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Reliability Sampling Design for the Lifetime Performance Index of Gompertz Lifetime Distribution under Progressive Type I Interval Censoring

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Abstract: In this artificial intelligence era, the constantly changing technology makes production techniques become sophisticated and complicated. Therefore, manufacturers are dedicated to improving the quality of products by increasing the lifetime in order to achieve the quality standards demanded by consumers. For products with lifetime following a Gompertz distribution, the lifetime performance index was used to measure the performance of manufacturing process under progressive type I interval censoring. The sampling design is investigated to reach the given level of significance and power level. When inspection interval length is fixed and the number of inspection intervals is not fixed, the required number of inspection intervals and sample size with minimum total cost are determined and tabulated. When the termination time is not fixed, the required number of inspection intervals, sample size, and equal interval length reaching the minimum total cost are determined and tabulated. The optimal parameter values are tabulated for the practical use of users. Finally, one practical example is given for the illustrative aim to show the implementation of this sampling design to collect data and the collected data are used to conduct the testing procedure to see if the process is capable.

Keywords: censored sample; Gompertz distribution; maximum likelihood estimator; process capability indices; testing algorithmic procedure; sampling design

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1. Introduction

Montgomery [1] presented the lifetime performance index C_L to measure the larger-the-better type quality characteristics—for example, lifetime, heat resistance, mpg for vehicles, etc. When the lifetime of products follows an exponential distribution, Tong et al. [2] used the uniformly minimum variance unbiased estimator (UMVUE) of C_L to construct a hypothesis testing procedure for the complete sample. However, there are many cases where the experimenters can not observe the lifetimes of all the items due to the budget restrictions or time limitation, etc. The censoring is thus arisen. For progressive censored data (see Balakrishnan and Aggarwala [3], Aggarwala [4]), Lee et al. [5] proposed a testing procedure for this lifetime performance index under progressive type II censoring. For step-stress accelerated life-testing data (see Wu et al. [6], Wu et al. [7]), Lee et al. [8] evaluated the lifetime performance index of exponential products. Wu and Lin [9] proposed a testing procedure for the lifetime performance index C_L for exponential products using the maximum likelihood estimator under progressive type I interval censoring. Wu and Hsieh [10] proposed a testing procedure for Gompertz products using this index under progressive type I interval censoring and the related testing procedure is summarized in Section 2. In Section 3, we propose a sampling design to determine the minimum sample size required to attain the pre-specified power level and level of significance. Under a specific cost structure, the total cost of the experiment is considered to be minimized.

When the termination time of the experiment is fixed or not fixed, the sampling design is determined separately to minimize the total cost in Section 3. The algorithms, figures, and tables are shown in this section. Our algorithms can help experimenters to setup the progressive type-I interval censoring scheme and then the experimenters can collect the progressive type-I interval censored sample under different considerations. We give an example to implement our proposed sampling design to collect the progressive type-I interval censored sample and utilize this sample to conduct the hypothesis test for the lifetime performance index C_L to see if the process is capable. Finally, the conclusions are made in Section 4.

2. The Introduction of the Testing Procedure for the Lifetime Performance Index

Suppose that the lifetime U of products follows a two-parameter Gompertz distribution with the probability density function (pdf) and the cumulative distribution function (cdf) as follows:

$$f_U(u) = \lambda e^{ku - \frac{\lambda}{k}(e^{ku} - 1)}, \quad u > 0, k > 0, \lambda > 0 \quad (1)$$

In addition,

$$F_U(u) = 1 - e^{-\frac{\lambda}{k}(e^{ku} - 1)}, \quad u > 0, k > 0, \lambda > 0 \quad (2)$$

The Gompertz distribution is usually used to describe the human mortality. It can also model the lifetime with increasing failure rate exponentially over time.

Considering the transformation $Y = \frac{1}{k}(e^{kU} - 1)$, $k > 0$, the new lifetime variable Y has an exponential distribution with scale parameter $1/\lambda$ and the pdf and cdf are given by

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y > 0, \lambda > 0 \quad (3)$$

In addition,

$$F_Y(y) = 1 - e^{-\lambda y}, \quad y > 0, \lambda > 0 \quad (4)$$

Montgomery [1] presented the lifetime performance index as

$$C_L = \frac{\mu - L}{\sigma}, \quad (5)$$

where μ is the process mean and σ is the process standard deviation, and L is the known lower specification limit. Supposing that the lower specification limit for U is L_U , then the lower specification limit for Y turns out to be $L = \frac{1}{k}(e^{kL_U} - 1)$.

The mean and the standard deviation of Y are $\mu = 1/\lambda$ and $\sigma = 1/\lambda$. Substitute μ and σ into Equation (5); then, we have the lifetime performance index $C_L = 1 - \lambda L$.

The conforming rate is obtained as

$$P_r = P(Y \geq L) = \exp\{-\lambda L\} = \exp\{C_L - 1\}, \quad -\infty < C_L < 1.$$

Apparently, the conforming rate is an increasing function of C_L .

The progressive type I interval censoring scheme is addressed as follows: We put n products on a life test at time 0. Determine the inspection times $(t_1, \dots, t_m)$ and the removal percentages of the remaining survival units at time $(t_1, \dots, t_m)$ as $(p_1, \dots, p_m)$. The termination time for the experiment is $t_m = T$ with $p_m = 1$. At the i th inspection time interval $(t_{i-1}, t_i]$, the number of failure units X_i is observed and then $R_i = [(n - \sum_{j=1}^i X_j - \sum_{j=1}^{i-1} R_j)p_i]$ units are randomly removed from the remaining $n - \sum_{j=1}^i X_j - \sum_{j=1}^{i-1} R_j$ survival units, where $[w]$ denotes the largest integer which is smaller than or equal to w , $i = 1, \dots, m$. Then, the observed number of failure units $(X_1, \dots, X_m)$ is regarded as the progressive type I interval censored sample under censoring scheme $(R_1, \dots, R_m)$ with removal percentages

$(p_1, \dots, p_m)$. Wu and Hsieh [10] obtained the maximum likelihood estimator (MLE) of λ denoted by $\hat{\lambda}$ as the numerical solution of the following log-likelihood equation:

$$\frac{d}{d\lambda} \ln L(\lambda) = \sum_{i=1}^m \frac{X_i}{k} \frac{(e^{kt_i} - e^{kt_{i-1}}) e^{-\lambda(e^{kt_i} - e^{kt_{i-1}})/k}}{1 - e^{-\lambda(e^{kt_i} - e^{kt_{i-1}})/k}} - \sum_{i=1}^m (R_i e^{kt_i} + X_i e^{kt_{i-1}}) / k \quad (6)$$

Its asymptotic variance is the reciprocal of the Fisher's information

$$\begin{aligned} I(\lambda) &= -E\left[\frac{d^2 \ln L(\lambda)}{d\lambda^2}\right] \\ &= \frac{n}{\lambda^2} \sum_{i=1}^m \frac{(-\frac{\lambda}{k}(e^{kt_i} - e^{kt_{i-1}}))^2}{1 - \exp(-\frac{\lambda}{k}(e^{kt_i} - e^{kt_{i-1}}))} \prod_{j=1}^{i-1} (1 - p_j) \prod_{k=1}^i \exp(-\frac{\lambda}{k}(e^{kt_i} - e^{kt_{i-1}})) \end{aligned} \quad (7)$$

Then, we have $\hat{\lambda} \xrightarrow[n \rightarrow \infty]{d} N(\lambda, I^{-1}(\lambda))$.

For equal interval lengths, $t_i - t_{i-1} = t$ and $t_i = it$, $i = 1, \dots, m$ are replaced in Equations (6) and (7).

By the property of the invariance of MLE, the MLE of C_L can be obtained as

$$\hat{C}_L = 1 - \hat{\lambda}L. \quad (8)$$

Suppose that c_0 is the desired level of lifetime performance index to make the process capable. Then, we want to test $H_0 : C_L \leq c_0$ (the process is not capable) vs. $H_a : C_L > c_0$ (the process is capable). Making use of the MLE of C_L as the test statistic under the level of significance α , the decision rule is to reject the null hypothesis if $\hat{C}_L > C_L^0$, where the critical value is $C_L^0 = 1 - L(\lambda_0 + Z_{1-\alpha} \sqrt{I^{-1}(\lambda_0)})$, $\lambda_0 = \frac{1-c_0}{L}$, and $Z_{1-\alpha}$ represents the α percentile of the standard normal distribution.

The test power $h(c_1)$ of this statistical hypothesis test at the point of $C_L = c_1 > c_0$ is

$$h(c_1) = \Phi\left(\frac{\lambda_0 - \lambda_1 + Z_{1-\alpha} \sqrt{I^{-1}(\lambda_0)}}{\sqrt{I^{-1}(\lambda_1)}}\right) \quad (9)$$

where $\Phi(\cdot)$ is the cdf for the standard normal distribution, $\lambda_0 = \frac{1-c_0}{L}$ and $\lambda_1 = \frac{1-c_1}{L}$.

3. Reliability Sampling Design

The reliability sampling design is very important for experimenters to plan the progressive type I interval censoring experiment. In Section 3.1, the required minimum sample size is determined under the control of the type II error (producers' risk). In Section 3.2, under the fixed total experimental time T , the required sample size and minimal number of inspection intervals are determined to reach the given power of the level α test for the testing procedure and also to minimize the total cost of experiment. In Section 3.3, for unfixed experimental termination time T , we determine the required sample size, minimal number of inspection intervals, and the equal length of interval so that the total experimental cost is minimized.

3.1. The Determination of Sample Size for Fixed m and T

In this section, the minimum sample size n is determined to reach the type II error β at c_1 under significance level α . Let

$$r(\lambda) = I(\lambda)/n = \frac{1}{\lambda^2} \sum_{i=1}^m \frac{(-\frac{\lambda}{k}(e^{kt_i} - e^{kt_{i-1}}))^2}{1 - \exp(-\frac{\lambda}{k}(e^{kt_i} - e^{kt_{i-1}}))} \prod_{j=1}^{i-1} (1 - p_j) \prod_{k=1}^i \exp(-\frac{\lambda}{k}(e^{kt_i} - e^{kt_{i-1}}))$$

which is independent of the sample size n .

Set the power to be $1 - \beta$ as

$$h(c_1) = \Phi\left(\frac{\lambda_0 - \lambda_1 + Z_{1-\alpha}\sqrt{I^{-1}(\lambda_0)}}{\sqrt{I^{-1}(\lambda_1)}}\right) = 1 - \beta \quad (10)$$

Solving the above equation, the minimum required sample size is obtained as

$$n = \left(\frac{Z_\beta\sqrt{r^{-1}(\lambda_1)} + Z_{1-\alpha}\sqrt{r^{-1}(\lambda_0)}}{\lambda_0 - \lambda_1}\right)^2 \quad (11)$$

Under $\alpha = 0.01, 0.05, 0.1$, $\beta = 0.25, 0.20, 0.15$, $L = 0.05$ and $T = 0.5$, the minimum required sample size is tabulated in Table 1, Table 2, Table 3 for $c_1 = 0.825(0.025)0.9$, $m = 5(1)8$ and $p = 0.05(0.025)0.1$ for testing $H_0 : C_L \leq 0.8$. The minimum required sample sizes are also displayed in Figure 1, Figure 2, Figure 3, Figure 4 for some typical cases. From Figure 1, Figure 2, Figure 3, Figure 4, we find that (1) the minimum required sample size decreases when c_1 increases for any combinations of α , β , m , and p overall speaking. (2) The minimum required sample size is a non-increasing function of probability of type II error β or a non-decreasing function of power $1-\beta$ at $\alpha = 0.05$ for fixed $m = 5$ and $p = 0.05$. (Other combinations of α , m , and p also have the same pattern); (3) the change in m does not show any trend on the minimum required sample size at $\alpha = 0.05$ for fixed $\beta = 0.2$ and $p = 0.05$. (Other combinations of α , β , and p also have the same pattern); (4) the minimum required sample size is a non-decreasing function of the removal percentage p for fixed α , β , and m , but the difference is not significant; (5) the minimum required sample size increases when the level of significance decreases for any combinations of β , m , and p . For example, the user wants to conduct the level 0.05 hypothesis testing of $H_0 : C_L \leq 0.8$ under power of 0.8 at $c_1 = 0.95$, $p = 0.05$, $L = 0.05$, $T = 0.5$, and $m = 5$ with $k = 4.47$, and the minimum required sample size is 8 with critical value 0.9234 from Table 2.

Figure 1. Minimum sample size for the test at $\alpha = 0.05$, $p = 0.05$, and $m = 5$.

Table 1. The minimum sample size for $c_1 = 0.825(0.025)0.9$, $m = 5(1)8$ and $p = 0.05(0.025)0.1$ under $\alpha = 0.01$, $L = 0.05$, $T = 0.5$, and $c_0 = 0.8$.

β	m	p	c_1					
			0.825	0.85	0.875	0.9	0.925	0.95
0.25	5	0.050	614	145	61	33	20	13
		0.075	628	149	63	33	20	13
		0.100	643	152	64	34	21	14
	6	0.050	609	144	61	32	20	13
		0.075	628	149	63	33	20	13
		0.100	647	153	65	35	21	14
	7	0.050	610	144	61	32	20	13
		0.075	633	150	63	34	21	14
		0.100	657	156	66	35	21	14
	8	0.050	614	145	61	33	20	13
		0.075	641	152	64	34	21	14
		0.100	670	159	67	36	22	14
0.2	5	0.050	677	158	66	35	21	14
		0.075	693	162	67	36	21	14
		0.100	709	166	69	36	22	14
	6	0.050	672	157	65	34	21	14
		0.075	693	162	68	36	21	14
		0.100	714	167	70	37	22	14
	7	0.050	673	158	66	35	21	14
		0.075	699	164	68	36	22	14
		0.100	725	170	71	37	22	15
	8	0.050	677	159	66	35	21	14
		0.075	708	166	69	36	22	14
		0.100	739	174	72	38	23	15
0.15	5	0.050	754	174	71	37	22	14
		0.075	772	178	73	38	23	14
		0.100	790	183	75	39	23	15
	6	0.050	749	173	71	37	22	14
		0.075	772	179	73	38	23	15
		0.100	796	184	76	39	23	15
	7	0.050	750	174	71	37	22	14
		0.075	779	180	74	39	23	15
		0.100	808	187	77	40	24	15
	8	0.050	755	175	72	37	22	14
		0.075	789	183	75	39	23	15
		0.100	824	191	79	41	24	16

Table 2. The minimum sample size for $c_1 = 0.825(0.025)0.9$, $m = 5(1)8$ and $p = 0.05(0.025)0.1$ under $\alpha = 0.05$, $L = 0.05$, $T = 0.5$, and $c_0 = 0.8$.

β	m	p	c_1					
			0.825	0.85	0.875	0.9	0.925	0.95
0.25	5	0.050	361	84	35	18	11	7
		0.075	369	86	36	19	11	7
		0.100	378	88	37	19	12	8
	6	0.050	358	83	35	18	11	7
		0.075	369	86	36	19	11	7
		0.100	381	89	37	19	12	8
	7	0.050	359	84	35	18	11	7
		0.075	372	87	36	19	11	7
		0.100	387	90	37	20	12	8
	8	0.050	361	84	35	18	11	7
		0.075	377	88	37	19	12	8
		0.100	394	92	38	20	12	8
0.2	5	0.050	409	94	38	20	12	8
		0.075	419	96	39	20	12	8
		0.100	429	99	40	21	12	8
	6	0.050	407	93	38	20	12	8
		0.075	419	96	39	20	12	8
		0.100	432	99	41	21	12	8
	7	0.050	407	94	38	20	12	8
		0.075	423	97	40	21	12	8
		0.100	439	101	41	21	13	8
	8	0.050	410	94	39	20	12	8
		0.075	428	99	40	21	12	8
		0.100	448	103	42	22	13	8
0.15	5	0.050	470	106	43	22	13	8
		0.075	481	109	44	22	13	8
		0.100	492	112	45	23	13	8
	6	0.050	467	106	43	22	13	8
		0.075	481	109	44	22	13	8
		0.100	496	113	45	23	13	8
	7	0.050	468	106	43	22	13	8
		0.075	486	110	44	23	13	8
		0.100	504	115	46	24	14	9
	8	0.050	470	107	43	22	13	8
		0.075	492	112	45	23	13	8
		0.100	514	117	47	24	14	9

Table 3. The minimum sample size for $c_1 = 0.825(0.025)0.9$, $m = 5(1)8$ and $p = 0.05(0.025)0.1$ under $\alpha = 0.01$, $L = 0.10$, $T = 0.5$, and $c_0 = 0.8$.

β	m	p	c_1					
			0.825	0.85	0.875	0.9	0.925	0.95
0.25	5	0.050	253	58	24	12	7	5
		0.075	259	60	24	13	8	5
		0.100	265	61	25	13	8	5
	6	0.050	251	58	24	12	7	5
		0.075	259	60	24	13	8	5
		0.100	267	62	25	13	8	5
	7	0.050	252	58	24	12	7	5
		0.075	261	60	25	13	8	5
		0.100	271	63	26	13	8	5
	8	0.050	253	58	24	12	7	5
		0.075	265	61	25	13	8	5
		0.100	277	64	26	14	8	5
0.2	5	0.050	294	66	27	14	8	5
		0.075	301	68	27	14	8	5
		0.100	308	70	28	14	8	5
	6	0.050	292	66	27	14	8	5
		0.075	301	68	28	14	8	5
		0.100	311	70	28	15	8	5
	7	0.050	293	66	27	14	8	5
		0.075	304	69	28	14	8	5
		0.100	316	72	29	15	9	5
	8	0.050	294	67	27	14	8	5
		0.075	308	70	28	14	8	5
		0.100	322	73	30	15	9	6
0.15	5	0.050	346	77	30	15	9	5
		0.075	354	79	31	16	9	6
		0.100	362	81	32	16	9	6
	6	0.050	344	77	30	15	9	5
		0.075	354	79	31	16	9	6
		0.100	365	82	32	16	9	6
	7	0.050	344	77	30	15	9	5
		0.075	358	80	32	16	9	6
		0.100	371	83	33	17	9	6
	8	0.050	346	77	31	15	9	5
		0.075	362	81	32	16	9	6
		0.100	379	85	34	17	10	6

Figure 2. Minimum sample size for the test at $\alpha = 0.05$, $\beta = 0.2$, and $p = 0.05$.

Figure 3. Minimum sample size for the test at $\alpha = 0.05$, $\beta = 0.2$, and $m = 5$.

Figure 4. Minimum sample size for the test at $\beta = 0.2$ under $p = 0.05$ and $m = 5$.

3.2. The Optimal m and n for Fixed T

The experimenters usually do not prefer a large number of inspection intervals m . Not like Section 3.1, the number of inspection intervals m is needed to be determined in this section. Let m_0 be the upper limit of m specified by the experimenters such that $m \leq m_0$. (m_0 is defaulted to be 100). The sample size n is the function of m . We would like to determine the optimal (m, n) to minimize the total cost occurred during the progressive type I interval censoring procedure. Similar to Huang and Wu [11], we consider the following costs:

1. Installation cost C_a : the cost for installing all test units;
2. Inspection cost C_I : the cost for implementing the inspection equipment;
3. Sample cost C_s : the cost for per test unit;
4. Operation cost C_o : the cost including the personnel cost, depreciation of test equipment, etc. It is proportional to the length of experimental time.

Considering all these costs, we have the total cost of

$$TC(m, n) = C_a + m C_I + n C_s + T C_o$$

$$= C_a + m C_I + \left(\frac{Z_\beta \sqrt{r^{-1}(\lambda_1)} + Z_{1-\alpha} \sqrt{r^{-1}(\lambda_0)}}{\lambda_0 - \lambda_1} \right)^2 C_s + T C_o. \quad (12)$$

We use a numeration method to search the optimal (m, n) , and the Algorithm 1 is given as follows:

Algorithm 1 Searching for the optimal (m, n) .

Step 1: Give the pre-assigned values of $c_0, c_1, \alpha, \beta, p, T, L$, and m_0 (the default value is 100) and the four costs $C_a = a C_o, C_s = b C_o, C_I = c C_o, C_o$.

Step 2: Compute $\lambda_0 = \frac{1-c_0}{L}$ and $\lambda_1 = \frac{1-c_1}{L}$.

Step 3: Set $m = 1$.

Step 4: Compute the sample size n as in Equation (11) and then compute the related total cost $TC(m, n)$ as in Equation (12).

Step 5: If $m < m_0$, then $m = m + 1$ and go to Step 4, else go to Step 6.

Step 6: The optimal solution of m^* is the minimum m value such that $TC^* = \min_{m \leq m_0} TC(m, n)$ is reached and then the corresponding sample size n^* is obtained.

Step 7: The critical value of $C_L^0 = 1 - L \left(\lambda_0 + Z_{1-\alpha} \sqrt{I^{-1}(\lambda_0)} \right)$ is calculated.

Consider $C_0 = 1$ and $a = b = c = 1$. For testing $H_0 : C_L \leq 0.8$ under $\alpha = 0.01$, $\beta = 0.25$, $p = 0.05$, $c_1 = 0.825$, $k = 4.47$, $L = 0.05$ and $T = 0.5$, we plot $m = 2(1)m_0$ against its corresponding total cost in Figure 5a. You can see that minimum total cost occurred when $m = 6$ with the total cost of 616.5. For different set up of parameters $\beta = 0.15$, $\alpha = 0.1$, $p = 0.1$, $c_1 = 0.9$, $k = 4.47$, $L = 0.05$, $T = 0.5$, another total cost curve with $m = 2(1)m_0$ is given in Figure 5b. You can see that it is a concave upward curve with some flats and the minimum total cost occurred when $m = 3$ with the total cost of 21.5.

Figure 5. (a) Total cost curve at $m = 2(1)m_0$; (b) total cost curve at $m = 2(1)m_0$.

The minimum suggested inspection intervals m^* and sample size n^* to attain minimal total cost $TC(m^*, n^*)$ under the constraint of $m < m_0$, where $m_0 = 20$ for testing $H_0 : C_L \leq 0.8$ are tabulated in Tables 4 and 5 at $\beta = 0.25, 0.20, 0.15$, $\alpha = 0.01, 0.05, 0.1$, $L = 0.05$, $T = 0.5$, $p = 0.05(0.025)0.1$ for $c_1 = 0.825, 0.850$, and $c_1 = 0.875, 0.90$, respectively. The corresponding critical values C_L^0 are also tabulated.

For example, suppose that the user wants to conduct the level 0.05 hypothesis testing of $H_0 : C_L \leq 0.8$ under $\beta = 0.8$ at $c_1 = 0.90$, $p = 0.05$ and $m_0 = 20$. From Table 5, the minimum required sample size is $n^* = 21$ and the minimum number of inspection intervals is $m^* = 3$. The total cost is calculated as $TC = 25.5$ and the critical value is 0.8799.

From Tables 4 and 5, we have the following findings:

- (1) When the level of significance α decreases, the minimum sample size n^* is increasing for fixed β and p ;
- (2) When c_1 increases, the minimum sample size n^* is decreasing for fixed α , β , and p ;
- (3) When the the probability of type II error β decreases, the minimum sample size n^* is increasing for fixed α , c_1 , and p ;
- (4) When c_1 increases, the minimum inspection intervals m^* is decreasing for fixed α , β , and p ;
- (5) When β decreases, the minimum inspection intervals m^* is non-increasing for fixed α , c_1 , and p ;
- (6) When c_1 increases, the minimum total cost is decreasing for fixed α , β , and p ;
- (7) When β decreases, the minimum total cost is increasing for fixed α , c_1 , and p ;
- (8) When p increases, the minimum total cost is increasing for fixed α , β , and c_1 .

3.3. The Optimal m , t , and n When the Termination Time T Is Not Fixed

We consider the case when the total life test time T is not fixed in this section. The inspection time t for each inspection interval is needed to be determined. The optimal (m, t, n) is specified to minimize the total cost occurred during the progressive type I interval censoring procedure. The total cost is

$$TC(m, t, n) = C_a + m C_I + n C_S + m t C_o$$

$$= C_a + m C_I + \left(\frac{Z_\beta \sqrt{r^{-1}(\lambda_1)} + Z_{1-\alpha} \sqrt{r^{-1}(\lambda_0)}}{\lambda_0 - \lambda_1} \right)^2 C_S + m t C_o \quad (13)$$

Table 4. The optimal (m^*, n^*) , total cost TC and critical value C_L^0 for $c_1 = 0.825, 0.85$ and $p = 0.05(0.025)0.1$ under $m_0 = 20, L = 0.05, T = 0.5$, and $c_0 = 0.8$.

c_1			0.825					0.85		
α	β	p	m^*	n^*	TC	C_L^0	m^*	n^*	TC	C_L^0
0.01	0.25	0.050	6	609	616.5	0.8199	5	145	151.5	0.8410
		0.075	5	628	634.5	0.8199	5	149	155.5	0.8409
		0.100	5	643	649.5	0.8199	4	153	158.5	0.8411
	0.20	0.050	6	672	679.5	0.8190	5	158	164.5	0.8393
		0.075	5	693	699.5	0.8190	5	162	168.5	0.8392
		0.100	5	709	715.5	0.8190	4	167	172.5	0.8393
	0.15	0.050	6	749	756.5	0.8180	5	174	180.5	0.8374
		0.075	5	772	778.5	0.8180	5	178	184.5	0.8374
		0.100	5	790	796.5	0.8180	4	184	189.5	0.8374
0.05	0.25	0.050	6	358	365.5	0.8184	5	84	90.5	0.8381
		0.075	5	369	375.5	0.8184	4	87	92.5	0.8382
		0.100	5	378	384.5	0.8184	4	89	94.5	0.8381
	0.20	0.050	6	407	414.5	0.8172	5	94	100.5	0.8360
		0.075	5	419	425.5	0.8172	5	96	102.5	0.8360
		0.100	5	429	435.5	0.8172	4	99	104.5	0.8361
	0.15	0.050	6	467	474.5	0.8161	5	106	112.5	0.8339
		0.075	5	481	487.5	0.8161	5	109	115.5	0.8338
		0.100	5	492	498.5	0.8161	4	112	117.5	0.8339
0.10	0.25	0.050	6	251	258.5	0.8171	4	59	64.5	0.8359
		0.075	5	259	265.5	0.8171	4	60	65.5	0.8358
		0.100	5	265	271.5	0.8171	4	61	66.5	0.8358
	0.20	0.050	6	292	299.5	0.8159	5	66	72.5	0.8335
		0.075	5	301	307.5	0.8159	4	69	74.5	0.8334
		0.100	5	308	314.5	0.8159	4	70	75.5	0.8334
	0.15	0.050	6	344	351.5	0.8146	5	77	83.5	0.8310
		0.075	5	354	360.5	0.8146	4	80	85.5	0.8310
		0.100	5	362	368.5	0.8146	4	81	86.5	0.8311

Table 5. The optimal (m^*, n^*) , total cost TC and critical value C_L^0 for $c_1 = 0.875, 0.90$ and $p = 0.05(0.025)0.1$ under $m_0 = 20, L = 0.05, T = 0.5$, and $c_0 = 0.8$.

c_1			0.875				0.90			
α	β	p	m^*	n^*	TC	C_L^0	m^*	n^*	TC	C_L^0
0.01	0.25	0.050	4	62	67.5	0.8635	4	33	38.5	0.8871
		0.075	4	64	69.5	0.8630	4	34	39.5	0.8864
		0.100	4	65	70.5	0.8630	4	34	39.5	0.8871
	0.20	0.050	4	67	72.5	0.8611	4	35	40.5	0.8845
		0.075	4	68	73.5	0.8611	4	36	41.5	0.8840
		0.100	4	70	75.5	0.8607	3	38	42.5	0.8847
	0.15	0.050	5	71	77.5	0.8586	4	38	43.5	0.8811
		0.075	4	74	79.5	0.8586	4	39	44.5	0.8807
		0.100	4	76	81.5	0.8582	4	39	44.5	0.8813
0.05	0.25	0.050	4	36	41.5	0.8589	3	20	24.5	0.8819
		0.075	4	36	41.5	0.8594	3	20	24.5	0.8822
		0.100	4	37	42.5	0.8590	3	20	24.5	0.8825
	0.20	0.050	4	39	44.5	0.8566	3	21	25.5	0.8799
		0.075	4	40	45.5	0.8563	3	22	26.5	0.8784
		0.100	3	42	46.5	0.8569	3	22	26.5	0.8787
	0.15	0.050	4	44	49.5	0.8533	3	23	27.5	0.8763
		0.075	4	44	49.5	0.8537	3	24	28.5	0.8750
		0.100	4	45	50.5	0.8535	3	24	28.5	0.8753
0.10	0.25	0.050	4	24	29.5	0.8562	3	13	17.5	0.8791
		0.075	3	26	30.5	0.8562	3	13	17.5	0.8794
		0.100	3	26	30.5	0.8564	3	14	18.5	0.8768
	0.20	0.050	4	27	32.5	0.8530	3	15	19.5	0.8737
		0.075	3	29	33.5	0.8532	3	15	19.5	0.8739
		0.100	4	28	33.5	0.8529	3	15	19.5	0.8742
	0.15	0.050	4	31	36.5	0.8495	3	16	20.5	0.8713
		0.075	3	33	37.5	0.8498	3	17	21.5	0.8695
		0.100	4	32	37.5	0.8495	3	17	21.5	0.8697

We use the numeration method to search the optimal (m, t, n) and the Algorithm 2 is given as follows:

Algorithm 2 Searching for the optimal (m, t, n) .

Step 1: Give the pre-assigned values of c_0, c_1, α, β and p, L and m_0 (the default value is 100) and the four costs $C_a = aC_0, C_s = bC_0, C_I = cC_0, C_o$.

Step 2: Compute $\lambda_0 = \frac{1-c_0}{L}$ and $\lambda_1 = \frac{1-c_1}{L}$.

Step 3: Set $m = 1$.

Step 4: Find the optimal solution t^* such that $TC(m, t, n)$ is minimized. Compute the sample size n as in Equation (11) and then compute the related total cost $TC(m, t^*, n)$ as in Equation (13).

Step 5: If $m < m_0$, then $m = m+1$ and go to Step 4, else go to Step 6.

Step 6: The optimal solution of m^* is the minimum m value such that $TC^{**} = \min_{m \leq m_0} TC(m, t^*, n)$ is reached and then the corresponding sample size n^* is obtained.

Step 7: The critical value of $C_L^0 = 1 - L \left(\lambda_0 + Z_{1-\alpha} \sqrt{I^{-1}(\lambda_0)} \right)$ is calculated.

Consider $C_0 = 1$ and $a = b = c = 1$. For testing $H_0 : C_L \leq 0.8$ under $\beta = 0.25$, $\alpha = 0.01$, $p = 0.05$, $c_1 = 0.825$, $k = 4.47$, $L = 0.05$, $T = 0.5$, we plot $m = 2(1)m_0$ against its corresponding total cost in Figure 6a. You can see that a minimum total cost occurred when $m = 6$ with the total cost of 616.476. For a different set up of parameters $\beta = 0.15$, $\alpha = 0.1$, $p = 0.1$, $c_1 = 0.9$, $k = 4.47$, $L = 0.05$, and $T = 0.5$, another total cost curve with $m = 1(1)m_0$ is given in Figure 6b. You can see that it is a concave upward curve with some flats and the minimum total cost occurred when $m = 3$ with the total cost of 21.309. For other combination of set ups, we also find the similar patterns.

Figure 6. (a) Total cost curve at $m = 2(1)m_0$; (b) total cost curve at $m = 1(1)m_0$.

The minimum suggested inspection intervals m^* , inspection interval time length t^* and sample size n^* to yield the minimum total cost $TC(m^*, t^*, n^*)$ under the constraint of $m < m_0$, where $m_0 = 20$ for testing $H_0 : C_L \leq 0.8$ are tabulated in Tables 6 and 7 at $\beta = 0.25$, 0.20, 0.15, $\alpha = 0.01$, 0.05, 0.1, $L = 0.05$, $p = 0.05(0.025)0.1$ for $c_1 = 0.825$, 0.850, and $c_1 = 0.875$, 0.90, respectively. The corresponding critical values C_L^0 are also tabulated.

For example, suppose that the user wants to conduct the level 0.05 hypothesis testing of $H_0 : C_L \leq 0.8$ under power of 0.8 at $c_1 = 0.825$, $p = 0.05$, and $m_0 = 20$. From Table 6, the minimum required sample size is 408, the minimum number of inspection intervals is 5, and the inspection interval time length is 0.08. The total cost is calculated as $TC = 414.416$ and the critical value is 0.8173.

For any other setup of the testing procedure, a software program is provided by authors for users to input L , T , m , c_0 , c_1 , α , β and output the minimum required sample size n ; or input L , T , c_0 , c_1 , α , β , m_0 to output the minimum suggested number of inspection m ; or input L , c_0 , c_1 , α , β , m_0 to output the minimum recommended number of inspection m and the equal length of time t for each inspection interval.

From Tables 6 and 7, we have the following findings:

- (1) When the level of significance α increases, the minimum sample size n^* is decreasing for fixed β and p ;
- (2) When c_1 decreases, the minimum sample size n^* is increasing for fixed α , β , and p ;
- (3) When the the probability of type II error β decreases, the minimum sample size n^* is increasing for fixed α , c_1 , and p ;
- (4) When c_1 decreases, the minimum inspection intervals is increasing for fixed α , β and p .
- (5) When c_1 increases, the minimum total cost is decreasing for fixed α , β , and p ;
- (6) When β decreases, the minimum total cost is increasing for fixed α , c_1 , and p ;
- (7) When p increases, the minimum total cost is non-decreasing for fixed α , β , and c_1 .

3.4. Example

The data in Lee and Wang [12] include the tumor-free times (200 days) by feeding unsaturated fat diets after injected by an identical amount of tumor cells of $n = 30$ rats $U_{(1)} \leq U_{(2)} \leq \dots \leq U_{(30)}$ is listed as follows:

Data of tumor-free times

0.3	0.315	0.315	0.315	0.33	0.33	0.33	0.34	0.35	0.35
0.385	0.385	0.42	0.455	0.455	0.47	0.49	0.505	0.525	0.54
0.545	0.56	0.56	0.575	0.63	0.715	0.765	0.805	0.82	0.89

Table 6. The optimal (m^*, t^*, n^*) , total cost TC and critical value C_L^0 for $c_1 = 0.825, 0.85$ and $p = 0.05(0.025)0.1$ under $m_0 = 20, L = 0.05, T = 0.5$, and $c_0 = 0.8$.

		0.825						0.85					
α	β	c_1	p	m^*	t^*	n^*	TC	C_L^0	m^*	t^*	n^*	TC	C_L^0
0.01	0.25	0.050	6	0.08	609	616.476	0.8199	4	0.10	146	151.385	0.8410	
		0.075	5	0.09	628	634.438	0.8199	4	0.10	149	154.407	0.8410	
		0.100	4	0.10	644	649.417	0.8199	4	0.10	153	158.390	0.8409	
	0.20	0.050	6	0.08	672	679.466	0.8190	4	0.10	159	164.400	0.8393	
		0.075	5	0.09	692	698.451	0.8190	4	0.10	163	168.392	0.8392	
		0.100	5	0.10	709	715.476	0.8190	4	0.11	166	171.421	0.8392	
	0.15	0.050	6	0.08	749	756.467	0.8180	5	0.08	174	180.410	0.8374	
		0.075	5	0.09	771	777.460	0.8180	4	0.10	179	184.418	0.8374	
		0.100	5	0.10	790	796.483	0.8180	4	0.10	183	188.419	0.8374	
0.05	0.25	0.050	5	0.08	359	365.425	0.8184	4	0.09	85	90.360	0.8381	
		0.075	5	0.09	369	375.438	0.8184	4	0.10	86	91.409	0.8382	
		0.100	4	0.10	379	384.404	0.8184	4	0.10	88	93.407	0.8381	
	0.20	0.050	5	0.08	408	414.416	0.8173	4	0.09	95	100.368	0.8360	
		0.075	5	0.09	419	425.438	0.8172	4	0.10	97	102.381	0.8359	
		0.100	4	0.10	430	435.410	0.8173	4	0.10	99	104.394	0.8360	
	0.15	0.050	5	0.09	468	474.430	0.8161	4	0.10	107	112.391	0.8338	
		0.075	5	0.09	481	487.444	0.8161	4	0.10	110	115.381	0.8337	
		0.100	5	0.10	492	498.496	0.8161	4	0.10	112	117.402	0.8338	
0.10	0.25	0.050	5	0.08	252	258.419	0.8171	4	0.09	59	64.354	0.8356	
		0.075	4	0.10	260	265.403	0.8171	4	0.09	60	65.374	0.8358	
		0.100	4	0.10	266	271.402	0.8171	4	0.10	61	66.398	0.8357	
	0.20	0.050	5	0.08	293	299.420	0.8159	4	0.09	67	72.376	0.8334	
		0.075	5	0.09	301	307.439	0.8158	4	0.09	69	74.368	0.8333	
		0.100	4	0.10	309	314.409	0.8159	4	0.10	70	75.395	0.8333	
	0.15	0.050	5	0.08	345	351.413	0.8146	4	0.09	78	83.368	0.8310	
		0.075	5	0.09	354	360.442	0.8146	4	0.10	79	84.411	0.8310	
		0.100	4	0.11	363	368.422	0.8146	4	0.10	81	86.404	0.8310	

Table 7. The optimal (m^*, t^*, n^*) , total cost TC and critical value C_L^0 for $c_1 = 0.875, 0.90$ and $p = 0.05(0.025)0.1$ under $m_0 = 20, L = 0.05, T = 0.5$, and $c_0 = 0.8$.

c_1			0.875					0.90				
α	β	p	m^*	t^*	n^*	TC	C_L^0	m^*	t^*	n^*	TC	C_L^0
0.01	0.25	0.050	4	0.09	62	67.356	0.8631	3	0.11	34	38.326	0.8863
		0.075	4	0.09	63	68.378	0.8633	3	0.10	35	39.314	0.8866
		0.100	4	0.10	64	69.407	0.8633	3	0.11	35	39.338	0.8865
	0.20	0.050	4	0.10	66	71.400	0.8609	3	0.11	36	40.341	0.8838
		0.075	4	0.09	68	73.379	0.8610	3	0.11	37	41.327	0.8834
		0.100	4	0.10	69	74.412	0.8609	3	0.11	38	42.319	0.8830
	0.15	0.050	4	0.10	72	77.390	0.8583	3	0.11	39	43.331	0.8805
		0.075	4	0.10	74	79.382	0.8582	4	0.10	38	43.419	0.8812
		0.100	4	0.11	75	80.423	0.8584	4	0.10	39	44.409	0.8810
0.05	0.25	0.050	3	0.11	36	40.337	0.8593	3	0.11	19	23.324	0.8816
		0.075	3	0.11	37	41.324	0.8590	3	0.10	20	24.296	0.8810
		0.100	3	0.11	38	42.316	0.8587	3	0.10	20	24.312	0.8818
	0.20	0.050	3	0.11	40	44.331	0.8562	3	0.10	21	25.313	0.8783
		0.075	3	0.11	41	45.324	0.8560	3	0.11	21	25.334	0.8783
		0.100	4	0.11	40	45.423	0.8565	2	0.14	23	26.279	0.8776
	0.15	0.050	4	0.10	43	48.384	0.8534	3	0.11	23	27.321	0.8741
		0.075	4	0.10	44	49.391	0.8534	3	0.11	23	27.343	0.8748
		0.100	4	0.10	45	50.398	0.8533	3	0.11	24	28.318	0.8739
0.10	0.25	0.050	3	0.10	25	29.313	0.8559	3	0.10	13	17.307	0.8776
		0.075	3	0.11	25	29.337	0.8559	3	0.11	13	17.325	0.8775
		0.100	3	0.10	26	30.314	0.8559	2	0.12	15	18.245	0.8769
	0.20	0.050	3	0.11	28	32.325	0.8524	3	0.11	14	18.343	0.8740
		0.075	3	0.10	29	33.309	0.8524	2	0.13	16	19.264	0.8729
		0.100	3	0.11	29	33.329	0.8524	2	0.13	16	19.270	0.8733
	0.15	0.050	3	0.11	32	36.326	0.8490	3	0.11	16	20.325	0.8693
		0.075	4	0.11	31	36.420	0.8496	2	0.13	18	21.266	0.8687
		0.100	3	0.11	33	37.335	0.8491	3	0.10	17	21.309	0.8691

Using the Gini-test (see Gail and Gastwirth [13]), the Gini-test value $G_{30} = \frac{\sum_{i=1}^{30-1} i(n-i+1)(Y_{(i+1)} - Y_{(i)})}{(30-1) \sum_{i=1}^{30} (n-i+1)(Y_{(i)} - Y_{(i-1)})}$, where $Y_{(i)} = \frac{1}{k}(e^{kU_{(i)}} - 1)$. The Gini-test value is a function of k , and the p -values versus k values are plotted in Figure 7. We assume the parameter k is known. The value of k is determined with the the maximum p -value for Gini-test. It can be seen that the value of k is chosen as $k = 4.47$ with the maximum p -value.

Figure 7. The p -values for Gini test versus k values.

Referring to Section 3.1, suppose that the lower specification limit is $L = 0.5$, the number of inspection is $m = 5$, the removal probability is $p = 0.05$, the target lifetime performance index is $c_0 = 0.8$, the level of significance is $\alpha = 0.1$, and the power is $1 - \beta = 0.80$ at $c_1 = 0.9$. The required sample size is determined as $n = 20$ from Table 2.

Based on this setup, we can start to conduct the testing procedure about C_L as follows:
 Step 1: Take a random sample of size $n = 20$ from the data set. Observe the progressive type I interval censored sample $(X_1, X_2, X_3, X_4, X_5) = (0, 0, 1, 7, 2)$ at the pre-set times $(t_1, t_2, t_3, t_4, t_5) = (0.1, 0.2, 0.3, 0.4, 0.5)$ with censoring schemes of $(R_1, R_2, R_3, R_4, R_5) = (2, 0, 1, 0, 7)$ and $(p_1, p_2, p_3, p_4, p_5) = (0.05, 0.05, 0.05, 0.05, 1)$.

Step 2: Find the MLE of λ as $\hat{\lambda} = 0.4270091$ and then obtain the test statistic $\hat{C}_L = 1 - \hat{\lambda}L = 1 - 0.4270091 \times 0.05 = 0.9786495$.

Step 3: For level of significance $\alpha = 0.05$, we can calculate the critical value $C_L^0 = 0.8780724$.

Step 4: Since $\hat{C}_L = 0.9786495 > C_L^0 = 0.8780724$, we should reject the null hypothesis $H_0 : C_L \leq 0.8$ and claim that the lifetime performance index of product meets the required level.

With respect to Section 3.2, m is not fixed and $T = 0.5$ is fixed, under the same consideration in the previous section except m and n are not determined. From Table 4, we can find that the optimal sampling design is $m^* = 3$, $n^* = 21$ with critical value $C_L^0 = 0.8799$ and resulting in the minimum total cost of 25.5 units under the cost setup of $C_0 = 1$ and $a = b = c = 1$.

We can start to conduct the testing procedure about C_L as follows:

Step 1: Take a random sample of size $n = 21$ from the data set. Observe the progressive type I interval censored sample $(X_1, X_2, X_3) = (0, 3, 9)$ at the pre-set times $(t_1, t_2, t_3) = (0.1667, 0.333, 0.5)$ with censoring schemes of $(R_1, R_2, R_3) = (0, 0, 9)$ and $(p_1, p_2, p_3) = (0.05, 0.05, 1)$.

Step 2: Find the MLE of λ as $\hat{\lambda} = 0.4026597$ and then obtain the test statistic $\hat{C}_L = 1 - \hat{\lambda}L = 1 - 0.4026597 \times 0.05 = 0.979867$.

Step 3: For level of significance $\alpha = 0.05$, we can calculate the critical value $C_L^0 = 0.8799$.

Step 4: Since $\hat{C}_L = 0.979867 > C_L^0 = 0.8799$, we conclude to reject the null hypothesis $H_0 : C_L \leq 0.8$ and claim that the lifetime performance index of product meets the required level.

With respect to Section 3.3, when m and t are not fixed under the same setup with the previous two cases, we can find that the optimal sampling design is $m^* = 3$, $n^* = 21$, and

$t^* = 0.11$ with critical value $C_L^0 = 0.8783$ and minimum total cost of 29.5 units under the cost setup of $C_0 = 1$ and $a = b = c = 1$ from Table 7.

The testing procedure about C_L can be implemented as follows:

Step 1: Take a random sample of size $n = 21$ from the data set. Observe the progressive type I interval censored sample $(X_1, X_2, X_3) = (0, 0, 4)$ at the pre-set times $(t_1, t_2, t_3) = (0.11, 0.22, 0.33)$ with censoring schemes of $(R_1, R_2, R_3) = (1, 2, 14)$ and $(p_1, p_2, p_3) = (0.05, 0.05, 1)$.

Step 2: Find the MLE of λ as $\hat{\lambda} = 0.2921072$ and then obtain the test statistic $\hat{C}_L = 1 - \hat{\lambda}L = 1 - 0.2921072 \times 0.05 = 0.9853946$.

Step 3: For level of significance $\alpha = 0.05$, we can calculate the critical value $C_L^0 = 0.8783$.

Step 4: Since $\hat{C}_L = 0.9853946 > C_L^0 = 0.8783$, we reached the same conclusion to support the alternative hypothesis.

4. Conclusions

For many practical cases, the collection of complete samples would be very costly or not be feasible. The progressive type I interval censoring is thus arisen. The progressive type-I interval censoring has the advantage of convenient collection of data for the quality personnel in the life test. For this type of censoring, we have developed a sampling design to determine the sample size for attaining the preassigned power for a level α test or to determine the number of inspection intervals and interval lengths with the minimum total experimental cost for the test of the lifetime performance index C_L when the lifetime of products is following the Gompertz distribution under two conditions. One condition is the consideration of fixed termination time of the experiment and the other condition is the consideration of unfixed termination time of the experiment. Our algorithms for the sampling design can help experimenters to setup the progressive type-I interval censoring scheme and to use the observed sample to conduct the hypothesis test to see if the lifetime performance index meets the desired target level. We have also given a real-life example to demonstrate our sampling design and the implementation on the hypothesis testing procedure regarding the lifetime performance index.

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