



Article Soft Frames in Soft Hilbert Spaces

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Abstract: In this paper, we use soft linear operators to introduce the notion of discrete frames on soft Hilbert spaces, which extends the classical notion of frames on Hilbert spaces to the context of algebraic structures on soft sets. Among other results, we show that the frame operator associated to a soft discrete frame is bounded, self-adjoint, invertible and with a bounded inverse. Furthermore, we prove that every element in a soft Hilbert space satisfies the frame decomposition theorem. This theoretical framework is potentially applicable in signal processing because the frame coefficients serve to model the data packets to be transmitted in communication networks.

Keywords: soft sets; soft inner product; soft Hilbert space; self-adjoint operator; soft frame

MSC: primary 03E72; 08A72; secondary 46B99



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1. Introduction

Discrete frame theory on Hilbert spaces is relatively recent. This theory was introduced by Duffin and Schaeffer in 1952 [1] as a tool to study problems related to non-harmonic Fourier series. Almost 30 years later, it was developed by Daubechies, Grossman and Meyer [2]; they used frames to find expansions in series of functions on $L_2(\mathbb{R})$ similar to the expansion in series that is done using orthonormal bases. As a result, the main advantage is that a good discrete frame can behave almost like an orthonormal basis; with an additional, they do not require the uniqueness of the coefficients when writing a Hilbert space vector as a linear combination of the frame elements (frame decomposition theorem). For this reason, in the literature, they are often called over-complete bases. Consequently, frames have turned out to be a powerful tool in signal processing, image processing, data understanding, sampling theory, and wavelet analysis [3] (see also [4–6]). Frame theory has been extended to the context of Krein's spaces, which are a generalization of Hilbert spaces that have a wide variety of applications in physics (see [7,8]).

Soft set theory was introduced by Molodtsov [9] in 1999 as a new mathematical tool for dealing with uncertainties while modeling problems in engineering, physics, computer science, economics, social science, and medical sciences. This theory began to receive special attention in 2002 when Maji et al. [10] applied the soft sets to decision-making problems, using rough mathematics, and later in 2003, with the definitions of various operations of soft sets [11]. Since then, research works in soft sets theory and its applications in various fields have been progressing rapidly because this theory is free from the many difficulties that have troubled the usual theoretical approaches. This is how research related to soft sets has been carried out in several directions among which we can mention the following: information systems, decision making, nonlinear neutral differential equations and algebraic structures, fuzzy sets, and rough sets, as we can see in the papers [12–16]. In particular, regarding mathematical analysis and its applications, the concepts and results of soft real sets, soft real numbers, soft complex sets, soft complex numbers, soft linear spaces, soft metric spaces, soft normed spaces, soft inner product, soft Hilbert spaces,

soft linear operator, soft linear functional, soft Banach algebra, soft topology, etc., have originated (see [11,17–26]). In the book [27], various applications of soft sets are discussed in problems related to data filling in incomplete information systems, and as is well known, frames are also a suitable tool to deal with these same types of problems, for which it is interesting to address frame theory together with soft sets theory to propose a new mathematical tool (soft frames) that is more efficient in the absence of partial information. This theoretical framework motivates the study of certain topics based on operator theory in spaces with a complete inner product (Hilbert spaces), which is potentially applicable in signal processing, where the frame coefficients serve to model the data packets to be transmitted; hence, the elimination of an element of the frame is equivalent to a packet that is not delivered in the communication, but this could be recovered using the frame decomposition theorem.

This manuscript was designed as follows. Section 2 corresponds to the preliminaries, where we cover all the theory of soft sets necessary to study soft frames, in addition to introducing the notions of soft inner product, soft operators, soft Hilbert spaces, etc. In Section 3, we develop the definition of soft frame on soft Hilbert spaces, and state the most important result: the soft frame decomposition theorem.

2. Preliminaries

Throughout this paper, *X* denotes a non-empty set (possibly without algebraic structure), $\mathcal{P}(X)$ the power set of *X*, and *A* a non-empty set of parameters.

Definition 1 ([9]). A soft set on X is a pair (F, A) where F is a mapping given by $F : A \to \mathcal{P}(X)$.

In this way, we can see a soft set as the following:

$$(F,A) := G_F^A = \{ (\lambda, F(\lambda)) : \lambda \in A, F(\lambda) \in \mathcal{P}(X) \},\$$

where G_F^A is the graph of *F* with respect to *A*.

Note that a soft set is determined by knowing $F(\lambda)$ for all $\lambda \in A$. Therefore, it is common to find in the literature that $F : A \to \mathcal{P}(X)$ is called a soft set on X, but it should not be a cause for confusion.

Example 1. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{\mathbb{Z}_4 = \lambda_1, \mathbb{Z}_6 = \lambda_2, \mathbb{Z}_8 = \lambda_3\}$. If $F : A \to \mathcal{P}(X)$ describes the generating elements of the cyclic group. Then, it is easy to see that $F(\lambda_1) = \{1, 3\}, F(\lambda_2) = \{1, 5\}, F(\lambda_3) = \{1, 3, 5, 7\}$. Hence, (F, A) is a soft set on X seen as follows:

 $(F, A) = \{ (\mathbb{Z}_4, \{1, 3\}), (\mathbb{Z}_6, \{1, 5\}), (\mathbb{Z}_8, \{1, 3, 5, 7\}) \}.$

Definition 2 ([23]). A soft set (F, A) on X is said to be a null soft set if $F(\lambda) = \emptyset$ for all $\lambda \in A$, and in this case we write $(F, A) := \Phi$.

If $F(\lambda) \neq \emptyset$ for some $\lambda \in A$, then the soft set (F, A) is said to be a non-null soft set on *X*.

Definition 3 ([23]). A soft set (F, A) on X is said to be an absolute soft set if $F(\lambda) = X$ for all $\lambda \in A$; in this case, we write $(F, A) := \check{X}$. This convention of absolute soft set is adopted throughout the present work.

Definition 4 ([23]). A soft element on X is a function $\epsilon : A \to X$. Now if $\epsilon(\lambda) \in F(\lambda)$ for all $\lambda \in A$, the soft element ϵ is said to belong to the soft set (F, A) on X, which we denote by $\epsilon \in F$.

is

$$F(\lambda) = \{ \epsilon(\lambda) : \epsilon \in F \}.$$

We denote the collection of all the soft elements of a soft set (F, A) by SE((F, A)); this

$$SE((F,A)) := \{ \epsilon : \epsilon \in F \} = \{ \epsilon : \epsilon(\lambda) \in F(\lambda), \forall \lambda \in A \}.$$

Definition 5 ([23]). *Let* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *, and let A be a non-empty set of parameters. Consider the set*

$$\mathfrak{B}(\mathbb{K}) := \{ B \in \mathcal{P}(\mathbb{K}) : B \neq \emptyset, B \text{ is bounded} \},\$$

then, a mapping $F : A \to \mathfrak{B}(\mathbb{K}) \subset \mathcal{P}(\mathbb{K})$ is called a soft \mathbb{K} -set. This is denoted by (F, A). Furthermore, if for all $\lambda \in A$ it is satisfied that $F(\lambda)$ is a singleton, then by identifying (F, A) with its corresponding soft element, we call this soft element a soft \mathbb{K} -number.

We denote the set of all the soft \mathbb{R} -numbers or soft real numbers by $\mathbb{R}(A)$. In addition, we use the symbols $\overline{\alpha}$, $\overline{\beta}$, etc., to denote soft \mathbb{K} -numbers such that they behave as constants, that is $\overline{\alpha}(\lambda) = \alpha$ for all $\lambda \in A$. Similarly, we symbolize the set of all soft \mathbb{C} -numbers or soft complex numbers by $\mathbb{C}(A)$.

Definition 6 ([23]). *The set* $\{\alpha \in \mathbb{R}(A) : \alpha(\lambda) \ge 0, \forall \lambda \in A\}$ *is called the set of all nonnegative soft real numbers and is denoted by* $\mathbb{R}(A)^*$.

Definition 7 ([23]). For two soft real numbers $\alpha : A \to \mathbb{R}$ and $\beta : A \to \mathbb{R}$ we define the following:

1. $\alpha \leq \beta$, if $\alpha(\lambda) \leq \beta(\lambda)$, for all $\lambda \in A$, 2. $\alpha \geq \beta$, if $\alpha(\lambda) \geq \beta(\lambda)$, for all $\lambda \in A$, 3. $\alpha \leq \beta$, if $\alpha(\lambda) < \beta(\lambda)$, for all $\lambda \in A$, 4. $\alpha \geq \beta$, if $\alpha(\lambda) > \beta(\lambda)$, for all $\lambda \in A$.

Note that if we assign to *X* a structure of vector space, it is interesting to think of the possible structure of $F(\lambda)$ as a subset of *X* for all $\lambda \in A$. This motivates the following important definition.

Definition 8 ([18,24]). Let X be a \mathbb{K} -vector space (typically $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), A be a nonempty set of parameters and (F, A) be a soft set on X. The soft set (F, A) is said to be a soft \mathbb{K} -vector space on X if $F(\lambda)$ is a vector subspace of X for all $\lambda \in A$.

The importance of the above definition is that it allows us to relate the usual linear algebra to soft set theory. Furthermore, it gives us the tools to define important concepts in the classical functional analysis, such as norm, inner product, Banach space, and Hilbert space, among others, but from this context.

Definition 9 ([18,24]). Let (F, A) be soft \mathbb{K} -vector space. A soft element of (F, A) is said to be a soft vector of (F, A). Similarly, a soft element $\epsilon : A \to \mathbb{K}$ is said to be a soft scalar, where \mathbb{K} is the scalar field.

Definition 10 ([18,24]). *Let* (*F*, *A*) *be a soft* \mathbb{K} *-vector space on X*. *A soft vector x of* (*F*, *A*) *is said to be a null soft vector if* $x(\lambda) = \theta$, $\forall \lambda \in A$, *where* θ *is the zero element of X*. *This is denoted by* Θ .

Definition 11 ([18,24]). Let (F, A) be a soft \mathbb{K} -vector space on X and x, y be two soft vectors of (F, A) and k be a soft scalar. Then, the addition x + y of x with y, and scalar multiplication $k \cdot x$ of k and x are defined by $(x + y)(\lambda) = x(\lambda) + y(\lambda)$, $(k \cdot x)(\lambda) = k(\lambda) \cdot x(\lambda)$. Evidently, x + y, $k \cdot x$ are soft vectors of (F, A).

Theorem 1 ([18,24]). Let X be a \mathbb{K} -vector space, A be a nonempty set of parameters and (F, A) be a soft \mathbb{K} -vector space on X. Then, we have the following:

- 1. $\overline{0} \cdot x = \Theta$, for all $x \in F$,
- 2. $k \cdot \Theta = \Theta$, for all soft scalar k,
- 3. $(-\overline{1}) \cdot x = -x$, for all $x \in F$.

Definition 12 ([19]). Let \check{X} be an absolute soft \mathbb{K} -vector space, then a mapping $\|\cdot\| : SE(\check{X}) \to \mathbb{R}(A)$ is said to be a soft norm on \check{X} if $\|\cdot\|$ satisfies the following conditions:

- (N1) $||x|| \geq \overline{0}$ for all $x \in \check{X}$,
- (N2) $||x|| = \overline{0}$ if and only if $x = \Theta$,
- (N3) $\|\alpha \cdot x\| = \|\alpha\| \|x\|$ for all $x \in \check{X}$ and for all soft scalar α ,
- (N4) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in \check{X}$.

The soft \mathbb{K} -vector space \check{X} with a soft norm $\|\cdot\|$ on \check{X} is said to be a soft \mathbb{K} -normed space and is denoted by $(\check{X}, \|\cdot\|, A)$ or $(\check{X}, \|\cdot\|)$.

Lemma 1 ([19]). Let $(\check{X}, \|\cdot\|, A)$ be a soft \mathbb{K} -normed space, then for all $x \in \check{X}$ and for $\lambda \in A$, we have $\|x\|(\lambda) = 0$ if and only if $x(\lambda) = \theta$.

Proof. Consider the soft scalar α defined by $\alpha(\mu) = 1$ if $\mu = \lambda$, $\alpha(\mu) = 0$ if $\mu \neq \lambda$. Then, note that $(\alpha \cdot x)(\mu) = \alpha(\mu)x(\mu) = 0 \cdot x(\mu) = \theta$ whenever $\mu \neq \lambda$; also $(\alpha \cdot x)(\mu) = x(\mu)$ for $\mu = \lambda$. Now, by (N3), we have $||x||(\lambda) = 0 \iff |\alpha||x|| = 0 \iff ||\alpha \cdot x|| = 0 \iff \alpha \cdot x = \Theta$ $\iff x(\lambda) = \theta$. \Box

Lemma 2 ([19]). Every soft norm $\|\cdot\|$ satisfies the following condition: (N5) For $\xi \in X$ and $\lambda \in A$, $\{\|x\|(\lambda) : x(\lambda) = \xi\}$ is a singleton.

The two lemmas above allow us to prove the following important theorem about soft normed spaces.

Theorem 2 ([19] Decomposition theorem). If $(X, \|\cdot\|, A)$ is a soft normed space, then for ech $\lambda \in A$, the mapping $\|\cdot\|_{\lambda} : X \to \mathbb{R}^+$ defined by $\|\xi\|_{\lambda} := \|x\|(\lambda)$ where x is such that $x(\lambda) = \xi$ is a norm on X for all $\lambda \in A$.

Definition 13 ([24]). If \check{X} is an absolute soft vector space, then a binary operation $\langle \cdot, \cdot \rangle : SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ is said to be a soft inner product on \check{X} if it satisfies the following conditions:

- (I1) $\langle x, x \rangle \geq \overline{0}$ for all $x \in X$ and $\langle x, x \rangle = \overline{0}$ if and only if $x = \Theta$,
- (I2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in \check{X}$,
- (I3) $\langle \alpha \cdot x, y \rangle = \alpha \cdot \langle x, y \rangle$ for all $x, y \in X$ and for all soft scalar α ,
- (I4) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in \check{X}$.

The soft vector space \check{X} with a soft inner product $\langle \cdot, \cdot \rangle$ on \check{X} is said to be a soft inner product space and is denoted by $(\check{X}, \langle \cdot, \cdot \rangle, A)$ or $(\check{X}, \langle \cdot, \cdot \rangle)$.

Lemma 3 ([24]). Every soft inner product $\langle \cdot, \cdot \rangle$ on \check{X} satisfies the following condition:

(15) For $(\xi, \eta) \in X \times X$ and $\lambda \in A$, $\{\langle x, y \rangle (\lambda) : x, y \in \check{X} \text{ such that } x(\lambda) = \xi, y(\lambda) = \eta\}$ is a singleton.

Theorem 3 ([24]). If $(\check{X}, \langle \cdot, \cdot \rangle, A)$ is a soft inner product space, then for all $\lambda \in A$, the mapping $\langle \cdot, \cdot \rangle_{\lambda} : X \times X \to \mathbb{C}$ is defined by $\langle \xi, \eta \rangle_{\lambda} := \langle x, y \rangle(\lambda)$ where x, y are such that $x(\lambda) = \xi$ and $y(\lambda) = \eta$ is an inner product on X.

Theorem 4 ([17] Cauchy-Schwarz inequality). Let $(\check{X}, \langle \cdot, \cdot \rangle, A)$ be a soft inner product space. Then, $|\langle x, y \rangle|^2 \cong \langle x, x \rangle \langle y, y \rangle$ for all $x, y \cong \check{X}$.

Definition 14 ([17]). A soft inner product space is said to be complete if it is complete with respect to the soft metric defined by the soft inner product. A complete soft inner product space is said to be a soft Hilbert space.

Definition 15 ([18,24]). Let $T : SE(\check{X}) \to SE(\check{Y})$ be an operator. Then, T is said to be soft linear if the following holds:

(L1) $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in \check{X}$,

(L2) T(cx) = cT(x), for all soft scalar c and all $x \in \check{X}$.

Lemma 4 ([18,24]). Every soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$ where \check{X}, \check{Y} are absolute soft normed spaces, and satisfies the following condition:

(L3) For $\xi \in X$ and $\lambda \in A$ the set $\{T(x)(\lambda) : x \in \check{X} \text{ tal que } x(\lambda) = \xi\}$ is a singleton.

Theorem 5 ([18,24]). Every soft linear operator can be decomposed into a parameterized family of linear operators. This is, if $T : SE(\check{X}) \to SE(\check{Y})$ is a soft linear operator, then the family $\{T_{\lambda} : \lambda \in A\}$ where $T_{\lambda} : X \to Y$ is defined by $T_{\lambda}(\xi) = T(x)(\lambda)$ for all $\xi \in X$ and $x \in \check{X}$ with $x(\lambda) = \xi$, is a family of linear operators.

Theorem 6 ([18,24]). Let $\{T_{\lambda} : \lambda \in A\}$ be a parameterized family of linear operators from X to Y. Then, the operator $T : SE(\check{X}) \to SE(\check{Y})$ given by $T(x)(\lambda) = T_{\lambda}(\xi)$ if $x(\lambda) = \xi$, $\lambda \in A$, is soft linear.

Definition 16 ([18,24]). Let $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator, where \check{X},\check{Y} are absolute soft normed spaces. The operator T is called bounded if there exists $M \ge \overline{0}$ such that $||T(x)|| \le M||x||, \forall x \in \check{X}.$

Definition 17 ([18,24]). Let $T : SE(\check{X}) \to SE(\check{Y})$ be a bounded soft linear operator. Then, the norm of the operator T, denoted by ||T||, is a soft real number defined as follows:

 $||T||(\lambda) := \inf\{t \in \mathbb{R} : ||T(x)||(\lambda) \le t ||x||(\lambda), \text{ for all } x \in \check{X}\}.$

Theorem 7 ([18,24]). Suppose that \check{X},\check{Y} are soft normed spaces and $T : SE(\check{X}) \to SE(\check{Y})$ is a bounded soft linear operator. Then, $||T(x)|| \leq ||T|| ||x||$ for all $x \in \check{X}$.

Definition 18 ([18]). Let $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator, where \check{X}, \check{Y} are absolute soft normed spaces. The operator T is said to be continuous at $x_0 \in \check{X}$ if for every sequence $\{x_n\}$ of soft elements of \check{X} with $x_n \to x_0$ as $n \to \infty$, we have $T(x_n) \to T(x_0)$ as $n \to \infty$. If T is continuous at each soft element of \check{X} , then T is said to be a continuous operator.

Theorem 8 ([18]). Let $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator, where \check{X}, \check{Y} are soft normed spaces. Then, T is bounded if and only if T is continuous.

Theorem 9 ([18,24]). Let \check{X} and \check{Y} be soft normed spaces and $T : SE(\check{X}) \to SE(\check{Y})$ be a continuous soft linear operator. Then, T_{λ} is continuous on X for all $\lambda \in A$.

Theorem 10 ([18,24]). Let \check{X} and \check{Y} be soft normed spaces and $T : SE(\check{X}) \to SE(\check{Y})$ be a bounded soft linear operator. Then, for all $\lambda \in A$, $||T||(\lambda) = ||T_{\lambda}||_{\lambda}$, where $||T_{\lambda}||_{\lambda}$ is the norm of the linear operator $T_{\lambda} : X \to Y$.

Theorem 11 ([24]). Suppose that \check{X} and \check{Y} are soft normed spaces and $\{T_{\lambda} : \lambda \in A\}$ is a family of continuous linear operators such that $T_{\lambda} : X \to Y$ for all λ . Then, the soft linear operator

 $T: SE(\check{X}) \to SE(\check{Y})$ defined by $T(x)(\lambda) = T_{\lambda}(x(\lambda)), \forall \lambda \in A$ is a continuous soft linear operator.

Definition 19 ([24]). Let $(\check{H}_1, \langle \cdot, \cdot \rangle_1)$, $(\check{H}_2, \langle \cdot, \cdot \rangle_2)$ be soft Hilbert spaces and $T : SE(\check{H}_1) \rightarrow$ $SE(\check{H}_2)$ be a bounded soft linear operator. The operator $T^* : SE(\check{H}_2) \to SE(\check{H}_1)$ is called the adjoint operator of T, if $\langle Tx, y \rangle_2 = \langle x, T^*y \rangle_1$, for all $x \in \check{H}_1$ and all $y \in \check{H}_2$.

Theorem 12 ([24]). Let \check{H} be a soft Hilbert space, $T : SE(\check{H}) \to SE(\check{H})$ be a continuous soft linear operator and T^{*} be the adjoint operator of T. Then, the following properties hold:

- 1. T^* is unique;
- 2. T^* is a soft linear operator;
- T^* is a continuous soft operator with $||T^*|| \leq ||T||$; 3.
- 4. $T^{**} = T;$
- $||T^*|| = ||T||;$ 5.
- $(T_1T_2)^* = T_2^*T_1^*$, where $T_1, T_2 : SE(\check{H}) \to SE(\check{H})$ are continuous soft linear operators; $||T^*T|| = ||T||^2$; 6.
- 7.
- $||TT^*|| = ||T||^2$ 8.
- $(T_1 + T_2)^* = T_1^* + T_2^*$, where $T_1, T_2 : SE(\check{H}) \to SE(\check{H})$ are continuous soft linear operators. 9.

Proposition 2 ([24]). Let \check{H} be a soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ be a continuous soft linear operator and T^* be the adjoint operator of T. Then T^*_{λ} defined by $T^*_{\lambda}(x(\lambda)) = (T^*(x))(\lambda)$ *is the adjoint operator of* T_{λ} *,* $\forall \lambda \in A$ *.*

Proposition 3 ([24]). Let \check{H} be a soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ be a continuous soft linear operator. Let $\{T_{\lambda}^*: \lambda \in A\}$ be a family of adjoint operators of T_{λ} . Then, the soft linear operator T^* defined by $T^*(x)(\lambda) = T^*_{\lambda}(x(\lambda)), \forall \lambda \in A, \forall x \in \check{H}$ is the adjoint operator of T.

Definition 20 ([24]). A continuous soft linear operator $T : SE(\check{H}) \to SE(\check{H})$ is called a selfadjoint soft linear operator if $T^* = T$.

Definition 21 ([17]). Let \check{H} be a soft Hilbert space. Then, a collection \mathcal{B} of soft vectors of \check{H} is said to be orthonormal if for all $x, y \in \mathcal{B}$, the following holds:

$$\langle x, y \rangle = \begin{cases} \overline{0} & \text{if } x \neq y \\ \\ \overline{1} & \text{if } x = y. \end{cases}$$

If the soft set \mathcal{B} contains only a countable number of soft vectors, then we arrange it in a sequence of soft vectors and call it an orthonormal sequence.

Theorem 13 ([17] Bessel inequality). Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be an orthonormal sequence on a soft Hilbert space \check{H} . Then for every $x \in \check{H}$, $\sum_{i=1}^{\infty} |\langle x, \alpha_i \rangle|^2 \leq ||x||^2$.

Theorem 14 ([17]). Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence on a soft Hilbert space \check{H} having a finite set of parameters A. Then, the infinite series

$$\sum_{i=1}^{\infty} c_i \alpha_i,$$

where $c_1, c_2, \cdots, c_n, \cdots$ are soft scalars, is convergent if and only if the series $\sum_{i=1}^{\infty} |c_i|^2$ is convergent.

Theorem 15 ([17]). Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence on a soft Hilbert space \check{H} having a finite set of parameters A. Then, for any $x \in \check{H}$,

$$\left(x-\sum_{i=1}^{\infty}\langle x,\alpha_i\rangle\alpha_i\right)\perp\alpha_j$$
, for all j

Definition 22 ([17]). Let \mathcal{B} be a non-null collection of orthonormal soft elements of \check{H} . Then, \mathcal{B} is said to be complete orthonormal if there exists a non-orthonormal set \mathcal{D} such that \mathcal{D} is a proper subset of \mathcal{B} . If the set \mathcal{B} contains only a countable number of soft elements then we call it a complete orthonormal sequence.

In the following theorem, we consider S the collection of all soft vectors x of \mathcal{H} such that $x(\lambda) \neq \theta$ for all $\lambda \in A$, together with the null soft vector Θ . In symbols,

$$S := \{ x \in \check{H} : x(\lambda) \neq \theta, \forall \lambda \in A \} \cup \{ \Theta \}.$$

Theorem 16 ([17]). Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in a soft Hilbert space H having a finite set of parameters A. Then, the following conditions are equivalent:

- 1. $\{\alpha_n\}_{n\in\mathbb{N}}$ is complete,
- 2. If for all $x \in \check{H}$ and for all $i \in \mathbb{N}$ we have $x \perp \alpha_i$ then $x = \Theta$,
- 3.
- For all $x \in \check{H}$, with $(x \sum_{i=1}^{\infty} \langle x, \alpha_i \rangle \alpha_i) \in S$, $x = \sum_{i=1}^{\infty} \langle x, \alpha_i \rangle \alpha_i$, For all $x \in \check{H}$, with $(x \sum_{i=1}^{\infty} \langle x, \alpha_i \rangle \alpha_i) \in S$, $||x||^2 = \sum_{i=1}^{\infty} |\langle x, \alpha_i \rangle|^2$. 4.

Corollary 1. Let G be a soft Banach algebra. If $x \in G$ and $\|\bar{e} - x\| < \bar{1}$, then there exists x^{-1} and $x^{-1} = \bar{e} + \sum_{j=1}^{\infty} (\bar{e} - x)^j$.

3. Soft Frames in Soft Hilbert Spaces

Definition 23 ([25]). Let $(\check{H}, \langle \cdot, \cdot \rangle_{\check{H}}, A)$ be a soft Hilbert space. We say that a self-adjoint soft linear operator T is positive if $\langle T(x), x \rangle_{\check{H}} \cong \overline{0}$ for all $x \in SE(\check{H})$. In notation, we write $T \cong \overline{0}$ Id. Also, $T \cong \overline{c}$ Id means that $\langle T(x), x \rangle_{\check{H}} \cong \langle \overline{c}$ Id $x, x \rangle_{\check{H}}$ for all $x \in SE(\check{H})$ and all soft scalar \overline{c} .

Proposition 4. Let $(\check{H}, \langle \cdot, \cdot \rangle_{\check{H}}, A)$ be a soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ be a selfadjoint soft linear operator. Then, $\langle Tx, x \rangle(\lambda) \in \mathbb{R}$ for all $x \in SE(\check{H})$ and all $\lambda \in A$.

Lemma 5. Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space and A be a finite set of parameters. Then, the mapping

$$\langle \cdot, \cdot \rangle : SE(\check{H}) \times SE(\check{H}) \to \mathbb{C}(A)$$

given by $\langle h_1, h_2 \rangle(\lambda) = \langle h_1(\lambda), h_2(\lambda) \rangle_H$ for all $\lambda \in A$ and all $h_1, h_2 \in \check{H}$ defines a soft inner product on *M*.

Theorem 17. Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space. If A is a finite set of parameters, we have that $(\check{H}, \langle \cdot, \cdot \rangle, A)$ is a soft Hilbert space, where $\langle \cdot, \cdot \rangle : SE(\check{H}) \times SE(\check{H}) \to \mathbb{C}(A)$ is given by $\langle h_1, h_2 \rangle(\lambda) = \langle h_1(\lambda), h_2(\lambda) \rangle_H$ for all $\lambda \in A$ and all $h_1, h_2 \in H$.

Proof. Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space. Let $\{h_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ be a Cauchy sequence in (\dot{H}, d) , where *d* is the soft metric on \dot{H} induced by $\langle \cdot, \cdot \rangle$. We must show that $\{h_n\}_{n \in \mathbb{N}}$ is convergent in $SE(\check{H})$. Indeed, we assert that for all $\lambda \in A$, $\{h_n(\lambda)\}_{n \in \mathbb{N}}$ is a Cauchy sequence, because given $\epsilon \geq 0$ —note that $\{h_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence—there exists $N \in$ \mathbb{N} such that $d(h_n, h_m) \approx \epsilon$ for all $m, n \geq N$, that is, $d(h_n, h_m)(\lambda) = d_H(h_n(\lambda), h_m(\lambda)) < \epsilon(\lambda)$ for all $\lambda \in A$ and all $m, n \geq N$, which proves the assertion. Thus, since (H, d) is complete, we have that $\{h_n(\lambda)\}_{n\in\mathbb{N}}$ is convergent for all $\lambda \in A$, for example, $h_n(\lambda) \to h_\lambda$ as $n \to \infty$. Now, given $\epsilon \geq \overline{0}$, there exists $N_{\lambda} \in \mathbb{N}$ such that $d_H(h_n(\lambda), h_{\lambda}) < \epsilon(\lambda)$ for all $n \geq N_{\lambda}$. Hence, since *A* is finite, taking $N = \max\{N_{\lambda} : \lambda \in A\}$ and making $\tilde{h}(\lambda) := h_{\lambda}$, the theorem is satisfied. \Box

Example 2. Let $X = \ell_2(\mathbb{N})$. It is well known that X is a Hilbert space with respect to the inner product $\langle x, y \rangle_{\ell_2} = \sum_{i=1}^{\infty} \xi_i \overline{\eta}_i$ for $x = \{\xi_i\}_{i=1}^{\infty}, y = \{\eta_i\}_{i=1}^{\infty}$ en $\ell_2(\mathbb{N})$. Now, let \widetilde{x} , \widetilde{y} be soft elements of the absolute soft vector space \check{X} . Then, $\widetilde{x}(\lambda) = \{\xi_i^{\lambda}\}_{i=1}^{\infty}, \widetilde{y}(\lambda) = \{\eta_i^{\lambda}\}_{i=1}^{\infty}$ are elements of $\ell_2(\mathbb{N})$. Thus, the mapping $\langle \cdot, \cdot \rangle : SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$, defined by $\langle \widetilde{x}, \widetilde{y} \rangle(\lambda) = \sum_{i=1}^{\infty} \xi_i^{\lambda} \overline{\eta}_i^{\overline{\lambda}} = \langle \widetilde{x}(\lambda), \widetilde{y}(\lambda) \rangle_{\ell_2}$ for all $\lambda \in A$, is a soft inner product on \check{X} . Therefore, $(\check{X}, \langle \cdot, \cdot \rangle, A)$, with A being a finite set of parameters, is a soft Hilbert space.

Next, we introduce the notion of soft discrete frame in soft Hilbert spaces. We study the pre-frame operator and the frame operator of a soft discrete frame. In addition, we establish the most important result, called the decomposition theorem of soft frames.

Definition 24. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of soft vectors of a soft Hilbert space \check{H} having a finite set of parameters. We say that $\{x_n\}_{n\in\mathbb{N}} \subseteq SE(\check{H})$ is a soft frame on \check{H} if there exist soft real numbers $\bar{b} \geq \bar{a} \geq \bar{0}$ such that the following holds:

$$\overline{a} \|h\|^{2} \cong \sum_{n \in \mathbb{N}} |\langle h, x_{n} \rangle|^{2} \cong \overline{b} \|h\|^{2}, \ \forall h \cong \check{H}.$$

$$(1)$$

The soft real numbers \overline{a} and \overline{b} are called bounds of the soft frame. These are not unique, as the optimal bounds are the largest possible value of \overline{a} and the smallest possible value of \overline{b} that satisfy (1). In the case that $\overline{a} = \overline{b}$, the soft frame is called tight.

Remark 1. The relationship between the previous definition and the discrete frames in usual Hilbert spaces is that every discrete frame on a Hilbert space induces a soft discrete frame on a soft Hilbert space, with respect to any finite set of parameters. The proof of this claim is as follows: Let $(H, \langle \cdot, \cdot \rangle_H)$ be any Hilbert space and let $\{f_n\}_{n \in \mathbb{N}} \subseteq H$ be a discrete frame on H with bounds $b \ge a > 0$. Then, if A is a finite set of parameters, we know that, by virtue of Theorem 17, $(\check{H}, \langle \cdot, \cdot \rangle_A)$ is a soft Hilbert space, where $\langle h_1, h_2 \rangle (\lambda) = \langle h_1(\lambda), h_2(\lambda) \rangle_H$ for all $\lambda \in A$ and all $h_1, h_2 \in \check{H}$, whereby, if for each $n \in \mathbb{N}$ we define $x_n : A \to H$ by $x_n(\lambda) = f_n$ for all $\lambda \in A$, we can affirm that $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ is a soft frame on \check{H} with bounds $\bar{b} \ge \bar{a} \ge \bar{0}$ defined by $\bar{b}, \bar{a} : A \to \mathbb{R}, \bar{b}(\lambda) = b, \forall \lambda \in A$ and $\bar{a}(\lambda) = a, \forall \lambda \in A$. Indeed, for each $h \in SE(\check{H})$ and each $\lambda \in A$, we have the following:

$$\begin{aligned} (\overline{a} \cdot ||h||^2)(\lambda) &= \overline{a}(\lambda) ||h||^2(\lambda) = a\langle h, h \rangle(\lambda) = a\langle h(\lambda), h(\lambda) \rangle_H \\ &\leq \sum_{n \in \mathbb{N}} |\langle h(\lambda), f_n \rangle_H|^2 = \sum_{n \in \mathbb{N}} |\langle h(\lambda), x_n(\lambda) \rangle_H|^2 \\ &= \left(\sum_{n \in \mathbb{N}} |\langle h, x_n \rangle|^2\right)(\lambda), \end{aligned}$$

where we have used that $\{f_n\}_{n \in \mathbb{N}} \subseteq H$ is a frame on H. On the other hand, we have the following:

$$\begin{aligned} (\bar{b} \cdot \|h\|^2)(\lambda) &= \bar{b}(\lambda)\|h\|^2(\lambda) = b\langle h(\lambda), h(\lambda) \rangle_H \\ &\geq \sum_{n \in \mathbb{N}} |\langle h(\lambda), f_n \rangle_H|^2 = \sum_{n \in \mathbb{N}} |\langle h(\lambda), x_n(\lambda) \rangle_H|^2 \\ &= \left(\sum_{n \in \mathbb{N}} |\langle h, x_n \rangle|^2\right)(\lambda). \end{aligned}$$

Thus, we have proved the following:

$$(\overline{a} \cdot \|h\|^2)(\lambda) \le \left(\sum_{n \in \mathbb{N}} |\langle h, x_n \rangle|^2\right)(\lambda) \le (\overline{b} \cdot \|h\|^2)(\lambda).$$

that is,

$$\overline{a} \|h\|^2 \cong \sum_{n \in \mathbb{N}} |\langle h, x_n \rangle|^2 \cong \overline{b} \|h\|^2, \ \forall h \in \check{H}.$$

Therefore, $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ *is a soft frame on* \check{H} *.*

Definition 25. Given a soft frame $\{x_n\}_{n \in \mathbb{N}}$ on a soft Hilbert space \check{H} having a finite set of parameters, we define the pre-frame operator by the following:

$$T: SE(\check{\ell}_2(\mathbb{N})) \to SE(\check{H}), \ T(\{\alpha_n\}_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} \alpha_n x_n.$$

Proposition 5. The pre-frame operator associated to $\{x_n\}_{n \in \mathbb{N}}$ is well defined and bounded.

Proof. Observe that $\alpha = {\alpha_n}_{n \in \mathbb{N}} \subseteq SE(\check{\ell}_2(\mathbb{N}))$. Then, we have the following:

$$\begin{aligned} \|T\alpha\| &= \sup_{\|y\|=\overline{1}} |\langle T\alpha, y\rangle| = \sup_{\|y\|=\overline{1}} \left|\sum_{n \in \mathbb{N}} \alpha_n \langle x_n, y\rangle\right| \\ & \widetilde{\leq} \sup_{\|y\|=\overline{1}} \sum_{n \in \mathbb{N}} |\alpha_n \langle x_n, y\rangle| \widetilde{\leq} \sup_{\|y\|=\overline{1}} \left(\sum_{n \in \mathbb{N}} |\alpha_n|^2\right)^{\frac{1}{2}} \left(\sum_{n \in \mathbb{N}} |\langle x_n, y\rangle|^2\right)^{\frac{1}{2}} \\ &= \|\alpha\| \sup_{\|y\|=\overline{1}} \left(\sum_{n \in \mathbb{N}} |\langle x_n, y\rangle|^2\right)^{\frac{1}{2}} \widetilde{\leq} \|\alpha\|\sqrt{\overline{b}} \sup_{\|y\|=\overline{1}} \|y\| \\ &= \|\alpha\|\sqrt{\overline{b}}. \end{aligned}$$

Therefore, $||T|| \leq \sqrt{\overline{b}}$. \Box

Proposition 6. The adjoint soft operator of *T* is given by $T^* : SE(\check{H}) \to SE(\check{\ell}_2(\mathbb{N})), T^*(x) = \{\langle x, x_n \rangle_{\check{H}}\}_{n \in \mathbb{N}}.$

Proof. Let $x \in SE(\check{H})$ and $y \in SE(\check{\ell}_2(\mathbb{N}))$, then $y = \{\xi_n\}_{n \in \mathbb{N}}$ with $y(\lambda) = \{\xi_n^{\lambda}\}_{n \in \mathbb{N}}$. Additionally, for all $\lambda \in A$, the following holds:

$$\begin{aligned} \langle T(y), x \rangle_{\check{H}}(\lambda) &= \left\langle \sum_{n \in \mathbb{N}} \xi_n x_n, x \right\rangle_{\check{H}}(\lambda) = \sum_{n \in \mathbb{N}} \xi_n \langle x_n, x \rangle_{\check{H}}(\lambda) \\ &= \sum_{n \in \mathbb{N}} \xi_n \overline{\langle x, x_n \rangle}_{\check{H}}(\lambda) = \left\langle \{\xi_n^\lambda\}_{n \in \mathbb{N}}, \{\langle x, x_n \rangle_{\check{H}}^\lambda\}_{n \in \mathbb{N}} \right\rangle_{\ell_2} \\ &= \left\langle \{\xi_n\}_{n \in \mathbb{N}}, \{\langle x, x_n \rangle_{\check{H}}\}_{n \in \mathbb{N}} \right\rangle_{\check{\ell}_2}(\lambda) \\ &= \langle y, T^*(x) \rangle_{\check{\ell}_2}(\lambda). \end{aligned}$$

Therefore, $T^*(x) = \{ \langle x, x_n \rangle_{\check{H}} \}_{n \in \mathbb{N}}$. \Box

Definition 26. We define the associated frame operator to the soft frame $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ by

$$S:TT^*:SE(\check{H})\to SE(\check{\ell}_2(\mathbb{N})).$$

Proposition 7. The associated frame operator to the soft frame $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ with bounds $b \geq a \geq \overline{0}$ is bounded and self-adjoint.

Proof. Note the following:

$$||S|| = ||TT^*|| \cong ||T|| ||T^*|| = ||T|| ||T|| = ||T||^2 \cong b$$

Hence, *S* is bounded. Additionally, the following holds:

$$S^* = (TT^*)^* = (T^*)^*T^* = TT^* = S.$$

Thus, *S* is self-adjoint. \Box

Proposition 8. Let $S : SE(\check{H}) \to SE(\check{\ell}_2(\mathbb{N}))$ be the associated frame operator to the soft frame $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$. Then

$$S(x) = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} x_n$$
, for all $x \in SE(\check{H})$.

Proof. Let $x \in SE(\check{H})$; then,

$$S(x) = TT^*(x) = T(\{\langle x, x_n \rangle_{\check{H}}\}_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} x_n.$$

Remark 2. Note that for all $x \in SE(\check{H})$ we have the following:

$$\begin{aligned} \langle S(x), x \rangle_{\check{H}} &= \left\langle \sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} x_n, x \right\rangle_{\check{H}} = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} \langle x_n, x \rangle_{\check{H}} \\ &= \sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} \overline{\langle x, x_n \rangle_{\check{H}}} = \sum_{n \in \mathbb{N}} |\langle x, x_n \rangle_{\check{H}}|^2. \end{aligned}$$

Hence, the frame condition can be written in the form

$$\overline{a} \|x\|^2 \ \cong \ \langle S(x), x \rangle_{\check{H}} \ \cong \ \overline{b} \|x\|^2, \text{ for all } x \in SE(\check{H}).$$

Lemma 6. If $\{x_n\}_{n\in\mathbb{N}} \subseteq SE(\check{H})$ is a soft frame on a soft Hilbert space \check{H} with bounds $\bar{b} \geq \bar{a} \geq \bar{0}$, then $\{x_n\}_{n\in\mathbb{N}}$ is complete on \check{H} .

Proof. Let $x \in SE(\check{H})$ be such that $\langle x, x_n \rangle_{\check{H}} = \overline{0}$ for all $n \in \mathbb{N}$. Then, by the frame condition, we have

$$\overline{0} \cong \overline{a} ||x||^2 \cong \sum_{n \in \mathbb{N}} |\langle x, x_n \rangle_{\check{H}}|^2 = \overline{0}$$

and so $x = \Theta$. \Box

Proposition 9. Given a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ in the soft Hilbert space \check{H} , the following statements are equivalent:

- 1. $\{x_n\}_{n\in\mathbb{N}}\subseteq SE(\check{H})$ is a soft frame with bounds $\bar{b} \geq \bar{a} \geq \bar{0}$,
- 2. $S(x) = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} x_n$ is a positive and bounded soft linear operator of $SE(\check{H})$ to $SE(\check{H})$, which satisfies $\bar{a}Id \cong S \cong \bar{b}Id$.

Proof. (i) \Rightarrow (ii) Suppose that $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ is a soft frame with bounds $\bar{b} \geq \bar{a} \geq \bar{0}$, then $S = TT^*$ is bounded by Proposition 7 and since

$$\langle \overline{a}Id(x), x \rangle_{\check{H}} = \overline{a} \langle x, x \rangle_{\check{H}} = \overline{a} ||x||^2, \quad \left\langle \overline{b}Id(x), x \right\rangle_{\check{H}} = \overline{b} \langle x, x \rangle_{\check{H}} = \overline{b} ||x||^2,$$

it follows by Remark 2 that $\langle S(x), x \rangle_{\check{H}} = \sum_{n \in \mathbb{N}} |\langle x, x_n \rangle_{\check{H}}|^2$. Thus,

$$\langle \overline{a}Id(x), x \rangle_{\check{H}} \cong \langle S(x), x \rangle_{\check{H}} \cong \langle \overline{b}Id(x), x \rangle_{\check{H}}$$

and hence, $\overline{a}Id \cong S \cong \overline{b}Id$.

(ii) \Rightarrow (i) Suppose that (*ii*) is true, that is,

$$\langle \overline{a}Id(x), x \rangle_{\check{H}} \cong \langle S(x), x \rangle_{\check{H}} \cong \langle \overline{b}Id(x), x \rangle_{\check{H}},$$

for all $x \in SE(\check{H})$. Then $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ is a soft frame with bounds $\bar{b} \geq \bar{a} \geq \bar{0}$. \Box

Lemma 7. Let $(\check{H}, \langle \cdot, \cdot \rangle_{\check{H}}, A)$ be a soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ be a positive soft linear operator. Then for any $\tilde{\epsilon} \geq \overline{0}$ the mapping

$$[\cdot, \cdot]_{\widetilde{\epsilon}} : SE(\check{H}) \times SE(\check{H}) \to \mathbb{C}(A),$$

defined by $[x, y]_{\tilde{\epsilon}} = \langle (T + \tilde{\epsilon}Id)x, y \rangle_{\check{H}}$ *for all* $x, y \in SE(\check{H})$ *, is a soft inner product.*

Proof. Indeed, let $x, y, z \in SE(\check{H})$ and α be soft scalars. Then, the following hold: (I1)

$$\begin{aligned} [x,x]_{\widetilde{\epsilon}} &= \langle (T+\widetilde{\epsilon}Id)x,x\rangle_{\check{H}} = \langle T(x)+\widetilde{\epsilon}Id(x),x\rangle_{\check{H}} \\ &= \langle T(x),x\rangle_{\check{H}} + \widetilde{\epsilon}\langle x,x\rangle_{\check{H}} \ \widetilde{\geq} \ \overline{0} \end{aligned}$$

because $T \ge \overline{0}$. Additionally,

$$[x,x]_{\tilde{\epsilon}} = \overline{0} \iff \langle T(x),x\rangle_{\check{H}} + \tilde{\epsilon}\langle x,x\rangle_{\check{H}} = \overline{0} \implies \langle x,x\rangle_{\check{H}} = \overline{0} \implies x = \Theta.$$

(I2)

$$\begin{split} [x,y]_{\widetilde{\epsilon}} &= \langle (T+\widetilde{\epsilon}Id)x,y\rangle_{\check{H}} = \langle T(x)+\widetilde{\epsilon}Id(x),y\rangle_{\check{H}} \\ &= \langle T(x),y\rangle_{\check{H}} + \widetilde{\epsilon}\langle x,y\rangle_{\check{H}} = \overline{\langle y,T(x)\rangle}_{\check{H}} + \widetilde{\epsilon}\overline{\langle y,x\rangle}_{\check{H}} \\ &= \overline{\langle y,T(x)\rangle_{\check{H}} + \widetilde{\epsilon}\langle y,x\rangle_{\check{H}}} = \overline{\langle T(y),x\rangle_{\check{H}} + \widetilde{\epsilon}\langle y,x\rangle_{\check{H}}} \\ &= \overline{\langle T(y),x\rangle_{\check{H}} + \langle \widetilde{\epsilon}Id(y),x\rangle_{\check{H}}} = \overline{\langle (T+\widetilde{\epsilon}Id)y,x\rangle_{\check{H}}} \\ &= \overline{[y,x]_{\widetilde{\epsilon}}}. \end{split}$$

(I3)

$$[\alpha x, y]_{\widetilde{\epsilon}} = \langle (T + \widetilde{\epsilon} Id)(\alpha x), y \rangle_{\check{H}} = \alpha \langle (T + \widetilde{\epsilon} Id)x, y \rangle_{\check{H}} = \alpha [x, y]_{\widetilde{\epsilon}}$$

(I4)

$$\begin{split} [x+z,y]_{\widetilde{\epsilon}} &= \langle (T+\widetilde{\epsilon}Id)(x+z),y\rangle_{\check{H}} \\ &= \langle (T+\widetilde{\epsilon}Id)x,y\rangle_{\check{H}} + \langle (T+\widetilde{\epsilon}Id)z,y\rangle_{\check{H}} \\ &= [x,y]_{\widetilde{\epsilon}} + [z,y]_{\widetilde{\epsilon}}. \end{split}$$

The following result is a version of the Cauchy–Schwarz inequality for positive soft linear operators. We will see its usefulness later.

Proposition 10. Let \check{H} be a soft Hilbert space. If $T : SE(\check{H}) \to SE(\check{H})$ is a positive soft linear operator; then,

$$|\langle T(x), y \rangle|^2 \cong \langle T(x), x \rangle \langle T(y), y \rangle$$
, for all $x, y \in SE(\check{H})$.

Proof. By Lemma 7, it is satisfied that for all $\tilde{\epsilon} > \bar{0}$, $[x, y]_{\tilde{\epsilon}} = \langle (T + \tilde{\epsilon}Id)x, y \rangle_{\tilde{H}}$ defines a soft inner product on \check{H} . Thus, by the Cauchy–Schwarz inequality, we have the following:

$$|[x,y]_{\widetilde{\epsilon}}| \cong [x,x]_{\widetilde{\epsilon}}^{\frac{1}{2}} [y,y]_{\widetilde{\epsilon}}^{\frac{1}{2}},$$

for all $\tilde{\epsilon} > \overline{0}$ and all $x, y \in SE(\check{H})$; this is,

$$|\langle (T+\widetilde{\epsilon}Id)x,y\rangle_{\check{H}}| \stackrel{\sim}{\leq} \langle (T+\widetilde{\epsilon}Id)x,x\rangle_{\check{H}}^{\frac{1}{2}} \langle (T+\widetilde{\epsilon}Id)y,y\rangle_{\check{H}'}^{\frac{1}{2}}$$

Thus, making $\tilde{\epsilon} \to \bar{0}$, we obtain the following

$$|\langle T(x), y \rangle|^2 \cong \langle T(x), x \rangle \langle T(y), y \rangle$$
, for all $x, y \in SE(\check{H})$.

Theorem 18. If $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ is a soft frame with bounds $\bar{b} \geq \bar{a} \geq \bar{0}$ for \check{H} , then the following statements are satisfied:

(i) The soft linear operator $S : SE(\check{H}) \to SE(\check{H})$ is invertible and

$$\overline{b}^{-1}Id \cong S^{-1} \cong \overline{a}^{-1}Id,$$

where $\overline{a}^{-1}(\lambda) = \frac{1}{a}$, for all $\lambda \in A$.

(*ii*) $\{S^{-1}(x_n)\}_{n\in\mathbb{N}}$ is a soft frame for \check{H} with bounds $\bar{a}^{-1} \cong \bar{b}^{-1} \cong \bar{0}$, called the dual soft frame of $\{x_n\}_{n\in\mathbb{N}}$.

Proof. (*i*) Since $\overline{a}Id \cong S \cong \overline{b}Id$ then

$$Id - \overline{b}^{-1}S = \overline{b}^{-1}(\overline{b}Id - S) \cong \overline{0},$$

and also,

$$-\overline{a}Id \stackrel{\sim}{\geq} -S \Leftrightarrow \overline{b}Id - \overline{a}Id \stackrel{\sim}{\geq} \overline{b}Id - S$$
$$\Leftrightarrow \overline{b}^{-1}(\overline{b} - \overline{a})Id \stackrel{\sim}{\geq} \overline{b}^{-1}(\overline{b}Id - S) \stackrel{\sim}{\geq} \overline{0}$$
$$\Leftrightarrow \frac{\overline{b} - \overline{a}}{\overline{b}}Id \stackrel{\sim}{\geq} Id - \overline{b}^{-1}S \stackrel{\sim}{\geq} \overline{0}.$$

In other words,

$$\left\langle \frac{\overline{b}-\overline{a}}{\overline{b}}Id(y),y\right\rangle_{\check{H}} \cong \left\langle (Id-\overline{b}^{-1}S)(y),y\right\rangle_{\check{H}} \cong \overline{0}$$

Then,

$$\|Id - \overline{b}^{-1}S\| = \sup_{\|y\| = \overline{1}} \left| \left\langle (Id - \overline{b}^{-1}S)(y), y \right\rangle_{\check{H}} \right| \stackrel{\sim}{=} \sup_{\|y\| = \overline{1}} \left| \left\langle \left(\frac{\overline{b} - \overline{a}}{\overline{b}}\right) Id(y), y \right\rangle_{\check{H}} \right| = \frac{\overline{b} - \overline{a}}{\overline{b}} \stackrel{\sim}{=} \overline{1}.$$

Therefore, by Corollary 1, it follows that *S* is invertible. In addition, observe that by Remark 2 and Cauchy–Schwarz inequality, we have the following:

$$\overline{0} \quad \widetilde{\leq} \quad \overline{a} \|S^{-1}x\|^2 = \langle \overline{a}Id(S^{-1}(x)), S^{-1}(x) \rangle_{\check{H}} \\ \widetilde{\leq} \quad \left\langle S(S^{-1}(x)), S^{-1}(x) \right\rangle_{\check{H}} = \left\langle x, S^{-1}(x) \right\rangle_{\check{H}} \\ = \quad \|x\|\|S^{-1}(x)\|.$$

Hence, $||S^{-1}|| \leq \overline{a}^{-1}$. Thus, the following holds:

$$\langle S^{-1}x, x \rangle_{\check{H}} \cong \|x\| \|S^{-1}(x)\| \cong \overline{a}^{-1} \|x\|^2 = \langle \overline{a}^{-1} Id(x), x \rangle_{\check{H}}.$$

Therefore, $S^{-1} \cong \overline{a}^{-1}Id$.

On the other hand, by Proposition 10 and frame condition, the following is satisfied:

$$\begin{split} \|x\|^4 &= \langle x, x \rangle_{\check{H}}^2 = \langle S(S^{-1}(x)), x \rangle_{\check{H}}^2 = \langle S^{-1}(x), S(x) \rangle_{\check{H}}^2 \\ & \stackrel{\sim}{\leq} \quad \langle S^{-1}(S(x)), S(x) \rangle_{\check{H}} \langle S^{-1}(x), x \rangle_{\check{H}} \\ &= \langle x, S(x) \rangle_{\check{H}} \langle S^{-1}(x), x \rangle_{\check{H}} \\ & \stackrel{\sim}{\leq} \quad \bar{b} \|x\|^2 \langle S^{-1}(x), x \rangle_{\check{H}}. \end{split}$$

Thus, $\langle S^{-1}(x), x \rangle_{\check{H}} \cong \overline{b}^{-1} ||x||^2 = \langle \overline{b}^{-1} Id(x), x \rangle_{\check{H}}$ and hence $S^{-1} \cong \overline{b}^{-1} Id$. In summary, we have proved that $\overline{b}^{-1} Id \cong S^{-1} \cong \overline{a}^{-1} Id$.

(*ii*) Since S^{-1} is positive and self-adjoint, we have the following:

$$\sum_{n\in\mathbb{N}}\langle x,S^{-1}x_n\rangle_{\check{H}}S^{-1}x_n = \sum_{n\in\mathbb{N}}\langle S^{-1}x,x_n\rangle_{\check{H}}S^{-1}x_n = S^{-1}\left(\sum_{n\in\mathbb{N}}\langle S^{-1}x,x_n\rangle_{\check{H}}x_n\right)$$
$$= S^{-1}(S(S^{-1}x)) = S^{-1}x$$

and as $\overline{b}^{-1}Id \cong S^{-1} \cong \overline{a}^{-1}Id$, by Proposition 9, we obtain that $\{S^{-1}(x_n)\}_{n\in\mathbb{N}}$ is a soft frame for \check{H} with bounds $\overline{a}^{-1} \cong \overline{b}^{-1} \cong \overline{0}$. \Box

Theorem 19 (Decomposition Theorem for soft frames). Let $\{x_n\}_{n \in \mathbb{N}} \subseteq SE(\check{H})$ be a soft frame with bounds $\bar{b} \cong \bar{a} \cong \bar{0}$ for \check{H} and S be the associated frame operator. Then, for all $x \in SE(\check{H})$, we have the following:

$$x = \sum_{n \in \mathbb{N}} \langle x, S^{-1} x_n \rangle_{\check{H}} x_n, \ x = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} S^{-1} x_n.$$

Proof. If $x \in SE(\check{H})$, then we have the following:

$$x = S(S^{-1}x) = \sum_{n \in \mathbb{N}} \langle S^{-1}x, x_n \rangle_{\check{H}} x_n = \sum_{n \in \mathbb{N}} \langle x, S^{-1}x_n \rangle_{\check{H}} x_n,$$

and also,

$$x = S^{-1}(S(x)) = S^{-1}\left(\sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} x_n\right) = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle_{\check{H}} S^{-1} x_n.$$

Note that Theorem 19 tells us that there is no restriction to write every element of a soft Hilbert space as a linear combination of the soft frame and the associated inverse soft frame operator, which differs from Theorem 16-(3).

4. Conclusions

In this article, we have introduced the concept of discrete frames in soft Hilbert spaces. In addition, we have studied the most important properties of frame theory, such as proving that the associated pre-frame operator to a soft frame is bounded. In addition, we have introduced a soft inner product that allows us to demonstrate a version of the Cauchy–Schwarz inequality for positive soft linear operators, a concept that we have also coined here. We have investigated some characterizations for soft frames and finished with the decomposition theorem for soft frames, which allows us to write or decompose every element of a soft Hilbert space, using the inverse of the associated frame operator. The theoretical framework established in this article can be used to extend the results given in [28] as well as the classical results of wavelet and dual frame theory to the context of soft Hilbert spaces. **Author Contributions:** Conceptualization, O.F., A.S. and J.S.; methodology, O.F., A.S. and J.S.; validation, O.F., A.S. and J.S.; formal analysis, O.F., A.S. and J.S.; investigation, O.F., A.S. and J.S.; resources, O.F., A.S. and J.S.; writing—original draft preparation, O.F., A.S. and J.S.; writing—review and editing, J.S.; visualization, J.S.; supervision, O.F., A.S. and J.S. All authors have read and agreed to the published version of the manuscript.

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