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Wavelets and Real Interpolation of Besov Spaces

Zhenzhen Lou [1,2], Qixiang Yang [3], Jianxun He [1,*] and Kaili He [1]

[1] School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, China; zhenzhenlou@e.gzhu.edu.cn (Z.L.); kailihe@e.gzhu.edu.cn (K.H.)
[2] School of Mathematics and Statistics, Qujing Normal University, Qujing 655011, China
[3] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China; qxyang@whu.edu.cn
[*] Correspondence: hejianxun@gzhu.edu.cn

Abstract: In view of the importance of Besov space in harmonic analysis, differential equations, and other fields, Jaak Peetre proposed to find a precise description of $(B^{s_0,q_0}_{p_0}, B^{s_1,q_1}_{p_1})_{\theta,r}$. In this paper, we come to consider this problem by wavelets. We apply Meyer wavelets to characterize the real interpolation of homogeneous Besov spaces for the crucial index p and obtain a precise description of $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$.

Keywords: real interpolation; besov space; meyer wavelet

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1. Introduction

Since the middle of 20th century, the study of interpolation space has greatly promoted the development of function space, operator theory, and developed a set of perfect mathematical theories. It greatly enriches the theory of harmonic analysis, see [1–4]. However, for a long time, only the real interpolation spaces of Lebesgue spaces have been studied thoroughly, their forms are known as Lorentz spaces, and there are a lot of literature about Lorentz spaces, see [2,5–9].

For the real interpolation of Besov spaces, we can refer to [9–16]. When the index p is fixed, it has been shown that $(B^{s_0,q_0}_{p}, B^{s_1,q_1}_{p})_{\theta,r}$ are still Besov spaces, see [4,9,16]. The interpolation for the index p is very different to which for the indices s and q. If $p_0 \neq p_1$, then $(B^{s,q}_{p_0}, B^{s,q}_{p_1})_{\theta,r}$ will fall outside of the scale of Besov spaces. J. Peetre proposed to consider the real interpolation of Besov spaces in [4]. For more than forty years, due to some inherent difficulties, little progress has been made in this regard.

In this paper, we consider the interpolation problem introduced in [4] for the crucial index p. Wavelets have localization of both frequency and spatial position, which provides a powerful tool for the study of the interpolation of Besov spaces. In this paper, we obtain a precise description of $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ by Meyer wavelets. Further, as $q = r$, we prove that $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,q}$ can fall into the Besov–Lorentz spaces in [17].

For Besov and Triebel–Lizorkin spaces, we use the characterization based on the Littlewood–Paley decomposition, see [9,18,19]. Given a function φ, such that its Fourier transform $\widehat{\varphi}(\xi) \in C^\infty_0(\mathbb{R}^n)$ and satisfies

$$\operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \; and \; \hat{\varphi}(\xi) = 1, \text{ if } |\xi| \leq \frac{1}{2}.$$

For $u \in \mathbb{Z}$, we define φ_u by

$$\varphi_u(x) = 2^{n(u+1)}\varphi(2^{u+1}x) - 2^{nu}\varphi(2^u x).$$

These functions $\{\varphi_u(x)\}_{u\in\mathbb{Z}}$ satisfy

$$\begin{cases} \text{supp } \hat{\varphi}_u \subset \{\xi \in \mathbb{R}^n, \ \frac{1}{2} \le 2^{-u}|\xi| \le 2\}; \\ |\hat{\varphi}_u(\xi)| \ge C > 0, \ \text{if } \frac{1}{2} < C_1 \le 2^{-u}|\xi| \le C_2 < 2; \\ |\partial^k \hat{\varphi}_u(\xi)| \le C_k 2^{-u|k|}, \ \text{for all } k \in \mathbb{N}^n; \\ \sum_{u=-\infty}^{+\infty} \hat{\varphi}_u(\xi) = 1, \ \text{for any } \xi \in \mathbb{R}^n. \end{cases}$$

Denote the space of all Schwartz functions on $\mathbb{R}^n$ by $\mathcal{S}(\mathbb{R}^n)$. The dual space of $\mathcal{S}(\mathbb{R}^n)$, namely, the space of all tempered distributions on $\mathbb{R}^n$, equipped with the weak-$*$ topology, is denoted by $\mathcal{S}'(\mathbb{R}^n)$. Denote the space of all polynomials on $\mathbb{R}^n$ by $P(\mathbb{R}^n)$. Let $f \in \mathcal{S}'(\mathbb{R}^n)\backslash P(\mathbb{R}^n)$. Define $f_u = \varphi_u * f$, the f_u is called the u-th dyadic block of the Littlewood–Paley decomposition of f. We recall the definition of $\dot{B}_p^{s,q}$ and $\dot{F}_p^{s,q}$.

Definition 1. *Given* $s \in \mathbb{R}$, $0 < q \le \infty$ *and* $u \in \mathbb{Z}$. *For* $f \in S'(\mathbb{R}^n)\backslash P(\mathbb{R}^n)$, *we define*

(i) *For* $0 < p \le \infty$, $f \in \dot{B}_p^{s,q}$, *if* $\left(\sum_u 2^{usq}\|f_u(x)\|_{L^p}^q\right)^{\frac{1}{q}} < \infty$.

(ii) *For* $0 < p < \infty$, $f \in \dot{F}_p^{s,q}$, *if* $\left\|\left(\sum_u 2^{usq}|f_u(x)|^q\right)^{\frac{1}{q}}\right\|_{L^p} < \infty$.

As $q = \infty$, *it should be replaced by the supremum norm.*

The definition of the above two spaces are independent of the selection of the functions φ, see [9].

Then, we recall some notations of Meyer wavelets. Let Ψ^0 be an even function in $C_0^\infty([-\frac{4\pi}{3}, \frac{4\pi}{3}])$ satisfying

$$\begin{cases} 0 \le \Psi^0(\xi) \le 1; \\ \Psi^0(\xi) = 1, \ \text{for } |\xi| \le \frac{2\pi}{3}. \end{cases}$$

Let

$$\Omega(\xi) = \sqrt{(\Psi^0(\frac{\xi}{2}))^2 - (\Psi^0(\xi))^2}.$$

Then, $\Omega(\xi)$ is an even function in $C_0^\infty([-\frac{8\pi}{3}, \frac{8\pi}{3}])$. It is easy to get

$$\begin{cases} \Omega(\xi) = 0, \ \text{for } |\xi| \le \frac{2\pi}{3}; \\ \Omega^2(\xi) + \Omega^2(2\xi) = 1 = \Omega^2(\xi) + \Omega^2(2\pi - \xi), \ \text{for } \frac{2\pi}{3} \le \xi \le \frac{4\pi}{3}. \end{cases}$$

Denote $\Psi^1(\xi) := \Omega(\xi)e^{-\frac{i\xi}{2}}$. For all $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \{0,1\}^n$, define

$$\hat{\Phi}^\epsilon(\xi) := \prod_{i=1}^n \Psi^{\epsilon_i}(\xi_i).$$

Furthermore, $\Gamma := \{(\epsilon, k), \epsilon \in \{0,1\}^n \setminus \{(0,...,0)\}, k \in \mathbb{Z}^n\}$ and

$$\Lambda := \{(\epsilon, j, k) : \epsilon \in \{0,1\}^n \setminus \{(0,...,0)\}, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

For $(\epsilon, j, k) \in \Lambda$, denote

$$\Phi_{j,k}^\epsilon(x) := 2^{\frac{jn}{2}}\Phi^\epsilon(2^j x - k).$$

For $f \in \mathcal{S}'$, let $a^{\epsilon}_{j,k} = \langle f, \Phi^{\epsilon}_{j,k}\rangle$. The following results are well-known, see [17,18,20].

Lemma 1. *The Meyer wavelets $\{\Phi^{\epsilon}_{j,k}\}_{(\epsilon,j,k)\in\Lambda}$ form an orthogonal basis in $L^2(\mathbb{R}^n)$, hence, for all $f \in L^2(\mathbb{R}^n)$, the following wavelet decomposition holds in L^2 sense,*

$$f = \sum_{(\epsilon,j,k)\in\Lambda} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}.$$

In this paper, we first give some precise descriptions of $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ with wavelets. Let $\chi(x)$ be the characteristic function on the unit cube $[0,1)^n$. For Borel set F in $\mathbb{R}^n$, denote $|F|$ the Lebesgue measure of F. Suppose that $j,u \in \mathbb{Z}$, $1 < p_0 < p_1 < \infty$ and $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$, denote

$$c_{j,n}(\tau) := \inf\left\{\lambda : \left|\left\{x \in \mathbb{R}^n : \sum_{(\epsilon,k)\in\Gamma} |a^{\epsilon}_{j,k}|\chi(2^j x - k) > 2^{-\frac{nj}{2}}\lambda\right\}\right| \leq \tau\right\},$$

$$b^{p_0,p_1}_{j,n,u} := \left(\int_0^{2^{u\alpha}} (c_{j,n}(\tau))^{p_0} d\tau\right)^{\frac{1}{p_0}} + 2^u\left(\int_{2^{u\alpha}}^{\infty} (c_{j,n}(\tau))^{p_1} d\tau\right)^{\frac{1}{p_1}}.$$

Theorem 1. *Given $\theta \in (0,1)$, $s \in \mathbb{R}$, $1 < p_0 < p_1 < \infty$, $0 < q,r \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. For $f = \sum_{(\epsilon,j,k)\in\Lambda} a^{\epsilon}_{j,k}\Phi_{j,k}$, we have*

(i) $f \in (\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{\infty})_{\theta,r}$ *if, and only if,*

$$\sum_u 2^{-ur\theta}\left\{\sum_j 2^{jsq}\left[\int_0^{2^{up_0}} (c_{j,n}(\tau))^{p_0} d\tau\right]^{\frac{q}{p_0}}\right\}^{\frac{r}{q}} < \infty;$$

(ii) $f \in (\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ *if and only if*

$$\sum_u 2^{-ur\theta}\left\{\sum_j 2^{jsq}\left[b^{p_0,p_1}_{j,n,u}\right]^q\right\}^{\frac{r}{q}} < \infty.$$

The above wavelet characterization is slightly complicated. Yang-Cheng-Peng [17] introduced Besov–Lorentz spaces. Further, when $q = r$, we can prove that $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,q}$ are just the Besov–Lorentz spaces defined in [17]. We have

Theorem 2. *Let $\theta \in (0,1)$, $s \in \mathbb{R}$, $0 < q \leq \infty$, $1 < p_0 < p_1 < \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $u \in \mathbb{Z}$ and $f = \sum_{(\epsilon,j,k)\in\Lambda} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}$. Then the following conditions are equivalent.*

(i) $f \in (\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,q}$ *if, and only if,*

$$\sum_j 2^{jsq}\left\{\sum_u 2^{uq}\left|\left\{x \in \mathbb{R}^n : \left|\sum_{(\epsilon,k)\in\Gamma} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)\right| > 2^u\right\}\right|^{\frac{q}{p}}\right\} < \infty.$$

(ii) $f \in (\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,q}$ *if, and only if,*

$$\sum_j 2^{jsq}\left(\sum_u 2^{uq}\left|\left\{x \in \mathbb{R}^n : 2^{\frac{nj}{2}}\sum_{(\epsilon,k)\in\Gamma} |a^{\epsilon}_{j,k}|\chi(2^j x - k) > 2^u\right\}\right|^{\frac{q}{p}}\right) < \infty.$$

Although the above main results still not solve the problem proposed by J. Peetre [4] thoroughly, we obtain a precise description of $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ by Meyer wavelets. The wavelet characterization of real interpolation spaces of Besov spaces provides people with an effective means to study the continuity of linear operators and bilinear operators on such spaces. We are using this point to study the well-posedness of non-linear fluid equations.

The plan of this paper is the following. In Section 2, we recall the general background of the real interpolation method and Lorentz spaces. Then we review wavelet characterization of $\dot{B}^{s,q}_{p}$ and $\dot{F}^{s,q}_{p}$. In Section 3, we give the proof of Theorem 1. Finally, in Section 4 we prove Theorem 2.

In this paper, $A \lesssim B$ means the estimation of the form $A \leq CB$ with some constant C independent of the main parameters, C may vary from line to line. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries on Real Interpolation and Wavelets

In this section, we present some preliminaries on real interpolation and wavelets.

2.1. K-Functional and Real Interpolation

The K-functional was introduced by J. Peetre in the process of dealing with real interpolation spaces, see [1,4]. If (A_0, A_1) is a pair of quasi-normed spaces which are continuously embedded in a Hausdorff space X, then the K-functional

$$K(t, f, A_0, A_1) := \inf_{f=f_0+f_1} \{\|f_0\|_{A_0} + t\|f\|_{A_1}\}$$

is defined for all $f = f_0 + f_1$, where $f_0 \in A_0$, $f_1 \in A_1$.

Definition 2. *Let* $0 < \theta < 1$ *and* $0 < q < \infty$. *We define*

$$\begin{gathered}(A_0, A_1)_{\theta,q,K} =: \\ \left\{ f : f \in A_0 + A_1, \|f\|_{(A_0,A_1)_{\theta,q,K}} = \left\{ \int_0^\infty \left[t^{-\theta} K(t, f, A_0, A_1)\right]^q \frac{dt}{t} \right\}^{\frac{1}{q}} < \infty \right\}.\end{gathered} \tag{1}$$

Further, we define

$$\begin{gathered}(A_0, A_1)_{\theta,\infty,K} =: \\ \left\{ f : f \in A_0 + A_1, \|f\|_{(A_0,A_1)_{\theta,\infty,K}} = \sup_t t^{-\theta} K(t, f, A_0, A_1) < \infty \right\}.\end{gathered} \tag{2}$$

Bergh-Löfström [1] has shown that the norms of the spaces $(A_0, A_1)_{\theta,q,K}$ in (1) and (2) have the following discrete representation.

Lemma 2. *Let* $0 < \theta < 1$. *Then,*

$$\|f\|_{(A_0,A_1)_{\theta,q,K}} \sim \begin{cases} \left[\sum\limits_{j\in\mathbb{Z}} 2^{-jq\theta} K(2^j, f, A_0, A_1)^q\right]^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup\limits_{j\in\mathbb{Z}} 2^{-j\theta} K(2^j, f, A_0, A_1), & q = \infty. \end{cases} \tag{3}$$

In the following part, we always use this form. For $x \in \mathbb{R}^n$ and function $f(x)$, the distribution function $\sigma_f(\lambda)$ and rearrangement function $f^*(\tau)$ are defined in the following way

$$\sigma_f(\lambda) = |\{x : |f(x)| > \lambda\}| \text{ and } f^*(\tau) = \inf\{\lambda : \sigma_f(\lambda) \leq \tau\}.$$

We review some results about K-functional, see [3].

Lemma 3. *Suppose that* $0 < p < \infty$ *and* $f \in L^p + L^\infty$. *Then*

$$K(t, f, L^p, L^\infty) \sim \left[\int_0^{t^p} (f^*(\tau))^p d\tau\right]^{\frac{1}{p}}.$$

Lemma 4. *If* $0 < p_0 < p_1 < \infty$ *and* $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$, *then*

$$K(t, f, L^{p_0}, L^{p_1}) \sim \left[\int_0^{t^\alpha} (f^*(\tau))^{p_0} d\tau\right]^{\frac{1}{p_0}} + t\left[\int_{t^\alpha}^\infty (f^*(\tau))^{p_1} d\tau\right]^{\frac{1}{p_1}}.$$

For $0 < p < \infty$, K-functional can be replaced to K_p functional, see [21]. Define K_p functional by

$$K_p := K_p(t, f, A_0, A_1) = \inf_{f=f_0+f_1} (\|f_0\|_{A_0}^p + t^p \|f_1\|_{A_1}^p)^{\frac{1}{p}},$$

and

$$\|f\|_{(A_0,A_1)_{\theta,q,K_p}} = \left[\int_0^\infty [t^{-\theta} K_p(t, f, A_0, A_1)]^q \frac{dt}{t}\right]^{\frac{1}{q}}.$$

We recall an important lemma about $K_p(t, f, A_0, A_1)$, see [21].

Lemma 5. *Let* (A_0, A_1) *be a couple of quasi-normed spaces. For any* $0 < p < \infty$, *we have*

$$\|f\|_{(A_0,A_1)_{\theta,q,K}} \sim \|f\|_{(A_0,A_1)_{\theta,q,K_p}}.$$

2.2. Lorentz Spaces and Lebesgue Spaces

In this subsection, we present first the definition of Lorentz spaces which are the generalization of Lebesgue spaces and then some relative lemmas.

Definition 3. *For* $1 \le p < \infty$ *and* $0 < r < \infty$, *the Lorentz spaces* $L^{p,r}$ *are defined as follows*

$$L^{p,r} = \left\{ f : \|f\|_{p,r} = \left[\frac{r}{p} \int_0^\infty \left(\tau^{\frac{1}{p}} f^*(\tau)\right)^r \frac{d\tau}{\tau}\right]^{\frac{1}{r}} < \infty \right\}.$$

For $r = \infty$,

$$L^{p,\infty} = \left\{ f : \|f\|_{p,\infty} = \sup_\tau \tau^{\frac{1}{p}} f^*(\tau) < \infty \right\}.$$

It is easy to see that $L^{p,p} = L^p$. Further, $L^{p,\infty}$ corresponds to the weak L^p spaces. The above definition depends on the rearrangement function $f^*(\tau)$. These spaces can be characterized by distribution function $\sigma_f(\lambda)$ also, see [2].

Lemma 6. *Let* $1 \le p < \infty$ *and* $0 < r \le \infty$. *Then, for any* $f \in L^{p,r}$, *one has*

$$\|f\|_{p,r} \sim \left[r \int_0^\infty \left(\lambda \sigma_f^{\frac{1}{p}}(\lambda)\right)^r \frac{d\lambda}{\lambda}\right]^{\frac{1}{r}} \textit{ and } \|f\|_{p,\infty} \sim \sup_\lambda \lambda \sigma_f^{\frac{1}{p}}(\lambda).$$

The above continuous integral can be written as the following discrete form, see [17].

Lemma 7. *Suppose that* $1 \le p < \infty$ *and* $0 < r < \infty$. *Then* $f \in L^{p,r}$, *if*

$$\left(\sum_u 2^{ru} |\{x \in \mathbb{R}^n : |f(x)| > 2^u\}|^{\frac{r}{p}}\right)^{\frac{1}{r}} < \infty,$$

as $r = \infty$, the L^r-norm should be replaced by the L^∞-norm.

The above Lorentz spaces are in fact real interpolation of Lebesgue spaces L^p, see [1].

Lemma 8. *Assume that $0 < p_0 < p_1 \leq \infty$, $0 < r \leq \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$(L^{p_0}, L^{p_1})_{\theta,r} = L^{p,r}, \text{ with } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

By Lemma 8, we get another characterization of $L^{p,r}$ as below.

Corollary 1. *Let all parameters be as defined in Lemma 8. Then,*

$$\|f\|_{p,r} \sim \left[\int_0^\infty \left(t^{-\theta} K(t, f, L^{p_0}, L^{p_1})\right)^r \frac{dt}{t}\right]^{\frac{1}{r}}.$$

2.3. Wavelet Characterization of $\dot{B}_p^{s,q}$ and $\dot{F}_p^{s,q}$

For any function $f(x)$ in $\dot{B}_p^{s,q}$ or $\dot{F}_p^{s,q}$ in Definition 1, the following wavelet decomposition holds in the sense of distribution,

$$f = \sum_{(\epsilon,j,k)\in\Lambda} a_{j,k}^\epsilon \Phi_{j,k}^\epsilon.$$

We recall the wavelet characterization of $\dot{B}_p^{s,q}$ and $\dot{F}_p^{s,q}$ in this subsection, see [16–18,20]. For any $s \in \mathbb{R}$ and $0 < q \leq \infty$, denote

$$S_{s,q}f(x) := \left(\sum_{(\epsilon,j,k)\in\Lambda} 2^{qj(s+\frac{n}{2})} |a_{j,k}^\epsilon|^q \chi(2^j x - k)\right)^{\frac{1}{q}}.$$

When $s = 0$ *and* $q = 2$, we denote $Sf := S_{s,q}f$.

Lemma 9. *Let $s \in \mathbb{R}$ and $0 < q \leq \infty$.*

(i) *For $0 < p < \infty$, $f \in \dot{F}_p^{s,q}(\mathbb{R}^n)$ if, and only if,*

$$\|S_{s,q}f\|_{L^p} < +\infty.$$

(ii) *For $0 < p \leq \infty$, $f \in \dot{B}_p^{s,q}(\mathbb{R}^n)$ if, and only if,*

$$\left[\sum_{j\in\mathbb{Z}} 2^{qj(s-\frac{n}{p}+\frac{n}{2})} \left(\sum_{(\epsilon,k)\in\Gamma} |a_{j,k}^\epsilon|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} < \infty.$$

It is easy to see that $\dot{F}_p^{0,2} = L^p$. In [17], Yang-Cheng-Peng proved the wavelet characterization of Lorentz spaces $L^{p,r}$.

Lemma 10. *Suppose that $1 \leq p < \infty$, $0 < r < \infty$ and $u \in \mathbb{Z}$. Then $f \in L^{p,r}$, if*

$$\left(\sum_u 2^{ru} |\{x \in \mathbb{R}^n : |Sf(x)| > 2^u\}|^{\frac{r}{p}}\right)^{\frac{1}{r}} < \infty,$$

as $r = \infty$, the L^r-norm should be replaced by the L^∞-norm.

Remark 1. *f and Sf can control each other by using good$-\lambda$ inequality. When the Fourier transform of f is supported on a ring, f and Sf can control each other. The distribution function $\sigma_f(\lambda)$ and rearrangement function $f^*(\tau)$ can be replaced by $\sigma_{Sf}(\lambda)$ and $(Sf)^*(\tau)$, see [17]. Without affecting the proof, these notations are not strictly distinguished in this paper.*

3. Proof of Theorem 1

In this section, we characterize $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{\infty})_{\theta,r}$, and $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ with wavelets. Now we come to prove Theorem 1.

Proof. Denote

$$\|f\|_p := \|f\|_{L^p}, \|f\|_{(A_0,A_1)_{\theta,q}} := \|f\|_{(A_0,A_1)_{\theta,q,K}}.$$

For any function f in $\dot{B}^{s,q}_p$, the following wavelet decomposition holds in the sense of distribution,

$$f = \sum_{(\epsilon,j,k)\in\Lambda} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}.$$

From Lemma 9, it follows that

$$\begin{aligned} K_q(t,f) := K_q(t,f,\dot{B}^{s,q}_{p_0},\dot{B}^{s,q}_{p_1}) = \inf\Bigg[& \sum_j 2^{jq(s-\frac{n}{p_0}+\frac{n}{2})}\Bigg(\sum_{(\epsilon,k)\in\Gamma}|x^{\epsilon}_{j,k}|^{p_0}\Bigg)^{\frac{q}{p_0}} \\ & + t^q\sum_j 2^{jq(s-\frac{n}{p_1}+\frac{n}{2})}\Bigg(\sum_{(\epsilon,k)\in\Gamma}|a^{\epsilon}_{j,k}-x^{\epsilon}_{j,k}|^{p_1}\Bigg)^{\frac{q}{p_1}}\Bigg]^{\frac{1}{q}}. \end{aligned}$$

Denote

$$x_j = \sum_{(\epsilon,k)\in\Gamma} x^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)\ ,\ a_j = \sum_{(\epsilon,k)\in\Gamma} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x).$$

By Lemma 9, we deduce that

$$\begin{aligned} \|x_j\|_{p_0} &= 2^{j(-\frac{n}{p_0}+\frac{n}{2})}\left\{\sum_k\left(\sum_{\epsilon}|x^{\epsilon}_{j,k}|^2\right)^{\frac{p_0}{2}}\right\}^{\frac{1}{p_0}} \\ &\sim 2^{j(-\frac{n}{p_0}+\frac{n}{2})}\left\{\sum_{(\epsilon,k)\in\Gamma}|x^{\epsilon}_{j,k}|^{p_0}\right\}^{\frac{1}{p_0}}, \\ \|a_j-x_j\|_{p_1} &= 2^{j(-\frac{n}{p_1}+\frac{n}{2})}\left\{\sum_k\left(\sum_{\epsilon}|a^{\epsilon}_{j,k}-x^{\epsilon}_{j,k}|^2\right)^{\frac{p_1}{2}}\right\}^{\frac{1}{p_1}} \\ &\sim 2^{j(-\frac{n}{p_1}+\frac{n}{2})}\left\{\sum_{(\epsilon,k)\in\Gamma}|a^{\epsilon}_{j,k}-x^{\epsilon}_{j,k}|^{p_1}\right\}^{\frac{1}{p_1}}. \end{aligned}$$

Hence,

$$
\begin{aligned}
K_q(t,a_j) &\sim \left[\sum_j 2^{jsq}\inf\Big(\|x_j\|_{p_0}^q + t^q\|a_j - x_j\|_{p_1}^q\Big)\right]^{\frac{1}{q}} \\
&\sim \left\{\sum_j 2^{jsq}\left[\inf(\|x_j\|_{p_0} + t\|a_j - x_j\|_{p_1})\right]^q\right\}^{\frac{1}{q}} \\
&= \left\{\sum_j 2^{jsq}\left[K\big(t, a_j, L^{p_0}, L^{p_1}\big)\right]^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|f\|_{(\dot{B}^{s,q}_{p_0},\dot{B}^{s,q}_{p_1})_{\theta,r}} &\sim \left\{\int_0^\infty [t^{-\theta}K_q(t,a)]^r \frac{dt}{t}\right\}^{\frac{1}{r}} \\
&\sim \left\{\int_0^\infty \left[t^{-\theta}\left\{\sum_j 2^{jsq}\left[K(t,a_j,L^{p_0},L^{p_1})\right]^q\right\}^{\frac{1}{q}}\right]^r \frac{dt}{t}\right\}^{\frac{1}{r}}.
\end{aligned} \tag{4}
$$

If $1 < p_0 < p_1 < \infty$, then $L^{p_0} = \dot{F}^{0,2}_{p_0}$ and $L^{p_1} = \dot{F}^{0,2}_{p_1}$. Applying Remark 1, we have

$$
(S^* f)(\tau) := (S^*_{0,2} f)(\tau) = \inf\{\lambda : |\{x \in \mathbb{R}^n : S_{0,2} f(x) > \lambda\}| \le \tau\}.
$$

For $a_j = \sum_{(\epsilon,k)\in\Gamma} a^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x)$, we have

$$
\begin{aligned}
Sa_j(x) := S_{0,2}a_j(x) &= \left(\sum_{(\epsilon,k)\in\Gamma} 2^{2j(0+\frac{n}{2})}|a^\epsilon_{j,k}|^2\chi(2^jx-k)\right)^{\frac{1}{2}} \\
&= \left(\sum_{(\epsilon,k)\in\Gamma} 2^{jn}|a^\epsilon_{j,k}|^2\chi(2^jx-k)\right)^{\frac{1}{2}} \\
&= 2^{\frac{jn}{2}}\sum_k\left(\sum_\epsilon |a^\epsilon_{j,k}|^2\right)^{\frac{1}{2}}\chi(2^jx-k) \\
&\sim 2^{\frac{jn}{2}}\sum_{(\epsilon,k)\in\Gamma} |a^\epsilon_{j,k}|\chi(2^jx-k).
\end{aligned}
$$

Thus,

$$
Sa_j(x) = 2^{\frac{jn}{2}}\sum_{(\epsilon,k)\in\Gamma} |a^\epsilon_{j,k}|\chi(2^jx-k). \tag{5}
$$

By (5), we deduce that

$$
\begin{aligned}
(S^*a_j)(\tau) &= \inf\{\lambda : |\{x\in\mathbb{R}^n : Sa_j(x) > \lambda\}| \le \tau\} \\
&= \inf\left\{\lambda : \left|\left\{x\in\mathbb{R}^n : 2^{\frac{jn}{2}}\sum_{(\epsilon,k)\in\Gamma}|a^\epsilon_{j,k}|\chi(2^jx-k) > \lambda\right\}\right| \le \tau\right\} \\
&= \inf\left\{\lambda : \left|\left\{x\in\mathbb{R}^n : \sum_{(\epsilon,k)\in\Gamma}|a^\epsilon_{j,k}|\chi(2^jx-k) > 2^{-\frac{jn}{2}}\lambda\right\}\right| \le \tau\right\}.
\end{aligned}
$$

Denote

$$c_{j,n}(\tau) := \inf\left\{\lambda : \left|\left\{x \in \mathbb{R}^n : \sum_{(\epsilon,k)\in\Gamma} |a^{\epsilon}_{j,k}|\chi(2^j x - k) > 2^{-\frac{jn}{2}}\lambda\right\}\right| \leq \tau\right\}. \tag{6}$$

Hence,

$$(S^* a_j)(\tau) = c_{j,n}(\tau). \tag{7}$$

Let us prove the theorem in two cases.

(i) For $p_1 = \infty$, by Remark 1 and Lemma 3, we have

$$K(t, a_j, L^{p_0}, L^\infty) \sim \left[\int_0^{t^{p_0}} (a^*_j(\tau))^{p_0} d\tau\right]^{\frac{1}{p_0}} \sim \left[\int_0^{t^{p_0}} [(S^* a_j)(\tau)]^{p_0} d\tau\right]^{\frac{1}{p_0}}.$$

By (6) and (7), we get

$$K(t, a_j, L^{p_0}, L^\infty) \sim \left[\int_0^{t^{p_0}} (c_{j,n}(\tau))^{p_0} d\tau\right]^{\frac{1}{p_0}}. \tag{8}$$

Applying (4), (8) and the discrete representation of the spaces $(A_0, A_1)_{\theta,q,K}$ which is described in Remark 3, we obtain

$$\|f\|^r_{(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{\infty})_{\theta,r}} \sim \sum_u 2^{-ur\theta}\left\{\sum_j 2^{jsq}\left[\int_0^{2^{up_0}} (c_{j,n}(\tau))^{p_0} d\tau\right]^{\frac{q}{p_0}}\right\}^{\frac{r}{q}}.$$

(ii) For $1 < p_0 < p_1 < \infty$, by Lemma 4, similar as we did in (i), we have

$$K(t, a_j, L^{p_0}, L^{p_1}) \sim \left[\int_0^{t^\alpha} (c_{j,n}(\tau))^{p_0} d\tau\right]^{\frac{1}{p_0}} + t\left[\int_{t^\alpha}^{\infty} (c_{j,n}(\tau))^{p_1} d\tau\right]^{\frac{1}{p_1}}, \tag{9}$$

where $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$. Denote

$$b^{p_0,p_1}_{j,n,u,} := \left(\int_0^{2^{u\alpha}} (c_{j,\lambda}(\tau))^{p_0} d\tau\right)^{\frac{1}{p_0}} + 2^u\left(\int_{2^{u\alpha}}^{\infty} (c_{j,\lambda}(\tau))^{p_1} d\tau\right)^{\frac{1}{p_1}}.$$

Combining (4) with (9) and using the discrete representation of the spaces $(A_0, A_1)_{\theta,q,K}$ which is described in Remark 3, we know that

$$\|f\|^r_{(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}} \sim \sum_u 2^{-ur\theta}\left\{\sum_j 2^{jsq}\left[b^{p_0,p_1}_{j,n,u}\right]^q\right\}^{\frac{r}{q}}.$$

The proof of Theorem 1 is complete. □

4. Proof of Theorem 2

Now we come to prove Theorem 2.

Proof. Applying Lemma 5, the same as we did in the proof of Theorem 1, we can also get

$$\|f\|_{(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}} \sim \left\{\int_0^\infty \left[t^{-\theta}\left\{\sum_j 2^{jsq}\left[K(t, a_j, L^{p_0}, L^{p_1})\right]^q\right\}^{\frac{1}{q}}\right]^r \frac{dt}{t}\right\}^{\frac{1}{r}},$$

where $f = \sum_{(\epsilon,j,k)\in\Lambda} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}$, $a_j = \sum_{(\epsilon,k)\in\Gamma} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}$. As $r = q$, we can write

$$\begin{aligned}
\|f\|_{(\dot{B}^{s,q}_{p_0},\dot{B}^{s,q}_{p_1})_{\theta,q}} &\sim \left\{\int_0^{\infty}\left[t^{-\theta q}\sum_j 2^{jsq}\left[K(t,a_j,L^{p_0},L^{p_1})\right]^q\right]\frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\sim \left\{\sum_j 2^{jsq}\left[\int_0^{\infty} t^{-\theta q}\left[K(t,a_j,L^{p_0},L^{p_1})\right]^q\frac{dt}{t}\right]\right\}^{\frac{1}{q}} \\
&= \left\{\sum_j 2^{jsq}\left(\left[\int_0^{\infty} t^{-\theta q}\left[K(t,a_j,L^{p_0},L^{p_1})\right]^q\frac{dt}{t}\right]^{\frac{1}{q}}\right)^q\right\}^{\frac{1}{q}} \\
&\sim \left\{\sum_j 2^{jsq}\|a_j\|^q_{(L^{p_0},L^{p_1})_{\theta,q}}\right\}^{\frac{1}{q}}.
\end{aligned}$$

Thus,

$$\|f\|_{(\dot{B}^{s,q}_{p_0},\dot{B}^{s,q}_{p_1})_{\theta,q}} \sim \left\{\sum_j 2^{jsq}\|a_j\|^q_{(L^{p_0},L^{p_1})_{\theta,q}}\right\}^{\frac{1}{q}}. \tag{10}$$

We will prove the theorem in two cases.

(i) For $a_j = \sum_{(\epsilon,k)\in\Gamma} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, using Lemma 7, we have

$$\|a_j\|_{(L^{p_0},L^{p_1})_{\theta,q}} = \|a_j\|_{p,q} = \left\{\sum_u 2^{uq}\left|\left\{x\in\mathbb{R}^n : \left|\sum_{(\epsilon,k)\in\Gamma} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)\right| > 2^u\right\}\right|^{\frac{q}{p}}\right\}^{\frac{1}{q}}. \tag{11}$$

From (10) and (11), it follows that

$$\|f\|^q_{(\dot{B}^{s,q}_{p_0},\dot{B}^{s,q}_{p_1})_{\theta,q}} \sim \sum_j 2^{jsq}\left\{\sum_u 2^{uq}\left|\left\{x\in\mathbb{R}^n : \left|\sum_{(\epsilon,k)\in\Gamma} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)\right| > 2^u\right\}\right|^{\frac{q}{p}}\right\}.$$

(ii) Applying Lemma 10, we obtain another equivalent form of $\|a_j\|_{p,q}$,

$$\begin{aligned}
\|a_j\|_{(L^{p_0},L^{p_1})_{\theta,q}} = \|a_j\|_{p,q} &= \left(\sum_u 2^{ru}\left|\{x\in\mathbb{R}^n : |Sa_j(x)| > 2^u\}\right|^{\frac{q}{p}}\right)^{\frac{1}{q}} \\
&= \left\{\sum_u 2^{uq}\left|\left\{x\in\mathbb{R}^n : 2^{\frac{nj}{2}}\sum_{(\epsilon,k)\in\Gamma}|a^{\epsilon}_{j,k}|\chi(2^j x - k) > 2^u\right\}\right|^{\frac{q}{p}}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{12}$$

Applying (10) and (12), we obtain that

$$\|f\|^q_{(\dot{B}^{s,q}_{p_0},\dot{B}^{s,q}_{p_1})_{\theta,q}} \sim \sum_j 2^{jsq}\left\{\sum_u 2^{uq}\left|\left\{x\in\mathbb{R}^n : 2^{\frac{nj}{2}}\sum_{(\epsilon,k)\in\Gamma}|a^{\epsilon}_{j,k}|\chi(2^j x - k) > 2^u\right\}\right|^{\frac{q}{p}}\right\}.$$

We finish the proof of Theorem 2. □

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References

1. Bergh, J.; Löfström, J. *Interpolation Spaces, An Introduction*; Springer: New York, NY, USA, 1976.
2. Cheng, M.; Deng, D.; Long, R. *Real Analysis*, 2rd ed.; High Eduacatin Press: Beijing, China, 2008.
3. Holmstedt, T. Interpolation of quasi-normed spaces. *Math. Scand.* **1970**, *10*, 177–199. [CrossRef]
4. Peetre, J. *New Thoughts on Besov Space*; Math Series; Duke University: Durham, NC, USA, 1976.
5. Blozinski, A.P. On a convolution theorem for $L(p,q)$ spaces. *Trans. Am. Math. Soc.* **1972**, *164*, 255–265.
6. Hunt, R.A. On $L(p,q)$ spaces. *Enseign. Math.* **1966**, *12*, 249–276.
7. Lorentz, G.G. Some new functional spaces. *Ann. Math.* **1950**, *51*, 37–55. [CrossRef]
8. O'Neil, R. Convolution operators and $L(p,q)$ spaces. *Duke Math. J.* **1963**, *30*, 129–142. [CrossRef]
9. Triebel, H. *Theory of Function Spaces*; Birkhauser Verlag: Boston, MA, USA, 1983.
10. Asekritova, I.; Kruglyak, N. Interpolation of Besov spaces in the nondiagonal case. *St. Petersb. Math. J.* **2007**, *18*, 511–516. [CrossRef]
11. Besoy, B.F.; Cobos, F.; Triebel, H. On Function Spaces of Lorentz–Sobolev type. *Math. Ann.* **2021**, 1–33. [CrossRef]
12. Devore, R.A.; Popov, V.A. Interpolation of Besov Spaces. *Trans. Am. Math. Soc.* **1988**, *305*, 397–414. [CrossRef]
13. Cobos, F.; Fernandez, D. *Hardy-Sobolev Spaces and Besov Spaces with a Function Parameter*; Springer: Berlin, Germany, 1988.
14. Devore, R.A.; Yu, X. K-functionals for Besov spaces. *J. Approx. Theory* **1991**, *67*, 38–50. [CrossRef]
15. Triebel, H. Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation. *Ark. Mat.* **1973**, *11*, 13–64. [CrossRef]
16. Yang, D. Real interpolations for Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. *Math. Nachr.* **2004**, *273*, 96–113. [CrossRef]
17. Yang, Q.; Cheng, Z.; Peng, L. Uniform characterization of function spaces by wavelets. *Acta Math. Sci. Ser. A Chin. Ed.* **2005**, *25*, 130–144.
18. Yang, Q. *Wavelet and Distribution*; Beijing Science and Technology Press: Beijing, China, 2002.
19. Yang, Q. *Introduction to Harmonic Analysis and Wavelets*; Wuhan University Press: Wuhan, China, 2012.
20. Meyer, Y. *Ondelettes et Opérateur, I et II*; Hermann: Paris, France, 1990.
21. Holmstedt, T.; Peetre, J. On certain functionals arising in the theory of interpolation spaces. *J. Funct. Anal.* **1969**, *4*, 88–94. [CrossRef]