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Article

Packing Three Cubes in D-Dimensional Space

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Abstract: Denote $V_n(d)$ the least number that every system of n cubes with total volume 1 in d-dimensional (Euclidean) space can be packed parallelly into some rectangular parallelepiped of volume $V_n(d)$. New results $V_3(5) \doteq 1.802803792$, $V_3(7) \doteq 2.05909680$, $V_3(9) \doteq 2.21897778$, $V_3(10) \doteq 2.27220126$, $V_3(11) \doteq 2.31533581$, $V_3(12) \doteq 2.35315527$, $V_3(13) \doteq 2.38661963$ can be found in the paper.

Keywords: packing of cubes; extreme

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1. Introduction

In 1966, L. Moser [1] raised the following problem: Determine the smallest number A so that any system of squares of total area 1 can be packed parallelly into some rectangle of area A. This problem can also be found in [2–6]. The problem has been extended to higher dimensions and has been studied for a specific number of squares (cubes). To distinguish the number of dimensions and cubes we denoted $V_n(d)$ the least number that every system of n cubes with total volume 1 in d-dimensional (Euclidean) space can be packed parallelly into some rectangular parallelepiped of volume $V_n(d)$. $V(d)$ denotes the maximum of all $V_n(d)$, $n = 1, 2, 3, \ldots$.

Some results are known in 2-dimensional space. Kleitman and Krieger [7] proved that every finite system of squares with total area 1, can be packed into the rectangle with sides of lengths $\frac{2}{\sqrt{3}}$ and $\sqrt{2}$, so $V(2) \leq \frac{4}{\sqrt{6}} \doteq 1.632993162$. After twenty years Novotný [8] showed that $V_3(2) = 1.2277589$ and $V(2) \geq \frac{2+\sqrt{3}}{3} > 1.244$. The exact results are known also for $n = 4, 5, 6, 7, 8$, Novotný [9] proved $V_4(2) = V_5(2) = \frac{2+\sqrt{3}}{3}$ and in Novotný [10] proved $V_6(2) = V_7(2) = V_8(2) = \frac{2+\sqrt{3}}{3}$. The estimate of $V(2)$ was improved by Novotný [11] $V(2) < 1.53$. Later, this result was improved by Hurgady [12] $V(2) \leq \frac{2867}{2048} < 1.4$.

In 3-dimensional space, the estimate of $V(3)$ was gradually improved. Meir and Moser [13] proved $V(3) \leq 4$ and later Novotný [14] proved $V(3) \leq 2.26$. The exact results are known for $n = 2, 3, 4, 5$: Novotný [15] $V_2(3) = \frac{4}{3}$, $V_3(3) = 1.44009951$, Novotný [16] $V_4(3) = 1.5196303266$, and in Novotný [14] proved $V_5(3) = V_4(3)$.

The results for $n = 3$ and $d = 4, 6, 8$ are known too: $V_3(4) = 1.63369662$, Bálint and Adamko in [17]; $V_3(6) = 1.94449161$, Bálint and Adamko in [18]; $V_3(8) = 2.14930609$, Sedliačková in [19].

Adamko and Bálint proved $\lim\limits_{d \to \infty} V_n(d) = n$ for $n = 5, 6, 7, \ldots$ in [20]. Theorem holds also for $n = 2, 3, 4$.

2. Main Results

The main part of this section is the proof of $V_3(5) \doteq 1.802803792$. We use the same method as [17,18]. At the end of the section, we offer (without proof) the values of $V_3(d)$ for $d \in \{7, 9, 10, 11, 12, 13\}$.

Theorem 1. *$V_3(5) \doteq 1.802803792$.*

Outline of the proof

1. We show, that there are only two important packing configurations. Their volumes are $W_1 = x^4(x+y+z)$ and $W_2 = x^3(x+y)(y+z)$, see Figure 1. Firstly, we need to find $min\{W_1, W_2\}$ for each $\{x, y, z\}$. The maximum from $min\{W_1, W_2\}$ is the final result, we denote it $max\ min\{W_1, W_2\}$;
2. Cubes with sides $x \doteq 0.946629932$, $y \doteq 0.690148624$, and $z \doteq 0.608279275$ have $W_1 = W_2 = 1.802803792$. We prove that this volume is sufficient for packing any three cubes with a total volume of 1 in dimension 5;
3. We obtain an estimation of the side size of the largest cube: $0.9445 \leq x \leq 0.9939$;
4. Using $1 = x^5 + y^5 + z^5$ and $1 > x \geq y \geq z > 0$, we obtain constraints $x^5 + y^5 \leq 1$ and $x^5 + 2y^5 \geq 1$;
5. $z = (1 - x^5 - y^5)^{1/5}$. Therefore, it is sufficient to work only with x and y. M is a region of $\{x, y\}$ bounded by constraints from steps 3 and 4. We obtain a curve C from $W_1 = W_2$, see Figure 2. Curve C divides the region M into continuous regions C_1 and C_2, see Figure 3;
6. We clarify:
   (a) $W_1(X) < W_2(X)$ holds for $X \in C_1$. Therefore, $max\ min\{W_1, W_2\} = max\ W_1(X)$, $X \in C_1$;
   (b) $W_1(X) > W_2(X)$ holds for $X \in C_2$. Therefore, $max\ min\{W1, W2\} = max\ W_2(X)$, $X \in C_2$;
7. We show that the asked maximum is on curve C;
   (a) We use critical points for region C_1;
   (b) We were unable to use critical points on the whole C_2, so we gradually numerically exclude subregions. We start with comparison of maximum of subregions and 1.8 (packing with $V_3(5) > 1.8$ exists).

Proof. Consider three cubes with edge lengths x, y, z in the 5-dimensional Euclidean space, where $1 > x \geq y \geq z > 0$ and the total volume $x^5 + y^5 + z^5 = 1$.

We are looking for the smallest volume of a parallelepiped containing all three cubes. Therefore, from several ways of packing, we can ignore the packing that leads in any circumstances to a larger volume.

Let X, Y, Z denote the cubes (sorted from the largest). We attach cubes X and Y to each other, for example, in the direction of the fifth dimension. Parallelepiped containing cubes X and Y has volume $x^4(x+y)$.

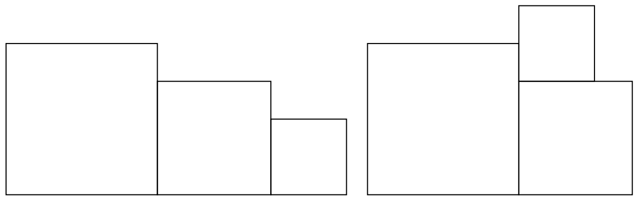
If we place the cube Z to the cube X in the direction of the fifth dimension, we receive volume $x^4(x+y+z)$. We obtain volume $x^3(x+y)(x+z)$ for other four directions.

If we place the cube Z to the cube Y in the direction of the fifth dimension, we receive again $x^4(x+y+z)$. We obtain (after appropriate shifting of the cube Y) volume $x^3(x+y)(y+z)$ for other four directions.

Because $x^3(x+y)(y+z) \leq x^3(x+y)(x+z)$, we can ignore packings that lead to the volume $x^3(x+y)(x+z)$.

If we start with cubes X and Z, or Z and Y, the same results are obtained.

Therefore, it is sufficient to consider only two cases of packing three cubes, see Figure 1a,b. In the first case, the volume $W_1 = x^4(x+y+z)$ is sufficient for packing, in the second case, the volume $W_2 = x^3(x+y)(y+z)$ is sufficient.



(**a**) Packing with volume W1 (**b**) Packing with volume W2

Figure 1. Two cases of packing three cubes.

We need to find $max\ min\{W_1, W_2\}$ under the conditions $x^5 + y^5 + z^5 = 1$, and $1 > x \geq y \geq z > 0$.

For three cubes with edge lengths $x \doteq 0.946629932$, $y \doteq 0.690148624$, $z \doteq 0.608279275$, a volume $W_1 = W_2 \doteq 1.802803792$ is necessary. Thus, $V_3(5) \geq 1.802803792$.

If $y + z \leq x$ than we can pack the cubes, as shown in Figure 1b, and volume $V_2(5)$ is sufficient, $V_2(5) \doteq 1.484663669$ for cubes with edge lengths $x \doteq 0.984432006$ and $y \doteq 0.596398035$.

Let us consider only the case that $y + z > x$. From $y^5 \leq z^5 + y^5 = 1 - x^5$ we find $y \leq \sqrt[5]{1 - x^5}$, and, therefore, $y + z \leq 2y \leq 2\sqrt[5]{1 - x^5}$. Then, $x < y + z \leq 2\sqrt[5]{1 - x^5}$ and, therefore, $x^5 < 2^5(1 - x^5)$. We attain the upper bound for x, $x \leq \frac{2}{\sqrt[5]{33}}$. $x \geq y \geq z$ and $x^5 + y^5 + z^5 = 1$, therefore, $x^5 \geq \frac{1}{3}$ and $x \geq \frac{1}{\sqrt[5]{3}}$. This implies that we can consider only $x \in \left[\frac{1}{\sqrt[5]{3}}, \frac{2}{\sqrt[5]{33}}\right]$, i.e., $0.8027 \leq x \leq 0.9939$.

Equality $W_1 = W_2$ holds if, and only if, $x^2 = y^2 + yz$. In this case, $z = \frac{x^2 - y^2}{y}$ and $W_1 = W_2 = x^5 + \frac{x^6}{y}$. When we substitute $z = \frac{x^2 - y^2}{y}$ into $x^5 + y^5 + z^5 = 1$ we find the curve C: $x^5y^5 + y^{10} - y^5 + (x^2 - y^2)^5 = 0$ (Figure 2).

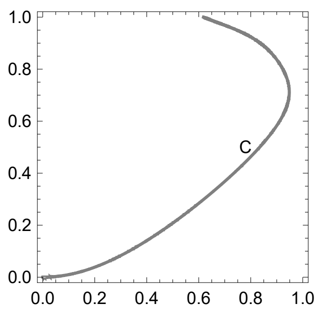


Figure 2. The curve C.

The interval for x can be reduced. If we choose $x \in [a, b]$, $0 < a < b < 1$, then $1 - b^5 \leq 1 - x^5 \leq 1 - a^5$. If $y = z$, then $1 - x^5 = y^5 + z^5 = 2y^5$ and, therefore, $y = \sqrt[5]{\frac{1 - x^5}{2}}$. The function $W_1 = x^4(x + y + z)$ has the greatest value if $y = z$, , i.e., $y = \sqrt[5]{\frac{1 - x^5}{2}}$. For $x \in [a, b]$, we find $W_1 \leq x^4(x + 2y) \leq b^4\left(b + 2\sqrt[5]{\frac{1 - a^5}{2}}\right)$.

Denote $W_1(a, b) = b^4\left(b + 2\sqrt[5]{\frac{1 - a^5}{2}}\right)$.

The inequality $W_1(a, b) < 1.8021$ is valid for the intervals: $x \in [0.8027, 0.9190]$, $x \in [0.9190, 0.9360]$, $x \in [0.9360, 0.9410]$, $x \in [0.9410, 0.9420]$, $x \in [0.9420, 0.9430]$, $x \in [0.9430, 0.9440]$, $x \in [0.9440, 0.9445]$, hence for the asked maximum holds $x \geq 0.9445$.

Therefore, we have shown that the asked $max\ min\{W_1, W_2\}$ will be attained for $x \in [0.9445, 0.9939]$.

From the assumption $0 < z \leq y \leq x < 1$ follows that $x^5 + y^5 \leq x^5 + y^5 + z^5 = 1$ and also $1 = x^5 + y^5 + z^5 \leq x^5 + 2y^5$.

Consider the closed region M determined by inequalities $0.9445 \leq x \leq 0.9939$, $x^5 + y^5 \leq 1$, $x^5 + 2y^5 \geq 1$. The curve C divides the region M into two open regions C_1, C_2, (Figure 3).

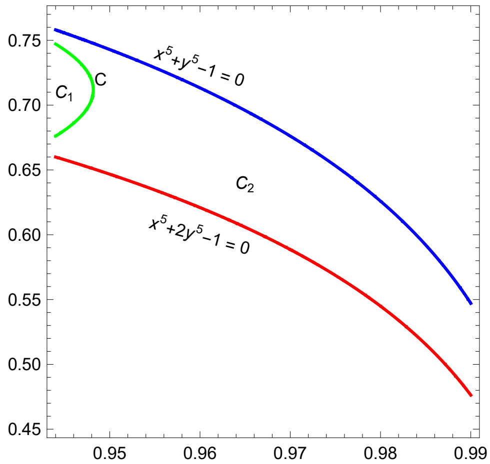


Figure 3. Regions C_1, C_2.

We are looking for *max min*$\{W_1, W_2\}$, when $W_1 = x^4(x+y+z)$, $W_2 = x^3(x+y)(y+z)$. From the condition $x^5 + y^5 + z^5 = 1$ we find

$$W_1 = W_1(x,y) = x^4(x + y + \sqrt[5]{1 - x^5 - y^5}), \tag{1}$$

$$W_2 = W_2(x,y) = x^3(x+y)(y + \sqrt[5]{1 - x^5 - y^5}). \tag{2}$$

Let $\bar{C}_i$ denote the closure of the set C_i. The functions W_1, W_2 are continuous on M and the equality $W_1 = W_2$ holds just in the points of the curve C.

Take the point $A_1 = (0.945, 0.70) \in C_1$. The inequality $W_1(X) < W_2(X)$ holds in every point $X \in C_1$, because of $W_1(A_1) < W_2(A_1)$. Therefore, for the asked maximum holds $\max\limits_{X \in \bar{C}_1} \min\{W_1(X), W_2(X)\} = \max\limits_{X \in \bar{C}_1}\{W_1(X)\}$.

Take the point $A_2 = (0.965, 0.65) \in C_2$. The inequality $W_1(X) > W_2(X)$ holds in every point $X \in C_2$, because of $W_1(A_2) > W_2(A_2)$. Therefore, for the asked maximum holds $\max\limits_{X \in \bar{C}_2} \min\{W_1(X), W_2(X)\} = \max\limits_{X \in \bar{C}_2}\{W_2(X)\}$.

On the compact set $\bar{C}_1$ the function (1) has its maximum in some point B.

It holds $\dfrac{\partial W_1}{\partial y} = x^4\left(1 - \dfrac{y^4}{\sqrt[5]{(1 - x^5 - y^5)^4}}\right)$. The equality $\dfrac{\partial W_1}{\partial y} = 0$ holds if $x^5 + 2y^5 - 1 = 0$ but the points of the curve $x^5 + 2y^5 - 1 = 0$ do not belong to the region $\bar{C}_1$. For every point $X \in C_1$ holds $\dfrac{\partial W_1}{\partial y} < 0$. Therefore, the point B must lie on the curve C.

For every point $X = (x,y)$, $x \in [a,b]$, $y \in [c,d]$ the inequality $z \leq \sqrt[5]{1 - a^5 - c^5}$ holds, and so $W_1 = x^4(x+y+z) \leq b^4(b + d + \sqrt[5]{1 - a^5 - c^5})$, $W_2 = x^3(x+y)(y+z) \leq b^3(b+d)(d + \sqrt[5]{1 - a^5 - c^5})$.

Denote

$$W_{11}(a,b,c,d) = b^4(b + d + \sqrt[5]{1 - a^5 - c^5}),$$
$$W_{22}(a,b,c,d) = b^3(b+d)(d + \sqrt[5]{1 - a^5 - c^5}).$$

Examine the region C_2.

For $x \in [0.9900, 0.9939]$, $y \in [0.43, 0.60]$ is $W_{22} < 1.8$. For $x \in [0.9850, 0.9900]$, $y \in [0.47, 0.60]$ is $W_{22} < 1.8$. For $x \in [0.9800, 0.9850]$ and, step by step, for $y \in [0.51, 0.56]$, $[0.56, 0.60]$, $[0.60, 0.65]$ is always $W_{22} < 1.8$.

For $x \in [0.975, 0.980]$ and, step by step, for $y \in [0.54, 0.60]$, $[0.60, 0.64]$, $[0.64, 0.7]$ is always $W_{22} < 1.8$.

For $x \in [0.970, 0.975]$, and, step by step, for $y \in [0.56, 0.61]$, $[0.61, 0.63]$, $[0.63, 0.65]$, $[0.65, 0.68]$ is always $W_{22} < 1.8$.

For $x \in [0.965, 0.970]$ and, step by step, for $y \in [0.58, 0.61]$, $[0.61, 0.63]$, $[0.63, 0.64]$, $[0.64, 0.65]$, $[0.65, 0.66]$, $[0.66, 0.67]$, $[0.67, 0.69]$, $[0.69, 0.75]$ is always $W_{22} < 1.8$.

For $x \in [0.960, 0.965]$ and, step by step, for $y \in [0.60, 0.62]$, $[0.62, 0.63]$, $[0.63, 0.635]$, $[0.635, 0.64]$, $[0.64, 0.644]$, $[0.644, 0.647]$, $[0.647, 0.65]$, $[0.65, 0.652]$, $[0.652, 0.654]$, $[0.654, 0.656]$, $[0.656, 0.658]$, $[0.658, 0.66]$, $[0.66, 0.662]$, $[0.662, 0.664]$, $[0.664, 0.666]$, $[0.666, 0.668]$, $[0.668, 0.670]$, $[0.670, 0.673]$, $[0.673, 0.677]$, $[0.677, 0.680]$, $[0.680, 0.685]$, $[0.685, 0.695]$, $[0.695, 0.72]$ is always $W_{22} < 1.8$.

For $x \in [0.955, 0.960]$ and, step by step, for $y \in [0.620, 0.630]$, $[0.630, 0.635]$, $[0.635, 0.638]$, $[0.638, 0.640]$, $[0.640, 0.641]$, $[0.641, 0.642]$, $[0.697, 0.698]$, $[0.698, 0.700]$, $[0.700, 0.703]$, $[0.703, 0.709]$, $[0.709, 0.724]$, $[0.724, 0.730]$ is always $W_{22} < 1.8$.

We do not exclude the region $x \in [0.9550, 0.9600]$, $y \in [0.642, 0.697]$ in this way, it is not effective.

We have

From (2): $\frac{\partial W_2}{\partial x} = \frac{x^2}{\sqrt[5]{(1-x^5-y^5)^4}}\left[(4x+3y)(y\sqrt[5]{(1-x^5-y^5)^4}+1-y^5)-5x^6-4x^5y\right]$ and $\frac{\partial W_2}{\partial y} = \frac{x^3}{\sqrt[5]{(1-x^5-y^5)^4}}\left[(x+2y)\sqrt[5]{(1-x^5-y^5)^4}+1-x^5-2y^5-xy^4\right]$.

$\frac{x^2}{\sqrt[5]{(1-x^5-y^5)^4}} > 0$ and $\frac{x^3}{\sqrt[5]{(1-x^5-y^5)^4}} > 0$, therefore, for every point $X = (x, y), x \in [a, b]$, $y \in [c, d]$ we have two inequalities:

$$(4x+3y)(y\sqrt[5]{(1-x^5-y^5)^4}+1-y^5)-x^5(5x+4y) \leq$$
$$\leq (4b+3d)(d\sqrt[5]{(1-a^5-c^5)^4}+1-c^5)-a^5(5a+4c)$$

and

$$(x+2y)\sqrt[5]{(1-x^5-y^5)^4}+1-x^5-2y^5-xy^4 \geq$$
$$\geq (a+2c)\sqrt[5]{(1-b^5-d^5)^4}+1-b^5-2d^5-bd^4$$

Denote

$$DW2x(a,b,c,d) = (4b+3d)(d\sqrt[5]{(1-a^5-c^5)^4}+1-c^5)-a^5(5a+4c),$$
$$DW2y(a,b,c,d) = (a+2c)\sqrt[5]{(1-b^5-d^5)^4}+1-b^5-2d^5-bd^4.$$

For $x \in [0.955, 0.960]$ and $y \in [0.642, 0.670]$ is $DW2x(a,b,c,d) < 0$ and, therefore, $\dfrac{\partial W_2}{\partial x} < 0$.

For $x \in [0.955, 0.960]$ and $y \in [0.670, 0.697]$ is also $DW2x(a,b,c,d) < 0$ and, therefore, $\dfrac{\partial W_2}{\partial x} < 0$.

Therefore, the asked maximum cannot be achieved for $x \in [0.955, 0.960]$.

For $x \in [0.950, 0.955]$ and, step by step, for $y \in [0.630, 0.636]$, $[0.636, 0.639]$, $[0.639, 0.640]$, $[0.640, 0.641]$, $[0.717, 0.718]$, $[0.718, 0.720]$, $[0.720, 0.725]$, $[0.725, 0.738]$, $[0.738, 0.750]$ is always $W_{22} < 1.8$.

We do not exclude the region $x \in [0.950, 0.955]$, $y \in [0.641, 0.717]$ in this way, it is not effective.

For $x \in [0.950, 0.955]$ and $y \in [0.641, 0.671]$ is $DW2y(a,b,c,d) > 0$ and, therefore, $\dfrac{\partial W_2}{\partial y} > 0$.

For $x \in [0.950, 0.955]$ and $y \in [0.671, 0.700]$ is $DW2x(a,b,c,d) < 0$ and, therefore, $\dfrac{\partial W_2}{\partial x} < 0$.

For $x \in [0.950, 0.955]$ and $y \in [0.700, 0.717]$ is $DW2x(a,b,c,d) < 0$ and, therefore, $\dfrac{\partial W_2}{\partial x} < 0$.

This implies that the asked maximum cannot be achieved for $x \in [0.950, 0.955]$.

For $x \in [0.9475, 0.9500]$ and, step by step, for $y \in [0.640, 0.649]$, $[0.649, 0.653]$, $[0.653, 0.655]$, $[0.655, 0.656]$, $[0.656, 0.657]$, $[0.719, 0.720]$, $[0.720, 0.722]$, $[0.722, 0.726]$, $[0.726, 0.735]$, $[0.735, 0.750]$ is always $W_{22} < 1.8$.

We do not exclude the region $x \in [0.9475, 0.9500]$, $y \in [0.657, 0.719]$ in this way, it is not effective.

For $x \in [0.9475, 0.9500]$ and $y \in [0.657, 0.684]$ is $DW2y(a,b,c,d) > 0$ and, therefore, $\dfrac{\partial W_2}{\partial y} > 0$.

For $x \in [0.9475, 0.9500]$ and $y \in [0.684, 0.719]$ is $DW2x(a,b,c,d) < 0$ and, therefore, $\dfrac{\partial W_2}{\partial x} < 0$.

This implies that the asked maximum cannot be achieved for $x \in [0.9475, 0.9500]$, see Figure 4.

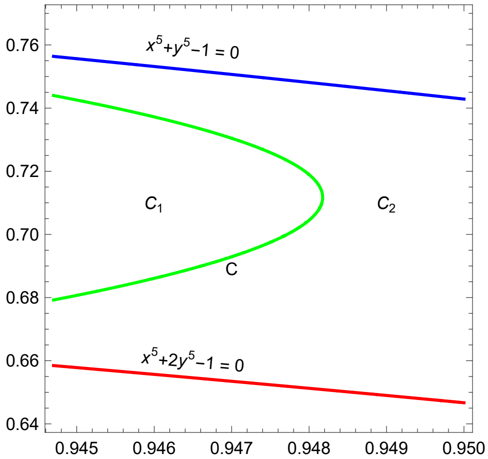


Figure 4. The Region M after the final reduction.

For $x \in [0.9445, 0.9475]$ and, step by step, for $y \in [0.650, 0.653]$, $[0.653, 0.655]$, $[0.655, 0.656]$, $[0.656, 0.657]$ is always $W_{22} < 1.8$.

For $x \in [0.9445, 0.9475]$ and $y \in [0.657, 0.690]$ is $DW2y(a,b,c,d) > 0$ and, therefore, $\dfrac{\partial W_2}{\partial y} > 0$.

For $x \in [0.9445, 0.9475]$ and $y \in [0.690, 0.700]$ is $DW2x(a,b,c,d) < 0$ and, therefore, $\dfrac{\partial W_2}{\partial x} < 0$.

For $x \in [0.9445, 0.9475]$ and, step by step, for $y \in [0.720, 0.726]$, $[0.726, 0.743]$, $[0.743, 0.760]$ is always $W_{11} < 1.8$.

So function (2) on the compact set $\bar{C}_2$ must achieve its maximum in some point of the curve C. It is the same point B as above.

We ask constrained maximum of the function

$$W(x,y) = x^5 + \frac{x^6}{y} \tag{3}$$

on the curve C

$$C(x,y) = x^5y^5 + y^{10} - y^5 + (x^2 - y^2)^5 = 0 \tag{4}$$

for $x \in [0.9445, 0.9475]$.

System of equations $\frac{\partial W}{\partial x}\frac{\partial C}{\partial y} - \frac{\partial W}{\partial y}\frac{\partial C}{\partial x} = 0$ and $C(x, y) = 0$ has the form

$$7x^6y^5 + 12xy^{10} - 6xy^5 + 5x^5y^6 + 10y^{11} - 5y^6 + (x^2 - y^2)^4(2x^3 - 10y^3 - 12xy^2) = 0,$$

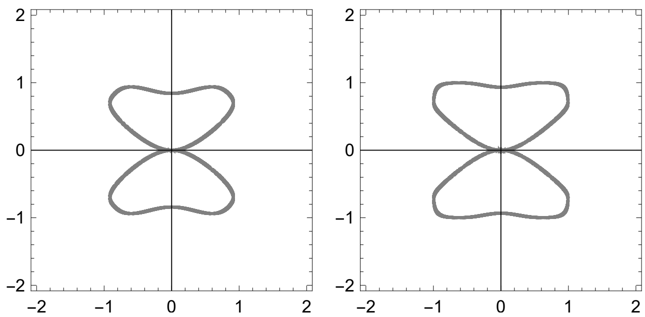
$$x^5y^5 + y^{10} - y^5 + (x^2 - y^2)^5 = 0.$$

The solution is $x \doteq 0.946629932, y \doteq 0.690148624$, and then $z \doteq 0.608279275$. □

If we generalize considerations from the proof, we will achieve the curve C: $x^dy^d + y^{2d} - y^d + (x^2 - y^2)^d = 0$, where d is dimension. The graph of the curve C depends on the parity of d, see Figures 5 and 6. Considering only the values $1 > x \geq y > 0$, the shape of the curve C is similar, regardless of parity, see Figure 2.

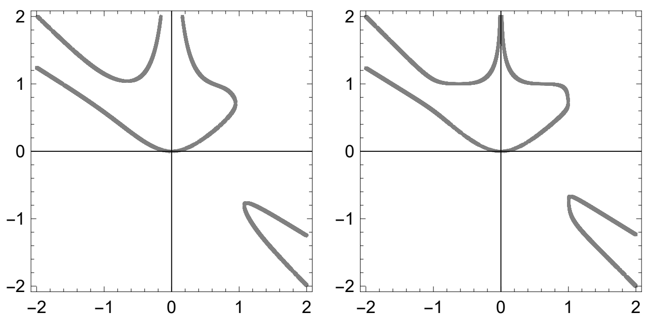
For $d \leq 10$ the asked maximum is achieved on the curve C. For dimensions 7, 9 and 10 the resultsare:

$$V_3(7) \doteq 2.05909680 \text{ and } x \doteq 0.978852925, y \doteq 0.703495386, z \doteq 0.658493716,$$
$$V_3(9) \doteq 2.21897778 \text{ and } x \doteq 0.991008397, y \doteq 0.704394561, z \doteq 0.689849087,$$
$$V_3(10) \doteq 2.27220126 \text{ and } x \doteq 0.993961280, y \doteq 0.702901846, z \doteq 0.702641521.$$



(**a**) Dimension 4 (**b**) Dimension 10

Figure 5. The curve C in even dimensions.



(**a**) Dimension 5 (**b**) Dimension 11

Figure 6. The curve C in odd dimensions.

Let P is intersection the constraint curve $x^d + 2y^d - 1 = 0$ and the curve C. If $d = 11$, then the constrained extreme on the curve C does not meet the required assumption $y \geq z$. Therefore, the asked maximum must be on the constraint curve to the left of point P or on the curve C above P, see Figure 7. The same situation occurs for $d = 12$ and $d = 13$.

$V_3(11) \doteq 2.31533581$ and $x \doteq 0.994989464$, $y = z \doteq 0.719809616$.
$V_3(12) \doteq 2.35315527$ and $x \doteq 0.995762712$, $y = z \doteq 0.734956999$,
$V_3(13) \doteq 2.38661963$ and $x \doteq 0.996369617$, $y = z \doteq 0.748358875$.

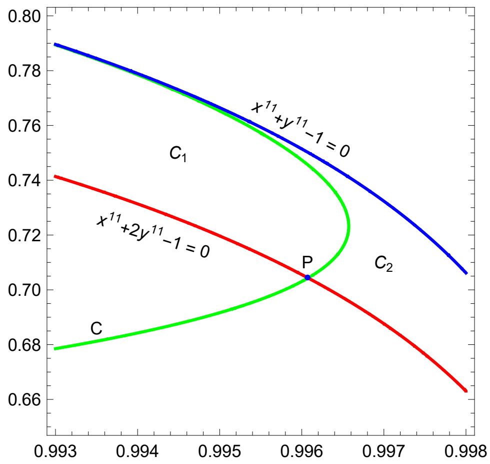


Figure 7. The regions C_1, C_2 and the curve C in 11-dimensional space.

3. Conclusions

The issue of packing squares is an old problem and even though there are multiple partial results, it remains unresolved. We investigated a modified problem: packing three cubes in 5-dimensional space. We also calculated results for dimensions $7, 9, 10, 11, 12, 13$.

Considering the previous results by [17–19], we can say that solution is located on the curve C for dimensions $4 \ldots 10$. It means, that there are two (different) packings that give (the same) the largest volume.

There seems to be only a single maximal packing for dimensions greater than 10. In this packing, two smallest cubes are the same. However, the paper confirms it only for dimensions 11, 12, 13.

There is a space for several improvements in our work: Is it possible to find a $V_3(d)$ without long numerical calculations? Is it true that two different maximum packings exist only for dimensions less than 11?

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