrak{s})^{\omega-2} \mathfrak{s}^k |g(\mathfrak{s})| d\mathfrak{s}$$

$$\leq \frac{(q+\epsilon_0)Q(\|x-y\|)}{\Delta\Gamma(\omega)} (p'(1) + (\omega-1)p(1) + a) \sum_{i=0}^{r-1} \|\tilde{g}_{i,n}\|,$$

and so

$$\|H_n x - H_n y\| \leq \lambda Q(\|x-y\|),$$

where $\lambda := \dfrac{(q+\epsilon_0)}{\Delta\Gamma(\omega)} (p'(1) + (\omega-1)p(1) + a) \sum_{i=0}^{r-1} \|\tilde{g}_{i,n}\|$. Further, define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(\mathfrak{t}) = \lambda Q(\mathfrak{t})$. Since Q is non-decreasing and $\lambda \in (0,1)$, so accordingly ψ is also non-decreasing and $\sum_{i=0}^{\infty} \psi^i(\mathfrak{t}) \leq Q^\infty(\mathfrak{t})$, where $Q^\infty(\mathfrak{t}) = \lim_{n\to\infty} Q^n(\mathfrak{t})$. Hence,

$$\alpha(x,y) d(H_n x, H_n y) \leq \psi(d(x,y)).$$

If $\alpha(x,y) = 0$, the last inequality is valid obviously and so H_n is α-ψ-contraction. Now by making use of Theorem 1, H_n admits a fixed-point in X for all $n \geq n_0$.

Now, choose $\{x_n\}_{n\geq n_0}$ so that $x_n(\mathfrak{t}) = H_n x_n(\mathfrak{t})$ and so

$$x_n(\mathfrak{t}) = \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} G(\mathfrak{t},\mathfrak{s}) g(\mathfrak{s}) \mathfrak{f}(x_n(\mathfrak{s})) d\mathfrak{s}$$

$$= \int_{[0,1] | \{\cup_{i=0}^{r} [\mathfrak{t}_i - \frac{1}{n}, \mathfrak{t}_i + \frac{1}{n}]\}} G(\mathfrak{t},\mathfrak{s}) g(\mathfrak{s}) \mathfrak{f}(x_n(\mathfrak{s})) d\mathfrak{s}.$$

G is continuous by terms of $\mathfrak{t}$ on $[0,1]$. Hence,

$$\lim_{\mathfrak{t}_k \to \mathfrak{t}} \frac{\partial x_n(\mathfrak{t}_k)}{\partial \mathfrak{t}} = \lim_{\mathfrak{t}_k \to \mathfrak{t}} \int_{[0,1] | \{\cup_{i=0}^{r} [\mathfrak{t}_i - \frac{1}{n}, \mathfrak{t}_i + \frac{1}{n}]\}} \frac{\partial G}{\partial \mathfrak{t}_k}(\mathfrak{t}_k,\mathfrak{s}) g(\mathfrak{s}) \mathfrak{f}(x_n(\mathfrak{s})) d\mathfrak{s}$$

$$= \int_{[0,1] | \{\cup_{i=0}^{r} [\mathfrak{t}_i - \frac{1}{n}, \mathfrak{t}_i + \frac{1}{n}]\}} \frac{\partial G}{\partial \mathfrak{t}}(\mathfrak{t},\mathfrak{s}) g(\mathfrak{s}) \mathfrak{f}(x_n(\mathfrak{s})) d\mathfrak{s}$$

for all $\mathfrak{t} \in [0,1]$. Thus, $\{x'_n\}_{n\geq n_0}$ is equi-continuous and $\{x_n\}_{n\geq n_0}$ admits the relative compactness on X. The Arzela–Ascoli theorem implies the existence of $x_0 \in X$ so that $\lim_{n\to\infty} x_n = x_0$. For $\mathfrak{t} \in [0,1]$, put

$$u_n(\mathfrak{t},\mathfrak{s}) = \chi_{\{\cup_{i=0}^{r-1} [\mathfrak{t}_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]\}}(\mathfrak{s}) G(\mathfrak{t},\mathfrak{s}) g(\mathfrak{s}) \mathfrak{f}(x_n(\mathfrak{s})),$$

where $\chi_E(\mathfrak{s}) = 1$ when $\mathfrak{s} \in E$ and $\chi_E(\mathfrak{s}) = 0$ when $\mathfrak{s} \notin E$. Since $x_n \to x_0$, so $\exists\, n_1 \in \mathbb{N}$ such that $\|x_n - x_0\| < \epsilon$, $\forall\, n \geq n_1$. Let $n \geq n_1 \geq n_0$. By using (7), we have

$$|u_n(\mathfrak{t},\mathfrak{s})| = |\chi_{\{\cup_{i=0}^{r-1} [\mathfrak{t}_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]\}}(\mathfrak{s}) G(\mathfrak{t},\mathfrak{s}) g(\mathfrak{s}) \mathfrak{f}(x_n(\mathfrak{s}))|$$

$$\leq \chi_{\{\cup_{i=0}^{r-1} [\mathfrak{t}_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]\}}(\mathfrak{s}) G(\mathfrak{t},\mathfrak{s}) |g(\mathfrak{s})| |\mathfrak{f}(x_n(\mathfrak{s})) - \mathfrak{f}(x_0(\mathfrak{s}))|$$

$$+\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})G(\mathfrak{t},\mathfrak{s})|g(\mathfrak{s})||\mathfrak{f}(x_0(\mathfrak{s}))|$$

$$\leq \frac{(p'(1)+(\omega-1)p(1)+a)}{\Delta\Gamma(\omega)}\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|M(\|x_n-x_0\|)$$

$$+\frac{(p'(1)+(\omega-1)p(1)+a)}{\Delta\Gamma(\omega)}\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|N(\|x_0\|)$$

$$\leq \frac{(p'(1)+(\omega-1)p(1)+a)(q+\epsilon)Q(\|x_n-x_0\|)}{\Delta\Gamma(\omega)}\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|$$

$$+\frac{(p'(1)+(\omega-1)p(1)+a)N(\|x_0\|)}{\Delta\Gamma(\omega)}\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|$$

$$\leq \frac{(p'(1)+(\omega-1)p(1)+a)(q+\epsilon)Q(\epsilon)}{\Delta\Gamma(\omega)}\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|$$

$$+\frac{(p'(1)+(\omega-1)p(1)+a)N(\|x_0\|)}{\Delta\Gamma(\omega)}\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k|g(\mathfrak{s})|.$$

Since $(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k g(\mathfrak{s})\in L^1[0,1]$, we obtain

$$\frac{(p'(1)+(\omega-1)p(1)+a)(q+\epsilon)Q(\epsilon)}{\Delta\Gamma(\omega)}\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k g(\mathfrak{s})\in L^1[0,1]$$

and

$$\frac{(p'(1)+(\omega-1)p(1)+a)N(\|x_0\|)}{\Delta\Gamma(\omega)}\chi_{\{\bigcup_{i=0}^{r-1}[\mathfrak{t}_i+\frac{1}{n},\mathfrak{t}_{i+1}-\frac{1}{n}]\}}(\mathfrak{s})(1-\mathfrak{s})^{\omega-2}\mathfrak{s}^k g(\mathfrak{s})\in L^1[0,1].$$

Hence $u_n(\mathfrak{t},.)\in L^1[0,1]$ for any $n\geq n_1$. The Lebesgue dominated theorem yields

$$x_0(\mathfrak{t})=\lim_{n\to\infty}x_n(\mathfrak{t})=\lim_{n\to\infty}\int_0^1\chi_{[0,1]|\{\bigcup_{i=0}^{r}[\mathfrak{t}_i-\frac{1}{n},\mathfrak{t}_i+\frac{1}{n}]\}}(\mathfrak{s})G(\mathfrak{t},\mathfrak{s})g(\mathfrak{s})\mathfrak{f}(x_n(\mathfrak{s}))d\mathfrak{s}$$

$$=\int_0^1 G(\mathfrak{t},\mathfrak{s})g(\mathfrak{s})\mathfrak{f}(x_0(\mathfrak{s}))d\mathfrak{s}$$

for all $\mathfrak{t}\in[0,1]$. Since $\dfrac{M(z)}{Q(z)}\to q$, so some $\delta>0$ exists provided that $0<z\leq\delta$ implies $M(z)<(q+\epsilon)Q(z)$. Since $Q(z)\to 0^+$, $\exists\,\delta'>0$ such that

$$0<z\leq\delta' \quad\Rightarrow\quad Q(z)<\epsilon.$$

On the other side, $x_n\to x_0$ gives the existence of some $n_2\in\mathbb{N}$ such that $\|x_n-x_0\|<\min\{\delta,\delta'\}$ for all $n\geq n_2$. Hence, we have

$$|\mathfrak{f}(x_n)-\mathfrak{f}(x_0)|\leq M(\|x_n-x_0\|)\leq(q+\epsilon)Q(z)<(q+\epsilon)\epsilon,$$

for all $n\geq n_2$. Thus, $\mathfrak{f}(x_n)\to\mathfrak{f}(x_0)$ as $x\to x_0$ and so H admits a fixed-point x_0, which will be a solution for the fractional strongly singular thermostat control BVP (1) and this ends the proof. □

4. Hybrid Version

To follow our study on the strongly singular models, we here consider the hybrid version of the fractional strongly singular thermostat control problem having the form

$$ {}^{c}D^{\omega}(g(\mathfrak{t})x(\mathfrak{t})) + g(\mathfrak{t})\mathfrak{f}(x(\mathfrak{t})) = 0, \tag{9} $$

with BCs

$$ \lim_{t\to 0^{+}} (g(\mathfrak{t})x(\mathfrak{t}))^{(j)}(\mathfrak{t}) = 0, \quad \forall\, j \in \{0,1,\ldots,n-1\} \quad \text{with } j \neq k, \tag{10} $$

and

$$ \lim_{\mathfrak{t}\to 1^{-}} \big(p(\mathfrak{t})g(\mathfrak{t})x(\mathfrak{t})\big)' + ag(\eta)x(\eta) = 0, \tag{11} $$

where $\eta \in (0,1)$, $a > 0$, $(\omega - k - 1)p(1) > a\eta^{k}$, $k \leq \omega - 2$, $g : [0,1] \to \mathbb{R}$ is singular at some points of $[0,1]$, $p : [0,1] \to [0,\infty)$ is differentiable in $\mathfrak{t} = 1$ and ${}^{c}D^{\omega}$ displays the Caputo derivative with given order ω.

Consider g as a singular function which may admit the strong singularity in the set $E = \{\mathfrak{t}_i\}_{i=0}^{r}$ with $\frac{1}{g} \in C[0,1]$ and $\frac{1}{g}(\mathfrak{s}) \neq 0$, for all $\mathfrak{s} \in [0,1]\backslash E$ and $\|\frac{1}{g}\| > 0$. As an example for such a function $g : [0,1] \to \mathbb{R}$, one can define $g(\mathfrak{t}) = \frac{1}{\mathfrak{t}^2}$. Then g involves the strong singularity in $\mathfrak{t} = 0$ and $\frac{1}{g}(\mathfrak{t}) \neq 0$ for all $\mathfrak{t} \neq 0$ and $\|\frac{1}{g}\| = 1$.

By applying a similar proof given in the Lemma 1, one can immediately conclude that x is a solution for the fractional hybrid strongly singular thermostat control problem (9)–(11) if and only if

$$ x(\mathfrak{t}) = \frac{1}{g(\mathfrak{t})} \int_0^1 G(\mathfrak{t},\mathfrak{s})g(\mathfrak{s})\mathfrak{f}(x(\mathfrak{s}))d\mathfrak{s}, \tag{12} $$

where $G(\mathfrak{t},\mathfrak{s})$ is given by (3). Before proceeding to prove the main theorem, we define a new space $\mathcal{Y}_g$ by

$$ \mathcal{Y}_g = \{x \in C[0,1] : x(\mathfrak{t}) = \frac{a_x(\mathfrak{t})}{g(\mathfrak{t})} \; for\ some\ a_x \in C[0,1]\}. $$

It is obvious that $\mathcal{Y}_g \neq \emptyset$. If $y \in \mathcal{Y}_g$, then

$$ \|\hat{a_y}\|_1 = \|(1-\mathfrak{t})^{\omega-2}a_y(\mathfrak{t})\|_1 = \int_0^1 (1-\mathfrak{s})^{\omega-2}|a_y(\mathfrak{s})|d\mathfrak{s} \leq \int_0^1 \|\hat{a}_y\|ds = \|\hat{a}_y\| $$

and

$$ \|\hat{a_y}\| = \sup_{\mathfrak{t}\in[0,1]} |(1-\mathfrak{t})^{\omega-2}a_y(\mathfrak{t})| \leq \sup_{\mathfrak{t}\in[0,1]} |a_y(\mathfrak{t})| = \|a_x\|. $$

Now, regard the space $\mathcal{Y}_g$ with the norm $\|\cdot\|_*$, where $\|x\|_* = \|\frac{1}{g}\|\|a_x\|$ for $x \in \mathcal{Y}_g$.

Lemma 2. *The space $\mathcal{Y}_g$ is Banach with the norm $\|\cdot\|_*$ defined above.*

Proof. Let $\{y_n\}$ be a Cauchy sequence contained in $\mathcal{Y}_g$. Then, for every $\epsilon > 0$, select some $n^* \in \mathbb{N}$ so that $\forall\, n, m \geq n^*$, we have $\|y_n - y_m\|_* < \epsilon$. Now, by definition of the space $\mathcal{Y}_g$, for $j = n, m$, take $\{a_{y_j}\}$ in $C[0,1]$ such that

$$ y_n(\mathfrak{t}) = \frac{a_{y_n}(\mathfrak{t})}{g(\mathfrak{t})}, \quad \text{and} \quad y_m(\mathfrak{t}) = \frac{a_{y_m}(\mathfrak{t})}{g(\mathfrak{t})} $$

for all $\mathfrak{t} \in [0,1]$ and $\|\frac{1}{g}\| \|a_{y_n} - a_{y_m}\| < \epsilon$. Thus

$$\|\hat{a}_{y_n} - \hat{a}_{y_m}\| < \frac{1}{\|\frac{1}{g}\|}\epsilon$$

for all $n, m \geq n^*$ and so $\{a_{y_n}\}$ is a Cauchy sequence contained in $X = C[0,1]$. We select $a_0 \in C[0,1]$ subject to $a_{x_n} \to a_0$. If $y_0(\mathfrak{t}) = \frac{a_0(\mathfrak{t})}{g(\mathfrak{t})}$, then $y_n(\mathfrak{t}) = \frac{a_n(\mathfrak{t})}{g(\mathfrak{t})} \to y_0(\mathfrak{t})$ with $y_0 \in \mathcal{Y}_g$. This means that $\mathcal{Y}_g$ is a Banach space. $\square$

To prove the next theorem, we define $H^* : \mathcal{Y}_g \to \mathcal{Y}_g$ by

$$H^* x(\mathfrak{t}) = \frac{1}{g(\mathfrak{t})} \int_0^1 G(\mathfrak{t}, \mathfrak{s}) g(\mathfrak{s}) f(x(\mathfrak{s})) d\mathfrak{s}, \tag{13}$$

where $G(\mathfrak{t}, \mathfrak{s})$ is given by (3). Note that in fact, we have $H^* x(\mathfrak{t}) = \frac{Hx(\mathfrak{t})}{g(\mathfrak{t})}$. One can check, by (12), that x_0 is a fixed-point of H^* iff x_0 is a solution for the fractional hybrid strongly singular thermostat control problem (9)–(11).

Theorem 3. *Assume that:*

(i) *a non-decreasing map* $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *exists with* $\lim_{z \to 0^+} \frac{\Lambda(z)}{z} = \lambda \in [0, \infty)$ *so that for* $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$*, we have*

$$|\mathfrak{f}(x) - \mathfrak{f}(y)| \leq \Lambda(|x - y|)$$

and $\lim_{z \to 0^+} \frac{\mathfrak{f}(z)}{z} = \theta \in [0, \infty)$, $\forall\, x, y, z \in \mathbb{R}$;

(ii) $\dfrac{(p'(1) + (\omega - 1)p(1) + a)(\max\{\theta, \lambda\})}{\Delta\Gamma(\omega)(k+1)} < 1.$

Then fractional hybrid strongly singular thermostat control problem (9)–(11) *admits a solution.*

Proof. Let $n \geq n_0$. We verify that H_n^* is continuous in the space $\mathcal{Y}_g$. For $x, y \in \mathcal{Y}_g$ and $\mathfrak{t} \in [0,1]$, we have

$$\begin{aligned}
|H_n x(\mathfrak{t}) - H_n y(\mathfrak{t})| &\leq \int_{[0,1] | \bigcup_{i=0}^{r} [\mathfrak{t}_i - \frac{1}{n}, \mathfrak{t}_i + \frac{1}{n}]} G(\mathfrak{t}, \mathfrak{s}) |g(\mathfrak{s})| |\mathfrak{f}(x(\mathfrak{s})) - \mathfrak{f}(y(\mathfrak{s}))| d\mathfrak{s} \\
&\leq \frac{1}{\Gamma(\omega)} \int_{[0,\mathfrak{t}] \cap \{\bigcup_{i=0}^{r-1} [\mathfrak{t}_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]\}} (\mathfrak{t} - \mathfrak{s})^{\omega - 1} |g(\mathfrak{s})| |\mathfrak{f}(x(\mathfrak{s})) - \mathfrak{f}(y(\mathfrak{s}))| d\mathfrak{s} \\
&+ \frac{\mathfrak{t}^k}{\Delta} \int_{\bigcup_{i=0}^{r-1} [\mathfrak{t}_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]} \Big(\frac{p'(1)(1 - \mathfrak{s})^{\omega - 1}}{\Gamma(\omega)} + \frac{p(1)(1 - \mathfrak{s})^{\omega - 2}}{\Gamma(\omega - 1)}\Big) |g(\mathfrak{s})| |\mathfrak{f}(x(\mathfrak{s})) - \mathfrak{f}(y(\mathfrak{s}))| d\mathfrak{s} \\
&+ \frac{a\mathfrak{t}^k}{\Delta\Gamma(\omega)} \int_{[0,\eta] \cap \{\bigcup_{i=0}^{r-1} [\mathfrak{t}_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]\}} (\eta - \mathfrak{s})^{\omega - 1} |g(\mathfrak{s})| |\mathfrak{f}(x(\mathfrak{s})) - \mathfrak{f}(y(\mathfrak{s}))| d\mathfrak{s} \\
&\leq \frac{1}{\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{[0,\mathfrak{t}] \cap [\mathfrak{t}_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]} (\mathfrak{t} - \mathfrak{s})^{\omega - 1} |g(\mathfrak{s})| \Lambda(|x(\mathfrak{s}) - y(\mathfrak{s})|) d\mathfrak{s} \\
&+ \frac{\mathfrak{t}^k}{\Delta} \sum_{i=0}^{r-1} \int_{[\mathfrak{t}_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]} \Big(\frac{p'(1)(1 - \mathfrak{s})^{\omega - 1}}{\Gamma(\omega)} + \frac{p(1)(1 - \mathfrak{s})^{\omega - 2}}{\Gamma(\omega - 1)}\Big) |g(\mathfrak{s})| \Lambda(|x(\mathfrak{s}) - y(\mathfrak{s})|) d\mathfrak{s} \\
&+ \frac{a\mathfrak{t}^k}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{[0,\eta] \cap [t_i + \frac{1}{n}, \mathfrak{t}_{i+1} - \frac{1}{n}]} (\eta - \mathfrak{s})^{\omega - 1} |g(\mathfrak{s})| \Lambda(|x(\mathfrak{s}) - y(\mathfrak{s})|) d\mathfrak{s}.
\end{aligned}$$

Let $\epsilon > 0$ be given. Since $\lim_{z\to 0^+} \dfrac{\Lambda(z)}{z} = \lambda \in [0,\infty)$, then some $\delta_0 > 0$ exists so that $\dfrac{\Lambda(z)}{z} < \lambda+\epsilon$ for all $z \in (0, \delta_0'(\epsilon)]$ with $\delta_0' \le \delta_0$. Put $\delta_0^*(\epsilon) = \min\{\delta_0(\epsilon), \epsilon\}$. For any $z \in (0, \delta_0^*(\epsilon)]$, we have $\Lambda(z) < (\lambda+\epsilon)z$. Let $x, y \in \mathcal{Y}_g$ be such that $\|x-y\| < \delta_0^*$. Select $a_x, a_y \in C[0,1]$ such that

$$x(\mathfrak{t}) = \frac{a_x(\mathfrak{t})}{g(\mathfrak{t})} \quad \text{and} \quad y(\mathfrak{t}) = \frac{a_y(\mathfrak{t})}{g(\mathfrak{t})},$$

for all $\mathfrak{t} \in [0,1]$. Thus,

$$\|x-y\| = \sup_{\mathfrak{t}\in[0,1]} \left| \frac{a_x(\mathfrak{t}) - a_y(\mathfrak{t})}{g(\mathfrak{t})} \right| \le \sup_{\mathfrak{t}\in[0,1]} |\frac{1}{g(\mathfrak{t})}| \sup_{\mathfrak{t}\in[0,1]} |a_x(\mathfrak{t}) - a_y(\mathfrak{t})| = \|x-y\|_* < \delta_0^*.$$

Hence

$$|x(\mathfrak{t}) - y(\mathfrak{t})| = \frac{|a_x(\mathfrak{t}) - a_y(\mathfrak{t})|}{|g(\mathfrak{t})|} < \delta_0^*$$

for all $\mathfrak{t} \in [0,1]$. Put $z(\mathfrak{t}) := |x(\mathfrak{t}) - y(\mathfrak{t})| < \delta_0^*$. For each $\mathfrak{t} \in [0,1]$, we have

$$\Lambda(|x(\mathfrak{t}) - y(\mathfrak{t})|) < (\lambda+\epsilon)|x(\mathfrak{t}) - y(\mathfrak{t})| = \frac{(\lambda+\epsilon)}{|g(\mathfrak{t})|}|a_x(\mathfrak{t}) - a_y(\mathfrak{t})|. \tag{14}$$

If $\|x-y\| < \delta^*$, then

$$\begin{aligned}
|H_n x(\mathfrak{t}) - H_n y(\mathfrak{t})| \le{}& \frac{1}{\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{[0,\mathfrak{t}]\cap[\mathfrak{t}_i+\frac{1}{n}, \mathfrak{t}_{i+1}-\frac{1}{n}]} (\mathfrak{t}-\mathfrak{s})^{\omega-1} |g(\mathfrak{s})| \frac{(\lambda+\epsilon)}{|g(\mathfrak{s})|} |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s} \\
&+ \frac{\mathfrak{t}^k}{\Delta} \sum_{i=0}^{r-1} \int_{[\mathfrak{t}_i+\frac{1}{n}, \mathfrak{t}_{i+1}-\frac{1}{n}]} (\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)} + \frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)}) |g(\mathfrak{s})| \frac{(\lambda+\epsilon)}{|g(\mathfrak{s})|} |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s} \\
&+ \frac{a\mathfrak{t}^k}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{[0,\eta]\cap[\mathfrak{t}_i+\frac{1}{n}, \mathfrak{t}_{i+1}-\frac{1}{n}]} (\eta-\mathfrak{s})^{\omega-1} |g(\mathfrak{s})| \frac{(\lambda+\epsilon)}{|g(\mathfrak{s})|} |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s} \\
\le{}& \frac{(\lambda+\epsilon)}{\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{[\mathfrak{t}_i+\frac{1}{n}, \mathfrak{t}_{i+1}-\frac{1}{n}]} (\mathfrak{t}-\mathfrak{s})^{\omega-1} |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s} \\
&+ \frac{(\lambda+\epsilon)\mathfrak{t}^k}{\Delta} \sum_{i=0}^{r-1} \int_{[\mathfrak{t}_i+\frac{1}{n}, \mathfrak{t}_{i+1}-\frac{1}{n}]} (\frac{p'(1)(1-\mathfrak{s})^{\omega-1}}{\Gamma(\omega)} + \frac{p(1)(1-\mathfrak{s})^{\omega-2}}{\Gamma(\omega-1)}) |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s} \\
&+ \frac{a(\lambda+\epsilon)\mathfrak{t}^k}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{[\mathfrak{t}_i+\frac{1}{n}, \mathfrak{t}_{i+1}-\frac{1}{n}]} (\eta-\mathfrak{s})^{\omega-1} |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s} \\
\le{}& \frac{(\lambda+\epsilon)}{\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} (1-\mathfrak{s})^{\omega-2} |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s} \\
&+ \frac{(\lambda+\epsilon)\mathfrak{t}^k}{\Delta} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} (\frac{p'(1) + (\omega-1)p(1)}{\Gamma(\omega)}) (1-\mathfrak{s})^{\omega-2} |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s} \\
&+ \frac{a(\lambda+\epsilon)\mathfrak{t}^k}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} (1-\mathfrak{s})^{\omega-2} |a_x(\mathfrak{s}) - a_y(\mathfrak{s})| d\mathfrak{s}
\end{aligned}$$

$$= \frac{(\lambda+\epsilon)}{\Gamma(\omega)}\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}}|\hat{a}_x(\mathfrak{s})-\hat{a}_y(\mathfrak{s})|d\mathfrak{s}$$

$$+\frac{(\lambda+\epsilon)(p'(1)+(\omega-1)p(1))\mathfrak{t}^k}{\Delta\Gamma(\omega)}\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}}|\hat{a}_x(\mathfrak{s})-\hat{a}_y(\mathfrak{s})|d\mathfrak{s}$$

$$+\frac{a(\lambda+\epsilon)\mathfrak{t}^k}{\Delta\Gamma(\omega)}\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}]}|\hat{a}_x(\mathfrak{s})-\hat{a}_y(\mathfrak{s})|d\mathfrak{s}$$

$$\leq \frac{(\lambda+\epsilon)}{\Gamma(\omega)}\|\hat{a}_x-\hat{a}_y\|\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}}d\mathfrak{s}$$

$$+\frac{(\lambda+\epsilon)(p'(1)+(\alpha-1)p(1))\mathfrak{t}^k}{\Delta\Gamma(\omega)}\|\hat{a}_x-\hat{a}_y\|\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}}d\mathfrak{s}$$

$$+\frac{a(\lambda+\epsilon)\mathfrak{t}^k}{\Delta\Gamma(\omega)}\|\hat{a}_x-\hat{a}_y\|\sum_{i=0}^{r-1}\int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}}d\mathfrak{s}.$$

As we checked, $\|x\|_* < \epsilon$ implies $\|\hat{a}_x\| < \|a_x\| < \frac{\epsilon}{\|\frac{1}{g}\|}$. If $\|x-y\|_* < \delta^* \leq \epsilon$, then

$$|H_nx(\mathfrak{t})-H_ny(\mathfrak{t})| \leq \frac{(\lambda+\epsilon)}{\Gamma(\omega)}\frac{\epsilon}{\|\frac{1}{g}\|}\sum_{i=0}^{r-1}(\mathfrak{t}_{i+1}-\mathfrak{t}_i)-\frac{2}{n}$$

$$+\frac{(\lambda+\epsilon)(p'(1)+(\omega-1)p(1))\mathfrak{t}^k}{\Delta\Gamma(\omega)}\frac{\epsilon}{\|\frac{1}{g}\|}\sum_{i=0}^{r-1}(\mathfrak{t}_{i+1}-\mathfrak{t}_i)-\frac{2}{n}$$

$$+\frac{a(\lambda+\epsilon)t^k}{\Delta\Gamma(\omega)}\frac{\epsilon}{\|\frac{1}{g}\|}\sum_{i=0}^{r-1}(\mathfrak{t}_{i+1}-\mathfrak{t}_i)-\frac{2}{n}$$

for all $\mathfrak{t}\in[0,1]$. So

$$\|H_nx-H_ny\| \leq (\Delta+p'(1)+(\omega-1)p(1)+a)\frac{(1-\frac{2}{n})(\omega+\epsilon)}{\|\frac{1}{g}\|\Delta\Gamma(\alpha)}\epsilon.$$

Thus $\|x-y\|_* < \delta^* \leq \epsilon$ implies

$$\|H_n^*x-H_n^*y\|_* = \|\frac{1}{g}\|\|H_nx-H_ny\| \leq (\Delta+p'(1)+(\omega-1)p(1)+a)\frac{(1-\frac{2}{n})(\lambda+\epsilon)}{\Delta\Gamma(\omega)}\epsilon.$$

This says that $x\to y$ in $\mathcal{Y}_g$ implies $H_n^*x\to H_n^*y$ and so H_n^* is continuous on the space $\mathcal{Y}_g$. Since $\lim_{z\to 0^+}\frac{|\mathfrak{f}(z)|}{z}=\theta\in[0,\infty)$, thus for any $\epsilon>0$, some $\delta_1(\epsilon)>0$ exists such that $|\mathfrak{f}(z)|<(\theta+\epsilon)z$ for all $\delta_1'(\epsilon)\leq\delta_1(\epsilon)$ and $z\in(0,\delta_1'(\epsilon)]$. Since by hypothesis (ii),

$$\frac{(p'(1)+(\omega-1)p(1)+a)\max\{\theta,\delta\}}{\Delta\Gamma(\omega)(k+1)}<1,$$

thus $\epsilon_0>0$ exists such that

$$\frac{(p'(1)+(\omega-1)p(1)+a)(\max\{\theta,\delta\}+\epsilon_0)}{\Delta\Gamma(\omega)(k+1)}<1.$$

Put $\tilde{r} := \min\{\frac{\delta^*(\epsilon_0)}{2}, \delta_1(\epsilon_0)\}$ and let $x \in \mathcal{Y}_g$. If $\|x\|_* \leq \tilde{r}$, then for all $\mathfrak{t} \in [0,1]$,

$$|x(\mathfrak{t})| = |\frac{a_x(\mathfrak{t})}{g(\mathfrak{t})}| \leq \sup |\frac{a_x(\mathfrak{t})}{g(\mathfrak{t})}| \leq \|x\|_* \leq \tilde{r}$$

and so $|\mathfrak{f}(x(\mathfrak{t}))| < (\theta + \epsilon_0)|x(\mathfrak{t})|$ for all $\mathfrak{t} \in [0,1]$. Consider $C = \{x \in \mathcal{Y}_g : \|x\|_* \leq \tilde{r}\}$. Define $\alpha : \mathcal{Y}_g^2 \to [0, \infty)$ by $\alpha(x,y) = 1$ when $x, y \in C$ and $\alpha(x,y) = 0$ otherwise. If $\alpha(x,y) \geq 1$, then

$$\|x\|_* \leq \tilde{r} \quad \text{and} \quad \|y\|_* \leq \tilde{r}.$$

Hence, we obtain

$$\begin{aligned}
|H_n x(\mathfrak{t})| &\leq \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} G(\mathfrak{t}, \mathfrak{s}) |g(\mathfrak{s})| |\mathfrak{f}(x(\mathfrak{s}))| d\mathfrak{s} \\
&\leq \frac{1}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} (p'(1) + (\omega - 1)p(1) + a)(1 - \mathfrak{s})^{\omega - 2} \mathfrak{s}^k |g(\mathfrak{s})| |\mathfrak{f}(x(\mathfrak{s}))| d\mathfrak{s} \\
&\leq \frac{(p'(1) + (\omega - 1)p(1) + a)}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} (1 - \mathfrak{s})^{\omega - 2} \mathfrak{s}^k |g(\mathfrak{s})| (\theta + \epsilon_0) |x(\mathfrak{s})| d\mathfrak{s} \\
&= \frac{(p'(1) + (\omega - 1)p(1) + a)(\theta + \epsilon_0)}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} (1 - \mathfrak{s})^{\omega - 2} \mathfrak{s}^k |g(\mathfrak{s})| \frac{a_x(\mathfrak{s})}{|g(\mathfrak{s})|} d\mathfrak{s}
\end{aligned}$$

and so

$$\begin{aligned}
|H_n x(\mathfrak{t})| &\leq \frac{(p'(1) + (\omega - 1)p(1) + a)(\theta + \epsilon_0)}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} \mathfrak{s}^k \|\hat{a}_x\| d\mathfrak{s} \\
&= \frac{(p'(1) + (\omega - 1)p(1) + a)(\theta + \epsilon_0)}{\Delta\Gamma(\omega)(k+1)} \|\hat{a}_x\| \sum_{i=0}^{r-1} \left((\mathfrak{t}_{i+1} - \tfrac{1}{n})^{k+1} - (\mathfrak{t}_i + \tfrac{1}{n})^{k+1} \right) \\
&\leq \frac{(p'(1) + (\omega - 1)p(1) + a)(\theta + \epsilon_0)}{\Delta\Gamma(\omega)(k+1)} \frac{\tilde{r}}{\|\frac{1}{g}\|} \leq \frac{\tilde{r}}{\|\frac{1}{g}\|}.
\end{aligned}$$

Hence, $\|H_n x\| \leq \frac{\tilde{r}}{\|\frac{1}{g}\|}$ and so $\|H_n^* x\|_* \leq \|\frac{1}{g}\| \|H_n x\| \leq \tilde{r}$. In view of the similar reasons, we obtain $\|H_n y\|_* \leq \tilde{r}$ and so $\alpha(H_n^* x, H_n^* y) \geq 1$. It is simple to observe that $x_0 \in C$ (and so $C \neq \emptyset$) and $\alpha(x_0, H_n^* x_0) \geq 1$. If $x, y \in C$, then $\alpha(x,y) = 1$. For every $\mathfrak{t} \in [0,1]$ and $n \geq n_0$, we have

$$\begin{aligned}
|H_n x(\mathfrak{t}) - H_n y(\mathfrak{t})| &\leq \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} G(\mathfrak{t}, \mathfrak{s}) |g(\mathfrak{s})| |\mathfrak{f}(x(\mathfrak{s})) - \mathfrak{f}(y(\mathfrak{s}))| d\mathfrak{s} \\
&\leq \frac{1}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i + \frac{1}{n}}^{\mathfrak{t}_{i+1} - \frac{1}{n}} (p'(1) + (\omega - 1)p(1) + a)(1 - \mathfrak{s})^{\omega - 2} \mathfrak{s}^k |g(\mathfrak{s})| \Lambda(|x(\mathfrak{s}) - y(\mathfrak{s})|) d\mathfrak{s}.
\end{aligned}$$

Since $x, y \in C$, $\|x - y\|_* \leq \|x\|_* + \|y\|_* \leq 2\tilde{r} \leq 2\frac{\delta^*}{2} = \delta^*$ and so by using (14), we find that

$$|H_n x(\mathfrak{t}) - H_n y(\mathfrak{t})| \le \frac{(p'(1) + (\omega-1)p(1) + a)}{\Delta\Gamma(\omega)} \sum_{i=0}^{r-1} \int_{\mathfrak{t}_i+\frac{1}{n}}^{\mathfrak{t}_{i+1}-\frac{1}{n}} \mathfrak{s}^k |g(\mathfrak{s})| \frac{\lambda+\epsilon_0}{|g(\mathfrak{s})|} |\hat{a}_x(\mathfrak{s}) - \hat{a}_y(\mathfrak{s})| d\mathfrak{s}$$

$$\le \frac{(p'(1) + (\omega-1)p(1) + a)(\lambda+\epsilon_0)}{\Delta\Gamma(\omega)} \|\hat{a}_x - \hat{a}_y\| \sum_{i=0}^{r-1} \int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} \mathfrak{s}^k d\mathfrak{s}$$

$$\le \frac{(p'(1) + (\omega-1)p(1) + a)(\lambda+\epsilon_0)}{\Delta\Gamma(\omega)(k+1)} \|\hat{a}_x - \hat{a}_y\|.$$

Thus, $|H_n x(\mathfrak{t}) - H_n y(\mathfrak{t})| \le \frac{(p'(1) + (\omega-1)p(1) + a)(\lambda+\epsilon_0)}{\Delta\Gamma(\omega)(k+1)} \frac{\|x-y\|_*}{\|\frac{1}{g}\|}$ and so

$$\|H_n x - H_n y\| \le \frac{(p'(1) + (\omega-1)p(1) + a)(\lambda+\epsilon_0)}{\Delta\Gamma(\omega)(k+1)} \frac{\|x-y\|_*}{\|\frac{1}{g}\|}$$

and

$$\|H_n^* x - H_n^* y\|_* \le \frac{(p'(1) + (\omega-1)p(1) + a)(\lambda+\epsilon_0)}{\Delta\Gamma(\omega)(k+1)} \frac{\|x-y\|_*}{\|\frac{1}{g}\|} \|\frac{1}{g}\|$$

$$= \frac{(p'(1) + (\omega-1)p(1) + a)(\lambda+\epsilon_0)}{\Delta\Gamma(\omega)(k+1)} \|x-y\|_*.$$

By setting

$$\gamma := \frac{(p'(1) + (\omega-1)p(1) + a)(\lambda+\epsilon_0)}{\Delta\Gamma(\omega)(k+1)},$$

we obtain $\alpha(x,y)d(H_n^* x, H_n^* y) \le \psi(d(x,y))$, in which $\psi(\mathfrak{t}) = \gamma\mathfrak{t}$. Further,

$$\sum_{i=0}^{\infty} \psi^i(\mathfrak{t}) = \frac{\gamma}{1-\gamma}\mathfrak{t}$$

for all $\mathfrak{t} \in [0,1]$. On the other side, the inequality $\alpha(x,y)d(H_n^* x, H_n^* y) \le \psi(d(x,y))$ is clearly valid whenever $\alpha(x,y) = 0$. Now, the conclusion of Theorem 1 gives the existence of a fixed point for H_n in $\mathcal{Y}_g$ for all $n \ge n_0$. Ultimately, by implementing a similar procedure in Theorem 2, we can find that H^* admits a fixed-point x_0, which is a solution for the fractional hybrid strongly singular thermostat control problem (9)–(11). □

5. Examples

In this situation, we examine our obtained results by presenting two examples.

Example 1. *Based on the given fractional thermostat model* (1)*, consider the following strongly singular thermostat control equation*

$$^{c}D^{\frac{5}{2}} x(\mathfrak{t}) + \frac{1}{2\mathfrak{t}(1-\mathfrak{t})} x(\mathfrak{t}) = 0 \tag{15}$$

with BCs

$$x'(0) = 0, \quad (\mathfrak{t}x(\mathfrak{t}))'|_{\mathfrak{t}=1} + x(\frac{1}{2}) = 0. \tag{16}$$

In view of the above model, we have data $\omega = \frac{5}{2}$, $k = 1$, $p(\mathfrak{t}) = \mathfrak{t}$, $a = 1$, $\eta = \frac{1}{2}$ *and*

$$g(\mathfrak{t}) = \frac{1}{2\mathfrak{t}(1-\mathfrak{t})} \quad \text{and} \quad \mathfrak{f}(x(\mathfrak{t})) = x(\mathfrak{t}).$$

Clearly, $\Delta = p'(1) + kp(1) + a\eta^k = \frac{5}{2}$ *and* $p(1) = 1 > \frac{1}{2} = a\eta^k > 0$*, where* Δ *is introduced in Theorem 2. On the other side, by considering the hypotheses of Theorem 2, by setting* $Q(z) = M(z) = N(z) = |z|$*, we have*

$$\lim_{z\to 0^+} \frac{|z|}{|z|} = 1 \in [0,\infty) \quad \text{and} \quad \lim_{z\to 0^+} \frac{|z|}{z} = 1 \in [0,\infty),$$

where $\theta = q = 1$ *and* $\lim_{z\to 0^+} Q(z) = 0$*. Further, we have*

$$|\mathfrak{f}(x) - \mathfrak{f}(y)| \le |x-y| = M(|x-y|), \quad |\mathfrak{f}(z)| \le |z| = N(z), \ \forall x,y,z \in \mathbb{R}.$$

If we assume $r = 1$*, then* $\|\tilde{g}\| = \|\tilde{g}_i\| = \int_0^1 (1-\mathfrak{s})^{2.5-2}\mathfrak{s}^k |g(\mathfrak{s})| d\mathfrak{s} = \int_0^1 \frac{1}{2\sqrt{1-\mathfrak{s}}} d\mathfrak{s} = \frac{1}{3}$ *and*

$$\frac{\max\{\theta,q\}}{\Delta\Gamma(\omega)}(p'(1) + (\omega-1)p(1) + a)\sum_{i=0}^{r-1}\|\tilde{g}_i\| = \frac{1}{\frac{5}{2}\Gamma(\frac{5}{2})}(1 + \frac{3}{2} + 1) \times \frac{1}{3} < 1.$$

At last, by Theorem 2, a solution is found to the fractional strongly singular model of thermostat control (15) *and* (16)*.*

Example 2. *Based on the given fractional hybrid thermostat model* (9)*, consider the following hybrid strongly singular thermostat control equation*

$$^cD^{\frac{5}{2}}(\frac{x(\mathfrak{t})}{\mathfrak{t}^2}) + \frac{1}{2\mathfrak{t}^2}x(\mathfrak{t}) = 0 \tag{17}$$

with boundary conditions

$$\lim_{\mathfrak{t}\to 0^+}(\frac{x(\mathfrak{t})}{\mathfrak{t}^2}) = 0, \quad \lim_{\mathfrak{t}\to 1^+}(\frac{x(\mathfrak{t})}{\mathfrak{t}})' + 4x(\frac{1}{2}) = 0, \tag{18}$$

where $\omega = \frac{5}{2}$, $k = 0$, $a = 1$, $\eta = \frac{1}{2}$, $p(\mathfrak{t}) = \mathfrak{t}$, $g(\mathfrak{t}) = \frac{1}{\mathfrak{t}^2}$, $\mathfrak{f}(x(\mathfrak{t})) = \frac{x(\mathfrak{t})}{2}$ *and we obtain* $\Delta = p'(1) + kp(1) + a\eta^k = 2$*. By assuming* $\Lambda(z) = \frac{1}{2}|z|$ *as a non-decreasing function, we have*

$$\lim_{z\to 0^+} \frac{\Lambda(z)}{z} = \frac{1}{2} \in [0,\infty), \quad \lim_{z\to 0^+} \frac{\mathfrak{f}(z)}{z} = \frac{1}{2} \in [0,\infty),$$

where $\theta = \lambda = \frac{1}{2}$*. On the other side, we have*

$$|\mathfrak{f}(x) - \mathfrak{f}(y)| \le \frac{1}{2}|x-y| = \Lambda(|x-y|), \ \forall x,y \in \mathbb{R}.$$

Take $r = 1$*, so*

$$\frac{(p'(1) + (\omega-1)p(1) + a)(\max\{\theta,\lambda\})}{\Gamma(\omega)(k+1)} = \frac{\frac{7}{2}}{4\Gamma(\frac{5}{2})} < 1.$$

Ultimately, by Theorem 3, the fractional hybrid strongly singular thermostat control problem (17) *and* (18) *involves a solution.*

6. Conclusions

This work is devoted to studying the existence of solutions for two different strongly singular versions of the thermostat control problem for the first time. In this way, we provided new techniques involving α-ψ-contractive operators, which are considered as the main novelty of the present study. For the hybrid version, we built a Banach space based on a function having strong singularity and proved the relevant results for the mentioned hybrid model of thermostat control. Ultimately, we proposed two illustrated examples for obtained results. This research work clarifies that we are able to investigate some qualitative aspects of more complicated strongly singular FBVPs describing real-world models and this encourages us to study other singular dynamical systems arising in different phenomena in nature and engineering. For future works, we can use these techniques for singular Langevin equations or singular pantograph systems modeled by different fractional operators having singular or non-singular kernels.

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Abbreviations

The following abbreviations are used in this manuscript:

FDE	Fractional Differential Equation
FBVP	Fractional Boundary Value Problem

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