Article

Asymptotic Stability of the Pexider–Cauchy Functional Equation in Non-Archimedean Spaces

Hamid Gharib ^{1,†}, Mohammad B. Moghimi ^{1,†}, Abbas Najati ^{1,†} and Jae-Hyeong Bae ^{2,*,†} ¹ Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran; hamidgharib.g@gmail.com (H.G.); mbfmoghimi@yahoo.com (M.B.M.); a.nejati@yahoo.com (A.N.)² Humanitas College, Kyung Hee University, Yongin 17104, Korea

* Correspondence: jhbae@khu.ac.kr

† These authors contributed equally to this work.

Abstract: In this paper, we investigated the asymptotic stability behaviour of the Pexider–Cauchy functional equation in non-Archimedean spaces. We also showed that, under some conditions, if $\|f(x+y) - g(x) - h(y)\| \leq \varepsilon$, then f, g and h can be approximated by additive mapping in non-Archimedean normed spaces. Finally, we deal with a functional inequality and its asymptotic behaviour.

Keywords: non-Archimedean space; Pexider–Cauchy equation; asymptotic stability

Citation: Gharib, H.; Moghimi, M.B.; Najati, A.; Bae, J.-H. Asymptotic Stability of the Pexider–Cauchy Functional Equation in Non-Archimedean Spaces. *Mathematics* **2021**, *9*, 2197. <https://doi.org/10.3390/math9182197>

Academic Editor: Ioannis K. Argyros

Received: 24 August 2021

Accepted: 7 September 2021

Published: 8 September 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The classical problem of the stability of homomorphisms was first posed by Ulam [1] in 1940 as follows: Given a group $(G_1, *)$, a metric group $(G_2, \diamond, d)$ and $\varepsilon > 0$. Does $\delta > 0$ exist such that if the function $f : G_1 \rightarrow G_2$ satisfies the inequality:

$$d(f(x * y), f(x) \diamond f(y)) < \delta, \quad x, y \in G_1,$$

then there exists a homomorphism $T : G_1 \rightarrow G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$? In 1941, Donald H. Hyers [2] gave a partial affirmative answer to this question under the assumption that $f : X \rightarrow Y$ is a function between Banach spaces X and Y . Indeed, D. H. Hyers' classical theorem states that, if X is a linear space and Y is a Banach space and, for some $\varepsilon \geq 0$, a function $f : X \rightarrow Y$ is ε -additive, meaning that:

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in X \quad (1)$$

then there exists a unique additive function $A : X \rightarrow Y$ such that f is ε -close to A , i.e., $\|f(x) - A(x)\| \leq \varepsilon$ for all $x \in X$. Ulam's question can be expressed in a more general way as follows: *when is it true that a function which approximately satisfies a functional equation (†) must be close to an exact solution of (†)?*

Aoki [3] and Th. M. Rassias [4] extended the Hyers' theorem by considering an unbounded Cauchy difference. They tried to weaken the condition for the bound of the norm of the Cauchy difference in (1) as follows:

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \quad x, y \in X$$

where $0 \leq p < 1$. In 1994, Găvruta [5] provided a generalization of Rassias' theorem by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. In recent decades, several stability problems for various functional equations and also for mappings with more general domains and ranges have been investigated by a number of mathematicians. We refer the interested reader the following books and surveys [6–10] and the references therein for more detailed information.

Let us recall (see, for instance, [11]) some basic definitions and facts concerning non-Archimedean normed spaces. By a non-Archimedean field, we mean a field $\mathbb{F}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{F}$ into $[0, +\infty)$ such that:

- (i) $|r| = 0$ if and only if $r = 0$;
- (ii) $|rs| = |r||s|$, $r, s \in \mathbb{F}$;
- (iii) $|r + s| \leq \max\{|r|, |s|\}$, $r, s \in \mathbb{F}$.

Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all integers n . An obvious example is the trivial valuation that can be defined on any field $\mathbb{F}$ by setting:

$$|x| := \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x \neq 0. \end{cases}$$

Example 1. For a given prime number p , any non-zero rational number x can be uniquely written as $x = \frac{a}{b} p^{n_x}$, where $a, b, n_x \in \mathbb{Z}$ and a, b are integers not divisible by p . Then, the function $|\cdot|_p : \mathbb{Q} \rightarrow [0, +\infty)$ defined by

$$|x| := \begin{cases} 0, & \text{if } x = 0; \\ p^{-n_x}, & \text{if } x \neq 0. \end{cases}$$

is a non-Archimedean (nontrivial) valuation on $\mathbb{Q}$.

Let X be a vector space over a non-Archimedean field $\mathbb{F}$ with a non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, +\infty)$ is called a non-Archimedean norm (valuation) if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{F}$ and all $x \in X$;
- (iii) the strong triangle inequality; namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad x, y \in X.$$

Then, $(X, \|\cdot\|)$ is called a non-Archimedean normed space. A sequence $\{x_n\}_{n=1}^\infty$ in X is called a Cauchy sequence if $\lim_{m,n \rightarrow \infty} \|x_n - x_m\| = 0$. A sequence $\{x_n\}_{n=1}^\infty$ in X is said to converge to $x \in X$ if $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$. By the strong triangle inequality (iii), we have

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\}, \quad n > m.$$

Then, the sequence $\{x_n\}_{n=1}^\infty$ in a non-Archimedean space is Cauchy if and only if $\{x_{n+1} - x_n\}_{n=1}^\infty$ converges to zero. By a Banach non-Archimedean space, we mean that every Cauchy sequence is convergent. It is necessary to remember that $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}$, and that the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$ and is called the p -adic number field. Indeed, p -adic numbers do not satisfy the Archimedean axiom which states: for each $x, y > 0$, there exists an integer n such that $nx > y$.

In [12], the authors showed that if $f : \mathbb{Q}_p \rightarrow \mathbb{R}$ is a continuous ε -additive function, then there exists a unique additive function $A : X \rightarrow Y$ such that f is ε -close to A . In [13,14], the authors investigated the stability of some functional equations in non-Archimedean normed spaces. In this paper, we investigated the asymptotic stability behaviour of the Pexider–Cauchy functional equation in non-Archimedean spaces. We will use the methods and ideas presented in [15–17].

2. Main Results

Throughout this section, we assume that X and Y are the non-Archimedean normed spaces over a non-Archimedean field $\mathbb{F}$ with a valuation $|\cdot|$.

Theorem 1. Let $\varepsilon \geq 0$ and $f, g, h : X \rightarrow Y$ satisfy the following asymptotic stability property:

$$\limsup_{\min\{\|x\|, \|y\|\} \rightarrow \infty} \|f(x+y) - g(x) - h(y)\| \leq \varepsilon. \quad (2)$$

If $|n| < 1$ for some integer n , then:

$$\|f(x+y) - f(x) - f(y) + f(0)\| \leq \varepsilon, \quad x, y \in X. \quad (3)$$

Proof. Let $\eta > \varepsilon$ be an arbitrary real number. Then, by (2) there exists $d > 0$ such that:

$$\|f(x+y) - g(x) - h(y)\| < \eta, \quad \|x\|, \|y\| \geq d. \quad (4)$$

Let $x, y \in X$ be arbitrary elements. Since $|n| < 1$ for some integer n , we obtain:

$$\lim_{m \rightarrow +\infty} \left\| \frac{1}{n^m} z \right\| = +\infty$$

for some $z \in X \setminus \{0\}$. Hence we can choose $u \in X$ such that $\|u\| \geq d + \|x\| + \|y\|$. It is easy to see that:

$$\min\{\|x-u\|, \|y+u\|, \|u\|\} \geq d.$$

Applying (4) and the strong triangle inequality, we obtain:

$$\begin{aligned} \|f(x+y) - f(x) - f(y) + f(0)\| &\leq \max \{ \|g(x-u) + h(u) - f(x)\|, \\ &\quad \|g(-u) + h(y+u) - f(y)\|, \\ &\quad \|f(x+y) - g(x-u) - h(y+u)\|, \\ &\quad \|f(0) - g(-u) - h(u)\| \} < \eta \end{aligned}$$

for all $x, y \in X$. Taking the limit as $\eta \rightarrow \varepsilon$ in the last inequality, we obtain (3): $\square$

Corollary 1. Let $\varepsilon \geq 0$ and $f : X \rightarrow Y$ satisfy the following asymptotic stability property:

$$\limsup_{\min\{\|x\|, \|y\|\} \rightarrow \infty} \|f(x+y) - f(x) - f(y)\| \leq \varepsilon. \quad (5)$$

If $|n| < 1$ for some integer n , then:

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in X. \quad (6)$$

Proof. Let $\eta > \varepsilon$ be an arbitrary real number. By (5), there exists: $d > 0$ such that

$$\|f(x+y) - f(x) - f(y)\| < \eta, \quad \|x\|, \|y\| \geq d. \quad (7)$$

By the proof of Theorem 1, we obtain:

$$\|f(x+y) - f(x) - f(y) + f(0)\| < \eta, \quad x, y \in X. \quad (8)$$

By using the strong triangle inequality, it follows from (7) and (8) that $\|f(0)\| < \eta$. Then:

$$\|f(x+y) - f(x) - f(y)\| \leq \max \{ \|f(x+y) - f(x) - f(y) + f(0)\|, \|f(0)\| \} < \eta$$

for all $x, y \in X$. Taking the limit as $\eta \rightarrow \varepsilon$ in the last inequality, we obtain (6): $\square$

Corollary 2. Let $|n| < 1$ for some integer n , and $f, g, h : X \rightarrow Y$ fulfil the following asymptotic property:

$$\lim_{\min\{\|x\|, \|y\|\} \rightarrow \infty} \|f(x+y) - g(x) - h(y)\| = 0.$$

Then, f is affine. Moreover, if $f = g = h$, then f is additive.

The following example shows that with the conditions of Corollary 2, functions g and h may not be affine even if $g = h$.

Example 2. Let $\mathbb{Q}$ be the field of rational numbers and let $|\cdot|_p$ be the p -adic absolute value, where $p > 2$ is a prime number. We define $f, g, h : \mathbb{Q} \rightarrow \mathbb{Q}$ by

$$f(x) = x \quad \text{and} \quad g(x) = h(x) = x + \frac{1}{1+x^2}, \quad x \in \mathbb{Q}.$$

Let $n > 1$ be an integer. For $|x|_p, |y|_p \geq p^{-n}$, we have:

$$\begin{aligned} |f(x+y) - g(x) - h(y)|_p &= \left| \frac{1}{1+x^2} + \frac{1}{1+y^2} \right|_p \\ &\leq \max \left\{ \frac{1}{|1+x^2|_p}, \frac{1}{|1+y^2|_p} \right\} \\ &\leq \frac{1}{p^{2n}-1} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Then:

$$\lim_{\min\{|x|_p, |y|_p\} \rightarrow \infty} |f(x+y) - g(x) - h(y)|_p = 0,$$

but g and h are not affine.

Example 3. Let $p > 2$ be a prime number. Suppose that a function $f : \mathbb{Q} \rightarrow \mathbb{Q}$ satisfies

$$f(1) = 1 \quad \text{and} \quad \lim_{\min\{|x|_p, |y|_p\} \rightarrow \infty} |f(x+y) - f(x) - f(y)|_p = 0.$$

Then f is the identity, i.e., $f(x) = x$ for all $x \in X$.

Theorem 2. Let $\varepsilon \geq 0$ and Y be a Banach non-Archimedean space. Suppose that $f, g, h : X \rightarrow Y$ are functions that satisfy:

$$\limsup_{\min\{\|x\|, \|y\|\} \rightarrow \infty} \|f(x+y) - g(x) - h(y)\| \leq \varepsilon. \quad (9)$$

If $|n| < 1$ for some integer n , then there exists a unique additive mapping $A : X \rightarrow Y$ such that:

$$\|f(x) - f(0) - A(x)\| \leq \varepsilon, \quad x \in X. \quad (10)$$

Proof. It follows by Theorem 1 that:

$$\|f(x+y) - f(x) - f(y) + f(0)\| \leq \varepsilon, \quad x, y \in X.$$

Defining $F : X \rightarrow Y$ by $F(x) := f(x) - f(0)$, we obtain:

$$\|F(x+y) - F(x) - F(y)\| \leq \varepsilon, \quad x, y \in X. \quad (11)$$

By ([13] [Theorem 2.1]), there is a unique additive mapping $A : X \rightarrow Y$ such that $\|F(x) - A(x)\| \leq \varepsilon$ for all $x \in X$ (see also [14,18]). This completes the proof. $\square$

By Corollary 1 and using the argument in Theorem 2, we obtain the following result.

Corollary 3. Let $\varepsilon \geq 0$ and Y be a Banach non-Archimedean space. Let $f : X \rightarrow Y$ be functions satisfying:

$$\limsup_{\min\{\|x\|, \|y\|\} \rightarrow \infty} \|f(x+y) - f(x) - f(y)\| \leq \varepsilon. \quad (12)$$

If $|n| < 1$ for some integer n , then there exists a unique additive mapping $A : X \rightarrow Y$ such that:

$$\|f(x) - A(x)\| \leq \varepsilon, \quad x \in X. \quad (13)$$

Theorem 3. Let $\varepsilon \geq 0$ and $f, g, h : X \rightarrow Y$ satisfy the following asymptotic stability property:

$$\limsup_{\|x\| + \|y\| \rightarrow \infty} \|f(x+y) - g(x) - h(y)\| \leq \varepsilon. \quad (14)$$

If $|n| < 1$ for some integer n , then:

$$\|f(x+y) - g(x) - h(y)\| \leq \varepsilon, \quad x, y \in X. \quad (15)$$

Proof. Let $\eta > \varepsilon$ be an arbitrary real number. Then, by (14) there exists: $d > 0$ such that

$$\|f(x+y) - g(x) - h(y)\| < \eta, \quad \|x\| + \|y\| \geq d. \quad (16)$$

Let $x, y \in X$ be arbitrary elements. Since $|n| < 1$ for some integer n , we can choose $u \in X$ such that $\|2u\| \geq d + \|x\| + \|y\|$. It is easy to verify that:

$$\begin{aligned} \|x - u\| + \|y + u\| &\geq d; \quad \|2u\| + \|x - u\| \geq d; \quad \|y\| + \|2u\| \geq d; \\ \|y + u\| + \|u\| &\geq d; \quad \|x\| + \|u\| \geq d. \end{aligned} \quad (17)$$

Then, (16) and (17) yield:

$$\begin{aligned} \|f(x+y) - g(x) - h(y)\| &\leq \max \{ \|f(x+y) - g(y+u) - h(x-u)\|, \\ &\quad \|f(x+u) - g(2u) - h(x-u)\|, \\ &\quad \|f(y+2u) - g(2u) - h(y)\|, \\ &\quad \|f(y+2u) - g(y+u) - h(u)\|, \\ &\quad \|f(x+u) - g(x) - h(u)\| \} < \eta \end{aligned}$$

for all $x, y \in X$. Taking the limit as $\eta \rightarrow \varepsilon$ in the last inequality, we obtain (15). $\square$

Corollary 4. Let $\varepsilon \geq 0$ and $f, g, h : X \rightarrow Y$ satisfy the following asymptotic property:

$$\lim_{\|x\| + \|y\| \rightarrow \infty} \|f(x+y) - g(x) - h(y)\| = 0 \quad (18)$$

If $|n| < 1$ for some integer n , then f, g and h are affine. Moreover, if $g(0) = h(0) = 0$, then $f = g = h$ are additive.

Theorem 4. Let $\varepsilon \geq 0$ and $f, g, h : X \rightarrow Y$ be functions satisfying (14). If $|n| < 1$ for some integer n , then there exists a unique additive mapping $A : X \rightarrow Y$ such that:

$$\|f(x) - f(0) - A(x)\| \leq \varepsilon, \quad \|g(x) - g(0) - A(x)\| \leq \varepsilon, \quad \|h(x) - h(0) - A(x)\| \leq \varepsilon$$

for all $x \in X$.

Proof. By Theorem 3, we have $\|f(x+y) - g(x) - h(y)\| \leq \varepsilon$ for all $x, y \in X$. Then, the result easily follows from ([13] [Theorem 2.4]) (see also [18]). $\square$

3. Results on the Hyperstability

Theorem 5. Let $\varphi : [0, +\infty) \rightarrow \mathbb{R}$ be a function such that $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$ and let $f, g, h : X \rightarrow Y$ satisfy:

$$\limsup_{\|x\|+\|y\| \rightarrow \infty} \varphi(\|x-y\|) \|f(x+y) - g(x) - h(y)\| < \infty. \quad (19)$$

If $|n| < 1$ for some integer n , then $f(x+y) = g(x) + h(y)$ for all $x, y \in X$, and consequently f, g and h are affine. Furthermore, $f - f(0) = g - g(0) = h - h(0)$.

Proof. According to (19), there exist constants $d > 0$ and $K > 0$ such that:

$$\varphi(\|x-y\|) \|f(x+y) - g(x) - h(y)\| < K, \quad \|x\| + \|y\| \geq d. \quad (20)$$

Let $\varepsilon > 0$ be arbitrary. We can choose $N > 0$ such that $\varphi(t) \geq \frac{K}{\varepsilon}$ for all $t \geq N$. Then, (20) implies that:

$$\|f(x+y) - g(x) - h(y)\| < \varepsilon, \quad \|x\| + \|y\| \geq d \quad \text{and} \quad \|x-y\| \geq N. \quad (21)$$

Let $x, y \in X$ with $\|x\| + \|y\| \geq 2d + 2N$. Since $|n| < 1$ for some integer n , we can choose $u \in X$ such that:

$$\min\{\|2u\|, \|3u\|\} \geq d + N + \|x\| + \|y\| + \|x-y\| + \|x-2y\|.$$

It is easy to verify that:

$$\begin{aligned} \min\{\|y \pm u\|, \|x \pm u\|, \|x-y+2u\|, \|u\|\} &\geq d, \\ \min\{\|y-x \pm 2u\|, \|3u-x\|, \|2u-y\|, \|x-u\|, \|x-2y+2u\|, \|x-2y+3u\|\} &\geq N. \end{aligned}$$

Since $\|x\| + \|y\| \geq 2d + 2N$, two cases arise according to whether $\|x\| \geq d + N$ or $\|y\| \geq d + N$. First, take the case $\|x\| \geq d + N$. Then, applying (21), we obtain:

$$\begin{aligned} \|f(x+y) - g(x) - h(y)\| &\leq \max \{ \|f(x+y) - g(x+u) - h(y-u)\|, \\ &\quad \|f(x+2u) - g(x+u) - h(u)\|, \\ &\quad \|f(x+u) - g(x-y+2u) - h(y-u)\|, \\ &\quad \|f(x+u) - g(x) - h(u)\|, \\ &\quad \|f(x+2u) - g(x-y+2u) - h(y)\| \} < \varepsilon. \end{aligned}$$

Now, consider the case $\|y\| \geq d + N$. Then, applying (21) yields:

$$\begin{aligned} \|f(x+y) - g(x) - h(y)\| &\leq \max \{ \|f(x+y) - g(y+u) - h(x-u)\|, \\ &\quad \|f(x+u) - g(2u) - h(x-u)\|, \\ &\quad \|f(y+2u) - g(2u) - h(y)\|, \\ &\quad \|f(y+2u) - g(y+u) - h(u)\|, \\ &\quad \|f(x+u) - g(x) - h(u)\| \} < \varepsilon. \end{aligned}$$

Therefore, $\|f(x+y) - g(x) - h(y)\| \leq \varepsilon$ for $\|x\| + \|y\| \geq 2d + 2N$. Then, by Theorem 3, we have $\|f(x+y) - g(x) - h(y)\| \leq \varepsilon$ for all $x, y \in X$. Since $\varepsilon > 0$ was arbitrary, this inequality implies that $f(x+y) = g(x) + h(y)$ for all $x, y \in X$. $\square$

Remark 1. Using the argument presented above, it can be concluded that Theorem 5 still holds if we have:

$$\limsup_{\|x\|+\|y\| \rightarrow \infty} \varphi(\max\{\|x\|, \|y\|\}) \|f(x+y) - g(x) - h(y)\| < \infty$$

instead of (20).

Theorem 6. Let $\varphi : [0, +\infty) \rightarrow \mathbb{R}$ be a function such that $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$ and let $f, g, h : X \rightarrow Y$ satisfy:

$$\limsup_{\min\{\|x\|, \|y\|\} \rightarrow \infty} \varphi(\|x - y\|) \|f(x + y) - g(x) - h(y)\| < \infty. \quad (22)$$

If $|n| < 1$ for some non-zero integer n , then f is affine.

Proof. According to (22), there exist constants $d > 0$ and $K > 0$ such that:

$$\varphi(\|x - y\|) \|f(x + y) - g(x) - h(y)\| < K, \quad \|x\|, \|y\| \geq d. \quad (23)$$

Let $\varepsilon > 0$ be arbitrary. We can choose $N > 0$ such that $\varphi(t) \geq \frac{K}{\varepsilon}$ for all $t \geq N$. Then, (23) implies that

$$\|f(x + y) - g(x) - h(y)\| < \varepsilon, \quad \|x\|, \|y\| \geq d \quad \text{and} \quad \|x - y\| \geq N. \quad (24)$$

Let $x, y \in X$ be arbitrary elements. Since $0 < |n| < 1$ for some integer n , we can choose $u \in X$ such that $\|2u\| \geq d + N + \|x\| + \|y\| + \|x - y\|$. It is easy to see that:

$$\begin{aligned} \min\{\|x - u\|, \|y + u\|, \|x - 2u\|, \|y + 2u\|, \|x - y - 2u\|, \|2u\|, \|u\|\} &\geq d, \\ \min\{\|x - 2u\|, \|y + 2u\|, \|x - y - 2u\|, \|2u\|\} &\geq N. \end{aligned}$$

Applying (24) and the strong triangle inequality, we obtain:

$$\begin{aligned} \|f(x + y) - f(x) - f(y) + f(0)\| &\leq \max\{\|g(x - u) + h(u) - f(x)\|, \\ &\quad \|g(-u) + h(y + u) - f(y)\|, \\ &\quad \|f(x + y) - g(x - u) - h(y + u)\|, \\ &\quad \|f(0) - g(-u) - h(u)\|\} < \varepsilon \end{aligned}$$

for all $x, y \in X$. Since $\varepsilon > 0$ was arbitrary, this inequality implies that $f(x + y) = f(x) + f(y) - f(0)$ for all $x, y \in X$. Then, $f - f(0)$ is additive, and so f is affine. $\square$

Remark 2. Based on the argument presented above, it can be concluded that the assertion of Theorem 6 is still true by considering:

$$\limsup_{\min\{\|x\|, \|y\|\} \rightarrow \infty} \varphi(\max\{\|x\|, \|y\|\}) \|f(x + y) - g(x) - h(y)\| < \infty$$

instead of (22).

4. Functional Inequalities

We assume that X and Y are non-Archimedean normed spaces over a non-Archimedean field $\mathbb{F}$ with a valuation $|\cdot|$.

Lemma 1. Let $0 < |k| < 1$ for some integer k , and $f : X \rightarrow Y$ with $f(0) = 0$ satisfy:

$$\|f(x + y) - f(x) - f(y)\| \leq \|f(x - y) - f(x) + f(y)\|, \quad x, y \in X. \quad (25)$$

Then, f is rational homogeneous, i.e., $f(rx) = rf(x)$ for all $x \in X$ and $r \in \mathbb{Q}$.

Proof. Letting $y = x$ in (25) and applying $f(0) = 0$, we obtain $f(2x) = 2f(x)$ for all $x \in X$. Now, we will prove by induction that $f(nx) = nf(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Suppose this result holds for all $1 \leq k < n$. Then, by letting $x = (n-1)y$ in (25), we obtain:

$$\|f(ny) - (n-1)f(y) - f(y)\| \leq \|(n-2)f(y) - (n-1)f(y) + f(y)\| = 0, \quad y \in X.$$

Hence, $f(ny) = nf(y)$ for all $y \in X$. Putting $y = -k^2x$ in (25), we obtain:

$$\|f(-x) + f(x)\| \leq |k|^2 \|f(x) + f(-x)\|, \quad x \in X.$$

Since $0 < |k| < 1$, we infer that $f(-x) = -f(x)$ for all $x \in X$. Therefore, $f(nx) = nf(x)$ for all $x \in X$ and $n \in \mathbb{Z}$. Let $x \in X$ and $r = \frac{m}{n}$ be a rational number. Then, $nf(rx) = f(nrx) = f(mx) = mf(x)$ for all $x \in X$. This implies that $f(rx) = rf(x)$, and the proof is completed. $\square$

The following example shows that the assumption $f(0) = 0$ is necessary in Lemma 1.

Example 4. Let $\mathbb{Q}_p$ be equipped with the p -adic norm $|\cdot|_p$, and $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ be defined by $f(x) = x + 3$ for all $x \in \mathbb{Q}_p$. Then, f satisfies (25), but f is not rational homogeneous.

Theorem 7. Let $|2| < 1, \varepsilon \geq 0$ and Y be a Banach non-Archimedean space. Suppose that the function $f : X \rightarrow Y$ satisfies:

$$\|f(x+y) - f(x) - f(y)\| \leq \|f(x-y) - f(x) + f(y)\| + \varepsilon, \quad \|x\|, \|y\| \geq d \quad (26)$$

for some $d > 0$. Then, there exists a unique rational homogeneous mapping $A : X \rightarrow Y$ such that:

$$\|f(x) - A(x)\| \leq \|f(0)\| + \varepsilon, \quad \|x\| \geq d. \quad (27)$$

Proof. Letting $y = x$ in (26), we obtain:

$$\|f(2x) - 2f(x)\| \leq \|f(0)\| + \varepsilon, \quad \|x\| \geq d.$$

Replacing x by $\frac{x}{2^{n+1}}$ in the above inequality and multiplying the resulting inequality by $|2|^n$ yields:

$$\left\| 2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) \right\| \leq |2|^n (\|f(0)\| + \varepsilon), \quad \|x\| \geq d, \quad n \geq 0. \quad (28)$$

Then:

$$\begin{aligned} \left\| 2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &= \left\| \sum_{k=m}^{n-1} \left[2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right) \right] \right\| \\ &\leq \max \left\{ \left\| 2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right) \right\| : m \leq k \leq n-1 \right\} \\ &\leq |2|^m (\|f(0)\| + \varepsilon), \end{aligned} \quad (29)$$

for all $\|x\| \geq d$ and $n > m \geq 0$. It follows from (28) that the sequence $\{2^n f(\frac{x}{2^n})\}_{n=1}^\infty$ is a Cauchy sequence for all $x \in X$. Since Y is a Banach non-Archimedean space, the sequence $\{2^n f(\frac{x}{2^n})\}_{n=1}^\infty$ is convergent. We define:

$$A : X \rightarrow Y, \quad A(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right).$$

Since $|2| < 1$, we obtain $A(0) = 0$. By the definition of A , (26) yields:

$$\|A(x+y) - A(x) - A(y)\| \leq \|A(x-y) - A(x) + A(y)\|, \quad x, y \in X.$$

By Lemma 1, A is rational homogeneous. Taking $m = 0$ and letting $n \rightarrow \infty$ in (29), we obtain (27) for all $\|x\| \geq d$. The uniqueness of A easily follows from (27). $\square$

Lemma 2. Let $0 < |k| < 1$ and $f : X \rightarrow Y$ with $f(0) = 0$ satisfy:

$$\left\| f\left(\frac{x+y}{k}\right) - \frac{f(x)+f(y)}{k} \right\| \leq \|f(x+y) - f(x) - f(y)\|, \quad x, y \in X. \quad (30)$$

Then, f is additive.

Proof. Letting $y = 0$ in (30), we obtain: $f\left(\frac{x}{k}\right) = \frac{f(x)}{k}$ for all $x \in X$. Then, (30) yields:

$$\|f(x+y) - f(x) - f(y)\| \leq |k| \|f(x+y) - f(x) - f(y)\|, \quad x, y \in X.$$

Then, $f(x+y) = f(x) + f(y)$ for all $x, y \in X$ because of $|k| < 1$. $\square$

Remark 3. Let $\mathbb{Q}_2$ be equipped with the two-adic norm $|\cdot|_2$. The function $f : \mathbb{Q}_2 \rightarrow \mathbb{Q}_2$ given by $f(x) = 2$ (for all $x \in \mathbb{Q}_2$) fulfils (30) with $k = 2$, but f is not additive.

For convenience, let us define:

$$\begin{aligned} \mathcal{A}_d &:= \{(x, y) \in X \times Y : \|x\| \geq d\}, \\ \mathcal{B}_d &:= \{(x, y) \in X \times Y : \|y\| \geq d\}, \\ \mathcal{C}_d &:= \{(x, y) \in X \times Y : \|x\| + \|y\| \geq d\} \end{aligned}$$

for some $d > 0$.

Theorem 8. Let $|k| < 1$ for some non-zero integer, $\varepsilon \geq 0$ and Y be a Banach non-Archimedean space. Suppose that $f : X \rightarrow Y$ satisfies:

$$\left\| f\left(\frac{x+y}{k}\right) - \frac{f(x)+f(y)}{k} \right\| \leq \|f(x+y) - f(x) - f(y)\| + \varepsilon \quad (31)$$

for all $(x, y) \in \mathcal{A}_d$ (respectively, $(x, y) \in \mathcal{B}_d$ or $(x, y) \in \mathcal{C}_d$), where $d > 0$ is a constant. Then, there exists a unique additive mapping $A : X \rightarrow Y$ such that:

$$\|f(x) - A(x)\| \leq \frac{|k|}{1-|k|} \max \left\{ \|f(0)\| + 2\varepsilon, \frac{\|f(0)\| + \varepsilon}{|k|} \right\}.$$

Proof. Since the both sides of (31) are symmetric in x and y , and $\mathcal{A}_d \subseteq \mathcal{C}_d$, it is sufficient to consider (31) for all $(x, y) \in \mathcal{A}_d$. Letting $y = 0$ in (31), we obtain:

$$\left\| f\left(\frac{x}{k}\right) - \frac{f(x)+f(0)}{k} \right\| \leq \|f(0)\| + \varepsilon, \quad \|x\| \geq d.$$

Then:

$$\left\| k^{n+1} f\left(\frac{x}{k^{n+1}}\right) - k^n f\left(\frac{x}{k^n}\right) \right\| \leq |k|^n (\|f(0)\| + \varepsilon), \quad \|x\| \geq d, \quad n \geq 0, \quad (32)$$

$$\left\| k^{n+1} f\left(\frac{x}{k^{n+1}}\right) - k^m f\left(\frac{x}{k^m}\right) \right\| \leq |k|^m (\|f(0)\| + \varepsilon), \quad \|x\| \geq d, \quad n > m \geq 0. \quad (33)$$

Similar to the process of proving Theorem 7, it can be concluded that the sequence $\{k^n f\left(\frac{x}{k^n}\right)\}_{n=1}^{\infty}$ is convergent and the function $A : X \rightarrow Y$ can be defined as follows:

$$A(x) := \lim_{n \rightarrow \infty} k^n f\left(\frac{x}{k^n}\right), \quad x \in X.$$

By the definition of A and (31), we obtain $A(0) = 0$ and:

$$\left\| A\left(\frac{x+y}{k}\right) - \frac{A(x)+A(y)}{k} \right\| \leq \|A(x+y) - A(x) - A(y)\|, \quad x, y \in X.$$

Hence, A is additive by Lemma 2. Taking $m = 0$ and letting $n \rightarrow \infty$ in (33), we obtain:

$$\|f(x) - A(x)\| \leq \|f(0)\| + \varepsilon, \quad \|x\| \geq d. \quad (34)$$

To achieve the inequality (8), suppose that x is a non-zero and arbitrary element of X . According to $0 < |k| < 1$, we can choose $y \in X$ such that $\min\{\|y\|, \|x+y\|\} \geq d$. Then, (34) yields:

$$\begin{aligned} \left\| f\left(\frac{x+y}{k}\right) - A\left(\frac{x+y}{k}\right) \right\| &\leq \|f(0)\| + \varepsilon, \\ \left\| \frac{f(y)}{k} - \frac{A(y)}{k} \right\| &\leq \frac{\|f(0)\| + \varepsilon}{|k|}, \\ \|f(x+y) - A(x+y)\| &\leq \|f(0)\| + \varepsilon. \end{aligned}$$

Using the strong triangle inequalities and the additivity of A , we obtain the following results from the above inequalities:

$$\begin{aligned} \left\| f\left(\frac{x+y}{k}\right) - \frac{f(y)}{k} - A\left(\frac{x}{k}\right) \right\| &\leq \frac{\|f(0)\| + \varepsilon}{|k|}, \\ \|f(x+y) - f(y) - A(x)\| &\leq \|f(0)\| + \varepsilon. \end{aligned}$$

Then:

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \max\{\|f(x+y) - f(y) - A(x)\|, \|A(x) - f(x)\|\} \\ &\leq \max\{\|f(0)\| + \varepsilon, \|A(x) - f(x)\|\}. \end{aligned}$$

According to this inequality and (31), we obtain:

$$\begin{aligned} \left\| \frac{f(x)}{k} - \frac{A(x)}{k} \right\| &\leq \max\left\{ \left\| f\left(\frac{x+y}{k}\right) - \frac{f(x)+f(y)}{k} \right\|, \left\| \frac{A(x)}{k} + \frac{f(y)}{k} - f\left(\frac{x+y}{k}\right) \right\| \right\} \\ &\leq \max\left\{ \|f(x+y) - f(x) - f(y)\| + \varepsilon, \frac{\|f(0)\| + \varepsilon}{|k|} \right\} \\ &\leq \max\left\{ \|f(0)\| + 2\varepsilon, \|A(x) - f(x)\| + \varepsilon, \frac{\|f(0)\| + \varepsilon}{|k|} \right\}. \end{aligned}$$

Hence:

$$\|f(x) - A(x)\| \leq \frac{|k|}{1-|k|} \max\left\{ \|f(0)\| + 2\varepsilon, \frac{\|f(0)\| + \varepsilon}{|k|} \right\}.$$

Obviously, the last inequality also holds true for $x = 0$, and results in A being unique. $\square$

Corollary 5. Let $|k| < 1$ be a Banach non-Archimedean space for some non-zero integer and Y . Suppose that $f : X \rightarrow Y$ with $f(0) = 0$ satisfies one of the following asymptotic properties:

- (i) $\lim_{\|x\|+\|y\|\rightarrow\infty} \left\| f\left(\frac{x+y}{k}\right) - \frac{f(x)+f(y)}{k} \right\| - \|f(x+y) - f(x) - f(y)\| = 0;$
- (ii) $\lim_{\|x\|\rightarrow\infty} \left\| f\left(\frac{x+y}{k}\right) - \frac{f(x)+f(y)}{k} \right\| - \|f(x+y) - f(x) - f(y)\| = 0;$
- (iii) $\lim_{\max\{\|x\|,\|y\|\}\rightarrow\infty} \left\| f\left(\frac{x+y}{k}\right) - \frac{f(x)+f(y)}{k} \right\| - \|f(x+y) - f(x) - f(y)\| = 0.$

Then, f is additive.

Proof. Let $\varepsilon > 0$ be an arbitrary real number. By the assumption (i), there exists $d > 0$ such that (31) holds. By Theorem 8, there exists an additive mapping $A_\varepsilon : X \rightarrow Y$ such that $\|f(x) - A_\varepsilon(x)\| \leq C\varepsilon$ for all $x \in X$, where the constant $C > 0$ is independent of ε . Therefore, by the strong triangle inequality and the additivity of A_ε , we obtain:

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \max\{\|f(x+y) - A_\varepsilon(x+y)\|, \\ &\quad \|A_\varepsilon(x) - f(x)\|, \|f(y) - A_\varepsilon(y)\|\} \\ &\leq C\varepsilon, \quad x, y \in X. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we infer that f is additive. For the other cases, the proof will be the same. $\square$

5. Conclusions

In this work, we studied the Pexider–Cauchy functional equation. We established a new strategy to study the Hyers–Ulam stability and hyperstability of the Pexider–Cauchy functional equation in the non-Archimedean normed spaces with restricted domains. We also showed that, subject to certain restrictions, every function which fulfils certain inequalities can be approximated by an additive function in non-Archimedean normed spaces. As a consequence, we used the results to investigate the asymptotic stability behaviour of the Cauchy and Pexider–Cauchy functional equations.

Author Contributions: Conceptualization, H.G., M.B.M., A.N. and J.-H.B.; methodology, H.G., M.B.M., A.N. and J.-H.B.; software, H.G., M.B.M., A.N. and J.-H.B.; validation, H.G., M.B.M., A.N. and J.-H.B.; formal analysis, H.G., M.B.M., A.N. and J.-H.B.; investigation, H.G., M.B.M., A.N. and J.-H.B.; resources, H.G., M.B.M., A.N. and J.-H.B.; data curation, H.G., M.B.M., A.N. and J.-H.B.; writing—original draft preparation, H.G., M.B.M., A.N. and J.-H.B.; writing—review and editing, H.G., M.B.M., A.N. and J.-H.B.; visualization, H.G., M.B.M., A.N. and J.-H.B.; supervision, H.G., M.B.M., A.N. and J.-H.B.; project administration, H.G., M.B.M., A.N. and J.-H.B.; funding acquisition, H.G., M.B.M., A.N. and J.-H.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ulam, S.M. *Problems in Modern Mathematics, Science Editions*; John Wiley Sons, Inc.: New York, NY, USA, 1964.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224.
3. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66.
4. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [[CrossRef](#)] [[PubMed](#)]
5. Găvruta, P. A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436. [[CrossRef](#)] [[PubMed](#)]
6. Czerwik, S. *Functional Equations and Inequalities in Several Variables*; World Scientific Publishing Co., Inc.: River Edge, NJ, USA, 2002. [[CrossRef](#)] [[PubMed](#)]
7. Forti, G.-L. Hyers–Ulam Stability of Functional Equations in Several Variables. *Aequationes Math.* **1995**, *50*, 143–190. [[CrossRef](#)]
8. Hyers, D.H.; Isac, G.; Rassias, T. *Stability of Functional Equations in Several Variables*; Birkhäuser: Boston, MA, USA, 1998. [[CrossRef](#)]
9. Jung, S.-M. *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer: New York, NY, USA; Dordrecht, The Netherlands; Heidelberg, Germany; London, UK, 2011. [[CrossRef](#)]
10. Kannappan, P. *Functional Equations and Inequalities with Applications*; Springer: Berlin/Heidelberg, Germany, 2009.

11. Khrennikov, A. *Non-Archimedean Analysis, Quantum Paradoxes, Dynamical Systems and Biological Models*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1997.
12. Arriola, L.M.; Beyer, W.A. Stability of the Cauchy functional equation over p -adic fields. *Real Anal. Exch.* **2006**, *31*, 125–132.
13. Mirmostafaei, A.K. Approximately additive mappings in non-Archimedean normed spaces. *Bull. Korean Math. Soc.* **2009**, *46*, 387–400. [[CrossRef](#)]
14. Moslehian, M.S.; Rassias, T.M. Stability of functional equations in non-Archimedean spaces. *Appl. Anal. Discrete Math.* **2007**, *1*, 325–334.
15. Khosravi, B.; Moghimi, M.B.; Najati, A. Asymptotic aspect of Drygas, quadratic and Jensen functional equations in metric abelian groups. *Acta Math. Hungar.* **2018**, *155*, 248–265.
16. Najati, A.; Kang, J.I.; Cho, Y.J. Local stability of the Pexiderized Cauchy and Jensen's equations in fuzzy spaces. *J. Inequal. Appl.* **2011**, *2011*, 78.
17. Najati, A.; Rassias, T.M. Stability of the Pexiderized Cauchy and Jensen's equations on restricted domains. *Commun. Math. Anal.* **2010**, *8*, 125–135.
18. Najati, A.; Cho, Y.J. Generalized Hyers–Ulam stability of the Pexiderized Cauchy functional equation in non-Archimedean spaces. *Fixed Point Theory Appl.* **2011**, *2011*, 309026.