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Asymptotics of the Sum of a Sine Series with a Convex Slowly Varying Sequence of Coefficients

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Abstract: We study the asymptotic behavior in a neighborhood of zero of the sum of a sine series $g(\boldsymbol{b}, x) = \sum_{k=1}^{\infty} b_k \sin kx$ whose coefficients constitute a convex slowly varying sequence $\boldsymbol{b}$. The main term of the asymptotics of the sum of such a series was obtained by Aljančić, Bojanić, and Tomić. To estimate the deviation of $g(\boldsymbol{b}, x)$ from the main term of its asymptotics $b_{m(x)}/x$, $m(x) = [\pi/x]$, Telyakovskiĭ used the piecewise-continuous function $\sigma(\boldsymbol{b}, x) = x \sum_{k=1}^{m(x)-1} k^2 (b_k - b_{k+1})$. He showed that the difference $g(\boldsymbol{b}, x) - b_{m(x)}/x$ in some neighborhood of zero admits a two-sided estimate in terms of the function $\sigma(\boldsymbol{b}, x)$ with absolute constants independent of $\boldsymbol{b}$. Earlier, the author found the sharp values of these constants. In the present paper, the asymptotics of the function $g(\boldsymbol{b}, x)$ on the class of convex slowly varying sequences in the regular case is obtained.

Keywords: sine series with monotone coefficients; convex sequence; slowly varying function

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1. Introduction

In this paper, we refine the asymptotics of the sum of a sine series with convex slowly varying coefficients, obtained by Aljančić, Bojanić, and Tomić [1] and strengthened by Telyakovskiĭ [2,3].

The result of Aljančić, Bojanić, and Tomić was generalized for more extensive classes of trigonometric series. In [4,5], results reducing the asymptotic behavior of trigonometric series with general monotone coefficients to ones with monotone coefficients were obtained. The asymptotics of trigonometric series with quasimonotone coefficients was studied in [6–9]. In the context of the indicated problem, in [10,11], quasimonotonicity and extensions of regularly varying and slowly varying coefficients were considered. Note that it is possible to improve the asymptotics for the last case using the results of the present paper. Along with the asymptotics of a trigonometric series with a slowly varying sequence of coefficients, it is interesting to study the asymptotics of the Fourier transform and the Laplace transform of a slowly varying function. The latter finds applications in the theory of entire functions (see, for example, [12]).

Consider a nonincreasing null sequence of positive numbers $\boldsymbol{b} = \{b_k\}_{k=1}^{\infty}$ and the function:

$$g(\boldsymbol{b}, x) = \sum_{k=1}^{\infty} b_k \sin kx.$$

The series $\sum_{k=1}^{\infty} b_k \sin kx$ is well known to converge everywhere. Its sum $g(\boldsymbol{b}, x)$ is continuous on $(0, 2\pi)$. We shall be concerned with the behavior of the function $g(\boldsymbol{b}, x)$ in a right-hand neighborhood of zero. The most principal is the situation when a sequence of the coefficients $\boldsymbol{b}$ is slowly varying. In this case, the sequence of partial sums of the sine series converges very slowly and its sum $g(\boldsymbol{b}, x)$ may not even be Lebesgue integrable.

Definition 1. *A sequence* $\{\beta_k\}_{k=1}^{\infty}$ *is called slowly varying (see [13,14]) if:*

$$\lim_{k\to\infty} \frac{\beta_{[\delta k]}}{\beta_k} = 1 \tag{1}$$

for any $\delta > 0$.

Throughout this paper, we use $[x]$ to denote the integer part of a number x.

The main term of the asymptotics of the function $g(\boldsymbol{b}, x)$ for the case in which a sequence of coefficients is slowly varying under the additional convexity condition ($b_k - 2b_{k+1} + b_{k+2} \geqslant 0$, $\forall k \in \mathbb{N}$) was obtained by Aljančić, Bojanić, and Tomić [1].

Theorem 1 ([1], (see Ch. V, §2, Theorem 2.17 in [15]))**.** *Let* $\boldsymbol{b}$ *be a convex slowly varying null sequence. Then, the following asymptotic formula holds:*

$$g(\boldsymbol{b}, x) \sim \frac{b_{m(x)}}{x}, \quad x \to +0. \tag{2}$$

Here and in the sequel, $m(x) = [\pi/x]$, $0 < x \leqslant \pi$, and the notation $f(x) \sim g(x)$, $x \to +0$, means that $\lim_{x\to+0} f(x)/g(x) = 1$.

Telyakovskiĭ [2,3] strengthened the result of Aljančić, Bojanić, and Tomić. He showed that it is convenient to compare the difference between the sum of a sine series and the main term of its asymptotics with the function:

$$\sigma(\boldsymbol{b}, x) = x \sum_{k=1}^{m(x)-1} \frac{k(k+1)}{2} \Delta b_k. \tag{3}$$

As usual, we define $\Delta b_k = b_k - b_{k+1}$. Telyakovskiĭ proved the following result.

Theorem 2 ([2,16])**.** *There are positive absolute constants* C_1 *and* C_2 *such that:*

$$C_1\, \sigma(\boldsymbol{b}, x) \leqslant g(\boldsymbol{b}, x) - \frac{b_{m(x)}}{x} \leqslant C_2\, \sigma(\boldsymbol{b}, x), \qquad x \in \left(0, \frac{\pi}{11}\right],$$

for any convex null sequence $\boldsymbol{b}$.

If a sequence $\boldsymbol{b}$ is slowly varying, then the order relation:

$$\Delta b_k = o\left(\frac{b_k}{k}\right), \quad k \to \infty,$$

holds (see Ch. II, §2, Theorem 2.4 in [14]), from which, in view of (3), the asymptotic formula is immediate (see Lemma 1):

$$\sigma(\boldsymbol{b}, x) = o\left(x \sum_{k=1}^{m(x)-1} k b_k\right) = o\left(\frac{b_{m(x)}}{x}\right).$$

Thus, Theorem 2 is an enhancement of Theorem 1 and gives the remainder term in the asymptotic (2).

In [17], the sharp values of the constants in Theorem 2 were found. Namely, the following results were obtained.

Theorem 3 ([17])**.** *For any convex null sequence* $\boldsymbol{b}$,

$$g(\boldsymbol{b}, x) < \frac{b_{m(x)}}{x} + \sigma(\boldsymbol{b}, x), \qquad x \in \left(0, \frac{\pi}{11}\right]. \tag{4}$$

*There exists a convex slowly varying null sequence **b** and a sequence of points* $\{x_l\}_{l=1}^{\infty}$, $x_l \to +0$, *such that:*

$$\lim_{l\to\infty} \frac{g(\boldsymbol{b}, x_l) - b_{m(x_l)}/x_l}{\sigma(\boldsymbol{b}, x_l)} = 1. \tag{5}$$

Theorem 4 ([17])**.** *For any convex null sequence **b**,*

$$g(\boldsymbol{b}, x) > \frac{b_{m(x)}}{x} + \frac{6(\pi-1)}{\pi^3}\sigma(\boldsymbol{b}, x) - \frac{\Delta b_{m(x)}}{\pi} - b_{m(x)}\left(\frac{1}{2}\cot\frac{x}{2} - \frac{1}{x}\right), \qquad x \in \left(0, \frac{\pi}{2}\right]. \tag{6}$$

*There exists a convex slowly varying null sequence **b** and a sequence of points* $\{x_l\}_{l=1}^{\infty}$, $x_l \to +0$, *such that:*

$$\lim_{n\to\infty} \frac{g(\boldsymbol{b}, x_l) - b_{m(x_l)}/x_l}{\sigma(\boldsymbol{b}, x_l)} = \frac{6(\pi-1)}{\pi^3}. \tag{7}$$

We remark that the last two terms in (6) are negative and do not exceed $O\left(xb_{m(x)}\right) = o(x)$, $x \to +0$.

Thus, for the class of all convex sequences $\boldsymbol{b}$, the following extreme problems are solved:

$$\max_{\boldsymbol{b}} \overline{\lim_{x\to+0}} \frac{g(\boldsymbol{b}, x) - b_{m(x)}/x}{\sigma(\boldsymbol{b}, x)} = 1, \tag{8}$$

$$\min_{\boldsymbol{b}} \underline{\lim_{x\to+0}} \frac{g(\boldsymbol{b}, x) - b_{m(x)}/x}{\sigma(\boldsymbol{b}, x)} = \frac{6(\pi-1)}{\pi^3}. \tag{9}$$

Note that in the examples verifying the accuracy of the limit relations (5) and (7), the sequence $\boldsymbol{b}$ is slowly varying. Thus, the validity of the relations (8) and (9) is preserved if the maximum and minimum are taken on the class of all convex slowly varying sequences $\boldsymbol{b}$.

In [18,19], the sharp two-sided estimate of the sum of a sine series with convex coefficients was obtained. In the examples verifying the accuracy of this estimate, the sequence of the coefficients is slowly varying. In the present paper, we supplement the two-sided estimate of the sum of a sine series with a convex sequence of coefficients (see (4) and (6)) with an asymptotic relation that refines Theorem 1 for the case in which the sequence $\{k\Delta b_k\}_{k=1}^{\infty}$ is slowly varying. Note (see Lemma 3) that the latter condition is slightly stronger than the condition that the sequence $\boldsymbol{b}$ is slowly varying, under which the relation (2) is true.

2. Preliminaries

In this section, we prove the necessary auxiliary results.

Definition 2. *A sequence* $\{\alpha_k\}_{k=1}^{\infty}$ *is called regularly varying with parameter p (see [13,14]) if:*

$$\lim_{k\to\infty} \frac{\alpha_{[\delta k]}}{\alpha_k} = \delta^p$$

for any $\delta > 0$.

Lemma 1. *Let **b** be a positive nonincreasing slowly varying sequence. If a positive sequence* $\{a_n\}_{n=1}^{\infty}$ *is such that the sequence* $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ *is regularly varying with parameter* $p > 0$, *then the following limit ratio holds:*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k b_k}{b_n \sum_{k=1}^{n} a_k} = 1.$$

Proof. It is sufficient to show that:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k(1 - b_k/b_n)}{\sum_{k=1}^{n} a_k} = \lim_{n\to\infty} \frac{b_n \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k b_k}{b_n \sum_{k=1}^{n} a_k} = 0. \tag{10}$$

For each $\varepsilon > 0$, we select $\delta > 0$ so that:

$$\delta^p < \frac{\varepsilon}{4}. \tag{11}$$

The sequence $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ is regularly varying, so we have:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{[\delta n]} a_k}{\sum_{k=1}^{n} a_k} = \delta^p. \tag{12}$$

In view of (11) and (12), there is N_1 such that, for all $n \geqslant N_1$,

$$\frac{\sum_{k=1}^{[\delta n]} a_k}{\sum_{k=1}^{n} a_k} < \frac{\varepsilon}{2}. \tag{13}$$

Since the sequence $\boldsymbol{b}$ is slowly varying, there is N_2 such that, for all $n > N_2$,

$$1 - \frac{b_{[\delta n]}}{b_n} < \frac{\varepsilon}{2}. \tag{14}$$

Using the monotonicity of the sequence $\boldsymbol{b}$ for all $n > \max\{N_1, N_2\}$ from (14) and (13), we have:

$$0 \leqslant \frac{\sum_{k=1}^{n} a_k(1 - b_k/b_n)}{\sum_{k=1}^{n} a_k} = \frac{\sum_{k=1}^{[\delta n]} a_k(1 - b_k/b_n)}{\sum_{k=1}^{n} a_k} + \frac{\sum_{k=[\delta n]+1}^{n} a_k(1 - b_k/b_n)}{\sum_{k=1}^{n} a_k}$$
$$< \frac{\sum_{k=1}^{[\delta n]} a_k}{\sum_{k=1}^{n} a_k} + \frac{\sum_{k=[\delta n]+1}^{n} a_k\left(1 - b_{[\delta n]}/b_n\right)}{\sum_{k=1}^{n} a_k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the limit relation (10), and therefore the lemma. □

Definition 3 ([13,14])**.** *A function $b(t)$ is called slowly varying if:*

$$\lim_{t\to\infty} \frac{b(At)}{b(t)} = 1 \tag{15}$$

for any $A > 0$.

Lemma 2. *Let $b(t)$ be a positive decreasing slowly varying function. Then, for any $A > 0$,*

$$\lim_{t\to\infty} \frac{\int_t^{At} b(u)/u\,du}{b(t)} = \ln A.$$

Proof. By the monotonicity and positivity of the function $b(t)$, we have:

$$b(At)\ln A \leqslant \int_t^{At} \frac{b(u)}{u}\,du \leqslant b(t)\ln A. \tag{16}$$

Now, the required limit relation follows from (15) and (16). □

Lemma 3. *Let $\boldsymbol{b}$ be a positive nonincreasing sequence. If the sequence $\{k\Delta b_k\}_{k=1}^{\infty}$ is slowly varying, then the sequence $\boldsymbol{b}$ is also slowly varying.*

Proof. The sequence $\boldsymbol{b}$ is monotone, and so, it is sufficient to show (see [14]) that:

$$\lim_{k\to\infty} \frac{b_{2k}}{b_k} = 1. \tag{17}$$

Since the sequence $k\Delta b_k$ is slowly varying, we have:

$$\lim_{k\to\infty} \frac{\Delta b_{2k}}{\Delta b_k} = \frac{1}{2} \lim_{k\to\infty} \frac{2k\,\Delta b_{2k}}{k\,\Delta b_k} = \frac{1}{2}, \tag{18}$$

$$\lim_{k\to\infty} \frac{\Delta b_{2k+2}}{\Delta b_k} = \frac{1}{2} \lim_{k\to\infty} \frac{(2k+2)\,\Delta b_{2k+2}}{k\,\Delta b_k} \cdot \frac{k}{k+1} = \frac{1}{2}. \tag{19}$$

In view of (18) and (19) and since the sequence $\boldsymbol{b}$ is monotone,

$$\lim_{k\to\infty} \frac{\Delta b_{2k+1}}{\Delta b_k} = \frac{1}{2}. \tag{20}$$

From (18) and (20), we have:

$$\lim_{k\to\infty} \frac{b_{2k} - b_{2k+2}}{b_k - b_{k+1}} = \lim_{k\to\infty} \frac{\Delta b_{2k} + \Delta b_{2k+1}}{\Delta b_k} = 1.$$

Now, (17) follows from this limit relation in view of Stolz's theorem. □

Lemma 4. *Let $b(t)$ be a positive decreasing differentiable function. If the function $-tb'(t)$ varies slowly, then the function $b(t)$ also varies slowly.*

Proof. To prove the lemma, it is sufficient to apply L'Hôpital's rule for an arbitrary $A > 0$:

$$\lim_{t\to\infty} \frac{b(At)}{b(t)} = \lim_{t\to\infty} \frac{Ab'(At)}{b'(t)} = \lim_{t\to\infty} \frac{-At\,b'(At)}{-t\,b'(t)} = 1.$$

□

Lemma 5. *Let $\psi(t)$ be a function continuous on $[0,\pi]$, and let $\{\beta_k\}_{k=0}^{\infty}$ be a positive nonincreasing slowly varying sequence. Then:*

$$\lim_{x\to+0} \frac{x}{\beta_{m(x)}} \sum_{k=0}^{m(x)-1} \psi\big((k+1/2)x\big)\beta_k = \int_0^{\pi} \psi(t)\,dt. \tag{21}$$

Proof. Let us first prove the limit relation (21) in the case when $\psi(t)$ is a piecewise constant function. Since both sides of the relation (21) depend linearly on $\psi(t)$, it suffices to verify (21) for the functions $\chi_c(t)$ of the form:

$$\chi_c(t) = \begin{cases} 1, & t \in [0,c], \\ 0, & t \in (c,\pi], \end{cases}$$

where c is some fixed number from the interval $(0,\pi)$. For such functions, the limit relation (21) has the form:

$$\lim_{x\to+0} \frac{x}{\beta_{m(x)}} \sum_{k=0}^{[c/x-1/2]} \beta_k = c. \tag{22}$$

The sequence $\{\beta_k\}_{k=0}^{\infty}$ is a nonincreasing slowly varying sequence; hence:

$$\lim_{x\to+0} \frac{\beta_{m(x)}}{\beta_{[c/x-1/2]}} = 1 \tag{23}$$

(see Ch. I, §1.8, Lemma 1.15 in [14]). Now, the limit relation (22) follows from (23) and Lemma 1 with $a_n = 1$, $b_n = \beta_n$. This proves the limit relation (21) for piecewise constant functions $\psi(t)$.

Now, let $\psi(t)$ be a continuous function on $[0, \pi]$. For any $\varepsilon > 0$, there is a piecewise constant function $\varphi(t)$ with the property:

$$\big|\psi(t) - \varphi(t)\big| \leqslant \frac{\varepsilon}{4\pi} \tag{24}$$

for all $t \in [0, \pi]$. As already proven, there exists $\delta_1 > 0$ such that for all x, $0 < x < \delta_1$,

$$\left|\frac{x}{\beta_m(x)} \sum_{k=0}^{m(x)-1} \varphi\big((k+1/2)x\big)\beta_k - \int_0^\pi \varphi(t)dt\right| < \frac{\varepsilon}{4}. \tag{25}$$

Again, taking into account Lemma 1, where $a_n = 1$, $b_n = \beta_n$, one can find $\delta_2 > 0$ such that for all x, $0 < x < \delta_2$,

$$\frac{x}{\beta_m(x)} \sum_{k=0}^{m(x)-1} \beta_k < 2\pi. \tag{26}$$

We set $\delta = \min\{\delta_1, \delta_2\}$. For all x, $0 < x < \delta$, from the inequalities (24)–(26), we have:

$$\left|\frac{x}{\beta_m(x)} \sum_{k=0}^{m(x)-1} \psi\big((k+1/2)x\big)\beta_k - \int_0^\pi \psi(t)dt\right| \leqslant \left|\frac{x}{\beta_m(x)} \sum_{k=0}^{m(x)-1} \Big(\psi\big((k+1/2)x\big) - \varphi\big((k+1/2)x\big)\Big)\beta_k\right|$$
$$+ \left|\frac{x}{\beta_m(x)} \sum_{k=0}^{m(x)-1} \varphi\big((k+1/2)x\big)\beta_k - \int_0^\pi \varphi(t)dt\right| + \left|\int_0^\pi \varphi(t)dt - \int_0^\pi \psi(t)dt\right| < \varepsilon.$$

□

Lemma 6. *Let $\{\beta_k\}_{k=1}^\infty$ be a positive slowly varying sequence. Furthermore, let the sequence $\{\beta_k/k\}_{k=1}^\infty$ be nonincreasing. Then:*

$$\lim_{x\to+0} \frac{1}{\beta_{m(x)}} \sum_{k=m(x)}^{\infty} \beta_k \frac{\cos(k+1/2)x}{k} = \int_\pi^{+\infty} \frac{\cos t}{t}\, dt.$$

Proof. For any $\varepsilon > 0$, we take the number N so that:

$$N > \frac{2(1+\varepsilon)}{\varepsilon}. \tag{27}$$

Then:

$$\left|\int_{\pi N}^{+\infty} \frac{\cos t}{t}\, dt\right| < \varepsilon. \tag{28}$$

For this N, we take η_1 so that, for all x, $0 < x < \eta_1$, the following inequalities are satisfied:

$$\left|\sum_{k=m(x)}^{Nm(x)-1} \frac{\cos(k+1/2)x}{k} - \int_\pi^{\pi N} \frac{\cos t}{t}\, dt\right| < \varepsilon, \tag{29}$$

$$\left|\sum_{k=m(x)}^{Nm(x)-1} \frac{|\cos(k+1/2)x|}{k} - \int_\pi^{\pi N} \frac{|\cos t|}{t}\, dt\right| < \varepsilon. \tag{30}$$

Since the sequence $\{\beta_k\}_{k=1}^{\infty}$ varies slowly, the limit relation $\lim_{n\to\infty} \beta_{[\lambda n]}/\beta_n = 1$ holds uniformly over $\lambda \in [1, N]$ (see Ch. I, § 1.2, Theorem 1.1 in [14]). Therefore, for some η_2 for all x, $0 < x < \eta_2$,

$$\max_{m(x)\leqslant k\leqslant Nm(x)} \left| \frac{\beta_k}{\beta_{m(x)}} - 1 \right| < \frac{\varepsilon}{\left(\int_{\pi}^{\pi N} |\cos t|/t\, dt + \varepsilon \right)}. \tag{31}$$

We set $\eta = \min\{\eta_1, \eta_2\}$. For any x, $0 < x < \eta$, we split the required sum into three sums:

$$\begin{aligned} \frac{1}{\beta_{m(x)}} \sum_{k=m(x)}^{\infty} \beta_k \frac{\cos(k+1/2)x}{k} &= \sum_{k=m(x)}^{Nm(x)-1} \frac{\cos(k+1/2)x}{k} \\ &+ \sum_{k=m(x)}^{Nm(x)-1} \left(1 - \frac{\beta_k}{\beta_{m(x)}}\right) \frac{\cos(k+1/2)x}{k} + \frac{1}{\beta_{m(x)}} \sum_{k=Nm(x)}^{\infty} \beta_k \frac{\cos(k+1/2)x}{k}. \end{aligned} \tag{32}$$

We estimate each term in (32). In view of (28) and (29), for the first term, we have:

$$\begin{aligned} &\left| \sum_{k=m(x)}^{Nm(x)-1} \frac{\cos(k+1/2)x}{k} - \int_{\pi}^{\infty} \frac{\cos t}{t}\, dt \right| \\ &\leqslant \left| \sum_{k=m(x)}^{Nm(x)-1} \frac{\cos(k+1/2)x}{k} - \int_{\pi}^{\pi N} \frac{\cos t}{t}\, dt \right| + \left| \int_{\pi N}^{\infty} \frac{\cos t}{t}\, dt \right| < 2\varepsilon. \end{aligned} \tag{33}$$

To estimate the second term on the right of (32), we use the inequalities (31) and (30) to obtain:

$$\begin{aligned} \left| \sum_{k=m(x)}^{Nm(x)-1} \left(1 - \frac{\beta_k}{\beta_{m(x)}}\right) \frac{\cos(k+1/2)x}{k} \right| &\leqslant \max_{m(x)\leqslant k\leqslant Nm(x)} \left| 1 - \frac{\beta_k}{\beta_{m(x)}} \right| \sum_{k=m(x)}^{Nm(x)-1} \frac{|\cos(k+1/2)x|}{k} \\ &< \frac{\varepsilon}{\int_{\pi}^{\pi N} |\cos t|/t\, dt + \varepsilon} \sum_{k=n}^{N} \frac{|\cos(k+1/2)x|}{k} < \frac{\varepsilon}{\int_{\pi}^{\pi N} |\cos t|/t\, dt + \varepsilon} \left(\int_{\pi}^{\pi N} \frac{|\cos t|}{t}\, dt + \varepsilon \right) = \varepsilon. \end{aligned} \tag{34}$$

For the third term, we use the standard estimation of the remainder of a trigonometric series with monotone coefficients, as well as the inequalities (31) and (27) and the inequality $\sin x > (2/\pi)x$, $x \in (0, \pi/2)$,

$$\frac{1}{\beta_{m(x)}} \left| \sum_{k=Nm(x)}^{\infty} \beta_k \frac{\cos(k+1/2)x}{k} \right| \leqslant \frac{\beta_{Nm(x)}}{\beta_m(x)} \cdot \frac{1}{Nm(x)\sin(x/2)} < \frac{\beta_{Nm(x)}}{\beta_{m(x)}} \cdot \frac{2}{N} < \frac{2(1+\varepsilon)}{N} < \varepsilon. \tag{35}$$

Combining (32)–(35), we arrive at the following estimate:

$$\left| \frac{1}{\beta_{m(x)}} \sum_{k=m(x)}^{\infty} \beta_k \frac{\cos(k+1/2)x}{k} - \int_{\pi}^{+\infty} \frac{\cos t}{t}\, dt \right| < 4\varepsilon.$$

The proof of the lemma is complete. □

3. Asymptotics of the Sum of a Sine Series in the Regular Case

In this section, we investigate the case when the sequence $\boldsymbol{b}$ is not only convex and slowly varying, but more or less regular. More precisely, we require that the sequence $\{k\Delta b_k\}_{k=1}^{\infty}$ be slowly varying. By Lemma 3, this condition implies that the sequence $\boldsymbol{b}$ is also slowly varying. It turns out that, with this additional requirement, for the sum of a sine series, the first two terms of the asymptotic expansion can be written down.

Theorem 5. *Let $\boldsymbol{b}$ be a non-negative convex null sequence. If the sequence $\{k\Delta b_k\}_{k=1}^{\infty}$ varies slowly, then:*

$$g(\boldsymbol{b},x)-\frac{b_{m(x)}}{x}\sim(\gamma+\ln\pi)\frac{m(x)\Delta b_{m(x)}}{x},\quad x\to+0. \tag{36}$$

Here and in the sequel, the Euler constant is denoted by γ.

Proof. Denote by:

$$\widetilde{D}_k(x)=\sum_{n=1}^{k}\sin nx=\frac{\cos(x/2)-\cos(k+1/2)x}{2\sin(x/2)} \tag{37}$$

the conjugate Dirichlet kernel. Applying the Abel transform to the sum of a sine series $g(\boldsymbol{b},x)$ and taking into account (37), we obtain:

$$\begin{aligned}g(\boldsymbol{b},x)&=\sum_{k=1}^{\infty}\Delta b_k\widetilde{D}_k(x)=\sum_{k=1}^{m(x)-1}\Delta b_k\widetilde{D}_k(x)+\sum_{k=m(x)}^{\infty}\Delta b_k\widetilde{D}_k(x)=\sum_{k=1}^{m(x)-1}\Delta b_k\widetilde{D}_k(x)+\frac{b_{m(x)}}{2}\cot\frac{x}{2}\\&-\frac{1}{2\sin(x/2)}\sum_{k=m(x)}^{\infty}\Delta b_k\cos(k+1/2)x=\frac{b_{m(x)}}{2}\cot\frac{x}{2}+\sum_{k=1}^{m(x)-1}\Delta b_k\frac{1-\cos(k+1/2)x}{2\sin(x/2)}\\&-\frac{1}{2\sin(x/2)}\sum_{k=m(x)}^{\infty}\Delta b_k\cos(k+1/2)x+\frac{1}{2}\tan\frac{x}{4}(b_1-b_{m(x)}).\end{aligned} \tag{38}$$

Applying Lemma 5 to the function $\psi(t)=(1-\cos t)/t$ and the sequence $\beta_k=k\Delta b_k$, we obtain:

$$\lim_{x\to+0}\frac{1}{m(x)\Delta b_{m(x)}}\sum_{k=1}^{m(x)-1}k\Delta b_k\frac{1-\cos(k+1/2)x}{k+1/2}=\int_0^{\pi}\frac{1-\cos t}{t}\,dt. \tag{39}$$

On the other hand,

$$\sum_{k=1}^{m(x)-1}k\Delta b_k\frac{1-\cos(k+1/2)x}{k+1/2}=\sum_{k=1}^{m(x)-1}\Delta b_k\big(1-\cos(k+1/2)x\big)-\frac{1}{2}\sum_{k=1}^{m(x)-1}\Delta b_k\frac{1-\cos(k+1/2)x}{k+1/2}. \tag{40}$$

Let us show that the second term on the right of (40) divided by $m(x)\Delta b_{m(x)}$ tends to zero. Indeed, since $1-\cos x<x^2/2$, $x>0$, we have:

$$\frac{1}{m(x)\Delta b_{m(x)}}\sum_{k=1}^{m(x)-1}\Delta b_k\frac{1-\cos(k+1/2)x}{k+1/2}\leqslant\frac{x^2}{2m(x)\Delta b_{m(x)}}\sum_{k=1}^{m(x)-1}(k+1/2)\Delta b_k.$$

At the same time, according to Lemma 1, we have:

$$\lim_{x\to+0}\frac{x^2}{m(x)\Delta b_{m(x)}}\sum_{k=1}^{m(x)-1}(k+1/2)\Delta b_k=\lim_{x\to+0}\frac{x^2\big(m^2(x)-1\big)\Delta b_{m(x)}}{m^3(x)\Delta b_{m(x)}}=0.$$

From (39) and (40), the equality follows:

$$\lim_{x\to+0}\frac{1}{m(x)\Delta b_{m(x)}}\sum_{k=1}^{m(x)-1}\Delta b_k\big(1-\cos(k+1/2)x\big)=\int_0^{\pi}\frac{1-\cos t}{t}\,dt. \tag{41}$$

Now, we apply Lemma 6 to the sequence $\beta_k = k\Delta b_k$. Note that the sequence $\beta_k/k = \Delta b_k$ is not increasing, since the sequence $\boldsymbol{b}$ is convex. We have:

$$\lim_{x\to+0} \frac{1}{m(x)\Delta b_{m(x)}} \sum_{k=m(x)}^{\infty} \Delta b_k \cos(k+1/2)x = \int_{\pi}^{+\infty} \frac{\cos t}{t}\, dt. \tag{42}$$

Finally, combining (38), (41), and (42), we conclude that:

$$\begin{aligned}
&\lim_{x\to+0} \frac{x}{m(x)\Delta b_{m(x)}} \left(g(\boldsymbol{b},x) - \frac{b_{m(x)}}{2} \cot\frac{x}{2} \right) = \frac{1}{2} \lim_{x\to+0} \frac{x\tan(x/4)}{m(x)\Delta b_{m(x)}} \left(b_1 - b_{m(x)} \right) \\
&\quad + \lim_{x\to+0} \frac{x}{2\sin(x/2)} \cdot \frac{1}{m(x)\Delta b_{m(x)}} \sum_{k=1}^{m(x)-1} \Delta b_k (1 - \cos(k+1/2)x) \\
&\quad + \lim_{x\to+0} \frac{x}{2\sin(x/2)} \cdot \frac{1}{m(x)\Delta b_{m(x)}} \sum_{k=m(x)}^{\infty} \Delta b_k \cos(k+1/2)x \\
&\quad = 0 + \int_0^{\pi} \frac{1-\cos t}{t}\, dt - \int_{\pi}^{+\infty} \frac{\cos t}{t}\, dt = \gamma + \ln\pi.
\end{aligned}$$

The proof of the theorem is complete. □

This theorem can be reformulated in terms of a comparison with the function $\sigma(\boldsymbol{b},x)$, which allows us to compare the asymptotics obtained above with the results of Theorems 3 and 4.

Corollary 1. *Let $\boldsymbol{b}$ be a non-negative convex null sequence. If the sequence $\{k\Delta b_k\}_{k=1}^{\infty}$ varies slowly, then:*

$$g(\boldsymbol{b},x) - \frac{b_{m(x)}}{x} \sim \frac{4}{\pi^2}(\gamma + \ln\pi)\sigma(\boldsymbol{b},x), \quad x \to +0.$$

Proof. We apply Lemma 1 with $a_k = k(k+1)/2$ (see also (3)). The following ordinal relations hold:

$$\frac{4}{\pi^2}\sigma(\boldsymbol{b},x) = \frac{4}{\pi^2} x \sum_{k=1}^{m(x)-1} \frac{k(k+1)}{2} \Delta b_k \sim \frac{4}{\pi^2} x m(x)\Delta b_{m(x)} \sum_{k=1}^{m(x)-1} \frac{k+1}{2} \sim \frac{m(x)\Delta b_{m(x)}}{x}, \quad x \to +0.$$

It remains to substitute the above asymptotics in the ordinal relation (36). □

Finally, a reformulation of the above result in the spirit of Theorem 1 allows us to write the second term of the asymptotic expansion of the sum of a sine series in a compact form.

Corollary 2. *Let $b(t)$ be a non-negative convex differentiable function that tends to zero as $t \to +\infty$. If the function $-tb'(t)$ varies slowly, then:*

$$\sum_{k=1}^{\infty} b(k)\sin kx - \frac{b(1/x)}{x} \sim -\gamma \frac{b'(1/x)}{x^2}.$$

Proof. Since the function $-tb'(t)$ is slowly varying, the following ordinal relation takes place:

$$m(x)\Delta b_{m(x)} \sim m(x) b'(m(x)) \sim \frac{1}{x} b'(1/x), \quad x \to +0. \tag{43}$$

On the other hand, by Lemma 2, we have:

$$b(1/x) - b(\pi/x) = \int_{1/x}^{\pi/x} b'(t)\, dt = \int_{1/x}^{\pi/x} \frac{-tb'(t)}{t}\, dt \sim -\frac{\ln\pi}{x} b(1/x), \quad x \to +0. \tag{44}$$

Substituting (43) and (44) in Theorem 5, we obtain the required asymptotics. □

Remark 1. *According to Lemma 4, the condition that the function $-tb'(t)$ is slowly varying implies that the function $b(t)$ is slowly varying. Thus, Corollary 2 is a refinement of Theorem 1 with an additional restriction on the sequence of the coefficients of a sine series.*

Example 1. *The condition that the function $-tb'(t)$ is slowly varying is satisfied for the majority of series. In particular, the following asymptotic expansion takes place:*

$$\sum_{k=1}^{\infty} \frac{\sin kx}{\ln(k+1)} = \frac{1}{x\ln(1/x)} + \frac{\gamma}{x\ln^2(1/x)} + o\left(\frac{1}{x\ln^2(1/x)}\right), \quad x \to +0.$$

Corollary 3. *Let $b(t)$ be a non-negative convex differentiable function that tends to zero as $t \to +\infty$. If the function $-tb'(t)$ varies slowly, then:*

$$\sum_{k=1}^{\infty} b(k)\sin kx = \frac{b(e^{-\gamma}/x)}{x} + o\left(\frac{b'(1/x)}{x^2}\right), \quad x \to +0.$$

Proof. The arguments are similar to those carried out in the proof of Corollary 2 and are based on the following asymptotic equalities:

$$b(1/x) - b\left(e^{-\gamma}/x\right) = \int_{1/x}^{e^{-\gamma}/x} b'(t)\,dt = \int_{1/x}^{e^{-\gamma}/x} \frac{-tb'(t)}{t}\,dt \sim \gamma\,\frac{b(1/x)}{x}, \quad x \to +0.$$

□

The last result shows that in Theorem 1, we can simply replace the argument of the function $b(t)$ with π/x by $e^{-\gamma}/x$ to obtain a more accurate approximation of the sum of a sine series.

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