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An Edgeworth Expansion for the Ratio of Two Functionals of Gaussian Fields and Optimal Berry–Esseen Bounds

Yoon-Tae Kim and Hyun-Suk Park *

Department of Statistics and Data Science Convergence Research Center, Hallym University, Chuncheon 200-702, Korea; ytkim@hallym.ac.kr

* Correspondence: hspark@hallym.ac.kr

Abstract: This paper is concerned with the rate of convergence of the distribution of the sequence $\{F_n/G_n\}$, where F_n and G_n are each functionals of infinite-dimensional Gaussian fields. This form very frequently appears in the estimation problem of parameters occurring in Stochastic Differential Equations (SDEs) and Stochastic Partial Differential Equations (SPDEs). We develop a new technique to compute the exact rate of convergence on the Kolmogorov distance for the normal approximation of F_n/G_n . As a tool for our work, an Edgeworth expansion for the distribution of F_n/G_n , with an explicitly expressed remainder, will be developed, and this remainder term will be controlled to obtain an optimal bound. As an application, we provide an optimal Berry–Esseen bound of the Maximum Likelihood Estimator (MLE) of an unknown parameter appearing in SDEs and SPDEs.

Keywords: Malliavin calculus; fourth moment theorem; Kolmogorov distance; multiple stochastic integral; Stein’s equation; Edgeworth expansion; stochastic (partial) differential equations; Berry–Esseen bound

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1. Introduction

Let $X = \{X(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, where $\mathfrak{H}$ is a real separable Hilbert space, and let $\{F_n, n \geq 1\}$ be a sequence of random variables of functionals of infinite-dimensional Gaussian fields associated with X . In recent years, there have been many efforts to find necessary and sufficient conditions for a normal approximation. The authors in [1] discovered a surprising central limit theorem, called the “fourth moment theorem”, for a sequence of random variables belonging to some Wiener chaos associated with X .

Theorem 1 (Fourth moment theorem). *Fix an integer $q \geq 2$ and consider a sequence of random variables $\{F_n, n \geq 1\}$ belonging to the q th Wiener chaos with $\mathbb{E}[F_n^2] = 1$ for all $n \geq 1$. Then $F_n \xrightarrow{\mathcal{L}} Z$ if and only if $\mathbb{E}[F_n^4] \rightarrow 3 = \mathbb{E}[Z^4]$, where Z is a standard Gaussian random variable and the notation $\xrightarrow{\mathcal{L}}$ denotes the convergence in distribution.*

Afterwards, the authors in [2] obtained the upper bound of the distances between two random variable F_n and Z by using the techniques based on the combination between Malliavin calculus (see, e.g., [3–6]) and Stein’s method for normal approximation (see, e.g., [7–9]). For instance, in the case of the Kolmogorov distance d_{Kol} , one obtained that

$$d_{Kol}(F_n, Z) \leq \sqrt{\mathbb{E}[(1 - \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}})^2]} := \varphi(F_n), \quad (1)$$

where DF_n and L^{-1} denote the Malliavin derivative of F_n and the pseudo-inverse of the Ornstein–Uhlenbeck generator, respectively (see Section 2). In the particular case where F_n is

an element in the q th Wiener chaos of X with $\mathbb{E}[F_n^2] = 1$, the upper bound (1) is expressed in the following form

$$d_{Kol}(F_n, Z) \leq \sqrt{\frac{q-1}{3q}(\mathbb{E}[F_n^4] - 3)}, \quad (2)$$

where $\mathbb{E}[F_n^4] - 3$ is just the fourth cumulant of F_n . Recall that the bound $\varphi(F_n)$ is optimal for the sequence $\{F_n\}$ with respect to some distance d if there exist constants $0 < c < C < \infty$, independent of n , such that

$$c\varphi(F_n) \leq d(F_n, Z) \leq C\varphi(F_n), \quad \text{for all } n \geq 1. \quad (3)$$

In [10], the authors prove that whenever the distance d is defined as the supremum over a class of twice continuously differentiable functions with a bounded second derivative, an optimal rate of convergence is given by the sequence

$$\max\{|\mathbb{E}[F_n^3]|, |\mathbb{E}[F_n^4] - 3|\}, \quad n \geq 1 \quad (4)$$

in the case where $\{F_n\}$ is a sequence of random variables belonging to a fixed Wiener chaos. The further work in [11] subsequent to this result is that the optimal rate in the total variation distance exactly coincides with the rate in the case of the distance related to the smooth test functions. When we considered the statistical estimation problem of parameters appearing in SDEs or SPDEs, quite often we encountered statistics of the form F_n/G_n . In these circumstances, how an upper bound $\varphi(n)$ in (1) is derived can not be applied directly. The authors of [12] developed a technique to obtain the upper and lower rates of the convergence on the Kolmogorov distance for a sequence $\{F_n/G_n\}$. Using both these rates, the authors obtain the optimal bound for the convergence rate of the normal approximation for MLE of the parameter occurring in SPDE. However, the paper [12] does not provide a complete answer to the optimal rates of convergence on the Kolmogorov distance because the convergence rates in the upper and lower bound derived in [12] may not match each other.

The aim of the present work is to find a complete answer for the optimal rate. To this end, a set of sufficient conditions will be derived, ensuring that the upper and lower bounds yield an optimal rate for the Kolmogorov distance. Our methodology to obtain the optimal bound is based on an Edgeworth expansion of $\mathbb{P}(F_n/G_n \leq z)$ derived from that for $\mathbb{P}(F_n \leq z)$ in [13] (for Edgeworth expansions, see, e.g., Chapter 2 in [14] and Chapter 5 in [15]). In that paper, the Edgeworth expansion with general order J is given by

$$\mathbb{P}(F_n \leq z) - \mathbb{P}(Z \leq z) = \sum_{s=2}^J H_s(z)\phi(z)g_s(F_n) + R_z^{(J+1)}(F_n),$$

where the quantity $g_s(F_n)$ can be expressed in terms of the cumulants of F_n , $H_s(z)$ is the Hermite polynomial of order s , and ϕ is the probability density function of a standard Gaussian distribution. It is easier and more extensive for newly developed techniques to apply to the estimation problems for parameters occurring in SDEs or SPDEs than to existing ones. Moreover, a set of sufficient conditions derived in this work is satisfied in most of the estimation problems for parameters in SDEs and SPDEs.

The rest of the paper is organized as follows. Section 2 reviews some basic notations, and the results of Malliavin calculus and Stein's method. In Section 3, the Edgeworth expansion with general order will be derived. Using this expansion, an optimal bound in Kolmogorov distance will be obtained in Section 4. Finally, as an application of our main results, in Section 5, we provide an optimal Berry–Esseen bound for the estimator of parameters in SDE and SPDE.

Throughout this paper, c (or C) stands for an absolute constant with possibly different values in different places.

2. Preliminaries

2.1. Malliavin Calculus

In this section, we recall some basic facts about Malliavin calculus for Gaussian processes. The reader is referred to [5,6] for a more detailed explanation. Suppose that $\mathfrak{H}$ is a real separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. Let $B = \{B(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $\mathbb{E}[B(h)B(g)] = \langle h, g \rangle_{\mathfrak{H}}$. For every $n \geq 1$, let $\mathbb{H}_n$ be the n th Wiener chaos of B , that is the closed linear subspace of $L^2(\Omega)$ generated by $\{H_n(B(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_n is the n th Hermite polynomial. We define a linear isometric mapping $I_n : \mathfrak{H}^{\odot n} \rightarrow \mathbb{H}_n$ by $I_n(h^{\otimes n}) = n!H_n(B(h))$, where $\mathfrak{H}^{\odot n}$ is the symmetric tensor product. It is well known that any square integrable random variable $F \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ ($\mathcal{G}$ denotes the σ -field generated by B) can be expanded into a series of multiple stochastic integrals:

$$F = \sum_{q=0}^{\infty} I_q(f_q),$$

where $f_0 = \mathbb{E}[F]$, the series converges in L^2 , and the functions $f_q \in \mathfrak{H}^{\odot q}$ are uniquely determined by F . Let $\{e_l, l \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. If $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$, defined by

$$f \otimes_r g = \sum_{l_1, \dots, l_r=1}^{\infty} \langle f, e_{l_1} \otimes \dots \otimes e_{l_r} \rangle_{\mathfrak{H}^{\otimes r}} \langle g, e_{l_1} \otimes \dots \otimes e_{l_r} \rangle_{\mathfrak{H}^{\otimes r}}. \quad (5)$$

The following formula for the product of the multiple stochastic integrals will be frequently used.

Proposition 1. Let $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$ be two symmetric functions. Then

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \quad (6)$$

Let $\mathfrak{S}$ be the class of smooth and cylindrical random variables F of the form

$$F = f(B(\varphi_1), \dots, B(\varphi_n)), \quad (7)$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathfrak{H}$, $i = 1, \dots, n$. The Malliavin derivative of F with respect to B is the element of $L^2(\Omega, \mathfrak{H})$, defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\varphi_1), \dots, B(\varphi_n)) \varphi_i. \quad (8)$$

We denote by $\mathbb{D}^{l,p}$ the closure of its associated smooth random variable class with respect to the norm

$$\|F\|_{l,p}^p = \mathbb{E}(|F|^p) + \sum_{k=1}^l \mathbb{E}(\|D^k F\|_{\mathfrak{H}^{\otimes k}}^p).$$

We denote by δ the adjoint of the operator D , also called the divergence operator. The domain of δ , denoted by $\text{Dom}(\delta)$, is an element $u \in L^2(\Omega; \mathfrak{H})$ such that

$$|\mathbb{E}(\langle D^l F, u \rangle_{\mathfrak{H}^{\otimes l}})| \leq C(\mathbb{E}|F|^2)^{1/2} \text{ for all } F \in \mathbb{D}^{l,2}.$$

If $u \in \text{Dom}(\delta)$, then $\delta(u)$ is the element of $L^2(\Omega)$ defined by the duality relationship

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u \rangle_{\mathfrak{H}}] \text{ for every } F \in \mathbb{D}^{1,2}.$$

Let $F \in L^2(\Omega)$ be a square integrable random variable. The operator L is defined through the projection operator J_n , $n = 0, 1, 2, \dots$, as $L = \sum_{n=0}^{\infty} -nJ_nF$, and is called the infinitesimal generator of the Ornstein–Uhlenbeck semigroup. The relationship between the operator D , δ , and L is given as follows: $\delta DF = -LF$; that is, for $F \in L^2(\Omega)$ the statement $F \in \text{Dom}(L)$ is equivalent to $F \in \text{Dom}(\delta D)$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom}(\delta)$), and in this case $\delta DF = -LF$. We also define the operator L^{-1} , which is the pseudo-inverse of L , as $L^{-1}F = \sum_{n=1}^{\infty} \frac{1}{n} J_n(F)$. Note that L^{-1} is an operator with values in $\mathbb{D}^{2,2}$ and $LL^{-1}F = F - \mathbb{E}[F]$ for all $F \in L^2(\Omega)$.

2.2. Stein's Method

In this paper, we focus only on the normal approximation of random variables with respect to the Kolmogorov distance defined as

$$d_{Kol}(F, Z) = \sup_{z \in \mathbb{R}} |\mathbb{P}(X \leq z) - \mathbb{P}(Z \leq z)|,$$

where Z is a standard Gaussian random variable. For fixed $z \in \mathbb{R}$, we consider the Stein equation

$$f'(x) - xf(x) = \mathbf{1}_{(-\infty, z]}(x) - \Phi(z), \quad (9)$$

where $\Phi(z) = \mathbb{P}(Z \leq z)$. It is well known (see, e.g., [7]) that for every fixed $z \in \mathbb{R}$, the function

$$\begin{aligned} f_z(x) &= e^{x^2/2} \int_{-\infty}^x [\mathbf{1}_{(-\infty, z]}(u) - \Phi(z)] e^{-u^2/2} du \\ &= \begin{cases} \sqrt{2\pi} e^{x^2/2} \Phi(x)(1 - \Phi(z)) & \text{if } x \leq z \\ \sqrt{2\pi} e^{x^2/2} \Phi(z)(1 - \Phi(x)) & \text{if } x > z, \end{cases} \end{aligned} \quad (10)$$

is a solution to the Stein Equation (9) such that $\|f_z\|_{\infty} \leq \sqrt{2\pi}/4$ and $\|f'_z\|_{\infty} \leq 1$. For given function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbb{E}[|h(Z)|] < \infty$, we define

$$(U_Z h)(w) = e^{\frac{1}{2}w^2} \int_{-\infty}^w (h(x) - \mathbb{E}[h(Z)]) e^{-\frac{1}{2}x^2} dx, \quad (11)$$

and write $(U_N \mathbf{1}_{(-\infty, z]})(w) = f_z(w)$.

The next lemma gives some properties of the function $U_Z h$ (see, e.g., [5]).

Lemma 1 (Stein's equation). *Given a bounded function $h : \mathbb{R} \rightarrow \mathbb{R}$, there exists an absolutely continuous function $U_Z h$ solving the Stein's equation:*

$$f'(x) - xf(x) = h(x) - \mathbb{E}[h(Z)] \text{ for all } x \in \mathbb{R} \quad (12)$$

for all x and satisfying

$$\|U_Z h\|_{\infty} \leq \sqrt{\frac{\pi}{2}} \|h - \mathbb{E}[h(Z)]\|_{\infty} \text{ and } \|(U_Z h)'\|_{\infty} \leq 2 \|h - \mathbb{E}[h(Z)]\|_{\infty}. \quad (13)$$

We define a function

$$(U_Z g)^{(i)}(x) = \begin{cases} (U_Z g)(x) & \text{for } i = 0, \\ (U_Z g)'(x) & \text{for } i = 1. \end{cases}$$

Let us set

$$(\Psi_{i_1, \dots, i_k} g)(x) = \underbrace{(U_Z \cdots (U_Z (U_Z g)^{(i_1)})^{(i_2)} \cdots)^{(i_k)}}_k(x) \text{ for } i_1, \dots, i_k \in \{0, 1\}.$$

We shall denote by H_q the q th Hermite polynomial defined as follows: $H_0(x) = 1$ and for $q \geq 1$

$$H_q(x) = (-1)^q e^{\frac{1}{2}x^2} \frac{d^q}{dx^q} e^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R}.$$

3. Edgeworth Expansion

Now we derive the J th order Edgeworth expansion of $\mathbb{P}(F/G \leq z)$ for $z \in \mathbb{R}$. We begin by the simple result, given in Lemma 2.3 in [10].

Lemma 2. Suppose that $F \in \mathbb{D}^{1,2}$ and $\mathbb{E}[|G|^2] < \infty$. Then $L^{-1}G \in \mathbb{D}^{2,2}$ and

$$\mathbb{E}[\langle DF, -DL^{-1}G \rangle_{\mathfrak{H}}] = \mathbb{E}[FG] - \mathbb{E}[F]\mathbb{E}[G]. \quad (14)$$

Before stating the Edgeworth expansion, we introduce some notations including the Gamma operators Γ (see [16] for a more detailed explanation). Let $F \in \mathbb{D}^{1,2}$. Define $\Gamma_0(F) = F$ and $\Gamma_1(F, F) = \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}$, and we write

$$\Gamma_j(F) = \langle DF, -DL^{-1}\Gamma_{j-1}(F) \rangle_{\mathfrak{H}}, \quad \text{for } j = 0, 1, \dots$$

Similarly, let $F \in \mathbb{D}^{1,2}$, we define $\Gamma_0^*(F) = F$ and $\Gamma_1^*(F) = \Gamma_1(F)$. If $\Gamma_j^*(F) \in \mathbb{D}^{1,2}$ for fixed $j \leq 1$, we write

$$\Gamma_{j+1}^*(F) = \langle -DL^{-1}F, D\Gamma_j^*(F) \rangle_{\mathfrak{H}}.$$

Let $F \in \mathbb{D}^{j,2^j}$ for a fixed integer $j \geq 1$. Then, for all $k = 1, \dots, j$, it is easy, from Lemma 4.2 in [16], to see that $\Gamma_k^*(F)$ is a well-defined element in $\mathbb{D}^{j-k,2^{j-k}}$, where we set by convention $\mathbb{D}^{0,1} = L^1(\Omega)$. For $F \in \mathbb{D}^{1,2}$, define an operator $D_F : \mathbb{D}^{1,2} \rightarrow L^2(\Omega)$ by $D_F^0(G) = G$,

$$D_F(G) = \langle -DL^{-1}F, DG \rangle_{\mathfrak{H}} \in L^1(\Omega),$$

and for $F, G \in \mathbb{D}^{k,2^k}$, we define

$$D_F^k(G) = \underbrace{D_F(D_F(\dots(D_F(G))\dots))}_k \in L^1(\Omega).$$

Let us define two sets $\mathfrak{D}(r, k)$ and $\mathfrak{D}_{\alpha, \beta}^*(r, k)$ for $r = 1, 2, \dots$ and $k = 1, \dots$ by

$$\mathfrak{D}(r, k) = \left\{ (i_1, \dots, i_k) \in \{0, 1\}^k : \sum_{l=1}^k i_l + k + 1 = r \right\},$$

and, for $\alpha = -1, 1, 2$ and $\beta = 1, 2$,

$$\mathfrak{D}_{(\alpha, \beta)}^*(J, k+1) = \left\{ (i_1, \dots, i_{k+1}) \in \mathfrak{D}(J + \beta, k+1) : (i_1, \dots, i_k) \in \mathfrak{D}(J + \alpha, k) \right\},$$

respectively. For the sake of simplicity, we write

$$\bar{\Gamma}_1(F)^m := \{1 - \Gamma_1(F)\}^m,$$

and sometimes abbreviate the argument F in the operator $\bar{\Gamma}_1(F)^m$ for $m = 0, 1$ and $\Gamma_j^*(F)$ for $j = 2, \dots$. Let us set

$$\begin{aligned} \mathfrak{g}_1(F) &= \mathbb{E}[\bar{\Gamma}_1(F)], \\ \mathfrak{g}_s(F) &= \frac{1}{s+1} \sum_{k=\lceil \frac{s-1}{2} \rceil}^{s-1} \sum_{i_1, \dots, i_k \in \mathfrak{D}(s,k)} (-1)^{\sum_{l=1}^k i_l + 2k} \\ &\quad \times \prod_{l=1}^k \frac{1}{\sum_{r=l}^k i_r + (k-l+1)} \\ &\quad \times \mathbb{E}[D_F^{1-i_k} (\bar{\Gamma}_1^{i_k} D_F^{1-i_{k-1}} (\bar{\Gamma}_1^{i_{k-1}} \dots (\bar{\Gamma}_1^{i_3} D_F^{1-i_2} (\bar{\Gamma}_1^{i_2} D_F^{1-i_1} (\bar{\Gamma}_1^{i_1+1}))) \dots))] \end{aligned}$$

for $s \geq 2$, where the notation $\lceil \cdot \rceil$ denotes the ceiling function defined by

$$\lceil x \rceil = \inf\{n \in \mathbb{N} : x \leq n\}.$$

Let us define

$$\begin{aligned} Q_{i_1, \dots, i_k}(A_z) &= (-1)^{\sum_{l=1}^k i_l + k} \mathbb{E}[\Psi_{i_1, \dots, i_k} f'_z(A_z) \\ &\quad \times D_F^{1-i_k} (\bar{\Gamma}_1^{i_k} D_{A_z}^{1-i_{k-1}} \dots (\bar{\Gamma}_1^{i_3} D_{A_z}^{1-i_2} (\bar{\Gamma}_1^{i_2} D_{A_z}^{1-i_1} (\bar{\Gamma}_1^{i_1+1}))) \dots)]. \end{aligned}$$

By just following the proof of Theorem 2 in [13], we obtain an Edgeworth expansion of F/G .

Theorem 2 (Edgeworth Expansion). *Let $F, G \in \mathbb{D}^{J+2, 2^{J+2}}$ have $\mathbb{E}[F] = 0$ and $G > 0$ a.s. Moreover, assume that F and G have an absolutely continuous law with respect to the Lebesgue measure. Then we have that, for $z \in \mathbb{R}$,*

$$\begin{aligned} &\mathbb{P}\left(\frac{F}{G} \leq z\right) - \mathbb{P}(Z \leq z) \\ &= \sum_{s=1}^J H_s(z) \phi(z) \mathfrak{g}_s(A_z) + R_z^{(J+1)}(A_z), \end{aligned} \quad (15)$$

where

$$A_z = \frac{F - z\bar{G}}{\mathbb{E}[G]}, \quad \bar{G} = G - \mathbb{E}[G],$$

and the remainder term is given by

$$\begin{aligned} R_z^{(J+1)}(A_z) &= \sum_{(i_1, \dots, i_{\lceil \frac{J-1}{2} \rceil}) \in \mathfrak{D}(J+1, \lceil \frac{J-1}{2} \rceil)} Q_{i_1, \dots, i_{\lceil \frac{J-1}{2} \rceil}}(A_z) \\ &\quad + \sum_{k=\lceil \frac{J-1}{2} \rceil}^{J-1} \sum_{(i_1, \dots, i_{k+1}) \in \bigcup_{\beta=1}^2 \mathfrak{D}_{(0,\beta)}^*(J, k+1)} Q_{i_1, \dots, i_{k+1}}(A_z) \\ &\quad + \sum_{k=\lceil \frac{J-2}{2} \rceil}^{J-2} \sum_{(i_1, \dots, i_{k+1}) \in \mathfrak{D}_{(-1,1)}^*(J, k+1)} Q_{i_1, \dots, i_{k+1}}(A_z). \end{aligned} \quad (16)$$

Here, if J is odd, then $\mathfrak{D}(J+1, \lceil \frac{J-1}{2} \rceil) = \emptyset$. In this case, the first term in $R_z^{(J+1)}(A_z)$ has vanished.

Proof. Since $G > 0$ a.s., we have that

$$\mathbb{P}\left(\frac{F}{G} \leq z\right) = \mathbb{P}\left(\frac{F - z(G - \mathbb{E}[G])}{\mathbb{E}[G]} \leq z\right). \quad (17)$$

For fixed $z \in \mathbb{R}$, obviously, $A_z \in \mathbb{D}^{1,2}$ with $\mathbb{E}[A_z] = 0$, and has an absolutely continuous law with respect to Lebesgue measure. By Stein's equation, we deduce, from (17), that, for every $z \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}\left(\frac{F}{G} \leq z\right) - \mathbb{P}(Z \leq z) &= \mathbb{P}(A_z \leq z) - \mathbb{P}(Z \leq z) \\ &= \mathbb{E}[f'_z(A_z) - A_z f_z(A_z)]. \end{aligned} \quad (18)$$

Applying the iterated method in the proof of Theorem 2 of the paper [13] to the expectation of (18), we get

$$\mathbb{P}\left(\frac{F}{G} \leq z\right) - \mathbb{P}(Z \leq z) = \sum_{s=1}^J H_s(z) \phi(z) \mathfrak{g}_s(A_z) + R_z^{(J+1)}(A_z). \quad (19)$$

Hence the proof of this theorem is completed. $\square$

Before stating and proving the main results in this section, we need to recall the multi-index notation. A multi-index is a vector of non-negative integers of the type $m = (m_1, \dots, m_d) \in \mathbb{N}^d$. Then we write

$$|m| = \sum_{i=1}^d m_i, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \partial^m = \partial_1^{m_1} \dots \partial_d^{m_d}, \quad x^m = \prod_{i=1}^d x_i^{m_i}.$$

For convention, we set $0^0 = 1$. For $n \geq 1$, we consider a vector of real-valued random variables $X_{[d]} = (X_1, \dots, X_d)$ such that $\mathbb{E}[|X_i|^d] < \infty$ for every $i = 1, \dots, d$. For every $b = \{i_1, \dots, i_k\} \subset [d] = \{1, \dots, d\}$, we write

$$X_b = (X_{i_1}, \dots, X_{i_k}) \text{ and } X^b = X_{i_1} \dots X_{i_k}.$$

The cumulant of the vector X_b is defined as

$$\kappa(X_b) = (-i)^k \frac{\partial^k}{\partial t_1 \dots \partial t_k} \log \mathbb{E} \left[\exp \left(i \sum_{\ell=1}^k t_\ell X_{i_\ell} \right) \right] \Big|_{t_1=\dots=t_k=0}. \quad (20)$$

Definition 1. Let $X_{[d]} = (X_1, \dots, X_d)$ be a $\mathbb{R}^d$ -valued random vector with $E[|X|^m] < \infty$ for some multi-index $m \in \mathbb{N}^d \setminus \{0\}$. The joint cumulant of order m of X is defined as

$$\kappa_m(X_1, \dots, X_d) = (-i)^{|m|} \partial^m \log \mathbb{E} \left[\exp \left(i \sum_{\ell=1}^d t_\ell X_\ell \right) \right] \Big|_{t_1=\dots=t_d=0}.$$

For $m = (m_1, \dots, m_d)$, we have that

$$\kappa_m(X) = \kappa(\underbrace{x_1, \dots, x_1}_{m_1}, \underbrace{x_2, \dots, x_2}_{m_2}, \dots, \underbrace{x_d, \dots, x_d}_{m_d}).$$

Next, we obtain J -th order, $J = 1, 2$, an Edgeworth expansion by expressing $\mathfrak{g}_s$, $s = 1, 2$, in terms of the cumulants of $(F, \bar{G})$. For $X_{[m]} = (X_1, \dots, X_m)$, we write $\kappa(X_{[m]}) = \kappa_m(X)$ if $X_i = X$ for $i = 1, \dots, m$. The next result contains the relation between cumulants and the moment associated with $X_{[m]}$ (see, e.g., [17]).

Lemma 3. For every $b \subseteq [n]$,

$$\mathbb{E}[X^b] = \sum_{\pi=\{b_1, \dots, b_k\} \in \mathcal{P}(b)} \prod_{i=1}^k \kappa(X_{b_i}), \quad (21)$$

$$\kappa(X_b) = \sum_{\pi=\{b_1, \dots, b_k\} \in \mathcal{P}(b)} (-1)^{k-1} (k-1)! \prod_{i=1}^k \mathbb{E}[X^{a_i}]. \quad (22)$$

The following result can be obtained from Lemma 6 in [13].

Lemma 4. Let F and G be random variables with $F, G \in \mathbb{D}^{3,2^3}$ with $\mathbb{E}[F] = 0$. Then

$$\mathfrak{g}_1(A_z) = -\frac{1}{2!} \kappa_2(A_z), \quad \mathfrak{g}_2(A_z) = -\frac{1}{3!} \kappa_3(A_z), \quad \mathfrak{g}_3(A_z) = -\frac{1}{4!} \kappa_4(A_z).$$

It is immediate, from Theorem 2 and Lemma 4, that the expansions with an explicit remainder will be given in the case where $J = 1, 2$. Let us set $X = (F, \bar{G})$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Corollary 1. Under the assumptions in Theorem 2, we have

(a) for $J = 1$,

$$\begin{aligned} & \mathbb{P}\left(\frac{F}{G} \leq z\right) - \mathbb{P}(Z \leq z) \\ &= \frac{H_1(z)\phi(z)}{2(\mathbb{E}[G])^2} \left\{ (\mathbb{E}[G])^2 - \sum_{\alpha=0}^2 \binom{2}{\alpha} (-z)^\alpha \kappa_{(2-\alpha)e_1 + \alpha e_2}(X) \right\} \\ & \quad + R_z^{(2)}(A_z), \end{aligned} \quad (23)$$

where the remainder term $R_z^{(2)}(A_z)$ is given by

$$R_z^{(2)}(A_z) = \mathbb{E}[\Psi_1 f'_z(A_z) \bar{\Gamma}_1^2(A_z)] + \mathbb{E}[\Psi_0 f'_z(F) \bar{\Gamma}_2^*(A_z)]. \quad (24)$$

(b) for $J = 2$,

$$\begin{aligned} & \mathbb{P}\left(\frac{F}{G} \leq z\right) - \mathbb{P}(Z \leq z) \\ &= \frac{H_1(z)\phi(z)}{2(\mathbb{E}[G])^2} \left\{ (\mathbb{E}[G])^2 - \sum_{\alpha=0}^2 \binom{2}{\alpha} (-z)^\alpha \kappa_{(2-\alpha)e_1 + \alpha e_2}(X) \right\} \\ & \quad - \frac{H_2(z)\phi(z)}{3!(\mathbb{E}[G])^3} \sum_{\alpha=0}^3 \binom{3}{\alpha} (-z)^\alpha \kappa_{(3-\alpha)e_1 + \alpha e_2}(X) \\ & \quad + R_z^{(3)}(A_z), \end{aligned} \quad (25)$$

where the remainder term $R_z^{(3)}(A_z)$ is given by

$$\begin{aligned} R_z^{(3)}(A_z) &= \mathbb{E}[\Psi_1 f'_z(A_z) \bar{\Gamma}_1^2(A_z)] + 2\mathbb{E}[\Psi_{0,1} f'_z(F) \bar{\Gamma}_1(A_z) \bar{\Gamma}_2^*(A_z)] \\ & \quad - \mathbb{E}[\Psi_{0,0} f'_z(F) \bar{\Gamma}_3^*(A_z)]. \end{aligned} \quad (26)$$

Proof. By homogeneity and multilinearity of the joint cumulant, we get

$$\kappa_q(A_z) = \frac{1}{(\mathbb{E}[G])^q} \sum_{r=0}^q \binom{q}{r} (-z)^{q-r} \kappa(\underbrace{F, \dots, F}_r, \underbrace{\bar{G}, \dots, \bar{G}}_{q-r}). \quad (27)$$

Using the formula (27) yields

$$\kappa_2(A_z) = \frac{1}{(\mathbb{E}[G])^2} \left(\kappa_2(F) + 2(-z)\kappa(F, \bar{G}) + z^2\kappa_2(\bar{G}) \right), \quad (28)$$

$$\kappa_3(A_z) = \frac{1}{(\mathbb{E}[G])^3} \left(\kappa_3(F) - 3z\kappa(F, F, \bar{G}) + 3z^2\kappa(F, \bar{G}, \bar{G}) + z^3\kappa_3(\bar{G}) \right). \quad (29)$$

By using Theorem 2 and Lemma 4, together with (28) and (29), we get the expansions (23) and (25). $\square$

Next, by using a formula for computing joint cumulants of the multiple stochastic integrals, the Edgeworth expansion of F/G , in the case where $F = I_2(f)$ and $\bar{G} = I_2(g)$, will be derived. For this, we introduce the diagrams and related notations (see [18] for a more detailed explanation). We will denote by $\Gamma(\pi, \sigma)$ the diagram of a pair of partitions (π, σ) . For $n_1, n_2, \dots, n_k \geq 1$, we write $n = \sum_{i=1}^k n_i$, and denote by π^* the partition of $[n]$ given by

$$\pi^* = \left\{ \{1, \dots, n_1\}, \{n_1 + 1, \dots, n_1 + n_2\}, \dots, \{n_1 + \dots + n_{k-1} + 1, \dots, n\} \right\}.$$

We define the function in n variables

$$f_1 \otimes_0 \dots \otimes_0 f_k = \prod_{i=1}^k f_i(x_{n_1+\dots, n_{i-1}+1}, \dots, x_{n_1+\dots+n_i}), \quad n_0 = 0,$$

and a set

$$\mathcal{M}_2([n], \pi^*) = \left\{ \sigma \in \mathcal{P} : \Gamma(\pi^*, \sigma) \text{ is Gaussain, non-flat and connected} \right\}.$$

Lemma 5. For any vector of functions $(f_1, \dots, f_k)$ such that $f_i \in L_s^2(\mathcal{Z}, d\lambda)$, $i = 1, \dots, k$, if $\mathcal{M}_2([n], \pi^*) = \emptyset$, then

$$\kappa(I_{n_1}(f_1), \dots, I_{n_k}(f_k)) = \sum_{\sigma \in \mathcal{M}_2([n], \pi^*)} \int_{\mathbb{R}^{n/2}} f_{\sigma,k}(x) d\lambda(x), \quad (30)$$

where the function $f_{\sigma,k}$ of $2/n$ variables is obtained by identifying x_i and x_j in the argument of $f_1 \otimes_0 \dots \otimes_0 f_k$ if and only if $i \sim_\sigma j$ (this notation means that i and j belong to the same block of σ).

Corollary 2. Under the assumptions in Theorem 2, we have that

(a) for $J = 1$,

$$\begin{aligned} & \mathbb{P}\left(\frac{F}{G} \leq z\right) - \mathbb{P}(Z \leq z) \\ &= \frac{H_1(z)\phi(z)}{2(\mathbb{E}[G])^2} \left\{ (\mathbb{E}[G])^2 - 2 \sum_{\sigma=(b_1, b_2) \in \mathcal{P}([2])} \int_{Z^2} \prod_{i \in b_1} f(z_i, z_{i+1}) \right. \\ & \quad \left. \times \prod_{i \in b_2} (-z)^{|b_2|} g(z_i, z_{i+1}) d\nu(z_1) \nu(dz_2) \right\} + R_z^{(2)}(A_z), \end{aligned} \quad (31)$$

where the remainder term $R_z^{(2)}(A_z)$ is given by (24).

(b) for $J = 2$,

$$\begin{aligned}
 & \mathbb{P}\left(\frac{F}{G} \leq z\right) - \mathbb{P}(Z \leq z) \\
 &= \frac{H_1(z)\phi(z)}{2(\mathbb{E}[G])^2} \left\{ (\mathbb{E}[G])^2 - 2 \sum_{\sigma=(b_1, b_2) \in \mathcal{P}([2])} \int_{Z^2} \prod_{i \in b_1} f(z_i, z_{i+1}) \right. \\
 &\quad \times \prod_{i \in b_2} (-z)^{|b_2|} g(z_i, z_{i+1}) dv(z_1) v(dz_2) \Big\} \\
 &\quad - \frac{8H_2(z)\phi(z)}{(3!\mathbb{E}[G])^3} \sum_{\sigma=(b_1, b_2) \in \mathcal{P}([3])} \int_{Z^3} \prod_{i \in b_1} f(z_i, z_{i+1}) \\
 &\quad \times \prod_{i \in b_2} (-z)^{|b_2|} g(z_i, z_{i+1}) dv(z_1) dv(z_2) v(dz_3) \\
 &\quad + R_z^{(3)}(A_z), \tag{32}
 \end{aligned}$$

where the remainder term $R_z^{(3)}(A_z)$ is given by (26).

Proof. Suppose that $\mathfrak{H} = L^2(Z, \mathcal{Z}, \nu)$, where ν is a σ -finite and non-atomic measure. By using Lemma 5, we obtain

$$\begin{aligned}
 & \kappa_n(I_2(f + (-z)g)) \\
 &= 2^{n-1}(n-1)! \int_{Z^n} (f + (-z)g)(z_1, z_2)(f + (-z)g)(z_2, z_3) \times \cdots \\
 &\quad \cdots \times (f + (-z)g)(z_{n-1}, z_n)(f + (-z)g)(z_n, z_1) \\
 &\quad \times dv(z_1) \cdots v(dz_n) \\
 &= 2^{n-1}(n-1)! \sum_{\sigma=(b_1, b_2) \in \mathcal{P}([n])} \int_{Z^n} \prod_{i \in b_1} f(z_i, z_{i+1}) \\
 &\quad \times \prod_{i \in b_2} (-z)^{|b_2|} g(z_i, z_{i+1}) dv(z_1) \cdots v(dz_n), \tag{33}
 \end{aligned}$$

where $\prod_{i \in b_k} = 1$ if $b_k = \emptyset$ and $z_{n+1} = z_1$. Since $A_z = I_2(f + (-z)g)/\mathbb{E}[G]$, the result follows from (33). $\square$

4. Optimal Berry–Esseen Bound

In this section, the main result in this paper will be obtained. Theorem 5 below is an interesting result in itself because it is an extension of the optimal bound (4) in the introduction, in some respect. We consider sequences $F_n = I_q(f_n)$ and $\bar{G}_n = I_q(g_n)$, $n \geq 1$, living in the q th Wiener chaos, and such that each F_n has the unit variance. In this case, we derive the optimal Berry–Esseen bound by using the two-term Edgeworth expansion given in Corollary 1. The following result provides some bounds on Gamma operators Γ_j .

Proposition 2. Let $F = I_q(f)$, where $f \in \mathfrak{H}^{\otimes q}$, with $\mathbb{E}[F^2] = 1$. Then we have

$$\begin{aligned}
 (a) \quad & \mathbb{E}[|1 - \Gamma_1(F)|^2] \leq C_q \kappa_4(F), \\
 (b) \quad & \mathbb{E}[|(1 - \Gamma_1(F))\Gamma_2^*(F)|] \leq C_q \sqrt{\kappa_4(F)} (|\kappa_3(F)| + \kappa_4(F)^{3/4}), \\
 (c) \quad & \mathbb{E}[|\Gamma_3^*(F)|] \leq C_q \kappa_4(F).
 \end{aligned}$$

Proof. By using Proposition 4.3 in [10], it is immediate to get results. $\square$

By regulating the remainder term in the expansion (15) of Theorem 2, the upper and lower bounds of the Kolmogorov distance between F_n/G_n and Z will be obtained. In this section, we assume that

$$\begin{aligned} (A_1) \quad & \lim_{n \rightarrow \infty} \kappa_{2e_2}(X_n) = 0, \\ (A_2) \quad & \lim_{n \rightarrow \infty} \kappa_{(3-\alpha)e_1+\alpha e_2}(X_n) = 0 \quad \alpha = 1, 2, 3, \\ (A_3) \quad & \lim_{n \rightarrow \infty} \kappa_{(4-\alpha)e_1+\alpha e_2}(X_n) = 0 \quad \alpha = 0, 1, 2, 3, 4. \end{aligned}$$

Theorem 3 (Upper bounds). Let $F_n = I_q(f_n)$ and $\bar{G}_n = I_q(g_n)$ for all $q \geq 2$, where $\bar{G}_n = G_n - \mathbb{E}[G_n]$, and let $X_n = (F_n, \bar{G}_n)$ such that $\mathbb{E}[G_n] = \kappa_{2e_1}(X_n) < \infty$. Assume that there exists a positive sequence $\{a_n\}$ such that $a_n \rightarrow \infty$, and

$$(i) \quad \sup_n \frac{|\kappa_{2e_2}(X_n)|}{|\kappa_{e_1+e_2}(X_n)|} < \infty, \quad (34)$$

$$(ii) \quad \sup_n \frac{a_n^\alpha |\kappa_{(3-\alpha)e_1+\alpha e_2}(X_n)|}{|\kappa_{3e_1}(X_n)|} < \infty, \quad \alpha = 1, 2, 3, \quad (35)$$

$$(iii) \quad \sup_n \frac{a_n^\alpha |\kappa_{(4-\alpha)e_1+\alpha e_2}(X_n)|^{3/4}}{|\kappa_{3e_1}(X_n)|} < \infty, \quad \alpha = 0, 1, 2, 3, 4. \quad (36)$$

Assume that

$$\sup_n \frac{\bar{\Phi}(a_n)}{M_n} < \infty, \quad (37)$$

where $\bar{\Phi}(x) = 1 - \Phi(x)$, and

$$M_n = \max \left\{ \frac{|\kappa_{e_1+e_2}(X_n)|}{(\mathbb{E}[G_n])^2}, \frac{|\kappa_{3e_1}(X_n)|}{(\mathbb{E}[G_n])^3} \right\}.$$

Then there exists a constant $C > 0$ such that

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq z\right) - \mathbb{P}(Z \leq z) \right| \leq C \max \left\{ M_n, \frac{|(\mathbb{E}[G_n])^2 - \kappa_{2e_1}(X_n)|}{(\mathbb{E}[G_n])^2} \right\}. \quad (38)$$

Proof. Obviously, it is sufficient to consider $z \geq 0$ to prove this theorem. From (23) of Corollary 1, one can deduce that

$$\begin{aligned} & \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq z\right) - \mathbb{P}(Z \leq z) \right| \\ & \leq \frac{|H_1(z)|\phi(z)}{2(\mathbb{E}[G_n])^2} \left\{ |(\mathbb{E}[G_n])^2 - \kappa_{2e_1}(X_n)| + 2|z||\kappa_{e_1+e_2}(X_n)| \right. \\ & \quad \left. + z^2|\kappa_{2e_2}(X_n)| \right\} + |R_z^{(2)}(A_{z,n})| \\ & \leq \frac{C}{(\mathbb{E}[G_n])^2} \left\{ |(\mathbb{E}[G_n])^2 - \kappa_{2e_1}(X_n)| + |\kappa_{e_1+e_2}(X_n)| \right. \\ & \quad \left. + |\kappa_{2e_2}(X_n)| \right\} + |R_z^{(2)}(A_{z,n})|, \end{aligned} \quad (39)$$

where

$$A_{z,n} = \frac{(1, -z) \cdot X_n}{\mathbb{E}[G_n]}.$$

Now the remainder term $R_z^{(2)}(A_{z,n})$ in (39) will be estimated. By using Lemma 1 and Proposition 2, the first expectation in (24) can be bounded as follows:

$$\begin{aligned} & |\mathbb{E}[\Psi_1 f'_z(A_{z,n}) \bar{\Gamma}_1(A_{z,n})^2]| \\ & \leq \|\Psi_1 f'_z\|_\infty \mathbb{E}[\bar{\Gamma}_1(A_{z,n})^2] \\ & \leq \frac{C}{(\mathbb{E}[G_n])^4} \sum_{\alpha=0}^4 |z|^\alpha |\kappa_{(4-\alpha)e_1+\alpha_2}(X_n)|. \end{aligned} \quad (40)$$

Similarly, the second term in the remainder (24) can be estimated as

$$\begin{aligned} & |\mathbb{E}[\Psi_0 f'_z(A_{z,n}) \Gamma_2^*(A_{z,n})]| \\ & \leq 2\|\Psi_0 f'_z\|_\infty \sqrt{\mathbb{E}[\Gamma_2^*(A_{z,n})^2]} \\ & \leq \frac{C}{|\mathbb{E}[G_n]|^3} \sum_{\alpha=0}^3 |z|^\alpha |\kappa_{(3-\alpha)e_1+\alpha e_2}(X_n)| \\ & \quad + \frac{C}{(\mathbb{E}[G_n])^3} \sum_{\alpha=0}^4 |z|^{3\alpha/4} |\kappa_{(4-\alpha)e_1+\alpha_2}(X_n)|^{3/4}. \end{aligned} \quad (41)$$

Combining (40) and (41), one can see that the right-hand side of (39) can be bounded as

$$\begin{aligned} & \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq z\right) - \mathbb{P}(Z \leq z) \right| \\ & \leq \frac{C}{\mathbb{E}[G_n]^2} \left\{ |(\mathbb{E}[G_n])^2 - \kappa_{2e_1}(X_n)| + |\kappa_{e_1+e_2}(X_n)| + |\kappa_{2e_2}(X_n)| \right\} \\ & \quad + \frac{C}{|\mathbb{E}[G_n]|^3} \sum_{\alpha=0}^3 |z|^\alpha |\kappa_{(3-\alpha)e_1+\alpha e_2}(X_n)| \\ & \quad + \frac{C}{(\mathbb{E}[G_n])^3} \sum_{\alpha=0}^4 (|z|^{3\alpha/4} \vee |z|^\alpha) |\kappa_{(4-\alpha)e_1+\alpha_2}(X_n)|^{3/4}. \end{aligned} \quad (42)$$

From (42), it follows that, for sufficiently large n ,

$$\begin{aligned} & \sup_{z \leq a_n} \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq z\right) - \mathbb{P}(Z \leq z) \right| \\ & \leq \frac{C}{(\mathbb{E}[G_n])^2} \left[|(\mathbb{E}[G_n])^2 - \kappa_{2e_1}(X_n)| + |\kappa_{e_1+e_2}(X_n)| \left\{ 1 + \sup_n \frac{|\kappa_{2e_2}(X_n)|}{|\kappa_{e_1+e_2}(X_n)|} \right\} \right] \\ & \quad + \frac{C|\kappa_{3e_1}(X_n)|}{|\mathbb{E}[G_n]|^3} \left\{ \sup_n \frac{\sum_{\alpha=1}^3 a_n^\alpha |\kappa_{(3-\alpha)e_1+\alpha e_2}(X_n)|}{|\kappa_{3e_1}(X_n)|} \right. \\ & \quad \left. + \sup_n \frac{\sum_{\alpha=0}^4 a_n^\alpha |\kappa_{(4-\alpha)e_1+\alpha e_2}(X_n)|^{3/4}}{|\kappa_{3e_1}(X_n)|} \right\} \\ & \leq C \max \left\{ M_n, \frac{|(\mathbb{E}[G_n])^2 - \kappa_{2e_1}(X_n)|}{(\mathbb{E}[G_n])^2} \right\}. \end{aligned} \quad (43)$$

On the other hand, it follows, from (43), that

$$\begin{aligned} & \sup_{z > a_n} \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq z\right) - \mathbb{P}(Z \leq z) \right| \\ & \leq \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq a_n\right) - \mathbb{P}(Z \leq a_n) \right| + 2\Phi(a_n) \\ & \leq C \max \left\{ M_n, \frac{|\mathbb{E}[G_n]|^2 - \kappa_{2e_1}(X_n)|}{(\mathbb{E}[G_n])^2} \right\}. \end{aligned} \quad (44)$$

Combining (43) and (44) completes the proof of this theorem. $\square$

Theorem 4 (Lower bounds). Let $F_n = I_q(f_n)$ and $\tilde{G}_n = I_q(g_n)$ for all $q \geq 2$, where $\tilde{G}_n = G_n - \mathbb{E}[G_n]$ and $\mathbb{E}[G_n] = \kappa_{2e_1}(X_n)$, and let $X_n = (F_n, \tilde{G}_n)$. Suppose that

$$(i) \quad \lim_{n \rightarrow \infty} \frac{\sum_{\alpha=0}^4 |\kappa_{(4-\alpha)e_1+\alpha e_2}(X_n)|}{(\mathbb{E}[G_n])^2 |\kappa_{e_1+e_2}(X_n)|} = 0, \quad (45)$$

$$(ii) \quad \lim_{n \rightarrow \infty} \frac{\sqrt{\sum_{\alpha=0}^4 |\kappa_{(4-\alpha)e_1+\alpha e_2}(X_n)|} \sum_{\alpha=0}^3 |\kappa_{(3-\alpha)e_1+\alpha e_2}(X_n)|}{\mathbb{E}[G_n] |\kappa_{e_1+e_2}(X_n)|} = 0, \quad (46)$$

$$(iii) \quad \lim_{n \rightarrow \infty} \frac{\kappa_{4e_1}(X_n)}{\mathbb{E}[G_n] |\kappa_{3e_1}(X_n)|} = 0. \quad (47)$$

Then there exists a constant $C > 0$ such that

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq z\right) - \mathbb{P}(Z \leq z) \right| \geq CM_n \quad (48)$$

Proof. Since $H_2(z) = 0$ for $z = 1$ and $z = -1$, we have, from (25), that

$$\begin{aligned} & \mathbb{P}\left(\frac{F_n}{G_n} \leq 1\right) - \mathbb{P}(Z \leq 1) \\ & = \frac{\phi(1)}{2(\mathbb{E}[G_n])^2} \left\{ \kappa_{e_1+e_2}(X_n) - \kappa_{2e_2}(X_n) \right\} + R_1^{(3)}(A_{1,n}), \end{aligned} \quad (49)$$

and

$$\begin{aligned} & \mathbb{P}\left(\frac{F_n}{G_n} \leq -1\right) - \mathbb{P}(Z \leq -1) \\ & = \frac{\phi(-1)}{2(\mathbb{E}[G_n])^2} \left\{ -\kappa_{e_1+e_2}(X_n) - \kappa_{2e_2}(X_n) \right\} + R_{-1}^{(3)}(A_{-1,n}). \end{aligned} \quad (50)$$

Subtracting (50) from (49) yields that

$$\begin{aligned} & \left\{ \mathbb{P}\left(\frac{F_n}{G_n} \leq 1\right) - \mathbb{P}(Z \leq 1) \right\} - \left\{ \mathbb{P}\left(\frac{F_n}{G_n} \leq -1\right) - \mathbb{P}(Z \leq -1) \right\} \\ & = \frac{\phi(1)}{(\mathbb{E}[G_n])^2} \kappa_{e_1+e_2}(X_n) + R_1^{(3)}(A_{1,n}) - R_{-1}^{(3)}(A_{-1,n}). \end{aligned} \quad (51)$$

By using Proposition 2, together with (51), we have that

$$\begin{aligned}
 & \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq z\right) - \mathbb{P}(Z \leq z) \right| \\
 & \geq \left| \left\{ \mathbb{P}\left(\frac{F_n}{G_n} \leq 1\right) - \mathbb{P}(Z \leq 1) \right\} - \left\{ \mathbb{P}\left(\frac{F_n}{G_n} \leq -1\right) - \mathbb{P}(Z \leq -1) \right\} \right| \\
 & \geq \frac{\phi(1)}{(\mathbb{E}[G_n])^2} |\kappa_{e_1+e_2}(X_n)| \left\{ 1 - c \left(\frac{\sum_{\alpha=0}^4 |\kappa_{(4-\alpha)e_1+\alpha e_2}(X_n)|}{(\mathbb{E}[G_n])^2 |\kappa_{e_1+e_2}(X_n)|} \right. \right. \\
 & \quad \left. \left. + \frac{\sqrt{\sum_{\alpha=0}^4 |\kappa_{(4-\alpha)e_1+\alpha e_2}(X_n)| \sum_{\alpha=0}^3 |\kappa_{(3-\alpha)e_1+\alpha e_2}(X_n)|}}{(\mathbb{E}[G_n])^2 |\kappa_{e_1+e_2}(X_n)|} \right) \right\}. \quad (52)
 \end{aligned}$$

On the other hand, since $H_1(z) = 0$ for $z = 0$, one immediately see, from the case of $J = 2$ in Corollary 1, that

$$\mathbb{P}\left(\frac{F_n}{G_n} \leq 0\right) - \mathbb{P}(Z \leq 0) = \frac{\phi(0)}{3! (\mathbb{E}[G_n])^3} \kappa_{3e_1}(X_n) + R_0^{(3)}(A_{1,n}). \quad (53)$$

By a similar estimate as for (52), we obtain

$$\begin{aligned}
 & \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq 0\right) - \mathbb{P}(Z \leq 0) \right| \\
 & \geq \frac{\phi(0)}{3! |\mathbb{E}[G_n]|^3} |\kappa_{3e_1}(X_n)| - |R_0^{(3)}(A_{1,n})| \\
 & \geq \frac{\phi(0)}{3! |\mathbb{E}[G_n]|^3} |\kappa_{3e_1}(X_n)| \left\{ 1 - c \left(\frac{\kappa_{4e_1}(X_n)}{|\mathbb{E}[G_n]| |\kappa_{3e_1}(X_n)|} \right. \right. \\
 & \quad \left. \left. + |\mathbb{E}[G_n]|^2 \sqrt{\kappa_{4e_1}(X_n)} \right) \right\}. \quad (54)
 \end{aligned}$$

Combining (52) and (54) yields the lower bound (48). $\square$

By using Theorems 3 and 4, we get an optimal bound for the normal approximation of F_n/G_n .

Theorem 5 (Optimal bound). Let $F_n = I_q(f_n)$ and $\bar{G}_n = I_q(g_n)$ for all $q \geq 2$, where $\bar{G}_n = G_n - \mathbb{E}[G_n]$ and $\mathbb{E}[G_n] = \kappa_{2e_1}(X_n)$, and let $X_n = (F_n, \bar{G}_n)$. Suppose that the assumptions (i), (ii) and (iii) in Theorems 3 and 4 are satisfied. Moreover, assume that

$$\sup_n \frac{\bar{\Phi}(a_n)}{M_n} < \infty, \quad (55)$$

and

$$\sup_n \frac{|(\mathbb{E}[G_n])^2 - \kappa_{2e_1}(X_n)|}{(\mathbb{E}[G_n])^2 M_n} < \infty. \quad (56)$$

Then there exist constants $0 < c < C < \infty$ such that

$$cM_n \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\frac{F_n}{G_n} \leq z\right) - \mathbb{P}(Z \leq z) \right| \leq CM_n. \quad (57)$$

5. Applications

In this section, we apply Theorem 5 to the estimation problem of parameters appearing in SDEs and SPDEs considered in previous works [12,19,20].

5.1. Stochastic Partial Differential Equation

In [21], the authors investigate asymptotic properties of the MLE for parameters occurring in parabolic SPDEs associated with the operator $A^\theta \equiv \theta A_1 + A_0$, and driven by a cylindrical Brownian motion W :

$$\frac{\partial u(t, x)}{\partial t} = A^\theta(x)u(t, x) + \frac{\partial}{\partial t}W(t, x), \quad (58)$$

$$u(0, x) = u_0(x), \quad (59)$$

$$D^\gamma u(t, x)|_{\partial G} = 0 \text{ for all multi-indices } \gamma \text{ such that } |\gamma| \leq m-1, \quad (60)$$

where A_0 and A_1 are partial differential operators of orders m_0 and m_1 , so that $\{A^\theta : \theta \in \Theta\}$ are operators of order $2m = \max(m_1, m_0)$. Problem (58)–(60) is understood in the sense of distributions. Some basic assumptions and definitions are described in Section 2 of the paper [21]. For the sake of completeness, we will briefly review these things. The main assumption is that the Equation (58) is diagonalizable. A complete orthonormal (in $L^2(G)$) system $\{h_i\}_{i=1}^\infty$ is a common system of eigenfunction of the operators A_0 and A_1 :

$$A_0 h_i = k_i h_i, \quad A_1 h_i = v_i h_i, \quad (61)$$

and hence $A^\theta h_i = \mu_i^\theta h_i$, where $\mu_i^\theta = \theta v_i + k_i$. For $s \geq 0$, set

$$\mathbb{H}^s = \left\{ u \in L^2(G) : \|u\|_s \equiv \left(\sum_{j=1}^\infty \lambda_j^{2s} |\langle u, h_j \rangle_{L^2(G)}|^2 \right)^{1/2} < \infty \right\}.$$

For $s < 0$, $\mathbb{H}^s$ is a closure of $L_2(G)$ in the norm $\|u\|_s$ defined for $s < 0$ by the same formula as for $s > 0$. For $s \in \mathbb{R}$, $\mathbb{H}^s$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_s$ associated with the norm $\|u\|_s$ and the functions $h_i^s = \lambda_i^{-s} h_i$, $i = 1, 2, \dots$, form an orthonormal basis in $\mathbb{H}^s$.

Proposition 3 (Huebner and Rozovskii). *For fixed $\alpha > d/2$ and $\theta \in \Theta$, assume that $u_0 \in \mathbb{H}^{-\alpha}$. Then $u^\theta(t, x) = \sum_{i=1}^\infty u_i^\theta(t) h_i^{-\alpha}(x)$ is the only solution of problem (58)–(60), where the Fourier coefficient $u_i^\theta(t)$ is a solution of the one-dimensional Ornstein-Uhlenbeck equation*

$$\begin{cases} du_i^\theta(t) &= \mu_i^\theta u_i^\theta(t) dt + \lambda_{i,\theta}^{-\alpha} dW_i(t), \\ u_i^\theta(0) &= \langle u_0, h_{i,\theta}^{-\alpha} \rangle_{-\alpha}, \end{cases} \quad (62)$$

where $W_i(t)$, $i = 1, 2, \dots$, are independent one-dimensional Wiener processes.

The projection of the solution $u(t, x)$ onto the subspace spanned by $\{h_1^{-\alpha}, \dots, h_N^{-\alpha}\}$ is given by

$$u^N(t, x) \equiv \sum_{i=1}^N u_i(t) h_i^{-\alpha}(x).$$

Let $\mathcal{B}_T^N$ be the Borel σ -algebra on $C([0, T]; \mathbb{R}^N)$, and P_θ^N be the measure on $\mathcal{B}_T^N$ generated by $u^{N,\theta}$. The MLE $\hat{\theta}^N$ corresponding to the likelihood ratio $\frac{dP_\theta^N}{dP_{\theta_0}^N}(u^N)$ is given by

$$\hat{\theta}^N = \theta_0 + \frac{\int_0^T \langle A_1 u^N(t), dW^N(t) \rangle_0}{\int_0^T \|A_1 u^N(t)\|_0^2 dt}. \quad (63)$$

Let $\mathcal{N}$ be an uniformly elliptic operator of the order m . Then define

$$\zeta_{\mathcal{N}} = \left(\frac{1}{2\pi} \int_{P_{\mathcal{N}}(x, \xi) < 1} dx d\xi \right)^{-m/d},$$

where $P_{\mathcal{N}}(x, \xi)$ is the principal symbol of the operator $\mathcal{N}$. In [21], the authors prove the problem of asymptotic normality of the MLE $\hat{\theta}^N$.

Theorem 6 (Huebner and Rozovskii). Assume that $m_1 \geq m - \frac{d}{2}$ and set

$$\Psi_N = \begin{cases} \frac{\zeta}{4\tau} TN^{2\tau} + o(N^{2\tau}) & \text{if } m_1 > m - d/2, \\ \frac{\zeta}{2} T \ln N + o(\ln N) & \text{if } m_1 = m - d/2, \end{cases} \quad (64)$$

where $\tau = (m_1 - m)/d + 1/2$ and

$$\zeta = \frac{\zeta_{A^\theta}^{2m/d}}{\zeta_{A_1}^{2m_1/d}}.$$

Then $\sqrt{\Psi_N}(\hat{\theta}^N - \theta)$ converges in distribution to a Gaussian random variable with zero mean and unit variance as $N \uparrow \infty$.

Obviously, the Fisher Information $I_N(\theta)$ is given by

$$I_N(\theta) = \sum_{i=1}^N \frac{v_i^2}{2\mu_i} |\lambda_i^\alpha u_i(0)|^2 (e^{2\mu_i T} - 1) - \frac{T}{2} \sum_{i=1}^N \frac{v_i^2}{\mu_i} + \sum_{i=1}^N \frac{v_i^2}{4\mu_i^2} (e^{2\mu_i T} - 1). \quad (65)$$

Since $u_0 \in L^2(G)$, we have $\langle u_0, h_n \rangle_0^2 = \lambda_i^{2\alpha} \langle u_0, h_n^{-\alpha} \rangle_{-\alpha}^2 = \lambda_i^{2\alpha} u_i(0) \rightarrow 0$ as $n \rightarrow \infty$. So the second sum in (65) is the main term of $I_N(\theta)$. This implies that we may assume $u_0 = 0$ to find the rate of convergence of the distribution of $\sqrt{I_N(\theta)}(\hat{\theta}^N - \theta_0)$. Define the kernels $f_2^{(i)}$ and $g_2^{(i)}$ for $i = 1, \dots, N$ as

$$\begin{aligned} f_2^{(i)}(s, t) &= v_i e^{\mu_i |t-s|}, \\ g_2^{(i)}(s, t) &= \frac{v_i^2}{2\mu_i} e^{-\mu_i(s+t)} \left(e^{2\mu_i T} - e^{2\mu_i(s \vee t)} \right). \end{aligned} \quad (66)$$

By Lemma 1 in [12], we can write

$$\sqrt{I_N(\theta)}(\hat{\theta}^N - \theta) = \frac{F_N}{\bar{G}_N + 1}, \quad (67)$$

where

$$F_N = \frac{\sum_{i=1}^N I_2(f_2^{(i)})}{2\sqrt{I_N(\theta)}} \quad \text{and} \quad \bar{G}_N = \frac{\sum_{i=1}^N I_2(g_2^{(i)})}{I_N(\theta)}.$$

Then we write $X_N = (F_N, \bar{G}_N)$.

Lemma 6. Assume that $m_1 > m - \frac{d}{2}$. Then there exists a constant $c = c(T, \theta, m, m_1, d)$, depending on T, θ, m, m_1 and d such that, for sufficiently large $N \in \mathbb{N}$,

$$\kappa_{2e_1}(X_N) = 1, \quad (68)$$

and for $\alpha = 0, 1, \dots, \beta$, where $\beta = 2, 3, 4$,

$$\kappa_{(\beta-\alpha)e_1+\alpha e_2}(X_N) = \begin{cases} c \left(N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}} + N^{-\frac{(\alpha+\beta-2)(2m+d)}{2d}} \right) \\ + o \left(N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}} + N^{-\frac{(\alpha+\beta-2)(2m+d)}{2d}} \right) & \text{if } m_1 \neq \frac{2(\alpha+\beta-1)m-d}{\alpha+\beta} \\ c N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}} \times \log N \\ + o \left(N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}} \times \log N \right) & \text{if } m_1 = \frac{2(\alpha+\beta-1)m-d}{\alpha+\beta} \end{cases}. \quad (69)$$

Proof. Direct computations yield that

$$\begin{aligned} \kappa_{2e_1}(X_N) &= \frac{1}{2I_N(\theta)} \sum_{i=1}^N \int_0^T \int_0^T f^{(i)}(z_1, z_2)^2 dz_1 dz_2 \\ &= 1. \end{aligned} \quad (70)$$

Using $e^{\mu_i|t-s|} = e^{-\mu_i(s+t)} e^{2\mu_i(s \vee t)}$, the function $g_2^{(i)}$ in (66) can be written as

$$\begin{aligned} g_2^{(i)}(s, t) &= \frac{v_i^2}{2\mu_i} e^{-\mu_i(s+t)} e^{2\mu_i T} - \frac{v_i}{2\mu_i} f_2^{(i)}(s, t) \\ &= -\frac{v_i^2}{2\mu_i} \left(e^{\mu_i|t-s|} - e^{-\mu_i(s+t)} e^{2\mu_i T} \right). \end{aligned} \quad (71)$$

Since $e^{2\mu_i T} e^{-\mu_i(s+t)} \leq e^{\mu_i|s-t|}$ for all $(s, t) \in [0, T]^2$, it is sufficient, from (71), to consider the term $\frac{v_i}{2\mu_i} f_2^{(i)}(s, t)$ in $g_2^{(i)}(s, t)$ as the main term of the asymptotics of $\kappa_{(\beta-\alpha)e_1+\alpha e_2}(X_N)$ for $\alpha = 0, 1, \dots, \beta$, where $\beta = 2, 3, 4$. Therefore, the main term is given as follows:

$$\begin{aligned} Q_N(\alpha, \beta) &= \frac{(-1)^\alpha}{I_N(\theta)^{\frac{\beta+\alpha}{2}}} \sum_{i=1}^N \frac{v_i^{\beta+\alpha}}{\mu_i^\alpha} \int_{[0,T]^\beta} \prod_{i=1}^\beta e^{\mu_i|z_i-z_{i+1}|} dz_1 \dots dz_\beta \\ &= \frac{\beta!(-1)^\alpha}{I_N(\theta)^{\frac{\beta+\alpha}{2}}} \sum_{i=1}^N \frac{v_i^{\beta+\alpha}}{\mu_i^\alpha} \int_{\{0 \leq z_1 \leq \dots \leq z_\beta \leq T\}} e^{2\mu_i(z_\beta-z_1)} dz_1 \dots dz_\beta \\ &= \frac{4!(-1)^\alpha}{2^{\alpha+\beta-1} I_N(\theta)^{\frac{\beta+\alpha}{2}}} \sum_{i=1}^N \frac{v_i^{\beta+\alpha}}{\mu_i^{\alpha+\beta-1}} + o \left(\frac{1}{I_N(\theta)^{\frac{\beta+\alpha}{2}}} \sum_{i=1}^N \frac{v_i^{\beta+\alpha}}{\mu_i^{\alpha+\beta-1}} \right). \end{aligned} \quad (72)$$

Here we define $z_{\beta+1} = z_1$. From (72), it follows that

$$\begin{aligned} \kappa_{(\beta-\alpha)e_1+\alpha e_2}(X_N) &= Q_N(\alpha, \beta) + o(Q_N(\alpha, \beta)) \\ &= \begin{cases} c \left(N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}} + N^{-\frac{(\alpha+\beta-2)(2m+d)}{2d}} \right) \\ + o \left(N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}} + N^{-\frac{(\alpha+\beta-2)(2m+d)}{2d}} \right) & \text{if } m_1 \neq \frac{2(\alpha+\beta-1)m-d}{\alpha+\beta} \\ c N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}} \times \log N \\ + o \left(N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}} \times \log N \right) & \text{if } m_1 = \frac{2(\alpha+\beta-1)m-d}{\alpha+\beta} \end{cases}. \end{aligned}$$

Hence the proof of this lemma is completed. $\square$

Now we derive an optimal Berry–Esseen bound for the MLE $\hat{\theta}_N$.

Theorem 7 (Optimal bound). Assume that $m_1 \geq \frac{4m}{3} - \frac{d}{3}$. Then there exist constants $0 < c_{T,\theta} < C_{T,\theta} < \infty$, depending on θ and T , such that, for sufficiently large $N \in \mathbb{N}$,

$$c_{T,\theta} N^{-\frac{2m+d}{2d}} \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}(\sqrt{I_N(\theta)}(\hat{\theta}_N - \theta) \leq z) - \mathbb{P}(Z \leq z) \right| \leq C_{T,\theta} N^{-\frac{2m+d}{2d}}. \quad (73)$$

Proof. From Lemma 6, compare the orders $N^{-\frac{(\alpha+\beta)(2m_1-2m+d)}{2d}}$ and $N^{-\frac{(\alpha+\beta-2)(2m+d)}{2d}}$. Obviously, for all $\alpha = 0, 1, \dots, \beta$, where $\beta = 2, 3, 4$ except in the case of $\alpha = 0$ and $\beta = 2$, we have that

$$\begin{aligned} & \frac{(\alpha+\beta)(2m_1-2m+d)}{2d} - \frac{(\alpha+\beta-2)(2m+d)}{2d} \\ & \geq \frac{2(\alpha+\beta)}{2d} \left[m_1 - \left\{ 2m - \min_{3 \leq \alpha+\beta \leq 8} \frac{1}{\alpha+\beta} (2m+d) \right\} \right] \\ & \geq \frac{2(\alpha+\beta)}{2d} \left\{ m_1 - \left(\frac{4m}{3} - \frac{d}{3} \right) \right\}, \end{aligned}$$

so that if

$$m_1 \geq \frac{4m}{3} - \frac{d}{3},$$

the main term in asymptotics of the cumulants is given by $N^{-\frac{(\alpha+\beta-2)(2m+d)}{2d}}$. It is clear, from (67), that $\mathbb{E}[G_N] = \kappa_{2e_1}(X_N) = 1$, which satisfies the condition (56) in Theorem 5. Now we prove the conditions (i), (ii) and (iii) in Theorem 5. It is obvious, from Lemma 6, that

$$(i) \quad \sup_N \frac{|\kappa_{2e_2}(X_N)|}{|\kappa_{e_1+e_2}(X_N)|} = \sup_N N^{-\frac{2m+d}{2d}} = 1, \quad (74)$$

$$(ii) \quad \sup_N \frac{a_N^\alpha |\kappa_{(3-\alpha)e_1+\alpha e_2}(X_N)|}{|\kappa_{3e_1}(X_N)|} = \sup_N a_N^\alpha N^{-\frac{\alpha(2m+d)}{2d}} \text{ for } \alpha = 1, 2, 3, \quad (75)$$

$$(iii) \quad \sup_N \frac{a_N^\alpha |\kappa_{(4-\alpha)e_1+\alpha e_2}(X_N)|^{3/4}}{|\kappa_{3e_1}(X_N)|} = \sup_N a_N^\alpha N^{-\frac{3(\alpha+1)}{4} \left(\frac{2m+d}{2d} \right)} \quad (76)$$

for $\alpha = 0, 1, 2, 3, 4$.

If we take $a_N = N^{k(\frac{2m+d}{2d})}$ for $k < \frac{15}{16}$, the right-hand sides in (75) and (77) meet the conditions (ii) and (iii) in Theorem 5. Moreover, the inequality

$$\bar{\Phi}(z) \leq \frac{1}{z\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad (77)$$

proves that the condition (37)

$$\sup_N \frac{\bar{\Phi}(a_N)}{M_N} < \infty$$

is satisfied. Thus the proof of this theorem is completed. $\square$

Remark 1. In Theorem 7 of the paper [12], the authors obtain an optimal bound (73) under the assumption $m_1 \geq \frac{3m}{2} - \frac{d}{4}$. However, in this paper, we assume that $m_1 \geq \frac{4m}{3} - \frac{d}{3}$. Note that

$$\frac{3m}{2} - \frac{d}{4} > \frac{4m}{3} - \frac{d}{3}.$$

This fact shows that the result given in this paper is an extension of the previous result.

5.2. Ornstein-Uhlenbeck Process

In this section, we find an optimal rate of convergence of the distribution of the MLE $\hat{\theta}_T$ of the unknown parameter $\theta \in \Theta \subseteq \mathbb{R}^+$ based on the observation $X = \{X_t, 0 \leq t \leq T\}$ given by

$$dX_t = -\theta X_t dt + dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \quad (78)$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion. Define kernels $f_2(s, t)$ and $g_2(s, t)$ for $t, s \in [0, T]$ as

$$\begin{aligned} f_2(s, t) &= e^{-\theta|t-s|}, \\ g_2(u, v) &= \frac{e^{\theta(u+v)}}{2\theta} \left(e^{-2\theta(u \vee v)} - e^{-2\theta T} \right). \end{aligned} \quad (79)$$

When the process $\{X_t, 0 \leq t \leq T\}$ can be observed, we can write

$$\sqrt{\frac{T}{2\theta}}(\hat{\theta}_T - \theta) = \frac{-F_T}{\bar{G}_T + Q_T}, \quad (80)$$

where F_T and $\bar{G}_T$ are the double stochastic integrals

$$F_T = \sqrt{\frac{\theta}{2T}} I_2(f_2), \quad \bar{G}_T = \frac{2\theta}{T} I_2(g_2) \quad (81)$$

and

$$Q_T = 1 - \frac{1}{2\theta T} (1 - e^{-2\theta T}).$$

Let us set $Y_T = (F_T, \bar{G}_T)$.

Lemma 7. For $\beta = 2, 3, 4$, we have that for $\alpha = 0, 1, \dots, \beta$,

$$\kappa_{(\beta-\alpha)e_1+\alpha e_2}(Y_T) = c_{\theta, \alpha, \beta} T^{-\frac{\beta+\alpha-2}{2}} + o\left(T^{-\frac{\beta+\alpha-2}{2}}\right). \quad (82)$$

Proof. For the proof, we will use the method in the proof of Lemma 6. Since $e^{-\theta|t-s|} = e^{\theta(t+s)-2\theta(s \vee t)}$, the kernel function g_2 in (79) can be written as

$$g_2(s, t) = \frac{1}{2\theta} \left(e^{-\theta|t-s|} - e^{-\theta(2T-t-s)} \right). \quad (83)$$

Obviously, $e^{-\theta|t-s|} \geq e^{-\theta(2T-t-s)}$ for all $s, t \in [0, T]$. This implies the fact that it is sufficient to consider the term $e^{-\theta|t-s|}$ in $g_2(s, t)$ as the main term of the asymptotics of $\kappa_{(\beta-\alpha)e_1+\alpha e_2}(Y_T)$ for $\alpha = 0, 1, \dots, \beta$, where $\beta = 2, 3, 4$. Hence the main term is given by

$$\begin{aligned} Q_N(\alpha, \beta) &= \left(\frac{\theta}{2}\right)^{\frac{\beta+\alpha}{2}} T^{-\frac{\alpha+\beta}{2}} \int_{[0, T]^\beta} \prod_{i=1}^{\beta} e^{-\theta|z_i - z_{i+1}|} dz_1 \dots dz_{\beta} \\ &= \beta! \left(\frac{\theta}{2}\right)^{\frac{\beta+\alpha}{2}} T^{-\frac{\alpha+\beta}{2}} \int_{\{0 \leq z_1 \leq \dots \leq z_{\beta} \leq T\}} e^{2\theta(z_{\beta} - z_1)} dz_1 \dots dz_{\beta} \\ &= c T^{-\frac{\alpha+\beta-2}{2}} + o\left(T^{-\frac{\alpha+\beta-2}{2}}\right). \end{aligned} \quad (84)$$

Here we define $z_{\beta+1} = z_1$. Therefore, the proof of this lemma is completed. $\square$

By using Theorem 5, we obtain an optimal Berry–Esseen bound for the sequence $\left\{ \sqrt{\frac{T}{2\theta}}(\hat{\theta}_T - \theta), T \geq 0 \right\}$.

Theorem 8 (Optimal bound). For sufficiently large $T > 0$, there exist constants $0 < c_\theta < C_\theta < \infty$ such that

$$c_\theta \frac{1}{\sqrt{T}} \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{T}{2\theta}} (\hat{\theta}_T - \theta) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq C_\theta \frac{1}{\sqrt{T}}. \quad (85)$$

Proof. For sufficiently large $n_0 > 0$, we have, from Lemma 7, that

$$(i) \quad \sup_{T \geq n_0} \frac{|\kappa_{2e_2}(Y_T)|}{|\kappa_{e_1+e_2}(Y_T)|} = c \sup_{T \geq n_0} \frac{1}{\sqrt{T}} < \infty, \quad (86)$$

$$(ii) \quad \sup_{T \geq n_0} \frac{a_T^\alpha |\kappa_{(3-\alpha)e_1+\alpha e_2}(Y_T)|}{|\kappa_{3e_1}(Y_T)|} = \sup_{T \geq n_0} a_T^\alpha T^{-\frac{\alpha}{2}} \text{ for } \alpha = 1, 2, 3, \quad (87)$$

$$(iii) \quad \sup_{T \geq n_0} \frac{a_T^\alpha |\kappa_{(4-\alpha)e_1+\alpha e_2}(Y_T)|^{3/4}}{|\kappa_{3e_1}(Y_T)|} = \sup_{T \geq n_0} a_T^\alpha T^{-\frac{2+3\alpha}{8}} \quad (88)$$

for $\alpha = 0, 1, 2, 3, 4$.

If we take $a_T = T^{k/2}$ for $k < \frac{7}{8}$, the right-hand sides in (87) and (89) satisfy the conditions (ii) and (iii) in Theorem 5, and obviously,

$$\sup_{T \geq n_0} \frac{\bar{\Phi}(a_T)}{M_T} < \infty.$$

Hence the proof of this theorem is completed. $\square$

5.3. α -Brownian Bridge

In [19], the authors find an optimal rate of convergence of the distribution of the MLE $\hat{\alpha}_t$ of the unknown parameter $\alpha \in (1/2, \infty)$ based on the observation $X = \{X_t, 0 \leq t \leq T\}$ given by the α -Brownian bridge

$$dX_t = -\alpha \frac{X_t}{T-t} dt + dW_t, \quad X_0 = 0, \quad 0 \leq t < T, \quad (89)$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion. Here we will derive an optimal Berry–Esseen bound using our result, Theorem 5. Define kernels $f_t(u, v)$ and $g_t(u, v)$ for $u, v \in [0, t]$ as

$$f_t(u, v) = \frac{\sqrt{2\alpha-1}}{\sqrt{|\log(T-t)|}} (T - (u \vee v))^{\alpha-1} (T - (u \wedge v))^{-\alpha} \mathbf{1}_{[0,t]^2}(u, v), \quad (90)$$

$$g_t(u, v) = \frac{(T-u)^{-\alpha} (T-v)^{-\alpha}}{|\log(T-t)|} [(T - (u \vee v))^{2\alpha-1} - (T-t)^{2\alpha-1}] \mathbf{1}_{[0,t]^2}(u, v). \quad (91)$$

Using the above kernel functions (90) and (91), we can write

$$\sqrt{\frac{|\log(T-t)|}{2\alpha-1}} (\alpha - \hat{\alpha}_t) = \frac{I_2(f_t)}{I_2(g_t) + b_t}, \quad (92)$$

where b_t is given by

$$b_t = 1 + \frac{\log(T)}{|\log(T-t)|} - \frac{1}{(2\alpha-1)|\log(T-t)|} \times \left(1 - \left(\frac{T-t}{T} \right)^{2\alpha-1} \right). \quad (93)$$

Let us set $D_t = (I_2(f_t), I_2(g_t))$ for $t \in [0, T]$.

Lemma 8. For $p = 2, 3, 4$, we have that for $q = 0, 1, \dots, p$,

$$\kappa_{(p-q)e_1+qe_2}(D_t) = \frac{c_{\alpha,p,q}}{|\log(T-t)|^{\frac{p+q-2}{2}}} + o\left(\frac{1}{|\log(T-t)|^{\frac{p+q-2}{2}}}\right). \quad (94)$$

Proof. The proof of this lemma can be done by a straightforward adaptation of the computations for the proof of Lemma 6. First, note that

$$(T - (u \vee v))^{\alpha-1} (T - (u \wedge v))^{-\alpha} = (T - u)^{-\alpha} (T - v)^{-\alpha} (T - (u \vee v))^{2\alpha-1}.$$

From this, the kernel function g_t in (91) can be written as

$$g_t(u, v) = \frac{2}{\sqrt{(2\alpha-1)|\log(T-t)|}} f_t(u, v) - \frac{1}{|\log(T-t)|} h_t(u, v), \quad (95)$$

where $h_t(u, v)$ is given by

$$h_t(u, v) = (T - u)^{-\alpha} (T - v)^{-\alpha} (T - t)^{2\alpha-1} \mathbf{1}_{[0,t]^2}(u, v).$$

Since $(T - (u \vee v))^{2\alpha-1} \geq (T - t)^{2\alpha-1}$ for all $u, v \in [0, t]$, it is sufficient to consider only the first term of $g_2(u, v)$ in (95) as the main term of the asymptotics of $\kappa_{(p-q)e_1+qe_2}(D_t)$ for $q = 0, 1, \dots, p$, where $p = 2, 3, 4$. Hence a direct computation yields that the asymptotics of the main term is given by

$$\begin{aligned} Q_t(p, q) &= (2\alpha-1)^{\frac{p-q}{2}} \frac{1}{|\log(T-t)|^{\frac{p+q}{2}}} \int_{[0,T]^p} \prod_{i=1}^p (T - (u_i \vee u_{i+1}))^{q-1} \\ &\quad \times (T - (u_i \wedge u_{i+1}))^{-\alpha} dz_1 \dots dz_p \\ &= \frac{c_{\alpha,p,q}}{|\log(T-t)|^{\frac{p+q-2}{2}}} + o\left(\frac{1}{|\log(T-t)|^{\frac{p+q-2}{2}}}\right). \end{aligned} \quad (96)$$

Here we define $z_{\beta+1} = z_1$. Therefore, the proof of this lemma is completed. $\square$

By using Theorem 5, we obtain an optimal Berry–Esseen bound for the distribution of (92).

Theorem 9 (Optimal bound). For all $t \in [0, T]$ sufficiently near $T > 0$ and $\alpha > 1/2$, there exist constants $0 < c_\alpha < C_\alpha < \infty$ such that

$$\begin{aligned} \frac{c_\alpha}{\sqrt{|\log(T-t)|}} &\leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\sqrt{\frac{|\log(T-t)|}{2\alpha-1}} (\alpha - \hat{\alpha}_t) \leq z\right) - \mathbb{P}(Z \leq z) \right| \\ &\leq \frac{C_\alpha}{\sqrt{|\log(T-t)|}}. \end{aligned} \quad (97)$$

Proof. Let T_0 be sufficiently near $T > 0$. Then we have, from Lemma 8, that

$$(i) \quad \sup_{t \geq T_0} \frac{|\kappa_{2e_2}(D_t)|}{|\kappa_{e_1+e_2}(D_t)|} = c \sup_{t \geq T_0} \frac{1}{\sqrt{|\log(T-t)|}} < \infty, \quad (98)$$

$$(ii) \quad \sup_{t \geq T_0} \frac{a_t^q |\kappa_{(3-q)e_1+ae_2}(D_t)|}{|\kappa_{3e_1}(D_t)|} = \sup_{t \geq T_0} \frac{a_t^q}{|\log(T-t)|^{\frac{q}{2}}} \text{ for } q = 1, 2, 3, \quad (99)$$

$$(iii) \quad \sup_{t \geq T_0} \frac{a_t^q |\kappa_{(4-q)e_1+ae_2}(D_t)|^{3/4}}{|\kappa_{3e_1}(D_t)|} = \sup_{t \geq T_0} \frac{a_t^q}{|\log(T-t)|^{\frac{2+3q}{8}}} \quad (100)$$

for $q = 0, 1, 2, 3, 4$.

If we take $a_t = |\log(T-t)|^{k/2}$ for $k < \frac{7}{8}$, the right-hand sides in (99) and (101) satisfy the conditions (ii) and (iii) in Theorem 5. A direct computation shows that

$$M_t = \frac{1}{\sqrt{|\log(T-t)|}} + o\left(\frac{1}{\sqrt{|\log(T-t)|}}\right),$$

so that using the inequality (77) gives

$$\begin{aligned} \sup_{t \geq T_0} \frac{\bar{\Phi}(a_t)}{M_t} &\leq \frac{1}{\sqrt{2\pi}} \sup_{t \geq T_0} \frac{e^{-\frac{a_t^2}{2}}}{a_t M_t} \\ &= \frac{1}{\sqrt{2\pi}} \sup_{t \geq T_0} |\log(T-t)|^{\frac{1-k}{2}} \exp\left(-\frac{|\log(T-t)|^k}{2}\right) \\ &< \infty. \end{aligned} \quad (101)$$

The proof of this theorem is completed. $\square$

6. Conclusions and Future Works

This paper developed a method to obtain an optimal Berry–Esseen bound for F/G , the form in which the estimator of a parameter in SDEs or SPDEs take place. To this end, we used the Edgeworth expansion of F/G derived in this paper. In order to achieve the optimal bound, all we have to do is to show that the conditions found in this paper are satisfied. The significant feature of our method is that the conditions of proof are met in many estimation problems of a parameter in SDEs or SPDEs. Moreover, we emphasize that Theorem 5 is an extension of the optimal bound given in (4) mentioned in the introduction, so it has drawn very meaningful results by itself. Future works will be carried out in two directions: (1) Applying our method to previous works such as [22,23], etc.; (2) finding new problems to apply our method to.

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