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Some New Oscillation Criteria of Even-Order Quasi-Linear Delay Differential Equations with Neutral Term

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Abstract: The neutral delay differential equations have many applications in the natural sciences, technology, and population dynamics. In this paper, we establish several new oscillation criteria for a kind of even-order quasi-linear neutral delay differential equations. Comparing our results with those in the literature, our criteria solve more general delay differential equations with neutral type, and our results expand the range of neutral term coefficient. Some examples are given to illustrate our conclusions.

Keywords: delay differential equation; quasi-linear; neutral; oscillation

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1. Introduction

Up to now, many academics have made essential contributions to the delay differential equations, because such equations have various applications in natural science and social science [1–5]. For example, when studying the London–Yorke model of measles transmission [6], some scholars considered the following delay equation

$$\dot{y}(t) = \rho(t)y(t)[y(t-12) - y(t-14) - 2\sigma] + \sigma, \quad (1)$$

where $\sigma > 0$.

In the study of lossless transmission lines in electrical networks, the following equation [7].

$$\dot{y}(t) - a\dot{y}(t - \sigma) = f(y(t), \dot{y}(t - \sigma)) \quad (2)$$

is derived.

Since the delay differential equation is obviously different from the ordinary differential equation, the method of studying ordinary differential equation cannot be used to that. Thus, researchers have focused attention on the qualitative theory of delay differential equations, where the oscillation and asymptotic behavior of the equation is an important branch.

Over the last several years, many researchers considered the oscillation of the non-linear delay differential equation as follows:

$$\left(a(t)\left(\chi^{(n-1)}(t)\right)^\eta\right)' + c(t)f(y(h(t))) = 0, \quad (3)$$

where $t \geq t_0$, $\chi(t) = y(t) + b(t)y(g(t))$, $a \in C^1([t_0, \infty), (0, \infty))$, $b \in C^n([t_0, \infty), (0, \infty))$, $c \in C([t_0, \infty), (0, \infty))$, $g \in C^n([t_0, \infty), \mathbb{R})$ is one-one mapping, $h \in C([t_0, \infty), \mathbb{R})$, $g(t) \leq t$, $h(t) \leq t$, $f(-v) = -f(v)$ and $\eta \in S = \{i | i = \frac{2m+1}{2k+1}, m, k \in N^*\}$. The neutral differential equation refers to a type of differential equation with a delay in the highest derivative. In the Equation (3), $g(t), h(t) \leq t$, and they are the delay terms.

Here, a function $y \in C^n([t_y, \infty), \mathbb{R})$ is a solution of (3) if it has the property $a(t)(\chi^{(n-1)}(t))^\eta \in C^1([t_y, \infty), \mathbb{R})$ and satisfies Equation (3) on (t_y, ∞) , where $t_y \geq t_0$. We only considered the proper solution of (3) which satisfies $\sup\{|y(t)| : t \geq \mathfrak{T}\} > 0$ for all $\mathfrak{T} \geq t_y$.

A proper solution of (3) is oscillatory if it has infinitely many zeros on $[t_y, \infty)$. That is, for any $t_1 \geq t_y$, there exists $t_2 \geq t_1$, s.t. $y(t_2) = 0$. Otherwise, it is called non-oscillatory. Equation (3) is oscillatory if all its proper solutions are oscillatory.

In the case where $n = 2$, authors [8–16] investigated the quasi-linear equation as follows:

$$\left(a(t)(\chi'(t))^\eta\right)' + c(t)y^\eta(h(t)) = 0, \quad (4)$$

where $\eta \in S$. They [10,13] studied the oscillation criteria of (4) for $b(t) \in [0, 1)$. In [9,12], academics derived some oscillation criteria for $\eta = 1$, $b(t) \in [0, b_0]$ and $b_0 < \infty$.

In [14,15], authors discussed the oscillation behavior of the Emden–Fowler equation

$$\left(a(t)(\chi'(t))^\eta\right)' + c(t)y^\theta(h(t)) = 0, \quad (5)$$

where $\eta, \theta \in S$. In [14], T. X. Li et al. mainly concerned oscillation behavior of (5) when $b(t) \in [0, b_0]$ and $b_0 < \infty$ holds. In [15], R. P. Agarwal et al. considered that when $b(t) \in [0, 1)$ holds.

Expanding this approach to all higher-order equations attracts the attention of more and more researchers. B. Baculiková et al. [17] discussed the following quasi-linear equation by using the comparison principles and Riccati transformation:

$$\left(\left(\chi^{(n-1)}(t)\right)^\eta\right)' + c(t)y^\eta(h(t)) = 0, \quad (6)$$

where n is even, $\eta \geq 1$, $b(t) \in [0, b_0]$ and $b_0 < \infty$.

If $\eta = 1$, then Equation (6) becomes

$$\chi^{(n)}(t) + c(t)y(h(t)) = 0. \quad (7)$$

They [18–21] studied the oscillation of (7) under the condition $b(t) \in [0, 1)$. In [22–24], authors investigated the oscillatory solutions of (7) where $b(t) \in [0, b_0]$ and $b_0 < \infty$.

Based on the above results of previous scholars, in this article, we are concerned with the following quasi-linear neutral delay differential equations of the form (i.e., Equation (3) when $f(v) = v^\eta$)

$$\left(a(t)(\chi^{(n-1)}(t))^\eta\right)' + c(t)y^\eta(h(t)) = 0, \quad (8)$$

where n is even. The study of quasi-linear differential equations has numerous applications, such as in the study of p -Laplace equations, porous medium problems, chemotaxis models, and so forth; see, e.g., the papers [25–27] for more details.

We establish some oscillation criteria of (8) by using the Riccati transformation technique and comparison method. Compared with the second-order results of [4,8–16], we extend Equations (4) and (5) to Equation (8), where n is even. For the results of [17–24,28–34], we get the oscillation criteria of the more general equations. In other word, $a(t)$ may not be 1, $\eta \in S$ and $b(t)$ is not only bounded, but also can be unbounded. Therefore, we complement and extend upon some results reported in literature. At the end of this paper, some examples are provided to exhibit our conclusions.

2. Auxiliary Lemmas

Throughout this paper, we will analyze the following situations of $a(t)$, $b(t)$, $g(t)$, and $h(t)$:

Hypothesis 1 (H1). $a(t) > 0$, $a'(t) \geq 0$, $\kappa(t) := \int_{t_0}^t a^{-\frac{1}{\eta}}(s)ds$, $\lim_{t \rightarrow \infty} \kappa(t) = \infty$;

Hypothesis 2 (H2). $0 \leq b(t) < 1$;

Hypothesis 3 (H3). $0 \leq b(t) \leq b_0 < \infty$;

Hypothesis 4 (H4). $b(t) > 1, h(t) \leq g(t)$;

Hypothesis 5 (H5). $g'(t) \geq g_0 > 0, h'(t) > 0$.

Before starting our major criteria, the following lemmas are needed.

Lemma 1 ([35]). Let $f \in C^n([t_0, \infty), (0, \infty))$. If $f^{(n)}(t)$ is eventually of one sign for all large t , then there exist a $t_x \geq t_0$ and an integer $l, 0 \leq l \leq n$ with $n+l$ even for $f^{(n)}(t) \geq 0$, or $n+l$ odd for $f^{(n)}(t) \leq 0$, such that

$l > 0$ yields $f^{(k)} > 0$ for $t \geq t_x, k = 0, 1, \dots, l-1$, and
 $l \leq n-1$ yields $(-1)^{l+k} f^{(k)}(t) > 0$ for $t \geq t_x, k = l, l+1, \dots, n-1$.

Lemma 2 ([36], Lemma 2.2.3). If f is as in Lemma 1, $f^{(n)}(t)f^{(n-1)}(t) \leq 0$ for $t \geq t_x$, and $\lim_{t \rightarrow \infty} f(t) \neq 0$, then for every constant $\lambda \in (0, 1)$, there exists $t_\lambda \in [t_x, \infty)$, such that

$$f(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} |f^{(n-1)}(t)|$$

holds on $[t_\lambda, \infty)$.

Lemma 3 ([37], Lemma 2.3). Let $g(u) = au - bu^{\frac{\beta+1}{\beta}}$, where a and b are positive constants, $\beta \in S$. Then, g attains its maximum value on $\mathbb{R}^+$ at $u^* = (\frac{\beta a}{(\beta+1)b})^\beta$ and

$$\max_{u \in \mathbb{R}^+} g = g(u^*) = \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{a^{\beta+1}}{b^\beta}.$$

Lemma 4 ([38], Lemmas 1 and 2). Assume that $\alpha \in (0, \infty)$ and $x_1 \geq 0$ and $x_2 \geq 0$. Then,

$$x_1^\alpha + x_2^\alpha \geq \frac{1}{2^{\alpha-1}} (x_1 + x_2)^\alpha \quad \text{if } \alpha \geq 1, \quad (9)$$

and

$$x_1^\alpha + x_2^\alpha \geq (x_1 + x_2)^\alpha \quad \text{if } 0 < \alpha < 1. \quad (10)$$

Lemma 5 ([39]). If a function f satisfies $f^{(i)} > 0, i = 1, 2, \dots, k$ and $f^{(k+1)} \leq 0$, then, for every l in $(0, 1)$, $\frac{f(t)}{f'(t)} \geq \frac{lt}{k}$.

Lemma 6. Let $y(t)$ be an eventually positive solution of (8). If (H1) and the hypotheses of Lemma 1 hold, then there exist $t_1 \geq t_0$, such that:

$$\chi(t) > 0, \quad \chi'(t) > 0, \quad \chi^{(n-1)}(t) > 0 \quad \text{and} \quad \chi^{(n)}(t) < 0, \quad t \geq t_1. \quad (11)$$

More precisely, $\chi(t)$ has the following two cases for $t \geq t_1$:

(I) For $l > 1, \chi(t) > 0, \chi'(t) > 0, \chi''(t) > 0, \chi^{(n-1)}(t) > 0, \chi^{(n)}(t) < 0$;
 (II) For $l = 1, \chi(t) > 0, \chi^{(i)}(t) > 0, \chi^{(i+1)}(t) < 0, \chi^{(n-1)}(t) > 0, \chi^{(n)}(t) < 0$,
 where $l, i \in L = \{j | j = 2m + 1 \leq n - 3, m \in N\}$.

Proof. The proof of (11) is similar to that of ([29], Lemma 2.3), and so we omit it. Furthermore, we can conclude that case (I) and (II) hold. $\square$

3. Main Results

Now, let us begin our main criteria. For simplicity, we use the following symbols:

$$\begin{aligned}(\Gamma'(t))_+ &:= \max(0, \Gamma'(t)), & (\varphi'(t))_+ &:= \max(0, \varphi'(t)), \\ B(t) &:= c(t)(1 - b(h(t)))^\eta, & B^*(t) &:= \left(\int_t^\infty B(s) ds \right)^{\frac{1}{\eta}}, \\ \Theta_1(t) &:= \frac{\delta}{(n-1)!} t^{n-1}, & \Theta_2(t) &:= \frac{\delta}{(n-2)!} t^{n-2}, \\ H(t) &:= \frac{a(t)(\Gamma'(t))_+^{\eta+1}}{(\eta+1)^{\eta+1}(\Gamma(t)\Theta_2(h(t))h'(t))^\eta}, \\ H^*(t) &:= \frac{a(t)\Gamma'(t)_+^{\eta+1}}{(\eta+1)^{\eta+1}(\Gamma(t)\Theta_2(g^{-1}(h(t)))\zeta(t)h'(t))^\eta}, \\ M(t) &:= \frac{1}{b(g^{-1}(t))} \left(1 - \frac{(g^{-1}(g^{-1}(t)))^{\frac{n-1}{l}}}{(g^{-1}(t))^{\frac{n-1}{l}} b(g^{-1}(g^{-1}(t)))} \right), \\ M^*(t) &:= \left(\int_t^\infty c(s)M^\eta(h(s))ds \right)^{\frac{1}{\eta}},\end{aligned}$$

where $\Gamma, \varphi \in C^1([t_0, \infty), (0, \infty))$, $\delta, l \in (0, 1)$, $\zeta(t) = (g^{-1})'(v)|_{v=h(t)}$ and g^{-1} is the inverse function of g .

Theorem 1. If $\eta > 0$, (H1), (H3), (H5) and

$$\int_{t_0}^\infty C(t)dt = \infty \quad (12)$$

hold, where $C(t) = \min\{c(t), c(g(t))\}$, then (8) is oscillatory.

Proof. Suppose towards a contradiction that (8) is not oscillatory, and let y be such a solution of (8). Then, we can clearly assume that $y > 0$ eventually positive. That is, $y(t) > 0$, $y(g(t)) > 0$, $y(h(t)) > 0$ for $t \in [t_1, \infty)$, where $t_1 \geq t_0$. By $\eta > 0$, we need to divide into two situations to discuss, that is $\eta \geq 1$ and $0 < \eta < 1$.

When $\eta \geq 1$ is satisfied, owing to Lemma 6, we have that (11) holds. According to Equation (8), we obtain

$$\left(a(t) \left(\chi^{(n-1)}(t) \right)^\eta \right)' = -c(t)y^\eta(h(t)) < 0, \quad t \geq t_1. \quad (13)$$

Thus, $a(t) \left(\chi^{(n-1)}(t) \right)^\eta$ is not increasing for $t \geq t_1$. Let $\Phi(t) = a(t) \left(\chi^{(n-1)}(t) \right)^\eta$, $\Psi(t) = y^\eta(h(t))$. By the definition of χ and (13), we get

$$(\Phi(t))' + c(t)\Psi(t) + \frac{b_0^\eta}{g'(t)}(\Phi(g(t)))' + b_0^\eta c(g(t))\Psi(g(t)) \leq 0,$$

which leads to

$$(\Phi(t))' + \frac{b_0^\eta}{g_0}(\Phi(g(t)))' + C(t)(\Psi(t) + b_0^\eta \Psi(g(t))) \leq 0. \quad (14)$$

According to Lemma 4 and (H3), we have

$$(\Phi(t))' + \frac{b_0^\eta}{g_0} (\Phi(g(t)))' + \frac{1}{2^{\eta-1}} C(t) \chi^\eta(h(t)) \leq 0. \quad (15)$$

Integrating (15) from t_1 to t , we obtain

$$\frac{1}{2^{\eta-1}} \int_{t_1}^t C(s) \chi^\eta(h(s)) ds \leq \Phi(t_1) - \Phi(t) + \frac{b_0^\eta}{g_0} (\Phi(g(t_1)) - \Phi(g(t))). \quad (16)$$

By $\chi'(t) > 0$ on $[t_1, \infty)$, we get $\chi(h(t)) > \alpha > 0$ on $[t_1, \infty)$. By virtue of (H1), (11) and (13), we know that $\Phi(t) > 0$, $\Phi'(t) < 0$, and so $\Phi(t)$ is bounded. Thus, the right of (16) is bounded, contrary to (12).

If $0 < \eta < 1$, the argument is analogous to that in the above discussion, so is omitted. This completes the proof. $\square$

Corollary 1. Let the hypotheses of Theorem 1 hold. If the following inequality

$$\left(\Phi(t) + \frac{b_0^\eta}{g_0} \Phi(g(t)) \right)' + \frac{1}{2^{\eta-1}} \frac{C(t)}{a(h(t))} \Theta_1^\eta(h(t)) \Phi(h(t)) \leq 0 \quad (17)$$

has no eventually positive solution, then (8) is oscillatory.

Proof. Similar to the proof of Theorem 1, we have (11), (15) and $\lim_{t \rightarrow \infty} \chi(t) \neq 0$. Thus, by Lemma 2, there exists $t_2 \geq t_1$, such that

$$\chi(t) \geq \Theta_1(t) \chi^{(n-1)}(t), \quad t \geq t_2. \quad (18)$$

That achieves

$$\left(\Phi(t) + \frac{b_0^\eta}{g_0} \Phi(g(t)) \right)' + \frac{1}{2^{\eta-1}} \frac{C(t)}{a(h(t))} \Theta_1^\eta(h(t)) \Phi(h(t)) \leq 0. \quad (19)$$

It is straightforward to know that $\Phi(t)$ is positive and satisfies (19). The proof is complete. $\square$

Corollary 2. If the hypotheses of Theorem 1 holds, and

$$\left(\Phi(t) + \frac{b_0^\eta}{g_0} \Phi(g(t)) \right)' + \frac{C(t)}{a(h(t))} \Theta_1^\eta(h(t)) \Phi(h(t)) \leq 0 \quad (20)$$

has no eventually positive solution, then (8) is oscillatory.

Theorem 2. Let $n \geq 4$ be even and (H1), (H2), (H5) hold. If there exist $\Gamma \in C^1([t_0, \infty), (0, \infty))$ and $\varphi \in C^1([t_0, \infty), (0, \infty))$, such that

$$\int_{t_1}^{\infty} [\Gamma(t) B(t) - H(t)] dt = \infty, \quad (21)$$

and

$$\int_{t_1}^{\infty} \left[\varphi(t) \int_t^{\infty} (\vartheta - t)^{n-4} \frac{B^*(\vartheta)}{(n-4)! a^{\frac{1}{\eta}}(\vartheta)} d\vartheta - \frac{(\varphi'(t))^2_+}{4\varphi(t)h'(t)} \right] dt = \infty, \quad (22)$$

then (8) is oscillatory.

Proof. Suppose that (8) is not oscillatory. Without loss of generality, assume that y is an eventually positive solution of (8). That means $y(t) > 0$, $y(g(t)) > 0$, $y(h(t)) > 0$ on $t \in [t_1, \infty)$. By the assumptions and Lemma 6, χ satisfies case (I) or case (II).

First, we consider case (I). Then, $\lim_{t \rightarrow \infty} \chi'(t) \neq 0$. From that and Lemma 2, we achieve

$$\chi'(t) \geq \Theta_2(t)\chi^{(n-1)}(t).$$

By $h(t) \leq t$ and the fact that $\chi^{(n-1)}(t)$ is not increasing, we obtain

$$\frac{\chi'(h(t))}{\chi^{(n-1)}(t)} \geq \frac{\chi'(h(t))}{\chi^{(n-1)}(h(t))} \geq \Theta_2(h(t)). \quad (23)$$

Owing to $\chi' > 0$ and the definition of χ , we have

$$y(h(t)) \geq (1 - b(h(t)))\chi(h(t)). \quad (24)$$

Let

$$\mu(t) := \Gamma(t) \frac{a(t) \left(\chi^{(n-1)}(t) \right)^\eta}{\chi^\eta(h(t))}, \quad t \geq t_1. \quad (25)$$

Thus, $\mu(t) > 0$ on $[t_1, \infty)$ and set

$$d(t) := \frac{(\Gamma'(t))_+}{\Gamma(t)}, \quad e(t) := \frac{\eta h'(t) \Theta_2(h(t))}{(\Gamma(t) a(t))^{\frac{1}{\eta}}}.$$

Then

$$\mu'(t) \leq -\Gamma(t)B(t) + d(t)\mu(t) - e(t)\mu^{\frac{\eta+1}{\eta}}(t). \quad (26)$$

By Lemma 3, we get

$$d(t)\mu(t) - e(t)\mu^{\frac{\eta+1}{\eta}}(t) \leq H(t).$$

Thus,

$$\mu'(t) \leq -\Gamma(t)B(t) + H(t).$$

This yields

$$\int_{t_1}^r [\Gamma(t)B(t) - H(t)] dt \leq \mu(t_1),$$

for all large enough r , which contradicts (21).

For the case (II), according to (8) and (24), we achieve

$$\left(a(t) \left(\chi^{(n-1)}(t) \right)^\eta \right)' + B(t) \chi^\eta(h(t)) \leq 0, \quad t \geq t_1. \quad (27)$$

Integrating (27) from t to ∞ , from $\chi'(t) > 0$ and (H5), we get

$$-\chi^{(n-1)}(t) + \chi(h(t)) \frac{B^*(t)}{a^{\frac{1}{\eta}}(t)} \leq 0. \quad (28)$$

Integrating (28) from t to ∞ , we derive

$$\chi^{(n-2)}(t) + \chi(h(t)) \int_t^\infty \frac{B^*(\theta)}{a^{\frac{1}{\eta}}(\theta)} d\theta \leq 0, \quad t \geq t_1. \quad (29)$$

Continuously, if we integrate (29) from t to ∞ for all $(n-4)$ times we obtain

$$\chi''(t) + \chi(h(t)) \int_t^\infty \frac{(\vartheta - t)^{n-4} B^*(\vartheta)}{(n-4)! a^{\frac{1}{\eta}}(\vartheta)} d\vartheta \leq 0, \quad t \geq t_1. \quad (30)$$

Now, let

$$\nu(t) := \varphi(t) \frac{\chi'(t)}{\chi(h(t))}, \quad t \geq t_1. \quad (31)$$

It is easy to verify that $\nu(t) > 0$ on $[t_1, \infty)$. Since χ' is decreasing and $h(t) \leq t$, according to Lemma 3, we get

$$\nu'(t) \leq -\varphi(t) \int_t^\infty \frac{(\vartheta - t)^{n-4} B^*(\vartheta)}{(n-4)! a^{\frac{1}{\eta}}(\vartheta)} d\vartheta + \frac{(\varphi'(t))^2_+}{4\varphi(t)h'(t)}. \quad (32)$$

This implies that

$$\int_{t_1}^r \left[\varphi(t) \int_t^\infty \frac{(\vartheta - t)^{n-4} B^*(\vartheta)}{(n-4)! a^{\frac{1}{\eta}}(\vartheta)} d\vartheta - \frac{(\varphi'(t))^2_+}{4\varphi(t)h'(t)} \right] dt \leq \nu(t_1)$$

for any r large enough. This contradicts our assumption (22), which completes the proof. $\square$

Theorem 3. Let $n \geq 4$ be even and (H1), (H4), (H5) hold. If there exist $\Gamma, \varphi \in C^1([t_0, \infty), (0, \infty))$ which satisfy

$$\int_{t_1}^\infty [c(t)\Gamma(t)M^\eta(h(t)) - H^*(t)]dt = \infty \quad (33)$$

and

$$\int_{t_1}^\infty \left[\varphi(t) \int_t^\infty \frac{(\vartheta - t)^{n-4} M^*(\vartheta)}{(n-4)! a^{\frac{1}{\eta}}(\vartheta)} d\vartheta - \frac{\varphi'(t)^2}{4\varphi(t)h'(t)\zeta(t)} \right] dt = \infty, \quad (34)$$

then (8) is oscillatory.

Proof. Just as the proof of Theorem 2, by the above assumptions, χ satisfies either case (I) or case (II).

Suppose that (I) holds. By the definition of χ and (H4), we get

$$y(t) \geq \frac{1}{b(g^{-1}(t))} \left(\chi(g^{-1}(t)) - \frac{\chi(g^{-1}(g^{-1}(t)))}{b(g^{-1}(g^{-1}(t)))} \right). \quad (35)$$

According to Lemma 5, we obtain

$$\frac{\chi(t)}{\chi'(t)} \geq \frac{lt}{n-1}, \quad (36)$$

where $l \in (0, 1)$, which leads to that $\frac{\chi(t)}{t^{\frac{n-1}{l}}}$ is non-increasing. By $g^{-1}(t) \leq g^{-1}(g^{-1}(t))$ and (35), we derive

$$y(t) \geq \chi(g^{-1}(t))M(t). \quad (37)$$

By (8) and (37), we gain

$$\left(a(t) \left(\chi^{(n-1)}(t) \right)^\eta \right)' + c(t) \chi^\eta(g^{-1}(h(t))) M^\eta(h(t)) \leq 0. \quad (38)$$

Then define

$$\mu^*(t) := \Gamma(t) \frac{a(t) \left(\chi^{(n-1)}(t) \right)^\eta}{\chi^\eta(g^{-1}(h(t)))}, \quad t \geq t_1. \quad (39)$$

Since $\mu^*(t) > 0$ on $[t_1, \infty)$, by (23) and Lemma 3, we have

$$(\mu^*)'(t) \leq -c(t)\Gamma(t)M^\eta(h(t)) + H^*(t). \quad (40)$$

This yields

$$\int_{t_1}^r [c(t)\Gamma(t)M^\eta(h(t)) - H^*(t)]dt \leq \mu^*(t_1)$$

for all large r , in contradiction with (33).

Then, assume that case (II) is true. Thus, we have that (38) holds. Integrating (38) from t to ∞ , by virtue of $\chi'(t) > 0$ and (H5), we obtain

$$-\chi^{(n-1)}(t) + \chi(g^{-1}(h(t))) \frac{M^*(t)}{a^{\frac{1}{\eta}}(t)} \leq 0. \quad (41)$$

Integrating (41) from t to ∞ , we obtain

$$\chi^{(n-2)}(t) + \chi(g^{-1}(h(t))) \int_t^\infty \frac{M^*(\vartheta)}{a^{\frac{1}{\eta}}(\vartheta)} d\vartheta \leq 0, \quad t \geq t_1. \quad (42)$$

Integrating (42) from t to ∞ for $(n-4)$ times, we achieve

$$\chi''(t) + \chi(g^{-1}(h(t))) \int_t^\infty \frac{(\vartheta - t)^{n-4} M^*(\vartheta)}{(n-4)! a^{\frac{1}{\eta}}(\vartheta)} d\vartheta \leq 0, \quad t \geq t_1. \quad (43)$$

Let

$$v^*(t) := \varphi(t) \frac{\chi'(t)}{\chi(g^{-1}(h(t)))}, \quad t \geq t_1. \quad (44)$$

Then $v^*(t) > 0$. Since $\chi'(t)$ is decreasing, by (H4), (43) and Lemma 3, we get

$$(v^*)'(t) \leq -\varphi(t) \int_t^\infty \frac{(\vartheta - t)^{n-4} M^*(\vartheta)}{(n-4)! a^{\frac{1}{\eta}}(\vartheta)} d\vartheta + \frac{\varphi'(t)^2}{4\varphi(t)h'(t)\zeta(t)}.$$

This implies that

$$\int_{t_1}^r \left[\varphi(t) \int_t^\infty \frac{(\vartheta - t)^{n-4} M^*(\vartheta)}{(n-4)! a^{\frac{1}{\eta}}(\vartheta)} d\vartheta - \frac{\varphi'(t)^2}{4\varphi(t)h'(t)\zeta(t)} \right] dt \leq v^*(t_1)$$

for all large r , which contradicts (34). The proof of the theorem is complete. $\square$

4. Examples

For $t \geq 1$, the following examples are given to verify our criteria.

Example 1.

$$\left(e^t \left(\left(x(t) + 3x\left(\frac{t}{2}\right) \right)^{(3)} \right)^3 \right)' + \frac{1}{t} x^3\left(\frac{t}{3}\right) = 0, \quad (45)$$

where $n = 4$, $a(t) = e^t$, $b(t) = 3$, $\eta = 3$, $c(t) = \frac{1}{t}$, $g(t) = \frac{t}{2}$, $h(t) = \frac{t}{3}$. Using Theorem 1, we know that $C(t) = c(t) = \frac{1}{t}$ and $\int_1^\infty \frac{1}{s} ds = \infty$. Thus, (45) is oscillatory.

Example 2.

$$\left(e^t \left(\left(x(t) + \frac{1}{2}x\left(\frac{t}{3}\right) \right)^{(3)} \right)^{\frac{1}{5}} \right)' + e^t x^{\frac{1}{5}}\left(\frac{t}{4}\right) = 0, \quad (46)$$

where $n = 4$, $a(t) = e^t$, $b(t) = \frac{1}{2}$, $\eta = \frac{1}{5}$, $c(t) = e^t$, $g(t) = \frac{t}{3}$, $h(t) = \frac{t}{4}$. Using Theorem 1, we know that $C(t) = c(g(t)) = e^{\frac{t}{3}}$ and $\int_1^\infty e^{\frac{s}{3}} ds = \infty$. Thus, (46) is oscillatory.

Example 3.

$$\left(t \left(x(t) + \frac{1}{2} x\left(\frac{t}{2}\right) \right)^{(3)} \right)' + \frac{c_0}{t^3} x\left(\frac{9t}{10}\right) = 0, \quad (47)$$

where $n = 4$, $a(t) = t$, $b(t) = \frac{1}{2}$, $\eta = 1$, $c(t) = \frac{c_0}{t^3}$, $g(t) = \frac{t}{2}$, $h(t) = \frac{9t}{10}$. Letting $\Gamma(t) = t^2$, $\varphi(t) = t$, $\delta = \frac{9^3-1}{9^3}$. Then,

$$B(t) = \frac{c_0}{2t^3},$$

$$H(t) = \frac{2000}{9^3 \delta t}.$$

Thus, $c_0 \geq \frac{4000}{9^3-1} \approx 5.5$, then

$$\int_{t_1}^\infty \left(\frac{c_0}{2} - \frac{2000}{9^3 \delta t} \right) t^{-1} dt \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

$$B^*(\vartheta) = \int_\vartheta^\infty \frac{c_0}{2} s^{-3} ds = \frac{c_0}{4} \vartheta^{-2}.$$

Thus, $c_0 \geq \frac{20}{9} \approx 2.2$, then

$$\int_{t_1}^\infty \left(\frac{c_0}{8} - \frac{5}{18} \right) t^{-1} dt \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

Thus, we can conclude that (47) is oscillatory if $c_0 \geq \frac{4000}{9^3-1} \approx 5.5$ when using Theorem 2.

Remark 1. Using the criteria of Theorem 2, we can obtain the same estimation as that in Example 3.1 [23]. Further, the results of [18–21,23] cannot solve (47) because of $a(t) = t$ and $\eta \in S$. Thus, our results complement and extend upon the results of previous papers on this topic.

Example 4.

$$\left(t \left(x(t) + 16x\left(\frac{t}{2}\right) \right)^{(3)} \right)' + \frac{c_0}{t^3} x\left(\frac{t}{3}\right) = 0, \quad (48)$$

where $n = 4$, $a(t) = t$, $b(t) = 16$, $\eta = 1$, $c(t) = \frac{c_0}{t^3}$, $g(t) = \frac{t}{2}$, $h(t) = \frac{t}{3}$, $h(t) < g(t)$. Letting $\Gamma(t) = t^2$, $\varphi(t) = t$, $\delta = \frac{27 \times 8 - 1}{27 \times 8}$, $l = \frac{10^4 - 1}{10^4}$,

$$M^\eta(h(t)) = \frac{1}{16} \left(1 - \frac{2^{\frac{3}{7}}}{16} \right) \approx \frac{1}{32} \quad \text{as } l = \frac{10^4 - 1}{10^4}.$$

$$H^*(t) = \frac{27}{4\delta t}.$$

Thus, if $c_0 \geq \frac{27 \times 8}{\delta} \approx 217.1$, then (33) $\rightarrow \infty$.

$$M^*(\vartheta) = \int_\vartheta^\infty \frac{c_0}{32} s^{-3} ds = \frac{c_0}{64} \vartheta^{-2}.$$

Thus, if $c_0 \geq \frac{3 \times 128}{8} = 48$, (34) $\rightarrow \infty$. Thus, we know that (48) is oscillatory if $c_0 \geq \frac{27 \times 8}{\delta} \approx 217.1$ when using Theorem 3.

Remark 2. Using the criteria of Theorem 3, we can get the same estimation as that in Example 3.2 [23]. Further, the results of [28–34] cannot solve (48) because of $a(t) = t$, $\eta > 0$ and $b(t) > 1$.

Remark 3. The results obtained in this article can be extended to the more general Equation (3) when assuming that $\frac{f(v)}{v^\eta} \geq q_0$ for all $v \neq 0$, where $q_0 > 0$ is a constant and the equation of the form

$$\left(a(t) \left| \chi^{(n-1)}(t) \right|^{\eta-1} \chi^{(n-1)}(t) \right)' + c(t) |y(h(t))|^{\eta-1} y(h(t)) = 0$$

where $\eta > 0$ is a constant.

Remark 4. In this article, the neutral term coefficient $b(t)$ not only can be bounded (i.e., $0 < b(t) < 1$) but also can be unbounded (i.e., $b(t) > 1$).

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