The Optimal Homotopy Asymptotic Method for Solving Two Strongly Fractional-Order Nonlinear Benchmark Oscillatory Problems

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Abstract: This paper proceeds from the perspective that most strongly nonlinear oscillators of fractional-order do not enjoy exact analytical solutions. Undoubtedly, this is a good enough reason to employ one of the major recent approximate methods, namely an Optimal Homotopy Asymptotic Method (OHAM), to offer approximate analytic solutions for two strongly fractional-order nonlinear benchmark oscillatory problems, namely: the fractional-order Duffing-relativistic oscillator and the fractional-order stretched elastic wire oscillator (with a mass attached to its midpoint). In this work, a further modification has been proposed for such method and then carried out through establishing an optimal auxiliary linear operator, an auxiliary function, and an auxiliary control parameter. In view of the two aforesaid applications, it has been demonstrated that the OHAM is a reliable approach for controlling the convergence of approximate solutions and, hence, it is an effective tool for dealing with such problems. This assertion is completely confirmed by performing several graphical comparisons between the OHAM and the Homotopy Analysis Method (HAM).

Keywords: strongly nonlinear oscillators; fractional-order derivatives; optimal homotopy asymptotic method

1. Introduction

It is common knowledge in the world of mathematical modeling that many phenomena and applications in engineering and physical sciences could be excellently outlined through using some mathematical tools that are offered by fractional calculus. For instance, it has recently been conclusively demonstrated that Fractional-order Differential Equations (FoDEs) play a vital role in affording precise descriptions for several nonlinear phenomena [1]. From this standpoint, many efforts and endeavors have been devoted, over a number of past decades, by lots of both physicists and mathematicians to assign explicit solutions for nonlinear FoDEs [2]. The nonlinear fractional-order oscillators are typically considered to be a significant exemplar of such equations. The strongly nonlinear oscillator, which is one of the major types of these oscillators, could be dealt with by means of three main schemes. Constructing new or using some special existent functions that relies on the nature of nonlinearity is the first scheme. On the other hand, the second scheme could be represented by appropriately rescaling the displacement, and then inserting a small parameter into motion equation. Whereas, the third scheme can be delineated by

introducing a further small parameter, and then transporting motion equation into a linear oscillator perturbation [3–5].

In general terms, the nonlinear fractional-order oscillators have been examined and explored by many researchers. In particular, Shen et al. studied the primary resonance of fractional-order van der Pol oscillator analytically and numerically while using the averaging method [6], and then used the incremental harmonic balance method to analyze some dynamical properties of fractional-order nonlinear oscillator [7]. The dynamical response of the fractional-order stochastic Duffing equation was explored by Xu et al. in [8]. Some novel dynamical features of fractional-order Duffing oscillator had been studied by Chen et al. in [9–11]. They proposed a new powerful bifurcation control approach that is based on the $PI^\lambda D^\mu$ controller [11]. However, obtaining accurate solutions of most of these nonlinear equations is considered to be an extremely difficult mission for lots of researchers. For addressing this problem, several semi-analytical and numerical methods have been widely established and implemented to solve such strongly nonlinear equations approximately, like Piecewise Variational Iteration Method (PVIM) [12], Perturbation-Incremental Method (PIM) [13], Generalized Averaging Method (GAM) [14], Homotopy Analysis Method (HAM) [15], Global Residue Harmonic Balance Method (GRHBM) [16], Improved Multiple-Scale Method (IMSM) [17], and many others. Despite the meaningful performance of all these methods, most of them provide series solutions with a small convergence region [18]. For example, Liao has, unfortunately, shown that the convergence region and rate of approximation series cannot be constantly ensured when using HAM [19,20]. From this point of view, there is a persistent necessity to evade all of these weaknesses and shortcomings. The way in which they can be overcome is by utilizing one of the most novel robust methods, called the Optimal Homotpy Asymptotic Method (OHAM).

The OHAM was recently proposed and developed by Marinca et al. as a generalization of the classical HAM [1,2,20–23]. Several solutions of significant nonlinear problems within lots of studies were then, consequently, constructed based on using this method (see [1,18,20,24–30]). In view of many of these studies, it was demonstrated that this method is a reliable, straightforward, and effective tool for offering accurate analytical approximate solutions to lots of strongly nonlinear problems [2,18,29]. Besides, it was revealed that its key characteristic is its ability to optimally control the convergence of approximate series solutions [2,18,29]. However, this work employs this method to provide approximate analytic solution for two strongly fractional-order nonlinear benchmark oscillatory problems through establishing an optimal auxiliary linear operator, an auxiliary function, and an auxiliary control parameter. These two nonlinear oscillators are: the fractional-order Duffing-relativistic oscillator and the fractional-order stretched elastic wire oscillator (with a mass attached to its midpoint). Nevertheless, the rest of this paper is arranged, as follows: the next section exhibits, briefly, the HAM along with a further modification of its scheme. Section 3 introduces the OHAM in order to construct approximate solutions for some strongly fractional-order nonlinear oscillatory problems. Section 4 demonstrates approximate solutions of the fractional-order Duffing-relativistic oscillator, and the fractional-order stretched elastic wire oscillator. Finally, the last section summarizes the main conclusions of this work.

2. The Homotopy Asymptotic Method

The HAM is a common analytical approach for solving both weakly and strongly nonlinear problems. In pursuance of this method, approximate series solutions are accurately obtained, even if these problems have fractional-order derivatives [15]. In this part, a modified approach of HAM is presented for the purpose of handling some types of nonlinear FoDEs that have the following general form:

$$D^\alpha u(t) = \mathcal{N}(t, u(t)), \qquad 1 < \alpha \leq 2, \qquad t > 0, \tag{1}$$

subject to the initial conditions

$$u(0) = u_0, \qquad u'(0) = u_1 \tag{2}$$

where $\mathcal{N}$ is a nonlinear operator, $u(t)$ is an unknown continuous function of the independent variable t, and D^α is the Caputo differential operator of order α that can be defined, as follows:

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t), \tag{3}$$

where $m - 1 < \alpha \le m$, $m \in \mathbb{N}$, and $f \in C^m(0, T]$. Here, D^m is the traditional integer-order differential operator of order m, and J^μ is the Riemann–Liouville integral operator of order $\mu = m - \alpha > 0$, which can be defined by:

$$J^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \eta)^{\mu-1} f(\eta)\, d\eta, \qquad t > 0. \tag{4}$$

For more insight regarding further properties that are associated with these two operators, Caputo and Riemann–Liouville operators, the reader may refer to [31]. However, in view of the HAM, the following homotopy can be established:

$$(1 - q)L[\Phi(t; q) - u_0] = qhH(t)\big(D^\alpha \Phi(t; q) - \mathcal{N}(t, \Phi(t; q))\big), \tag{5}$$

where $q \in [0, 1]$ is the embedding parameter, $h \ne 0$ is a non zero auxiliary parameter, u_0 is an initial guess, $H(t) \ne 0$ is an auxiliary function, L is an auxiliary linear operator, and $\Phi(t; q)$ is an unknown function. Observe that homotopy (5) becomes simply $L[\Phi(t; 0) - u_0] = 0$ when $q = 0$, whereas it returns back to its original nonlinear form that is given in (1) when $q = 1$. Therefore, as q differs from 0 up to 1, $\Phi(t; q)$ differs from the initial guess u_0 up to the exact solution $u(t) = \Phi(t; 1)$ that is constructed for (1). Regardless, $\Phi(t; q)$ could be expanded with respect to q by using Taylor series as follows:

$$\Phi(t; q) = u_0 + \sum_{m=1}^{\infty} q^m u_m(t). \tag{6}$$

Note that, whenever the series $u_0 + \sum_{m=1}^{\infty} q^m u_m(t)$ converges at $q = 1$, then the following homotopy series solution could be established:

$$u(t) = \Phi(t; 1) = u_0 + \sum_{m=1}^{\infty} u_m(t), \tag{7}$$

which should satisfy (1). In the same vein, one can track the same procedure that was established in [2,18,29] for the purose of identifying each term of u_m's that given in series (6). Now, substituting series (6) in homotopy (5) and then equating the coefficients of the similar powers of q yields the following m^{th}-order deformation equation:

$$L(u_m(t) - \chi_m u_{m-1}(t)) = hH(t)\mathcal{R}[u_{m-1}(t)], \qquad m \ge 1, \tag{8}$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \tag{9}$$

and

$$\mathcal{R}[u_{m-1}(t)] = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \Big[D^\alpha \Phi(t; q) - \mathcal{N}(t, \Phi(t; q)) \Big]_{q=0}. \tag{10}$$

In light of the previous considerations, a further modification has been proposed for the HAM that can be employed simply and directly for obtaining series solutions for nonlinear FoDEs. It has clearly appeared that the success of this modification relies on the favorable choice for each of the auxiliary parameter h, the auxiliary function $H(t)$,

and the auxiliary linear operator L. For more details regarding the proper selection of the auxiliary function $H(t)$, and the auxiliary control parameter h, the reader may refer to the references [32–34].

3. An OHAM for Fractional-Order Nonlinear Oscillators

This section targets introducing the OHAM for the purpose of generally establishing approximate solutions for the strongly fractional-order nonlinear oscillatory problems that can be expressed by the following form [35]:

$$D^\alpha u(t) + f(u(t)) = 0, \tag{11}$$

subject to the following initial conditions:

$$u(0) = u_0, \qquad u'(0) = 0, \tag{12}$$

where D^α is the Caputo operator of order $1 < \alpha \leq 2$, f is a nonlinear function, and $u(t)$ is an unknown continuous function of the independent variable t. First of all, we set out to rewrite the nonlinear oscillator that is given in (11) to be in the following form:

$$F(D^\alpha u(t), u(t)) = 0, \tag{13}$$

where F is a nonlinear function. The idea of constructing our proposed algorithm initially relies on choosing an optimal auxiliary linear operator by taking into account that the nonlinear function F can be written by a Taylor series at $t = 0$. Therefore, making a linearization of the function F at $t = 0$ yields the following linear approximation:

$$F(D^\alpha u(t), u(t)) \cong F(D^\alpha u(0), u(0)) + \frac{\partial F}{\partial D^\alpha u}(D^\alpha u(0), u(0))D^\alpha u(t) + \frac{\partial F}{\partial u}(D^\alpha u(0), u(0))u(t). \tag{14}$$

Accordingly, solving straightforwardly the algebraic equation $F(D^\alpha u(0), u(0)) = 0$ for $D^\alpha u(0)$ leads us to design an optimal auxiliary linear operator L in the form:

$$L[u(t)] = D^\alpha u(t) + k(u_0)u(t), \tag{15}$$

where the constant $k(u_0)$, which only depends on u_0, can be computed according to the following formula:

$$k(u_0) = \frac{\frac{\partial F}{\partial u}(u_0^\alpha, u_0)}{\frac{\partial F}{\partial D^\alpha u}(u_0^\alpha, u_0)}, \tag{16}$$

where $u_0^\alpha = D^\alpha u(0)$. One should observe that the designed linear operator is an optimal operator in the sense that the approximation $L[u(t)] = D^\alpha u(t) + k(u_0)u(t)$ is the best linear approximation to the function $F(D^\alpha u(t), u(t))$ near $t = 0$ [35]. In a subsequent step, the optimal approach of HAM for the nonlinear fractional-order oscillator problem that is given in (11) can be established by employing the linear operator given in (15), as proposed in the following homotopy:

$$(1-q)\big[D^\alpha + qk(u_0)hH(t)\big]\Big[\Phi(t;q) - u_0\Big] = qhH(t)F\big(D^\alpha \Phi(t;q), \Phi(t;q)\big). \tag{17}$$

It is worth noting that the proposed approach divides the linear operator $L[u(t)]$ into two main parts, namely $D^\alpha[u(t)]$ and $k(u_0)[u(t)]$, and it furthermore embeds them into the homotopy as $(D^\alpha + qk(u_0)hH(t))[u(t)]$. Besides, it utilizes u_0 as an initial approximation to simplify computations. However, the last step that allows for us to successfully implement OHAM considers that the nonlinear fractional-order oscillatory problem that is given in (11) has an approximate solution of the form: $u(t) = u_0 + \sum_{m=1}^\infty u_m(t)$. This solution can be easily obtained, so that the solution components u_m's should satisfy the following m^{th}-order deformation equation:

$$D^\alpha\big(u_m(t) - \chi_m u_{m-1}(t)\big) + k(u_0)hH(t)\big(\chi_m u_{m-1}(t) - \chi_{m-1}u_{m-2}(t)\big) = hH(t)\mathcal{R}[u_{m-1}(t)], \tag{18}$$

where

$$\mathcal{R}[u_{m-1}(t)] = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[F\big(D^\alpha \Phi(t;q), \Phi(t;q)\big) \right]_{q=0}, \tag{19}$$

and where χ_m is previously defined in (9), such that $m \geq 1$.

4. Test Problems

This section employs the OHAM to provide approximate analytic solutions for two strongly fractional-order nonlinear benchmark oscillatory problems, namely: the fractional-order Duffing-relativistic oscillator, and the fractional-order stretched elastic wire oscillator (with a mass attached to its midpoint). All of the theoretical findings in this section have been numerically performed using the MATLAB software package.

Example 1. *Consider the following fractional-order Duffing-relativistic oscillator:*

$$D^\alpha u(t) + \delta u(t) + \gamma u^3(t) = 0, \tag{20a}$$

subject to the following initial conditions:

$$u(0) = A, \qquad u'(0) = 0, \tag{20b}$$

where $1 < \alpha \leq 2$, δ is a constant, and γ is a positive non-dimensional coefficient of nonlinearity that does need to be small [3]. If one selects the linear operator L to be as $L = D^\alpha$, then the standard homotopy will be established as:

$$(1-q)D^\alpha \left[\Phi(t;q) - A \right] = qhH(t)\big(D^\alpha \Phi(t;q) + \delta \Phi(t;q) + \gamma \Phi^3(t;q)\big). \tag{21}$$

Thus, taking $H(t) = 1$ makes all of the components of the standard HAM solution to be gained by collecting the terms with similar powers of q via the following equation:

$$(1-q)D^\alpha\left[u_0 + qu_1(t) + q^2u_2(t) + \cdots\right] = qh\Big(D^\alpha\big(u_0 + qu_1(t) + q^2u_2(t) + \cdots\big)$$
$$+ \delta\big(u_0 + qu_1(t) + q^2u_2(t) + \cdots\big) + \gamma\big(u_0 + qu_1(t) + q^2u_2(t) + \cdots\big)^3\Big). \tag{22}$$

The optimal linear operator then has the following form:

$$\mathrm{L}[u(t)] = D^\alpha u(t) + (\delta + 3\gamma A^2)u(t), \tag{23}$$

and, moreover, the optimal homotopy, when $u_0 = A$, will be of the form:

$$(1-q)\left[D^\alpha + q(\delta + 3\gamma A^2)hH(t)\right]\left[\Phi(t;q) - A\right] = qhH(t)\big(D^\alpha\Phi(t;q) + \delta\Phi(t;q) + \gamma\Phi^3(t;q)\big). \tag{24}$$

Consequently the OHAM's solution can be obtained as $u(t) = A + \sum_{m=1}^{\infty} u_m(t)$, in which all components $u_m(t)$ of that solution satisfy the following m^{th}-order deformation equation:

$$D^\alpha\big(u_m(t) - \chi_m u_{m-1}(t)\big) + (\delta + 3\gamma A^2)hH(t)\big(\chi_m u_{m-1}(t) - \chi_{m-1}u_{m-2}(t)\big) = hH(t)\mathcal{R}[u_{m-1}(t)], \tag{25}$$

where

$$\mathcal{R}[u_{m-1}(t)] = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[D^\alpha\Phi(t;q) + \delta\Phi(t;q) + \gamma\Phi^3(t;q) \right]_{q=0}, \tag{26}$$

and where $m \geq 1$.

Again, taking $H(t) = 1$ makes, this time, all of the components of the OHAM's solution to be obtained by collecting the terms with similar powers of q via other equations that could be expressed by:

$$(1-q)\big[D^\alpha + q(\delta+3\gamma A^2)h\big]\big[u_{0+}qu_1(t)+q^2u_2(t)+\cdots\big] = qh\Big(D^\alpha\big(u_0+qu_1(t)+q^2u_2(t)+\cdots\big)$$
$$+\delta\big(u_0+qu_1(t)+q^2u_2(t)+\cdots\big)+\gamma\big(u_0+qu_1(t)+q^2u_2(t)+\cdots\big)^3\Big), \tag{27}$$

which, consequently, implies the following recursive states:

$$\begin{cases} D^\alpha u_1(t) = h(\delta u_0 + \gamma u_0^3), \\ D^\alpha u_2(t) = D^\alpha u_1(t) - (\delta+3\gamma A^2)hu_1(t) + h\big(D^\alpha u_1(t)+\delta u_1(t)+3\gamma u_0^2 u_1(t)\big) \\ \vdots \\ D^\alpha u_k(t) = D^\alpha u_{k-1}(t) - (\delta+3\gamma A^2)h(u_{k-1}(t)-u_{k-2}(t)) + h\mathcal{R}[u_{k-1}(t)] \end{cases} \tag{28}$$

subject to the initial conditions:

$$u_0(0) = A, \ u_m'(0) = 0, \tag{29}$$

where $m = 1, 2, \cdots$. Applying the operator J^α on (28) implies:

$$\begin{cases} u_1(t) = \dfrac{hA(\delta+\gamma A^2)}{\Gamma(\alpha+1)}t^\alpha, \\ u_2(t) = \dfrac{Ah(1+h)(\delta+\gamma A^2)}{\Gamma(\alpha+1)}t^\alpha. \\ \vdots \end{cases} \tag{30}$$

In a similar manner, we can obtain the rest of all the components using MATLAB software code. In addition, the series solutions expression can be then written in the form:

$$u(t) \simeq u_0 + \sum_{m=1}^{N} u_m(t) = u_0 + u_1(t) + u_2(t) + \dots, \tag{31}$$

or

$$u(t) \simeq A + Ah(2+h)(\delta+\gamma A^2)\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right) + Ah(1+h)^2(\delta+\gamma A^2)\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)$$
$$+ h(\delta+3\gamma A^2)(Ah(1+h)(\delta+\gamma A^2)\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right) \tag{32}$$
$$+ 3\gamma A^3 h^2(\delta+\Gamma A^2)^2\left(\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\right) + \cdots.$$

In connection with the selection of the value of parameter h or the so-called the convergent-control parameter h, in Figure 1 we draw its corresponding h-curves according to different values of A, δ, γ and α. In view of such a figure, we deduce the convergence interval that guarantees, in turn, a convergence of the approximate solution $u(t)$. Here, such an interval is deduced to be as $[-3, 3]$, so that the value of h can be chosen within this scope. For more simplification, one may choose the auxiliary function $H(t)$ to be, e.g., equal 1. However, Figure 2 shows approximate solutions $u(t)$ for problem (20) by using the OHAM for different values of A, α, h, δ, and γ. For more effective practice, we perform some graphical comparisons between the OHAM and the HAM, as shown in Figure 3. Obviously, these comparisons reveal that the approximate solutions obtained by such methods are very close to each other, which confirms the efficient and robustness of the OHAM. The reader may refer to the references [33,34,36] to obtain a complete overview about the h-curves and how they can be utilized to determine the admissible values of the parameter h.
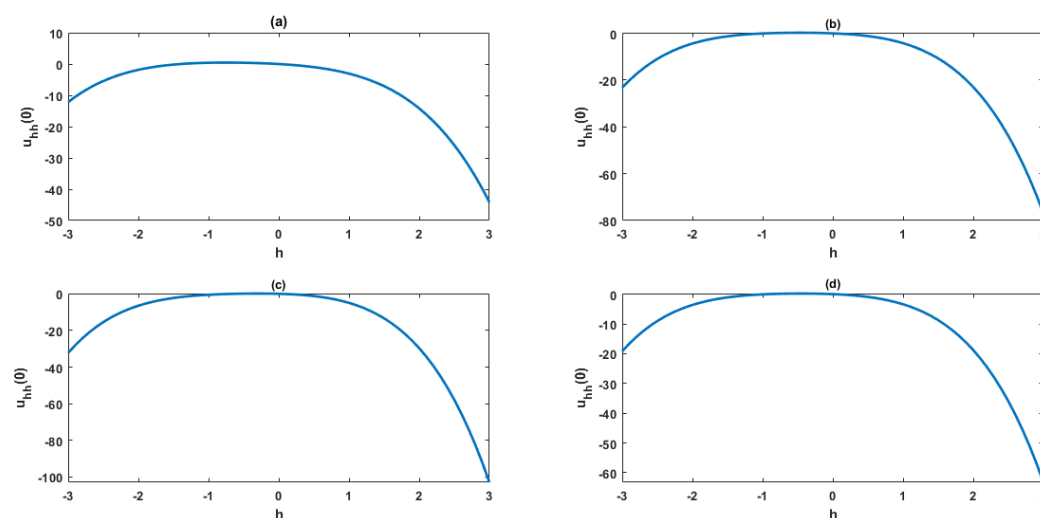
Figure 1. Plots of several h–curves according different values: (**a**) $A = 0.75$, $\gamma = \delta = 1$, (**b**) $A = 1.25$, $\gamma = 0.3$, $\delta = 0.5$, (**c**) $A = 2$, $\gamma = \delta = 1$, (**d**) $A = 1.5$, $\gamma = \delta = 1$.
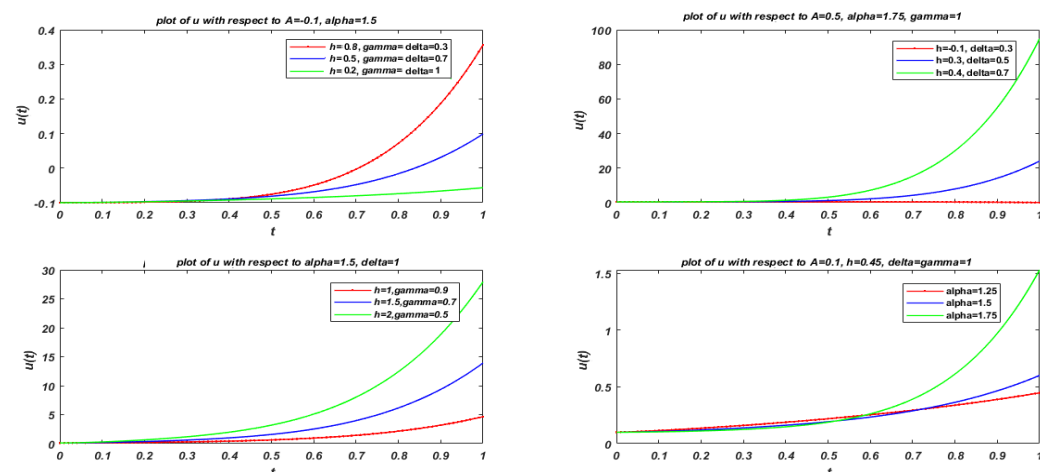

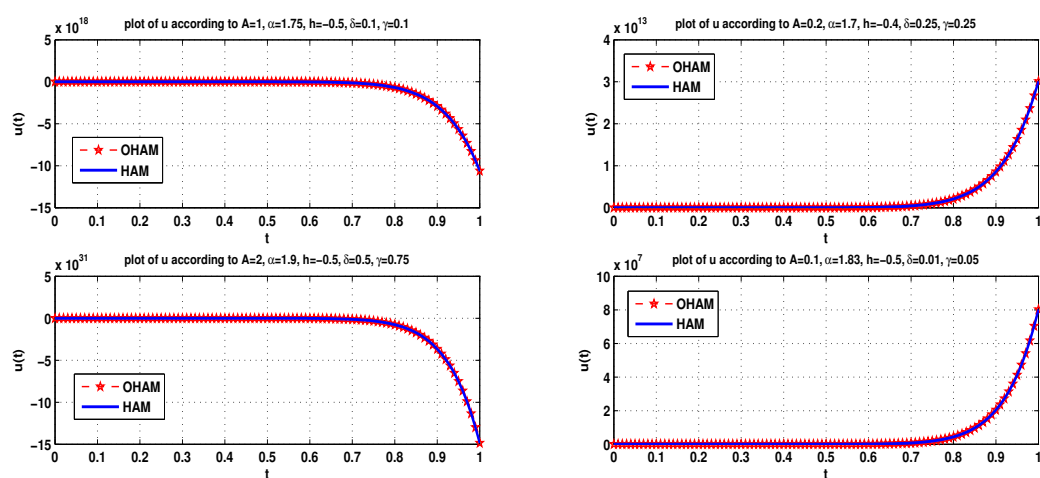**Figure 2.** Plots of the Optimal Homotopy Asymptotic Method's (OHAM's) solutions $u(t)$ according to different values of A, δ, γ and α.

Figure 3. Graphical comparisons between the OHAM and Homotopy Analysis Method (HAM) for different values of A, λ, α and h.

Example 2. *Consider the following nonlinear fractional-order problem that represents the motion equation of the stretched elastic wire oscillator (with a mass that is attached to its midpoint):*

$$D^\alpha u(t) + u(t) - \frac{\lambda u(t)}{\sqrt{1 + u^2(t)}} = 0, \tag{33a}$$

subject to the following initial conditions:

$$u(0) = A, \qquad u'(0) = 0, \tag{33b}$$

where $0 < \lambda \leq 1$ and $1 < \alpha \leq 2$.

The optimal linear operator here is of the following form:

$$\mathrm{L}[u(t)] = D^\alpha u(t) + \left(1 - \lambda\left(1 + A^2\right)^{-\frac{1}{2}} + \lambda A^2\left(1 + A^2\right)^{-\frac{3}{2}}\right) u(t). \tag{34}$$

Furthermore, the optimal homotopy, when $u_0 = A$, is then of the form:

$$(1-q)\left(D^\alpha + q\left(1 - \lambda\left(1 + A^2\right)^{-\frac{1}{2}} + \lambda A^2\left(1 + A^2\right)^{-\frac{3}{2}}\right)hH(t)\right)[\Phi(t;q) - A]$$
$$= qhH(t)[D^\alpha\Phi(t;q) + \Phi(t;q) - \lambda\Phi(t;q)\left(1 + (\Phi(t;q))^2\right)^{-\frac{1}{2}}]. \tag{35}$$

Consequently, the OHAM's solution can be formulated as $u(t) = A + \sum_{m=1}^{\infty} u_m(t)$, in which u_m's hold the following m^{th}-order deformation equation:

$$D^\alpha(u_m(t) - \chi_m u_{m-1}(t)) + \left(1 - \lambda\left(1 + A^2\right)^{-\frac{1}{2}} + \lambda A^2\left(1 + A^2\right)^{-\frac{3}{2}}\right)$$
$$\times hH(t)(\chi_m u_{m-1}(t) - \chi_{m-1} u_{m-2}(t)) = hH(t)\mathcal{R}[u_{m-1}(t)], \tag{36}$$

where

$$\mathcal{R}[u_{m-1}(t)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[D^\alpha\Phi(t;q) + \Phi(t;q) - \lambda\Phi(t;q)\left(1 + (\Phi(t;q))^2\right)^{-\frac{1}{2}}]|_{q=0}, \tag{37}$$

and where $m \geq 1$.

Now, taking $H(t) = 1$ allows for one to gain all the components of the OHAM's solution by collecting the terms with similar powers of q via the following equation:

$$(1-q)\left[D^\alpha + q\left(1 - \lambda\left(1 + A^2\right)^{-\frac{1}{2}} + \lambda A^2\left(1 + A^2\right)^{-\frac{3}{2}}\right)h\right]\left(u_0 + qu_1(t) + q^2u_2(t) + \cdots\right)$$
$$= qhD^\alpha\left(u_0 + qu_1(t) + q^2u_2(t) + \cdots\right) + \left(u_0 + qu_1(t) + q^2u_2(t) + \cdots\right) \tag{38}$$
$$- \lambda\left(u_0 + qu_1(t) + q^2u_2(t) + \cdots\right)\left(1 + \left(u_0 + qu_1(t) + q^2u_2(t) + \cdots\right)^2\right)^{-\frac{1}{2}}.$$

This leads us to establish the following recursive states:

$$\begin{cases} D^\alpha u_1(t) = h[D^\alpha u_0 + u_0 - \lambda u_0(1 + u_0^2)], \\ D^\alpha u_2(t) = (1+h)D^\alpha u_1(t) - hku_1(t) + h[u_1(t) - \lambda N_1(u_0, u_1(t))], \\ \vdots \\ D^\alpha u_k(t) = (1+h)D^\alpha u_{k-1}(t) - hku_{k-1}(t) + h[u_{k-1}(t) - \lambda N_{k-1}(u_0, u_1(t), \ldots, u_{k-1}(t))], \end{cases} \tag{39}$$

where $k = 1 - \lambda\left(1 + A^2\right)^{-\frac{1}{2}} + \lambda A^2\left(1 + A^2\right)^{-\frac{3}{2}}$, and where

$$N_{k-1}(u_0, u_1(t), \ldots, u_{k-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[-\lambda\left(u_0 + qu_1(t) + q^2 u_2(t) + q^3 u_3(t) + \cdots\right)\right.$$
$$\left. \times\left(1 + \left(u_0 + qu_1(t) + q^2 u_2(t) + q^3 u_3(t) + \cdots\right)^2\right)^{-\frac{1}{2}}\right]\Bigg|_{q=0}, \tag{40}$$

subject to the following initial conditions:

$$u_0(0) = A, \ u'_m(0) = 0, \tag{41}$$

where $m = 1, 2, \ldots$.

Now, applying J^α on (39) yields:

$$\begin{cases} u_1(t) = & Ah(1 - \frac{\lambda}{\sqrt{1+A^2}})\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2(t) = & Ah(1+h)(1 - \frac{\lambda}{\sqrt{1+A^2}})\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2\lambda A^3 h^2}{(1+A^2)^{\frac{3}{2}}}(1 - \frac{\lambda}{\sqrt{1+A^2}})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ \vdots & \end{cases} \tag{42}$$

In a similar manner, the rest of other components can be obtained, and then the series solutions expression will be, as given before, in (31). That is;

$$u(t) \simeq A + Ah(2+h)\left(1 - \frac{\lambda}{\sqrt{1+A^2}}\right)\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)$$
$$+ \left(\frac{2\lambda A^3 h^2}{(1+A^2)^{3/2}}\right)\left(1 - \frac{\lambda}{\sqrt{1+A^2}}\right)\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right) + \cdots. \tag{43}$$

Similarly to Example 1, Figure 4 illustrates several h-curves in accordance with different values of A, λ, and α. Based on this figure, one may candidate the interval $[-3, 3]$ to be also the interval of convergence. The value of the parameter h can be, then, chosen from such interval. On the other hand, the auxiliary function $H(t)$ can be chosen once again 1. Taking the previous data into account when carrying out the OHAM via MATLAB software code generates the results that are shown in Figure 5, which represents the approximate solutions for problem (33) according to different values of A, α, h, and λ. For more insight, Figure 6 shows some graphical comparisons that are performed between the OHAM and the HAM. Such comparisons reveal the influence and impact of the method under consideration.
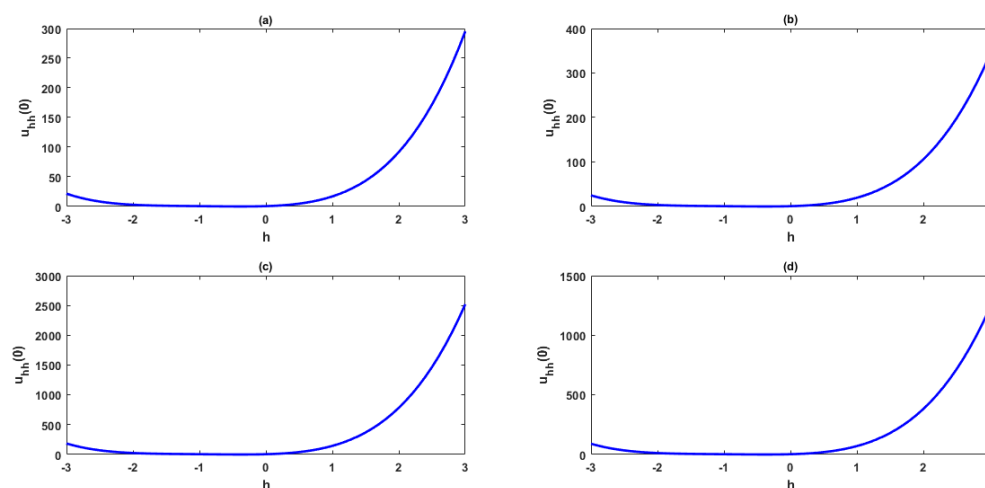

Figure 4. Plots of several h–curves according different values: (**a**) $A = 0.75$, $\lambda = 0.5$, (**b**) $A = 1.25$, $\lambda = 0.5$, (**c**) $A = 2$, $\lambda = 0.9$, and (**d**) $A = 1.5$, $\lambda = 0.95$.
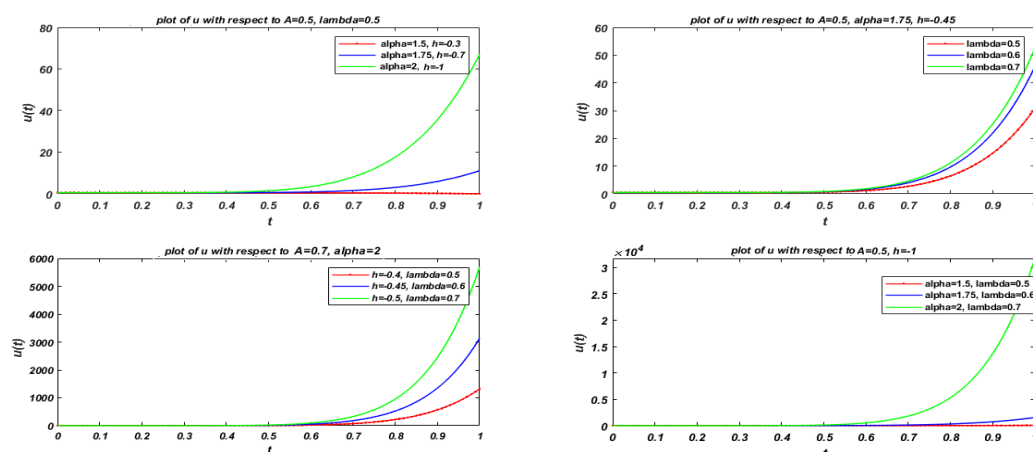
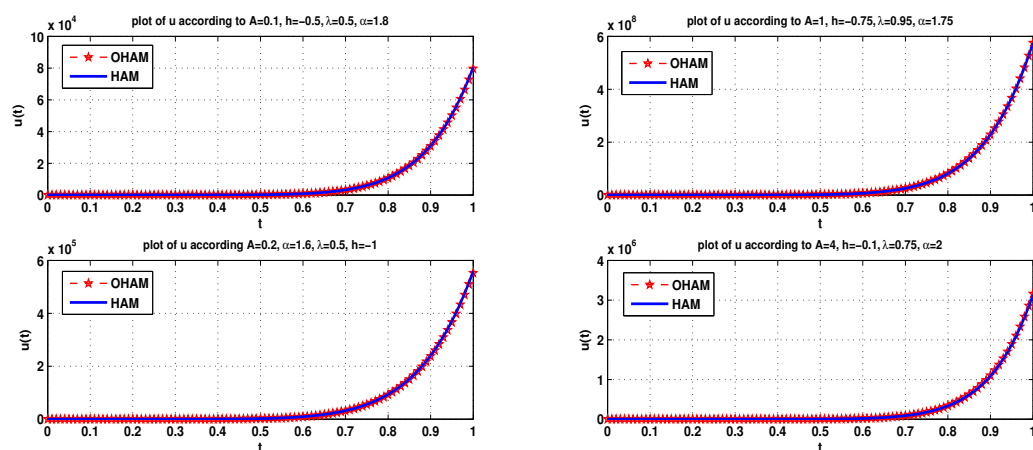**Figure 5.** Plots of the OHAM's solutions $u(t)$ according to different values of A, δ, γ and α.

Figure 6. Graphical comparisons between the OHAM and HAM for different values of A, λ, α and h.

5. Conclusions

In this paper, a further modification for an Optimal Homotopy Asymptotic Method (OHAM) has been successfully implemented to tsolvewo strongly fractional-order non-linear benchmark oscillatory problems, namely: the fractional-order Duffing-relativistic oscillator and the fractional-order stretched elastic wire oscillator (with a mass attached to its midpoint). Such a modification has been performed by establishing an optimal auxiliary linear operator, an auxiliary function, and an auxiliary control parameter. The proposed scheme has shown its reliability in comparison with the approximate solutions that were obtained using HAM, and its efficiency in handling the considered problems.

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