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On a Coupled System of Random and Stochastic Nonlinear Differential Equations with Coupled Nonlocal Random and Stochastic Nonlinear Integral Conditions

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Abstract: It is well known that Stochastic equations had many useful applications in describing numerous events and problems of real world, and the nonlocal integral condition is important in physics, finance and engineering. Here we are concerned with two problems of a coupled system of random and stochastic nonlinear differential equations with two coupled systems of nonlinear nonlocal random and stochastic integral conditions. The existence of solutions will be studied. The sufficient condition for the uniqueness of the solution will be given. The continuous dependence of the unique solution on the nonlocal conditions will be proved.

Keywords: stochastic processes; stochastic differential equation; coupled system; nonlocal stochastic integral conditions

MSC: 34A12; 34A30; 34D20; 34F05; 60H10

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra and P is a probability measure.

The aim of this article is to extend the results of A.M.A. El-Sayed [1,2] on the stochastic fractional calculus operators defined on $C([0,T], L_2(\Omega))$ and the solution of stochastic differential equations subject to nonlocal integral conditions which have been considered in [3,4].

Moreover, we motivate the coupled system of integral equations in reflexive Banach space by A.M.A. El-Sayed and H.H.G.Hashem [5] to the coupled systems with random memory on the space of all second order stochastic process.

The continuous dependence of a unique solution has been studied on the random initial data and the random function which ensures the stability of the solution.

Nonlocal problem of differential equation have been studied by many authors (see for example [6–8]).

Let $Z(t;\omega) = Z(t)$, $t \in [0,T]$, $\omega \in \Omega$ be a second order stochastic process, i.e., $E(Z^2(t)) < \infty$, $t \in [0,T]$.

Let $C = C([0,T], L_2(\Omega))$ be the space of all second order stochastic processes which is mean square (m.s) continuous on $[0,T]$. The norm of $Z \in C([0,T], L_2(\Omega))$ is given by

$$\|Z\|_C = \sup_{t\in[0,T]} \|Z(t)\|_2, \qquad \|Z(t)\|_2 = \sqrt{E(Z^2(t))}.$$

Let $T \geq 1$. In this paper we study the existence of solutions $(x,y) \in C([0,T], L_2(\Omega))$ of the problem of the coupled system of random and stochastic differential equations

$$\frac{dx(t)}{dt} = f_1(t, y(\phi_1(t))), \qquad t \in (0, T], \tag{1}$$
$$dy(t) = f_2(t, x(\phi_2(t)))dW(t), \qquad t \in (0, T] \tag{2}$$

subject to each one of the two nonlinear nonlocal stochastic integral conditions

$$x(0) + \int_0^\tau h_1(s, y(s))dW(s) = x_0, \qquad y(0) + \int_0^\eta h_2(s, x(s))ds = y_0 \tag{3}$$

and

$$x(0) + \int_0^\tau h_1(s, x(s))dW(s) = x_0, \qquad y(0) + \int_0^\eta h_2(s, y(s))ds = y_0 \tag{4}$$

where x_0 and y_0 are two second order random variables.

Let $X = C([0,T], L_2(\Omega)) \times C([0,T], L_2(\Omega))$ be the class of all ordered pairs (x, y), $x, y \in C$ with the norm

$$\|(x, y)\|_X = \max\{ \|x\|_C, \|y\|_C\} = \max\{ \sup_{t\in[0,T]} \|x(t)\|_2, \sup_{t\in[0,T]} \|y(t)\|_2\}. \tag{5}$$

Let $\phi_i : [0,T] \to [0,T]$ be continuous functions such that $\phi_i(t) \le t$ and consider the following assumptions

Assumption 1. *$f_i : [0,T] \times L_2(\Omega) \to L_2(\Omega)$, $i = 1,2$ are measurable in $t \in [0,T]$ for all $x \in L_2(\Omega)$ and continuous in $x \in L_2(\Omega)$ for all $t \in [0,T]$. There exist two bounded measurable functions $m_i : [0,T] \to R$ and two positive constants b_i such that*

$$\|f_i(t, x)\|_2 \le |m_i(t)| + b_i\|x(t)\|_2, \quad i = 1,2. \tag{6}$$

Assumption 2. *$h_i : [0,T] \times L_2(\Omega) \to L_2(\Omega)$, $i = 1,2$ are measurable in $t \in [0,T]$ for all $x \in L_2(\Omega)$ and continuous in $x \in L_2(\Omega)$ for all $t \in [0,T]$. There exist two bounded measurable functions $k_i : [0,T] \to R$ and two positive constants c_i such that*

$$\|h_i(t, x)\|_2 \le |k_i(t)| + c_i\|x(t)\|_2, \quad i = 1,2. \tag{7}$$

Assumption 3. $M = \max\{\sup_{t\in[0,T]} |m_1(t)|, \sup_{t\in[0,T]} |m_2(t)|\}$, $b = \max\{b_1, b_2\}$.

Assumption 4. $K = \max\{\sup_{t\in[0,T]} |k_1(t)|, \sup_{t\in[0,T]} |k_2(t)|\}$, $c = \max\{c_1, c_2\}$.

Assumption 5. $(b + c)T < 1$.

Now, integrating the two random and stochastic differential Equations (1) and (2) (see [1,2,9–14]) and using the nonlocal conditions (3) and (4) the following Lemma can be proven.

Lemma 1. *The integral representations of the solutions of the nonlocal problems (1) and (2) with conditions (3) and (1) and (2) with conditions (4) are given by*

$$x(t) = x_0 - \int_0^\tau h_1(s, y(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \tag{8}$$
$$y(t) = y_0 - \int_0^\eta h_2(s, x(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s). \tag{9}$$

and

$$x(t) = x_0 - \int_0^\tau h_1(s, x(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \tag{10}$$
$$y(t) = y_0 - \int_0^\eta h_2(s, y(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s). \tag{11}$$

respectively.

2. Solutions of the Problem (1)–(3)

Define the mapping $(F(x,y))(t) = (F_1 y, F_2 x)(t), \quad t \in [0,T]$ where $(F_1 y)(t), \ (F_2 x)(t)$ are given by the following stochastic integral equations

$$(F_1 y)(t) = x_0 - \int_0^\tau h_1(s, y(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \tag{12}$$

$$(F_2 x)(t) = y_0 - \int_0^\eta h_2(s, x(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s). \tag{13}$$

Consider the set Q such that

$$Q = \{x, y \in \ C([0,T], L_2(\Omega)), (x,y) \in X : ||(x,y)||_X = \ \max\{||x(t)||_2, ||y(t)||_2\} \le r.\}$$

Now, we have the following two lemmas

Lemma 2. $F : Q \to Q$.

Proof. Let $y \in Q, \quad \|y(t)\|_2 \le r_1$, then

$$\begin{aligned}
\|(F_1 y)(t)\|_2 &\le \|x_0\|_2 + \| \int_0^\tau h_1(s, y(s))dW(s)\|_2 + \| \int_0^t f_1(s, y(\phi_1(s)))ds\|_2 \\
&\le \|x_0\|_2 + \sqrt{\int_0^\tau \|h_1(s, y(s))\|_2^2 ds} + \int_0^t \|f_1(s, y(\phi_1(s)))\|_2 ds \\
&\le \|x_0\|_2 + \sqrt{\int_0^\tau (|k_1(s)| + c_1\|y(s)\|_2)^2 ds} + \int_0^t (|m_1(S)| + b_1\|y(s)\|_2)ds \\
&\le \|x_0\|_2 + (K + cr_1)\sqrt{T} + (M + br_1)T < \|x_0\|_2 + (K + cr_1)T + (M + br_1)T = r_1
\end{aligned}$$

where

$$r_1 = \frac{\|x_0\|_2 + KT + MT}{1 - (b+c)T} > 0.$$

Let $x \in Q, \quad \|x(t)\|_2 \le r_2$, then

$$\begin{aligned}
\|(F_2 x)(t)\|_2 &\le \|y_0\|_2 + \| \int_0^\eta h_2(s, x(s))ds\|_2 + \| \int_0^t f_2(s, x(\phi_2(s)))dW(s)\|_2 \\
&\le \|y_0\|_2 + \int_0^\eta \|h_2(s, x(s))\|_2 ds + \sqrt{\int_0^t \|f_2(s, x(\phi_2(s)))\|_2^2 ds} \\
&\le \|y_0\|_2 + \int_0^\eta (|k_2(s)| + c_2\|x(s)\|_2)ds + \sqrt{\int_0^t (|m_2(t)| + b_2\|x\|_2)^2 ds} \\
&\le \|y_0\|_2 + (K + cr_2)T + (M + br_2)T < \|y_0\|_2 + (K + cr_2)T + (M + br_2)T = r_2
\end{aligned}$$

where

$$r_2 = \frac{\|y_0\|_2 + KT + MT}{1 - (b+c)T} > 0.$$

Let $r = \max\{r_1, r_2\}, \quad (x,y) \in Q$, then

$$\begin{aligned}
\|F(x,y)\|_X &= \|(F_1 y, F_2 x)\|_X \\
&= \max\{\|(F_1 y)\|_C, \|(F_2 x)\|_C\} < r.
\end{aligned}$$

This proves that $F : Q \to Q$ and the class of functions $\{F(x,y)\}$ is uniformly bounded on Q. $\square$

Lemma 3. *The class of functions* $\{F(x,y)\}$ *is equicontinuous on* Q.

Proof. Let $x, y \in Q$, $t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
\|(F_1y)(t_2) - (F_1y)(t_1)\|_2 &= \| \int_0^{t_2} f_1(s, y(\phi_1(s)))ds - \int_0^{t_1} f_1(s, y(\phi_1(s)))ds\|_2 \\
&\leq \int_{t_1}^{t_2} \|f_1(s, y(\phi_1(s)))\|_2 \\
&\leq (M + b\|y\|_C)(t_2 - t_1) \qquad (14)
\end{aligned}$$

This proves the equicontinuity of the class $\{F_1y\}$ and

$$\begin{aligned}
\|(F_2x)(t_2) - (F_2x)(t_1)\|_2 &= \| \int_0^{t_2} f_2(s, x(\phi_2(s)))dW(s) - \int_0^{t_1} f_2(s, x(\phi_2(s)))dW(s)\|_2 \\
&\leq \sqrt{\int_{t_1}^{t_2} \|f_2(s, x(\phi_2(s)))\|_2^2 ds} \\
&\leq (M + b\|x\|_C)\sqrt{(t_2 - t_1)}. \qquad (15)
\end{aligned}$$

This proves the equicontinuity of the class $\{F_1x\}$.

Now

$$\begin{aligned}
(F(x,y))(t_2) - F(x,y))(t_1)) &= ((F_1y)(t_2), (F_2x)(t_2)) - ((F_1y)(t_1), (F_2x)(t_1)) \\
&= ((F_1y)(t_2) - (F_1y)(t_1)), ((F_2x)(t_2) - (F_2x)(t_1))),
\end{aligned}$$

then from (14) and (15), we can deduce the equicontinuity of the class $\{F(x,y)\}$ on Q. □

2.1. Existence Theorem

Now, we have the following existence theorem

Theorem 1. *Let the Assumptions 1–5 be satisfied, then there exists at least one solution* $(x,y) \in X$ *of the problem (1)–(3).*

Proof. Firstly, from the results of Lemmas 2 and 3 and Arzela–Ascoli Theorem [9] we deduce that the closure of FQ is a compact subset.

Let $(x_n, y_n) \in Q$ be such that

$$L.i.m_{n\to\infty}(x_n, y_n) = (x, y) \qquad w.p.1.$$

where $L.i.m$ denotes the limit in the mean square sense of the continuous second order process ([1,2,9]).

Now,

$$\begin{aligned}
L.i.m_{n\to\infty}F(x_n, y_n) &= (L.i.m_{n\to\infty}F_1y_n, L.i.m_{n\to\infty}F_2x_n) \\
&= (L.i.m_{n\to\infty}\{x_0 - \int_0^\tau h_1(s, y_n(s))dW(s) + \int_0^t f_1(s, y_n(\phi_1(s)))ds\}, \\
&\quad L.i.m_{n\to\infty}\{y_0 - \int_0^\eta h_2(s, x_n(s))ds + \int_0^t f_2(s, x_n(\phi_2(s)))dW(s)\}) \\
&= (x_0 - \int_0^\tau h_1(s, L.i.m_{n\to\infty}y_n(s))dW(s) + \int_0^t f_1(s, L.i.m_{n\to\infty}y_n(\phi_1(s)))ds, \\
&\quad y_0 - \int_0^\eta h_2(s, L.i.m_{n\to\infty}x_n(s))ds + \int_0^t f_2(s, L.i.m_{n\to\infty}x_n(\phi_2(s)))dW(s))
\end{aligned}$$

$$
\begin{aligned}
&= (x_0 - \int_0^{\tau} h_1(s, y(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \\
&\quad y_0 - \int_0^{\eta} h_2(s, x(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s)) \\
&= (F_1 y, F_2 x) \ = \ F(x, y).
\end{aligned}
$$

Applying stochastic Lebesgue dominated convergence Theorem the operator $F : Q \to Q$ is continuous.

Finally, applying Schauder Fixed Point Theorem [9], we can deduce that there exists at least one solution $(x, y) \in Q$ of the problem (1)–(3) such that $x, y \in C([0, T], L_2(\Omega))$. □

2.2. Uniqueness Theorem

Replace the assumptions $(A1)$ and $(A2)$ by (A^*1) and (A^*2), respectively, such that (A^*1) The functions $f_i : [0, T] \times L_2(\Omega) \to L_2(\Omega), \quad i = 1, 2$ are measurable in $t \in [0, T]$ for all $x \in L_2(\Omega)$ and satisfy the Lipschitz condition with respect to the second argument

$$\|f_i(t, u) - f_i(t, v)\|_2 \leq b\|u - v\|_2.$$

(A^*2) The functions $h_i : [0, T] \times L_2(\Omega) \to L_2(\Omega), \quad i = 1, 2$ are measurable in $t \in [0, T]$ for all $x \in L_2(\Omega)$ and satisfy the Lipschitz condition with respect to the second argument

$$\|h_i(t, u) - h_i(t, v)\|_2 \leq c\|u - v\|_2.$$

Remark 1. *Let the assumptions (A^*1) and (A^*2) be satisfied, then we can obtain*

$$\|f_i(t, u)\|_2 - \|f_i(t, 0)\|_2 \leq \|f_i(t, u) - f_i(t, 0)\|_2 \leq b\|u\|_2,$$

$$\|f_i(t, u)\|_2 \leq \|f_i(t, 0)\|_2 + b\|u\|_2 \leq M + b\|u\|_2$$

and

$$\|h_i(t, u)\|_2 \leq \|h_i(t, 0)\|_2 + c\|u\|_2 \leq K + c\|u\|_2.$$

Theorem 2. *Let the assumptions $(A^*1) - (A^*2)$ and $(A3) - (A5)$ be satisfied, then the solution of problem (1)–(3) is unique.*

Proof. Let (x_1, y_1) and (x_2, y_2) be two solutions of the problem (1)–(3), then

$$
\begin{aligned}
(x_i(t), y_i(t)) = (x_0 \quad &- \quad \int_0^{\tau} h_1(s, y(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \\
y_0 \quad &- \quad \int_0^{\eta} h_2(s, x(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s)), \quad i = 1, 2 \qquad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
\|x_1(t) - x_2(t)\|_2 \ &\leq \ \| \int_0^{\tau} [h_1(s, y_2(s)) - h_1(s, y_1(s))]dW(s)\|_2 + \| \int_0^t (f_1(s, y_1) - f_1(s, y_2))ds\|_2 \\
&\leq \ \sqrt{\int_0^{\tau} c^2\|y_2 - y_1\|_C^2 ds} + Tb\|y_1 - y_2\|_C \leq T\sqrt{c}\|y_1 - y_2\|_C + Tb\|y_1 - y_2\|_C \\
&\leq \ T(b + c)\|y_1 - y_2\|_C, \\
&\leq \ T(b + c) \max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}
\end{aligned}
$$

and

$$\begin{aligned}
\|y_1(t) - y_2(t)\|_2 &\leq \int_0^{\eta} \|h_2(s, x_2(s)) - h_2(s, x_1(s))\|_2 ds + \sqrt{\int_0^t b^2 \|x_1(s)) - x_2(s)))\|_2^2 ds} \\
&\leq \sqrt{T} b \|x_1 - x_2\|_C + cT\|x_2 - x_1\|_C \\
&\leq T(b+c)\|x_1 - x_2\|_C, \\
&\leq T(b+c)\max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|(x_1, y_1) - (x_2, y_2)\|_X &= \|(x_1 - x_2), (y_1, y_2)\|_X \\
&= \max\{\|(x_1 - x_2)\|_C, \|(y_1, y_2)\|_C\} \\
&\leq T(b+c)\max\{\|x_1 - x_2\|_C, \|y_2 - y_1\|_C\} \\
&\leq T(b+c)\|(x_1, y_1) - (x_2, y_2)\|_X.
\end{aligned}$$

This implies that

$$(1 - T(b+c))\|(x_1, y_1) - (x_2, y_2)\|_X \leq 0$$

and

$$\|(x_1, y_1) - (x_2, y_2)\|_X = 0,$$

then $(x_1, y_1) = (x_2, y_2)$ and the solution of the problem (1)–(3) is unique. □

2.3. Continuous Dependence

Theorem 3. *Let the assumptions of Theorem 2 be satisfied. Then the solution (16) of the problem (1)–(3) depends continuously on the two random data* (x_0, y_0).

Proof. Let $(\hat{x}, \hat{y})$ be the solution of the coupled system

$$\begin{aligned}
\hat{x}(t) &= \hat{x}_0 - \int_0^{\tau} h_1(s, \hat{y}(s)) dW(s) + \int_0^t f_1(s, \hat{y}(\phi_1(s))) ds \\
\hat{y}(t) &= \hat{y}_0 - \int_0^{\eta} h_2(s, \hat{x}(s)) ds + \int_0^t f_2(s, \hat{x}(\phi_2(s))) dW(s),
\end{aligned}$$

such that $\|(x_0, y_0) - (\hat{x}_0, \hat{y}_0)\|_X < \delta_1$, then

$$\begin{aligned}
\|x - \hat{x}\|_C &\leq \|x_0 - \hat{x}_0\|_C + T(b+c)\|y - \hat{y}\|_C \\
&\leq \delta_1 + T(b+c)\|y - \hat{y}\|_C \\
&\leq \delta_1 + T(b+c)\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\
\|y - \hat{y}\|_C &\leq \|y_0 - \hat{y}_0\|_C + T(b+c)\|x - \hat{x}\|_C, \\
&\leq \delta_1 + T(b+c)\|x - \hat{x}\|_C \\
&\leq \delta_1 + T(b+c)\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\}.
\end{aligned}$$

Then

$$\begin{aligned}
\|(x, y) - (\hat{x}, \hat{y})\|_X &= \|(x - \hat{x}, y - \hat{y})\|_X \\
&= \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\
&\leq \delta_1 + T(b+c)\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\
&\leq \delta_1 + T(b+c)\|(x, y) - (\hat{x}, \hat{y})\|_X.
\end{aligned}$$

This implies that

$$\|(x,y)-(\hat{x},\hat{y})\|_X \le \frac{\delta_1}{1-T(b+c)} = \epsilon$$

which completes the proof. □

Theorem 4. *The solution (16) of the problem (1)–(3) depends continuously on the two random functions h_1 and h_2.*

Proof. Let $(\hat{x},\hat{y})$ be the solutions of the coupled system

$$\begin{aligned}
\hat{x}(t) &= x_0 - \int_0^\tau h_1^*(s,\hat{y}(s))dW(s) + \int_0^t f_1(s,\hat{y}(\phi_1(s)))ds, \\
\hat{y}(t) &= y_0 - \int_0^\eta h_2^*(s,\hat{x}(s))ds + \int_0^t f_2(s,\hat{x}(\phi_2(s)))dW(s)
\end{aligned}$$

such that $\|h_i^*(s,.) - h(s,.)\|_2 \le \delta_2, \quad i=1,2,$ then

$$\begin{aligned}
\|x(t)-\hat{x}(t)\|_2 &= \|\int_0^\tau [h_1^*(s,\hat{y}(s)) - h_1(s,y(s))]dW(s) + \int_0^t [f_1(s,y(\phi_1(s))) - f_1(s,\hat{y}(\phi_1(s)))]ds\|_2 \\
&\le \sqrt{\int_0^\tau \|h_1^*(s,\hat{y}(s)) - h_1(s,y(s))\|_2^2 ds} + \int_0^t \|f_1(s,y(\phi_1(s))) - f_1(s,\hat{y}(\phi_1(s)))\|_2 ds \\
&\le \sqrt{\int_0^\tau [\|h_1^*(s,\hat{y}(s)) - h_1^*(s,y(s))\|_2 + \|h_1^*(s,y(s)) - h_1(s,y(s))\|_2]^2 ds} \\
&+ \int_0^t \|f_1(s,y(\phi_1(s))) - f_1(s,\hat{y}(\phi_1(s)))\|_2 ds \\
&\le \sqrt{\int_0^\tau (c\|y(s)-\hat{y}(s)\|_2 + \delta_2)^2 ds} + \int_0^t b\|y(s)-\hat{y}(s)\|_2 ds \\
&\le (c\sqrt{T} + bT)\|y-\hat{y}\|_C + \delta_2\sqrt{T} \\
&\le T(b+c)\max\{\|x-\hat{x}\|_C, \|y-\hat{y}\|_C\} + \delta_2 T
\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}
\|y(t)-\hat{y}(t)\|_2 &= \|\int_0^\eta [h_2^*(s,\hat{x}(s)) - h_2(s,x(s))]ds + \int_0^t [f_2(s,x(\phi_2(s))) - f_2(s,\hat{x}(\phi_2(s)))]dW(s)\|_2 \\
&\le (cT + b\sqrt{T})\|x-\hat{x}\|_C + \delta_2 T \\
&\le T(b+c)\|x-\hat{x}\|_C + \delta_2 T \\
&\le T(b+c)\max\{\|x-\hat{x}\|_C, \|y-\hat{y}\|_C\} + \delta_2 T \\
&\le T(b+c)\max\{\|x-\hat{x}\|_C, \|y-\hat{y}\|_C\} + \delta_2 T.
\end{aligned}$$

Now

$$\begin{aligned}
\|(x,y)-(\hat{x},\hat{y})\|_X &= \max\{\|x-\hat{x}\|_C, \|y-\hat{y}\|_C \\
&\le T(b+c)\max\{(\|x-\hat{x}\|_C, \|y-\hat{y}\|_C\} + \delta_2 T \\
&\le T(c+b)\|(x,y)-(\hat{x},\hat{y})\|_X + \delta_2 T.
\end{aligned}$$

This implies that

$$\|(x,y)-(\hat{x},\hat{y})\|_X \le \frac{\delta_2 T}{1-T(b+c)} = \epsilon$$

which completes the proof. □

3. Solutions of the Problem (1), (2) and (4)

Define the mapping $L(x,y) = (L_1x, L_2y)$ where $L_1x,\ L_2y$ are given by the following stochastic integral equations

$$L_1x(t) = x_0 - \int_0^\tau h_1(s, x(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \tag{17}$$

$$L_2y(t) = y_0 - \int_0^\eta h_2(s, y(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s). \tag{18}$$

Lemma 4. $L : Q \to Q$.

Proof. Let $x,\ y \in Q,$, then we obtain

$$\begin{aligned}
\|L_1x(t)\|_2 &\leq \|x_0\|_2 + \|\int_0^\tau h_1(s, x(s))dW(s)\|_2 + \|\int_0^t f_1(s, y(\phi_1(s)))ds\|_2 \\
&\leq \|x_0\|_2 + \sqrt{\int_0^\tau \|h_1(s, x(s))\|_2^2 ds} + \int_0^t \|f_1(s, y(\phi_1(s)))\|_2 ds \\
&\leq \|x_0\|_2 + \sqrt{\int_0^\tau (|k_1(s)| + c_1\|x(s)\|_2)^2} \int_0^t (|m_1(s)| + b_1\|y(s)\|_2)ds \\
&\leq \|x_0\|_2 + K\sqrt{T} + MT + c\sqrt{T}\|x\|_C + bT\|y\|_C) \\
&\leq \|x_0\|_2 + \|y_0\|_2 + (K+M)T + 2rT(b+c)
\end{aligned}$$

and

$$\begin{aligned}
\|L_2y(t)\|_2 &\leq \|y_0\|_2 + \|\int_0^\eta h_2(s, y(s))ds\|_2\| \int_0^t f_2(s, x(\phi_2(s)))dW(s)\|_2 \\
&\leq \|y_0\|_2 = \int_0^\eta \|h_2(s, y(s))\|_2 ds + \sqrt{\int_0^t \|f_2(s, x(\phi_2(s)))\|_2^2 ds} \\
&\leq \|y_0\|_2 + \int_0^\eta (|k_2(s)| + c_2\|y(s)\|_2)ds + \sqrt{\int_0^t (|m_2(t)| + b_2\|x\|_2)^2 ds} \\
&\leq \|y_0\|_2 + KT + M\sqrt{T} + cT\|y\|_C + b\sqrt{T}\|x\|_C \\
&\leq \|y_0\|_2 + (K+M)T + T(b+c)\|y\|_C + T(b+c)\|x\|_C \\
&\leq \|x_0\|_2 + \|y_0\|_2 + (K+M)T + 2rT(b+c).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|L(x,y)\|_X &= \|(L_1x, L_2y)\|_X \\
&= \max\{\|L_1x(t)\|_C, \|L_2y(t)\|_C\} \\
&\leq \|x_0\|_2 + \|y_0\|_2 + (K+M)T + 2rT(b+c) = r
\end{aligned}$$

where

$$r = \frac{\|x_0\|_2 + \|y_0\|_2 + (K+M)T}{1 - T(b+c)},$$

then the class $\{L(x,y)\}$ is uniformly bounded and $L(x,y) : Q \to Q$. □

Lemma 5. *The class of function* $\{L(x,y)(t)\}$ *,* $t \in [0,T]$ *is equicontinuous.*

Proof. Let $x,\ y \in Q,\quad t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| < \delta,$ then

$$\begin{aligned} \|L_1x(t_2) - L_1y(t_1)\|_2 &= \|\int_0^{t_2} f_1(s, y(\phi_1(s)))ds - \int_0^{t_1} f_1(s, y(\phi_1(s)))ds\|_2 \\ &\leq \int_{t_1}^{t_2} \|f_1(s, y(\phi_1(s)))\|_2 ds \\ &\leq (M + b\|y\|_C)(t_2 - t_1) \end{aligned} \tag{19}$$

and

$$\begin{aligned} \|L_2x(t_2) - L_2x(t_1)\|_2 &= \|\int_0^{t_2} f_2(s, x(\phi_2(s)))dW(s) - \int_0^{t_1} f_2(s, x(\phi_2(s)))dW(s)\|_2 \\ &\leq \sqrt{\int_{t_1}^{t_2} \|f_2(s, x(\phi_2(s)))\|_2^2 ds} \\ &\leq (M + b\|x\|_C)\sqrt{(t_2 - t_1)}. \end{aligned} \tag{20}$$

However,

$$\begin{aligned} L(x(t_2), y(t_2)) - L(x(t_1), y(t_1)) &= (L_1x(t_2), L_2y(t_2)) - (L_1x(t_1), L_2y(t_1)) \\ &= ((L_1x(t_2) - L_1x(t_1)), (L_2y(t_2) - L_2y(t_1))), \end{aligned}$$

then from (19) and (20), we deduce the equicontinuity of the class $\{L(x,y)(t)\}$ on Q. □

3.1. Existence Theorem

Now, we have the following existence theorem

Theorem 5. *Let the Assumptions (A1)–(A5) be satisfied, then there exists at least one solution $(x,y) \in X$ of the problem (1), (2) and (4).*

Proof. Let $\{(x_n, y_n)\} \in Q$ be such that

$$(x_n, y_n) \to (x, y) \qquad w.p.1.$$

Using Lemmas 1–3, then applying stochastic Lebesgue dominated convergence Theorem [9], we can obtain

$$\begin{aligned} L.i.m_{n\to\infty} L(x_n, y_n) &= (L.i.m_{n\to\infty} L_1x_n, L.i.m_{n\to\infty} L_2y_n) \\ &= (L.i.m_{n\to\infty}\{x_0 - \int_0^{\tau} h_1(s, x_n(s))dW(s) + \int_0^t f_1(s, y_n(\phi_1(s)))ds\}, \\ &\quad L.i.m_{n\to\infty}\{y_0 - \int_0^{\eta} h_2(s, y_n(s))ds + \int_0^t f_2(s, x_n(\phi_2(s)))dW(s)\}) \\ &= (x_0 - \int_0^{\tau} h_1(s, L.i.m_{n\to\infty} x_n(s))dW(s) + \int_0^t f_1(s, L.i.m_{n\to\infty} y_n(\phi_1(s)))ds, \\ &\quad y_0 - \int_0^{\eta} h_2(s, L.i.m_{n\to\infty} y_n(s))ds + \int_0^t f_2(s, L.i.m_{n\to\infty} x_n(\phi_2(s)))dW(s)) \\ &= (x_0 - \int_0^{\tau} h_1(s, x(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \\ &\quad y_0 - \int_0^{\eta} h_2(s, y(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s)) \\ &= (L_1x, L_2y) \ = \ L(x,y). \end{aligned}$$

This proves that the operator $L : Q \to Q$ is continuous. □

Then by the Arzela–Ascoli Theorem [9], the closure of LQ is a compact subset of X, then applying Schauder Fixed Point Theorem [9], there exists at least one solution $(x,y) \in X$ of the problem (1), (2) and (4) such that $x, y \in C([0,T], L_2(\Omega))$.

3.2. Uniqueness Theorem

Theorem 6. *Let the assumptions (A^*1)–(A^*2) and $(A3)$–$(A5)$ be satisfied then the solution of problem (1), (2) and (4) is unique.*

Proof. Let (x_1, y_1) and (x_2, y_2) be two solutions of the problem (1), (2) and (4) on the form

$$\begin{aligned}(x(t), y(t)) = (x_0 &- \int_0^\tau h_1(s, x(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \\ y_0 &- \int_0^\eta h_2(s, y(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s)), \end{aligned} \tag{21}$$

then we can obtain

$$\begin{aligned}\|x_1(t) - x_2(t)\|_2 &\leq c\sqrt{T}\|x_1 - x_2\|_C + bT\|y_1 - y_2\|_C < cT\|x_1 - x_2\|_C + bT\|y_1 - y_2\|_C \\ &\leq (b+c)T\|x_1 - x_2\|_C + (b+c)T\|y_1 - y_2\|_C \\ &\leq (b+c)T \, max\{\|x_1 - x_2\|_C, \ \|y_1 - y_2\|_C\}. \end{aligned} \tag{22}$$

Similarly, we can obtain

$$\|y_1(t) - y_2(t)\|_2 \leq (b+c)T \, max\{\|x_1 - x_2\|_C, \ \|y_1 - y_2\|_C\}. \tag{23}$$

Hence from (22) and (23)

$$\begin{aligned}\|(x_1, y_1) - (x_2, y_2)\|_X &= \|(x_1 - x_2), (y_1 - y_2)\|_X \\ &\leq \max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\} \\ &\leq (b+c)T \, max\{\|x_1 - x_2\|_C, \ \|y_1 - y_2\|_C\}. \end{aligned}$$

This implies that

$$(1 - (b+c)T)\|(x_1, y_1) - (x_2, y_2)\|_X \leq 0.$$

Then

$$\|(x_1, y_1) - (x_2, y_2)\|_X = 0$$

and $(x_1, y_1) = (x_2, y_2)$ which proves that the solution of the problem (1), (2) and (4) is unique. □

3.3. Continuous Dependence

Theorem 7. *The solution (16) of the problem (1)–(2) and (4) depends continuously on the two random data (x_0, y_0).*

Proof. Let $(\hat{x}, \hat{y})$ be the solution of the coupled system

$$\begin{aligned}\hat{x}(t) &= \hat{x}_0 - \int_0^\tau h_1(s, \hat{x}(s))dW(s) + \int_0^t f_1(s, \hat{y}(\phi_1(s)))ds \\ \hat{y}(t) &= \hat{y}_0 - \int_0^\eta h_2(s, \hat{y}(s))ds + \int_0^t f_2(s, \hat{x}(\phi_2(s)))dW(s), \end{aligned}$$

such that $\|(x_0, y_0) - (\hat{x_0}, \hat{y_0})\|_X < \delta_3$. Then we have

$$\begin{aligned} x(t) - \hat{x}(t) &= x_0 - \hat{x_0} - \int_0^\tau [h_1(s, \hat{x}(s)) - h_1(s, x(s))]dW(s) \\ &+ \int_0^t [f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))]ds \end{aligned}$$

and

$$\begin{aligned} \|x(t) - \hat{x}(t)\|_2 &\leq \|x_0 - \hat{x_0}\|_C + c\sqrt{T}\|x - \hat{x}\|_C + bT\|y - \hat{y}\|_C \\ &\leq \|x_0 - \hat{x_0}\|_C + cT\|x - \hat{x}\|_C + bT\|y - \hat{y}\|_C \\ &\leq \|x_0 - \hat{x_0}\|_2 + cTmax\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + bTmax\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\ &\leq max\{\|x_0 - \hat{x_0}\|_2, \|y_0 - \hat{y_0}\|_2\} + (b + c)Tmax\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\}. \end{aligned}$$

By the same way we can obtain

$$\|y(t) - \hat{y}(t)\|_2 \leq max\{\|x_0 - \hat{x_0}\|_2, \|y_0 - \hat{y_0}\|_2\} + (b + c)Tmax\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\}$$

and

$$\begin{aligned} \|(x, y) - (\hat{x}, \hat{y})\|_X &= \max\{\|(x - \hat{x}\|_C, \|(y - \hat{y}\|_C\} \\ &\leq max\{\|x_0 - \hat{x_0}\|_2, \|y_0 - \hat{y_0}\|_2\} + (b + c)Tmax\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\ &\leq \delta_3 + (b + c)Tmax\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \end{aligned}$$

which gives our result

$$\|(x, y) - (\hat{x}, \hat{y})\|_X \leq \frac{\delta_3}{1 - T(b + c)} = \epsilon$$

and completes the proof. □

Theorem 8. *The solution (16) of the problem (1), (2) and (4) depends continuously on the two random functions h_1 and h_2.*

Proof. Let $(\hat{x}, \hat{y})$ be the solutions of the coupled system of stochastic integral Equations (1), (2) and (4) such that

$$\begin{aligned} \hat{x}(t) &= x_0 - \int_0^\tau h_1^*(s, \hat{x}(s))dW(s) + \int_0^t f_1(s, \hat{y}(\phi_1(s)))ds \\ \hat{y}(t) &= y_0 - \int_0^\eta h_2^*(s, \hat{y}(s))ds + \int_0^t f_2(s, \hat{x}(\phi_2(s)))dW(s). \end{aligned}$$

Let $\|h_i^*(t, u(t)) - h(t, u(t))\|_2 \leq \delta_4, \quad i = 1, 2$ then

$$\begin{aligned} \|x(t) - \hat{x}(t)\|_2 &= \| \int_0^\tau [h_1^*(s, \hat{x}(s)) - h_1(s, x(s))]dW(s) + \int_0^t [f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))]ds\|_2 \\ &\leq \sqrt{\int_0^\tau \|h_1^*(s, \hat{x}(s)) - h_1(s, x(s))\|_2^2 ds} + \int_0^t \|f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))\|_2 ds \\ &\leq \sqrt{\int_0^\tau [\|h_1^*(s, \hat{x}(s)) - h_1^*(s, x(s))\|_2 + \|h_1^*(s, x(s)) - h_1(s, x(s))\|_2]^2 ds} \\ &+ \int_0^t \|f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))\|_2 ds \\ &\leq \sqrt{\int_0^\tau (c\|x(s) - \hat{x}(s)\|_2 + \delta_4)^2 ds} + \int_0^t b\|y(s) - \hat{y}(s)\|_2 ds \end{aligned}$$

$$\begin{aligned}
&\leq \quad c\sqrt{T}\|x-\hat{x}\|_C + bT\|y-\hat{y}\|_C + \delta_4\sqrt{T}) \\
&\leq \quad cT\|x-\hat{x}\|_C + bT\|y-\hat{y}\|_C + \delta_4 T. \\
&\leq \quad cT\ max\{\|x-\hat{x}\|_C,\ \|y-\hat{y}\|_C\} + bT\ \ max\{\|x-\hat{x}\|_C,\ \|y-\hat{y}\|_C\} + \delta_4 T \\
&\leq \quad (b+c)T\ \ max\{\|x-\hat{x}\|_C,\ \|y-\hat{y}\|_C\} + \delta_4 T.
\end{aligned}$$

Similarly we can obtain

$$\|y-\hat{y}\|_C \leq (b+c)T\ \ max\{\|x-\hat{x}\|_C,\ \|y-\hat{y}\|_C\} + \delta_4 T$$

and

$$\|(x,y)-(\hat{x},\hat{y})\|_X = \max\{\|x-\hat{x}\|_C, \|y-\hat{y}\|_C\} \leq (b+c)T\ \ max\{\|x-\hat{x}\|_C,\ \|y-\hat{y}\|_C\} + \delta_4 T.$$

This implies that

$$\|(x,y)-(\hat{x},\hat{y})\|_X \leq \frac{\delta_4 T}{1-T(b+c)} = \epsilon$$

which completes the proof. □

Example 1. *Consider the coupled system*

$$\begin{aligned}
\frac{dx}{dt}(t) &= \frac{a(t)+y(t)}{5(1+\|y(t)\|_2)}, \qquad t \in (0,1] \\
dy(t) &= \frac{tx(t)}{2(1+\|x\|_2)} dW(t), \qquad t \in (0,1]
\end{aligned} \tag{24}$$

subject to

$$x_0 = \int_0^\tau \frac{e^{-s}y(s)}{120+s^2} dW(s), \quad y_0 = \int_0^\eta \frac{x(s)}{\sqrt{s+36}} ds \tag{25}$$

where

$$\|f_1(t,y(t))\|_2 \leq \frac{1}{5}[|a(t)| + \|y(t)\|_2], \quad \|f_2(t,x(t))\|_2 \leq \frac{1}{2\|x(t)\|_2}$$

and

$$\|h_1(t,y(t)\|_2 \leq \frac{\|y(t)\|_2}{120}, \quad \|h_2(t,x(t)\|_2 \leq \frac{\|x(t)\|_2}{6}.$$

Easily, the coupled system (24) with nonlocal integral conditions (25) satisfies all the Assumptions 1–5 of Theorem 1. with $b = \frac{1}{2}$, $c = \frac{1}{6}$, then there exists at least one solution of the system (24) on $[0,1]$.

4. Conclusions

Here, we proved the existence of solutions of a coupled system of random and stochastic nonlinear differential equations with coupled nonlocal random and stochastic nonlinear integral conditions. The sufficient conditions for the uniqueness of the solution have been given. The continuous dependence of the unique solution has been studied.

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