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Article

Regular and Intra-Regular Semigroups in Terms of *m*-Polar Fuzzy Environment

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Abstract: The central objective of the proposed work in this research is to introduce the innovative concept of an m-polar fuzzy set (m-PFS) in semigroups, that is, the expansion of bipolar fuzzy set (BFS). Our main focus in this study is the generalization of some important results of BFSs to the results of m-PFSs. This paper provides some important results related to m-polar fuzzy subsemigroups (m-PFSSs), m-polar fuzzy ideals (m-PFIs), m-polar fuzzy generalized bi-ideals (m-PFGBIs), m-polar fuzzy bi-ideals (m-PFBIs), m-polar fuzzy quasi-ideals (m-PFQIs) and m-polar fuzzy interior ideals (m-PFIIs) in semigroups. This research paper shows that every m-PFBI of semigroups is the m-PFGBI of semigroups, but the converse may not be true. Furthermore this paper deals with several important properties of m-PFIs and characterizes regular and intra-regular semigroups by the properties of m-PFIs and m-PFBIs.

Keywords: m-PF subsemigroups; m-PF generalized bi-ideals; m-PF bi-ideals; m-PF quasi-ideals; m-PF interior ideals

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1. Introduction

In 2014, Chen et al. [1] presented the m-PFS as an expansion of the BFS. The mathematical theories of a 2-polar fuzzy set and BFS are equivalent, and we can see that one connected to the other. The BFS is expanded into an m-PFS by applying the notion of one-to-one correspondence. Sometimes, different things are monitored in different ways. The m-PFS is effective in assigning degrees of membership to various objects in multi-polar data. The m-PFS gives only a positive degree of membership to each element. The m-PFS has an extensive range of implementations in real world problems related to the multi-agent, multi-objects, multi-polar information, multi-index and multi-attributes. This theory is applicable when a company decides to construct an item, a country elects its political leaders, or a group of friends wants to visit a country, with various options. It can be used in decision making, co-operative games, disease diagnosis, to discuss the confusions and conflicts of communication signals in wireless communications and as a model to define clusters or categorization and multi-relationships. In sum, an m-PFS on H is a mapping $I : H \to [0,1]^m$.

Here, we will make a model-based example on m-PFS, and use it to conveniently select an appropriate employee in a company. Here, the selection of an employee is based on 4-PFS with their four qualities, which are honesty, punctuality, communication skills, and being hardworking. Let $H = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of five employees in a company. We shall characterize them according to four qualities in the form of 4-PFS, given in Table 1:

Table 1. Table of qualities in persons with their membership values.

	Honesty	Punctual	Communication	Hardworking
a_1	0.6	0.5	0.8	1
a_2	1	0.8	0.5	0.4
a_3	0.5	1	1	0.8
a_4	0.8	0.5	1	0.7
a_5	1	0.5	0	0.6

Therefore, we attain a 4-PFS $\eta : H \to [0,1]^4$ of H such that

$$\begin{aligned}
\eta(a_1) &= (0.6, 0.5, 0.8, 1) \\
\eta(a_2) &= (1, 0.8, 0.5, 0.4) \\
\eta(a_3) &= (0.5, 1, 1, 0.8) \\
\eta(a_4) &= (0.8, 0.5, 1, 0.7) \\
\eta(a_5) &= (1, 0.5, 0, 0.6).
\end{aligned}$$

Figure 1 is the graphical representation of 4-PFS:

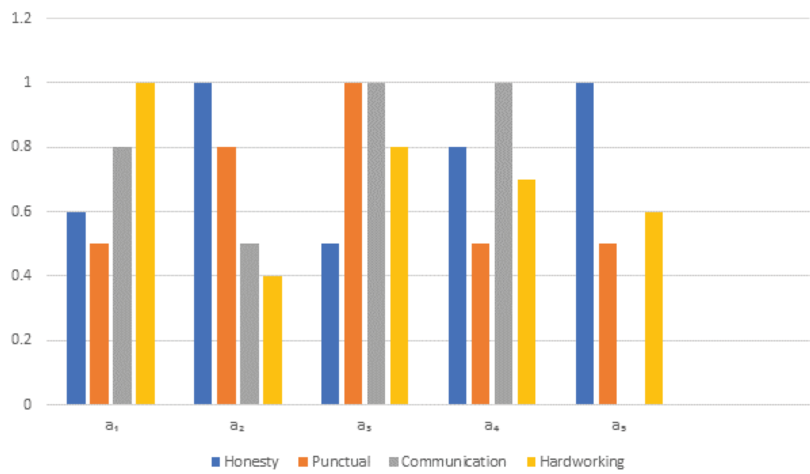

Figure 1. Graphical representation of 4-polar fuzzy subset.

Here, 1 represents good remarks, 0.5 represents average and 0 represents bad remarks. Similarly, we can solve any other problem with uncertainty in multiple directions.

Zhang [2] proposed that the function is mapped to the interval $[-1, 1]$ rather than $[0, 1]$ in BFS theory. Lee [3] coined the term bipolar fuzzy ideals. BFS is useful for solving uncertain problems with two poles of a situation: positive and negative pole. For more applications of BFS, see [4–9]. In medical science, environmental research, and engineering, we may find data or information that are ambiguous or complicated. All mathematical equations and techniques in classical mathematics are exact, they cannot deal with such problems. Many tools have been developed to deal with such issues. After extensive effort, Zadeh [10] was the first to propose fuzzy set theory as a solution to such complicated issues. This idea is used in a variety of areas, including logic, measure theory, topological space, ring theory, group theory and real analysis. The theory of fuzzy group was first intitated by Rosenfeld [11]. Kuroki [12] and Mordeson [13] have extensively explored fuzzy semigroups.

Semigroups are very useful in many applications containing dynamical systems, control problems, partial differential equations, sociology, stochastic differential equations, biology, etc. Some examples of semigroups are the collection of all mappings of a set, under the composition of functions (termed a full transformation monoid) and the set of natural numbers $\mathbb{N}$ under either addition or multiplication. The word "semigroup" was introduced to provide a title for some structures that were not groups but were created through the expansion of consequences. The proper semigroup theory was initiated by the working of Russian mathematician Anton Kazimirovich Suschkewitsch [14]. Quasi-ideals in semigroups were introduced by Otto Steinfeld [15].

The study of m-PF algebraic structures began with the concept of m-PF Lie subalgebras [16]. After that, the m-PF Lie ideals were studied in Lie algebras [17]. In 2017, Sarwar and Akram worked on new applications of m-PF matroids [18]. In 2019, Ahmad and Al-Masarwah introduced the concept of m-PF (commutative) ideals and m-polar (α, β)-fuzzy ideals in BCK/BCI-algebras [19,20]. To continue their work, they introduced a new aspect of generalized m-PF ideals and studied the normalization of m-PF subalgebras in [21,22]. Recently, Muhiuddin and Al-Kadi presented interval-valued m-PF BCK/BCI-Algebras [23]. Shabir et al. [24] studied regular and intra-regular semirings in terms of BFIs. Then, Bashir et al. [25,26] studied regular ordered ternary semigroups and semirings in terms of BFIs. Shabir et al. extended the work of [24], initiated the concept of m-PFIs in LA-semigroups and characterized the regular LA-semigroups according to the properties of these m-PFIs [27]. By extending the work of [24,27], the concept of m-PFIs in semigroups was introduced and characterizations of regular and intra-regular semigroups according to the properties of m-PFIs are given in this paper.

This paper is charaterized as follows: We present some basic concepts related to m-PFS in Section 2. The major part of this paper is Section 3, the m-PFSSs, m-PFIs (left, right), m-PFBIs, m-PFGBIs, m-PFQIs, m-PFIIs of semigroups are discussed with examples. In Section 4, the regular and intra-regular semigroups are characterized by the properties of m-PFIs. A comparison between this research and previous work is shown in Section 5. In Section 6, we also talk about the conclusions and future work.

The list of acronyms used in the research article is given in Table 2.

Table 2. List of acronyms.

Acronyms	Representation
BFS	Bipolar fuzzy set
BFIs	Bipolar fuzzy ideals
m-PFS	m-Polar fuzzy set
m-PFSs	m-Polar fuzzy subsets
m-PFSSs	m-Polar fuzzy subsemigroups
m-PFIs	m-Polar fuzzy ideals
m-PFGBIs	m-Polar fuzzy generalized bi-ideals
m-PFBIs	m-Polar fuzzy bi-ideals
m-PFQIs	m-Polar fuzzy quasi-ideals
m-PFIIs	m-Polar fuzzy interior ideals

2. Preliminaries

In this Section, we have studied the fundamental but essential definitions and preliminary findings based on semigroups that are important in their own right. These are necessary for later Sections.

If a groupoid $(P, \cdot)$ satisfies the associative property, then it is called a semigroup. Throughout this paper, P will denote the semigroup, unless specified otherwise. A non-empty subset H of P is called a subsemigroup of P if $ab \in H$ for every $a, b \in H$. In this paper, subsets mean non-empty subsets. A subset H of P is called a left ideal (resp. right ideal) of P if $PH \subseteq H$ (resp. $HP \subseteq H$). If H is left and right ideal, then H is called a two-sided ideal or ideal of P [28] .

A subset H of P is called a generalized bi-ideal of P if $HPH \subseteq H$. The subsemigroup H of P is called a bi-ideal of P if $HPH \subseteq H$. A subset H of P is called a quasi-ideal of P if $HP \cap PH \subseteq H$. The subsemigroup H of P is called an interior ideal of P if $(PH)P \subseteq H$ [28].

A fuzzy subset η of P is a mapping from P to closed interval $[0,1]$, *that is* $\eta : P \to [0,1]$ [10]. A bipolar fuzzy subset η of P is a mapping from P to closed interval $[-1,1]$ written as $\eta = (P, \eta^-, \eta^+)$, where $\eta^- : P \to [-1,0]$ and $\eta^+ : P \to [0,1]$. It can differentiate between unrelated and contrary components of fuzzy problems. A natural one-to-one correspondence exists among the BFS and 2-polar fuzzy set ($[0,1]^2$-set). When data for real world complex situations come from m factors ($m \geq 2$), then m-PFS is used to deal with such problems. An m-PFS (or a $[0,1]^m$-set) on P is a function $\eta = P \to [0,1]^m$. More generally, the m-PFS is the m-tuple of membership degree function of P that is $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$, where $\eta_\kappa : P \to [0,1]$ is the mapping for every $\kappa \in \{1,2,\ldots,m\}$. Here, $\mathbf{0} = (0,0,\ldots,0)$ is the smallest value in $[0,1]^m$ and $\mathbf{1} = (1,1,\ldots,1)$ is the largest value in $[0,1]^m$ [1].

The set of all m-PFSs of P is represented by $m(P)$. We define relation $\leq$ on $m(P)$ as follows: For any m-PFSs $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ and $m' = (m_1, m_2, \ldots, m_m)$ of P, $\eta \leq m'$ means that $\eta_\kappa(a) \leq m'_\kappa(a)$ for every $a \in P$ and $\kappa \in \{1,2,\ldots,m\}$. The symbols $\eta \wedge m'$ and $\eta \vee m'$ mean the following m-PFSs of P. $(\eta \wedge m')(a) = \eta(a) \wedge m'(a)$ and $(\eta \vee m')(a) = \eta(a) \vee m'(a)$ that is $(\eta_\kappa \wedge m'_\kappa)(a) = \eta_\kappa(a) \wedge m'_\kappa(a)$ for each $a \in P$ and $\kappa \in \{1,2,\ldots,m\}$; $(\eta_\kappa \vee m'_\kappa)(a) = \eta_\kappa(a) \vee m'_\kappa(a)$ for each $a \in P$ and $\kappa \in \{1,2,\ldots,m\}$. For two m-PFSs $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ and $m' = (m_1, m_2, \ldots, m_m)$, the product of $\eta \circ m' = (\eta_1 \circ m_1, \eta_2 \circ m_2, \ldots, \eta_m \circ m_m)$ is defined as

$$(\eta_\kappa \circ m'_\kappa)(a) = \begin{cases} \bigvee\limits_{a=st} \{\eta_\kappa(s) \wedge m_\kappa(t), & \text{if } a = st \text{ for some } s,t \in P; \\ 0, & \text{otherwise;} \end{cases}$$

for all $\kappa \in \{1,2,\ldots,m\}$. The next example shows the product of m-PFSs η and m' of P for $m = 4$.

Example 1. *Consider the semigroup $P = \{\imath, \jmath, \ell, \hbar\}$ given in Table 3.*

Table 3. Table of multiplication of P.

$\cdot$	$\imath$	$\jmath$	ℓ	$\hbar$
$\imath$	$\imath$	$\imath$	$\imath$	$\imath$
$\jmath$	$\imath$	$\imath$	$\imath$	$\imath$
ℓ	$\imath$	$\imath$	$\jmath$	$\imath$
$\hbar$	$\imath$	$\imath$	$\imath$	$\jmath$

We define 4-PFSs $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ and $m' = (m_1, m_2, m_3, m_4)$ as follows:

$\eta(\imath) = (0.2, 0.1, 0, 0.4)$, $\eta(\jmath) = (0.7, 0.5, 0.1, 0)$, $\eta(\ell) = (0.1, 0.3, 0.7, 0.4)$, $\eta(\hbar) = (0, 0, 0, 0.1)$ and

$m'(\imath) = (0.7, 0.3, 0, 0.4)$, $m'(\jmath) = (0.2, 0, 0, 0.1)$, $m'(\ell) = (0.2, 0.2, 0.4, 0)$, $m'(\hbar) = (0.2, 0.3, 0, 0)$.

By defintion, we obtain

$(\eta_1 \circ m_1)(\imath) = 0.7$, $(\eta_1 \circ m_1)(\jmath) = 0.1$, $(\eta_1 \circ m_1)(\ell) = 0$, $(\eta_1 \circ m_1)(\hbar) = 0$;
$(\eta_2 \circ m_2)(\imath) = 0.3$, $(\eta_2 \circ m_2)(\jmath) = 0.1$, $(\eta_2 \circ m_2)(\ell) = 0$, $(\eta_2 \circ m_2)(\hbar) = 0$;
$(\eta_3 \circ m_3)(\imath) = 0.1$, $(\eta_3 \circ m_3)(\jmath) = 0.4$, $(\eta_3 \circ m_3)(\ell) = 0$, $(\eta_3 \circ m_3)(\hbar) = 0$;
$(\eta_4 \circ m_4)(\imath) = 0.4$, $(\eta_4 \circ m_4)(\jmath) = 0.0$, $(\eta_4 \circ m_4)(\ell) = 0$, $(\eta_4 \circ m_4)(\hbar) = 0$.

Hence, the product of $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ and $m' = (m_1, m_2, m_3, m_4)$ is defined by

$(\eta \circ m')(\imath) = (0.7, 0.3, 0.1, 0.4)$, $(\eta \circ m')(\jmath) = (0.1, 0.1, 0.4, 0)$, $(\eta \circ m')(\ell) = (0,0,0,0)$, $(\eta \circ m')(\hbar) = (0,0,0,0)$.

Definition 1. *Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ be an m-PFS of P.*

1. *Define* $\eta_t = \{a \in P | \eta(a) \geq t\}$ *for all* t*, where* $t = (t_1, t_2, \ldots, t_m) \in (0,1]^m$*, that is,* $\eta_\kappa(a) \geq t_\kappa$ *for all* $\kappa \in \{1, 2, \ldots, m\}$*. Then,* η_t *is called t-cut or a level set.*
2. *The support of* $\eta : P \to [0,1]^m$ *is defined as the set* $Supp(\eta) = \{a \in P | \eta(a) > (0, 0, \ldots, 0)$ *m-tuple*$\}$*, that is* $\eta_\kappa(a) > 0$ *for all* $\kappa \in \{1, 2, \ldots, m\}$*.*

Definition 2. *An m-PFS* $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ *of* P *is called an m-PFSS of* P *if, for all* $a, b \in P$*,* $\eta(ab) \geq \min\{\eta(a), \eta(b)\}$*, that is,* $\eta_\kappa(ab) \geq \min\{\eta_\kappa(a), \eta_\kappa(b)\}$ *for all* $\kappa \in \{1, 2, \ldots, m\}$*.*

Definition 3. *An m-PFS* $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ *of* P *is called an m-PFI left (resp. right) of* P *for all* $a, b \in P$*,* $\eta(ab) \geq \eta(b)$ *(resp.* $\eta(ab) \geq \eta(a)$*), that is* $\eta_\kappa(ab) \geq \eta_\kappa(b)$ *(resp.* $\eta_\kappa(ab) \geq \eta_\kappa(a)$*) for all* $\kappa \in \{1, 2, \ldots, m\}$*.*

An m-PFS η of P is called an m-PFI of P if η is both an m-PFI (left) and m-PFI (right) of P.

The example given below is of 4-PFI of P.

Example 2. *Let* $P = \{\imath, \jmath, \ell, \hbar\}$ *be a semigroup given in Table 4.*

Table 4. Table of multiplication of P.

$\cdot$	$\imath$	$\jmath$	ℓ	$\hbar$
$\imath$	$\imath$	$\imath$	$\imath$	$\imath$
$\jmath$	$\imath$	$\jmath$	ℓ	$\imath$
ℓ	$\imath$	$\imath$	$\imath$	$\imath$
$\hbar$	$\imath$	$\hbar$	$\imath$	$\imath$

We define a 4-PFS $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ of P as follows:

$\eta(\imath) = (0.7, 0.6, 0.6, 0.4)$, $\eta(\jmath) = (0.2, 0, 0, 0.1)$, $\eta(\ell) = (0.5, 0.4, 0.3, 0.1)$, $\eta(\hbar) = (0.5, 0.4, 0.3, 0.1)$.

Clearly, $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ is both 4-PFIs (left and right) of P. Hence η is a 4-PFI of P.

Definition 4. *Let a subset* H *of* P*. Then, the m-polar characteristic function* $C_H : H \to [0,1]^m$ *is defined as*

$$C_H(h) = \begin{cases} (1, 1, \ldots, 1), & m\text{-tuple if } h \in H; \\ (0, 0, \ldots, 0), & m\text{-tuple if } h \notin H. \end{cases}$$

3. Characterization of Semigroups by *m*-Polar Fuzzy Sets

This is the most essential portion, because here we make our major contributions. With the help of several lemmas, theorems, and examples, the notions of m-PFSSs and m-PFIs of semigroups are explained in this section. We have proved that every m-PFBI of P is m-PFGBI, but the converse does not hold. For LA-semigroups, Shabir et al. [27] has proved this result. We have generalized the results in Shabir et al. [27] for semigroups. In whole paper, δ is an m-PFS of P that maps each element of P on (1, 1, . . . , 1).

Lemma 1. *Consider two subsets* H *and* I *of* P*. Then*

1. $C_H \wedge C_I = C_{H \cap I}$;
2. $C_H \vee C_I = C_{H \cup I}$;
3. $C_H \circ C_I = C_{HI}$.

Proof. The proof of (1) and (2) are obvious.

(3): Case 1: Let $a \in HI$. This implies that $a = hi$ for some $h \in H$ and $i \in I$. Therefore, $C_{HI}(a) = (1, 1, \ldots, 1)$. Since $h \in H$ and $i \in I$, we have $C_H(h) = (1, 1, \ldots, 1)$ or $C_I(i) = (1, 1, \ldots, 1)$. Now,

$$\begin{aligned}(C_H \circ C_I)(a) &= \bigvee_{a=bc} \{C_H(b) \wedge C_I(c)\} \\ &\geq C_H(h) \wedge C_I(i) \\ &= (1,1,\ldots,1).\end{aligned}$$

Therefore, $C_H \circ C_I = C_{HI}$. Case 2: If $a \notin HI$. This implies that $C_{HI}(a) = (0,0,\ldots,0)$, since $a \notin hi$ for every $h \in H$ and $i \in I$. Therefore

$$\begin{aligned}(C_H \circ C_I)(a) &= \bigvee_{a=hi} \{C_H(h) \wedge C_I(i)\} \\ &= (0,0,\ldots,0).\end{aligned}$$

Hence $C_H \circ C_I = C_{HI}$. □

Lemma 2. *Let H be a subset of P. Then, the given statements hold.*

1. *H is a subsemigroup of P if, and only if, C_H is an m-PFSS of P;*
2. *H is a left ideal (resp. right) of P if and only if C_H is an m-PFI left (resp. right) of P.*

Proof. (1) Consider H as the subsemigroup of P. We have to show that $C_H(ab) \geq C_H(a) \wedge C_H(b)$ for all $a, b \in P$. Now, we consider some cases:

Case 1: Let $a, b \in H$. Then, $C_H(a) = C_H(b) = (1,1,\ldots,1)$. As H is a subsemigroup of P, so $ab \in H$ implies that $C_H(ab) = (1,1,\ldots,1)$. Hence $C_H(ab) \geq C_H(a) \wedge C_H(b)$.

Case 2: Let $a \in H$, $b \notin H$. Then, $C_H(a) = (1,1,\ldots,1), C_H(b) = (0,0,\ldots,0)$. Hence, $C_H(ab) \geq (0,0,\ldots,0) = C_H(a) \wedge C_H(b)$.

Case 3: Let $a, b \notin H$. Then, $C_H(a) = C_H(b) = (0,0,\ldots,0)$. Clearly, $C_H(ab) \geq (0,0,\ldots,0) = C_H(a) \wedge C_H(b)$.

Case 4: Let $a \notin H$, $b \in H$. Then, $C_H(a) = (0,0,\ldots,0)$ and $C_H(b) = (1,1,\ldots,1)$. Clearly, $C_H(ab) \geq (0,0,\ldots,0) = C_H(a) \wedge C_H(b)$.

Conversely, let C_H be an m-PFSS of P. Let $a, b \in H$. Then, $C_H(a) = C_H(b) = (1,1,\ldots,1)$. By definition, $C_H(ab) \geq C_H(a) \wedge C_H(b) = (1,1,\ldots,1) \wedge (1,1,\ldots,1) = (1,1,\ldots,1)$, we have $C_H(ab) = (1,1,\ldots,1)$. This implies that $ab \in H$, that is H is a subsemigroup of P.

(2) Suppose that H is the left ideal of P. We have to show that $C_H(ab) \geq C_H(b)$ for every $a, b \in P$. Now, consider the two cases:

Case 1: Let $b \in H$ and $a \in P$. Then, $C_H(b) = (1,1,\ldots,1)$. Since H is a left ideal of P, $ab \in H$ implies that $C_H(ab) = (1,1,\ldots,1)$. Hence $C_H(ab) \geq C_H(b)$.

Case 2: Let $b \notin H$ and $a \in P$. Then, $C_H(b) = (0,0,\ldots,0)$. Clearly, $C_H(ab) \geq C_H(b)$.

Conversely, let C_H be an m-PFI (left) of P. Let $a \in P$ and $b \in H$. Then, $C_H(b) = (1,1,\ldots,1)$. By definition, $C_H(ab) \geq C_H(b) = (1,1,\ldots,1)$, we have $C_H(ab) = (1,1,\ldots,1)$. This implies that $ab \in H$, that is H is a left ideal of P.

In the same way, we can show that H is right ideal of P if, and only if, C_H is an m-PFI (right) of P. Therefore, H is an ideal of P if, and only if, C_H is an m-PFI of P. □

Lemma 3. *For m-PFS $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ of P, the following properties hold.*

1. *η is an m-PFSS of P if, and only if, $\eta \circ \eta \leq \eta$;*
2. *η is an m-PFI (left) of P if, and only if, $\delta \circ \eta \leq \eta$;*
3. *η is an m-PFI (right) of P if, and only if, $\eta \circ \delta \leq \eta$;*
4. *η is an m-PFI of P if, and only if, $\delta \circ \eta \leq \eta$ and $\eta \circ \delta \leq \eta$, where δ is the m-PFS of P that maps each element of P on $(1,1,\ldots,1)$.*

Proof. (1) Assume that $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ is an m-PFSS of P, that is, $\eta_\kappa(ab) \geq \eta_\kappa(a) \wedge \eta_\kappa(b)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Let $p \in P$. If p is not expressible as $p = ab$ for some $a, b \in P$;

then, $(\eta \circ \eta)(p) = 0$. Hence, $(\eta \circ \eta)(p) \leq \eta(p)$. However, if p is expressible as $p = ab$ for some $a, b \in P$, then

$$\begin{aligned}
(\eta_\kappa \circ \eta_\kappa)(p) &= \bigvee_{p=ab} \{\eta_\kappa(a) \wedge \eta_\kappa(b)\} \\
&\leq \bigvee_{p=ab} \{\eta_\kappa(ab)\} \\
&= \eta_\kappa(p) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Hence, $\eta \circ \eta \leq \eta$. Conversely, let $\eta \circ \eta \leq \eta$ and $a, b \in P$. Then

$$\begin{aligned}
\eta_\kappa(ab) &\geq (\eta_\kappa \circ \eta_\kappa)(ab) \\
&= \bigvee_{ab=uv} \{\eta_\kappa(u) \wedge \eta_\kappa(v)\} \\
&\geq \eta_\kappa(a) \wedge \eta_\kappa(b) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Hence, $\eta_\kappa(ab) \geq \eta_\kappa(a) \wedge \eta_\kappa(b)$. Thus, η is m-PFSS of P.

(2) Assume that $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ is m-PFI (left) of P, that is, $\eta_\kappa(ab) \geq \eta_\kappa(b)$ for all $\kappa \in \{1, 2, \ldots, m\}$ and $a, b \in P$. Let $p \in P$. If p is not expressible as $p = ab$ for some $a, b \in P$, then $(\delta \circ \eta)(p) = 0$. Hence, $\delta \circ \eta \leq \eta$. However, if p is expressible as $p = ab$ for some $a, b \in P$, then

$$\begin{aligned}
(\delta_\kappa \circ \eta_\kappa)(p) &= \bigvee_{p=ab} \{\delta_\kappa(a) \wedge \eta_\kappa(b)\} \\
&= \bigvee_{p=ab} \{\eta_\kappa(b)\} \\
&\leq \bigvee_{p=ab} \eta_\kappa(ab) \\
&= \eta_\kappa(p) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Hence $\delta \circ \eta \leq \eta$. Conversely, let $\delta \circ \eta \leq \eta$ and $a, b \in P$. Then,

$$\begin{aligned}
\eta_\kappa(ab) &\geq (\delta_\kappa \circ \eta_\kappa)(ab) \\
&= \bigvee_{ab=uv} \{\delta_\kappa(u) \wedge \eta_\kappa(v)\} \\
&\geq \{\delta_\kappa(a) \wedge \eta_\kappa(b)\} \\
&= \eta_\kappa(b) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Hence, $\eta(ab) \geq \eta(b)$. Thus, η is m-PFI (left) of P.

(3) This can be proved similarly to the proof of part (2) of Lemma 3.

(4) The proof of this follows from parts (2) and (3) of Lemma 3. □

Lemma 4. *The given statements are true in P.*

1. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ and $m' = (m_1, m_2, \ldots, m_m)$ be two m-PFSSs of P. Then, $\eta \wedge m'$ is also an m-PFSS of P;
2. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ and $m' = (m_1, m_2, \ldots, m_m)$ be two m-PFIs of P. Then, $\eta \wedge m'$ is also an m-PFI of P.

Proof. Straightforward. □

Proposition 1. *Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ be an m-PFS of P. Then, η is an m-PFSS (resp. m-PFI) of P if, and only if, $\eta_t = \{a \in P | \eta(a) \geq t\} \neq \phi$ is a subsemigroup (resp. ideal) of P for all $t \in (t_1, t_2, \ldots, t_m) \in (0, 1]^m$.*

Proof. Let η be an m-PFSS of P. Let $a, b \in \eta_t$. Then, $\eta_\kappa(a) \geq t_\kappa$ and $\eta_\kappa(b) \geq t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$. As η is an m-PFSS of P, this implies $\eta_\kappa(ab) \geq \eta_\kappa(a) \wedge \eta_\kappa(b) \geq t_\kappa \wedge t_\kappa = t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$. Therefore, $ab \in \eta_t$. Then η_t is a subsemigroup of P.

Conversely, let $\eta_t \neq \phi$ be a subsemigroup of P. On the contrary, let us consider that η is not an m-PFSS of P. Suppose $a, b \in P$ such that $\eta_\kappa(ab) < \eta_\kappa(a) \wedge \eta_\kappa(b)$ for some $\kappa \in \{1, 2, \ldots, m\}$. Take $t_\kappa = \eta_\kappa(a) \wedge \eta_\kappa(b)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Then, $a, b \in \eta_t$ but $ab \notin \eta_t$, there is a contradiction. Hence, $\eta_\kappa(ab) \geq \eta_\kappa(a) \wedge \eta_\kappa(b)$. Thus, η is an m-PFSS of P. Other cases can be proved on the same lines. □

Now, we define the m-PFGBI of a semigroup.

Definition 5. *An m-PFS $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ of P is called an m-PFGBI of P if for all $a, b, c \in P$, $\eta(abc) \geq \eta(a) \wedge \eta(c)$, that is $\eta_\kappa(abc) \geq \eta_\kappa(a) \wedge \eta_\kappa(c)$ for all $\kappa \in \{1, 2, \ldots, m\}$.*

Lemma 5. *A subset H of P is generalized bi-ideal of P if and only if C_H is an m-PFGBI of P.*

Proof. This Lemma 5 can be proved similarly to the proof of Lemma 2. □

Lemma 6. *An m-PFS η of P is m-PFGBI of P if and only if, $\eta \circ \delta \circ \eta \leq \eta$, where δ is the m-PFS of P that maps each element of P on $(1, 1, \ldots, 1)$.*

Proof. Suppose $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ is the m-PFGBI of P, that is, $\eta_\kappa(abc) \geq \eta_\kappa(a) \wedge \eta_\kappa(c)$ for all $\kappa \in \{1, 2, \ldots, m\}$ and $a, b, c \in P$. Let $p \in P$. If p is not expressible as $p = ab$ for some $a, b \in P$, then $(\eta \circ \delta \circ \eta)(p) = 0$. Hence, $\eta \circ \delta \circ \eta \leq \eta$. However, if p is expressible as $p = ab$ for some $a, b \in P$. Then

$$
\begin{aligned}
(\eta_\kappa \circ \delta_\kappa \circ \eta_\kappa)(p) &= \bigvee_{p=ab} \{(\eta_\kappa \circ \delta_\kappa)(a) \wedge \eta_\kappa(b)\} \\
&= \bigvee_{p=ab} \{ \bigvee_{a=uv} \{\eta_\kappa(u) \wedge \delta_\kappa(v)\} \wedge \eta_\kappa(b)\} \\
&= \bigvee_{p=ab} \{ \bigvee_{a=uv} \{\eta_\kappa(u) \wedge \eta_\kappa(b)\}\} \\
&\leq \bigvee_{p=ab} \{ \bigvee_{a=uv} \{\eta_\kappa(uv)b)\}\} \\
&= \bigvee_{p=ab} \{\eta_\kappa(ab)\} \text{ for all } \kappa \in \{1, 2, \ldots, m\}. \\
&= \eta_\kappa(p) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}
$$

Hence, $\eta \circ \delta \circ \eta \leq \eta$. Conversely, let $\eta \circ \delta \circ \eta \leq \eta$ and $a, b, c \in P$. Then,

$$
\begin{aligned}
\eta_\kappa(abc) &\geq (\eta_\kappa \circ \delta_\kappa) \circ \eta_\kappa)((ab)c) \\
&= \bigvee_{(ab)c=uv} \{(\eta_\kappa \circ \delta_\kappa)(u) \wedge \eta_\kappa(v)\} \\
&\geq (\eta_\kappa \circ \delta_\kappa)(ab) \wedge \eta_\kappa(c) \\
&= \bigvee_{(ab)=xy} \{(\eta_\kappa(x) \wedge \delta_\kappa)(y))\} \wedge \eta_\kappa(c) \\
&= \{(\eta_\kappa(a) \wedge \delta_\kappa)(b))\} \wedge \eta_\kappa(c) \\
&= \eta_\kappa(a) \wedge \eta_\kappa(c) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}
$$

Hence, $\eta(abc) \geq \eta(a) \wedge \eta(c)$. Thus, η is m-PFGBI of P. □

Proposition 2. *Assume that $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ is an m-PFS of P. Then, η is an m-PFGBI of P if, and only if, $\eta_t = \{a \in P | \eta(a) \geq t\} \neq \phi$ is a generalized bi-ideal of P for all $t = (t_1, t_2, \ldots, t_m) \in (0, 1]^m$.*

Proof. Let η be an m-PFGBI of P. Let $a, c \in \eta_t$ and $b \in P$. Then, $\eta_\kappa(a) \geq t_\kappa$ and $\eta_\kappa(c) \geq t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$. Since η is m-PFGBI of P, we have $\eta_\kappa(abc) \geq \eta_\kappa(a) \wedge \eta_\kappa(c) \geq t_\kappa \wedge t_\kappa = t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$. Therefore, $abc \in \eta_t$. That is η_t is a GBI of P.

Conversely, assuming that $\eta_t \neq \phi$ is a GBI of P. On the contrary, assume that η is not m-PFGBI of P. Suppose $a, b, c \in P$, such that $\eta_\kappa(abc) < \eta_\kappa(a) \wedge \eta_\kappa(c)$ for some $\kappa \in \{1, 2, \ldots, m\}$. Take $t_\kappa = \eta_\kappa(a) \wedge \eta_\kappa(c)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Then, $a, c \in \eta_t$ but $abc \notin \eta_t$, which is a contradiction. Hence, $\eta_\kappa(abc) \geq \eta_\kappa(a) \wedge \eta_\kappa(c)$, that is, η is m-PFGBI of P. □

Next, we define the m-PFBI of a semigroup.

Definition 6. *A subsemigroup $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ of P is called an m-PFBI of P if for all $a, b, c \in P, \eta(abc) \geq \eta(a) \wedge \eta(c)$ that is, $\eta_\kappa(abc) \geq \eta_\kappa(a) \wedge \eta_\kappa(c)$ for all $\kappa \in \{1, 2, \ldots, m\}$.*

Lemma 7. *A subset H of P is a bi-ideal of P if, and only if, C_H is an m-PFBI of P.*

Proof. Follows from Lemmas 2 and 5. □

Lemma 8. *An m-PFSS η of P is an m-PFBI of P if and only if, $\eta \circ \delta \circ \eta \leq \eta$, where δ is the m-PFS of P, which maps each element of P on $(1, 1, \ldots, 1)$.*

Proof. Follows from Lemma 6. □

Proposition 3. *Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ be a subsemigroup of P. Then η is an m-PFBI of P if and only if, $\eta_t = \{a \in P | \eta(a) \geq t\} \neq \phi$ is a bi-ideal of P for all $t = (t_1, t_2, \ldots, t_m) \in (0, 1]^m$.*

Proof. Follows from Proposition 2. □

Remark 1. *Every m-PFBI of P is an m-PFGBI of P.*

The Example 3 illustrates that the converse of above Remark may not be true.

Example 3. *Let $P = \{\imath, \jmath, \ell, \hbar\}$ be a semigroup given in Table 5.*

Table 5. Table of multiplication of P.

$\cdot$	$\imath$	$\jmath$	ℓ	$\hbar$
$\imath$	$\imath$	$\imath$	$\imath$	$\imath$
$\jmath$	$\imath$	$\imath$	$\imath$	$\imath$
ℓ	$\imath$	$\imath$	$\jmath$	$\imath$
$\hbar$	$\imath$	$\imath$	$\imath$	$\jmath$

We define a 4-PFS $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ of P as follows: $\eta(\imath) = (0.1, 0.3, 0.3, 0.4)$, $\eta(\jmath) = (0, 0, 0, 0)$, $\eta(\ell) = (0, 0, 0, 0)$, $\eta(\hbar) = (0.5, 0.6, 0.7, 0.8)$. Then, simple calculations show that the η is a 4-PFGBI of P.

Now, $\eta(\jmath) = \eta(\hbar\hbar) = (0, 0, 0, 0) \ngeq (0.5, 0.6, 0.7, 0.8) = \eta(\hbar) \wedge \eta(\hbar)$. Therefore, η is not a bi-ideal of P. Next, we define the m-PFQI of a semigroup.

Definition 7. *An m-PFS $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ of P is called an m-PFQI of P if $(\eta \circ \delta) \wedge (\delta \circ \eta) \leq \eta$, that is $(\eta_\kappa \circ \delta_\kappa) \wedge (\delta_\kappa \circ \eta_\kappa) \leq \eta_\kappa$, for all $\kappa \in \{1, 2, \ldots, m\}$.*

Lemma 9. *A subset H of P is a quasi ideal of P if and only if C_H is an m-PFQI of P.*

Proof. Let H be a quasi ideal of P, that is $HP \cap PH \subseteq H$. We show that $(C_H \circ \delta) \wedge (\delta \circ C_H) \leq C_H$, that is, $((C_H \circ \delta) \wedge (\delta \circ C_H))(h) \leq C_H(h)$ for every $h \in P$. We study the following cases:

Case 1: If $h \in H$ then $C_H(h) = (1, 1, \ldots, 1) \geq ((C_H \circ \delta) \wedge (\delta \circ C_H))(h)$. Hence $(C_H \circ \delta) \wedge (\delta \circ C_H) \leq C_H$.

Case 2: If $h \notin H$ then $h \notin HP \cap PH$. This implies that $h \neq bc$ and $h \neq fe$ for some $b \in H, c \in P, f \in P$ and $e \in H$. Therefore, either $(C_H \circ \delta)(h) = (0, 0, \ldots, 0)$ or $(\delta \circ C_H)(h) = (0, 0, \ldots, 0)$ that is $((C_H \circ \delta) \wedge (\delta \circ C_H))(h) = (0, 0, \ldots, 0) \leq C_H(h)$. Hence $(C_H \circ \delta) \wedge (\delta \circ C_H) \leq C_H$.

Conversely, let $n \in HP \cap PH$. Then $n = be$ and $n = af$, where $a, e \in P$ and $b, f \in H$. Since C_H is an m-PFQI of P, we have

$$\begin{aligned} C_H(n) &\geq ((C_H \circ \delta) \wedge (\delta \circ C_H))(n) \\ &= (C_H \circ \delta)(n) \wedge (\delta \circ C_H)(n) \\ &= \{\bigvee_{n=wv} \{(C_H(w) \wedge \delta(v)\}\} \wedge \{\bigvee_{n=pq} \{\delta(p) \wedge C_H(q)\}\} \\ &\geq \{C_H(b) \wedge \delta(e)\} \wedge \{\delta(a) \wedge C_H(f)\} \text{ since } n = be \text{ and } n = af \\ &= (1, 1, \ldots, 1). \end{aligned}$$

Therefore, $C_H(n) = (1, 1, \ldots, 1)$. Hence, $n \in H$. □

Proposition 4. *An m-PFS $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ of P is an m-PFQI of P if, and only if, $\eta_t = \{a \in P | \eta(a) \geq t\} \neq \phi$ is a quasi ideal of P for all $t = (t_1, t_2, \ldots, t_m) \in (0, 1]^m$.*

Proof. Let η is an m-PFQI of P. To show that $\eta_t P \cap P\eta_t \subseteq \eta_t$. Let $n \in \eta_t P \cap P\eta_t$. Then, $n \in \eta_t P$ and $n \in P\eta_t$. Therefore, $n = ba$ and $n = sd$ for some $a, s \in P$ and $b, d \in \eta_t$. Therefore, $\eta_\kappa \geq t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$.

Now

$$\begin{aligned} (\eta_\kappa \circ \delta_\kappa)(n) &= \bigvee_{n=uv} \{\eta_\kappa(u) \wedge \delta_\kappa(v)\} \\ &\geq \eta_\kappa(b) \wedge \delta_\kappa(a) \text{ because } n = ba \\ &= \eta_\kappa(b) \wedge 1 \\ &= \eta_\kappa(b) \\ &\geq t_\kappa. \end{aligned}$$

So

$$(\eta_\kappa \circ \delta_\kappa)(n) \geq t_\kappa \text{ for all } \kappa \in \{1, 2, \ldots, m\}.$$

Now

$$\begin{aligned} (\delta_\kappa \circ \eta_\kappa)(n) &= \bigvee_{n=uv} \{\delta_\kappa(u) \wedge \eta_\kappa(v)\} \\ &\geq \delta_\kappa(s) \wedge \eta_\kappa(d) \text{ because } n = sd \\ &= 1 \wedge \eta_\kappa(d) \\ &= \eta_\kappa(d) \\ &\geq t_\kappa. \end{aligned}$$

So

$$(\delta_\kappa \circ \eta_\kappa)(n) \geq t_\kappa \text{ for all } \kappa \in \{1, 2, \ldots, m\}.$$

Therefore, $((\eta_\kappa \circ \delta_\kappa) \wedge (\delta_\kappa \circ \eta_\kappa))(n) = (\eta_\kappa \circ \delta_\kappa)(n) \wedge (\delta_\kappa \circ \eta_\kappa)(n) \geq t_\kappa \wedge t_\kappa = t_\kappa$ for all $\kappa \in \{1, 2, \ldots, m\}$. So, $((\eta \circ \delta) \wedge (\delta \circ \eta))(n) \geq t$. Since $\eta(n) \geq ((\eta \circ \delta) \wedge (\delta \circ \eta))(n) \geq t$, so $n \in \eta_t$. Hence, η_t is a quasi ideal of P.

Conversely, consider that η is not quasi ideal of P. Let $n \in P$ be such that $\eta_\kappa(n) < (\eta_\kappa \circ \delta_\kappa)(n) \wedge (\delta_\kappa \circ \eta_\kappa)(n)$ for some $\kappa \in \{1, 2, \ldots, m\}$. Choose $t_\kappa \in (0, 1]$, such that $t_\kappa = (\eta_\kappa \circ \delta_\kappa)(n) \wedge (\delta_\kappa \circ \eta_\kappa)(n)$ for all $\kappa \in \{1, 2, \ldots, m\}$. This implies that $n \in (\eta_\kappa \circ \delta_\kappa)_{t_\kappa}$ and $n \in (\delta_\kappa \circ \eta_\kappa)_{t_\kappa}$ but $n \notin (\eta_\kappa)_{t_\kappa}$ for some κ. Hence, $n \in (\eta \circ P)_t$ and $n \in (P \circ \eta)_t$ but $n \notin (\eta)_t$, which is a contradiction. Hence, $(\eta \circ \delta) \wedge (\delta \circ \eta) \leq \eta$. □

Lemma 10. *Every m-PF one-sided ideal of P is an m-PFQI of P.*

Proof. This proof follows from Lemma 3. □

In the next example, it is shown that the converse of the above Lemma may not be true.

Example 4. *Consider the semigroup $P = \{\imath, \jmath, \ell\}$ given in Table 6.*

Table 6. Table of multiplication of P.

$\cdot$	$\imath$	$\jmath$	ℓ
$\imath$	$\imath$	$\jmath$	ℓ
$\jmath$	$\jmath$	$\jmath$	$\jmath$
ℓ	ℓ	$\jmath$	ℓ

Define a 3-PFS $\eta = (\eta_1, \eta_2, \eta_3)$ of P as follows: $\eta(\imath) = (0.3, 0.3, 0.4)$, $\eta(\jmath) = (0.7, 0.8, 0.9)$, $\eta(\ell) = (0, 0, 0)$.

Then, simple calculations show that $\eta_\jmath$ is QI of P. Therefore, by using Proposition 4, η is 3-PFQI of P. Now,

$\eta(\ell) = \eta(\imath\ell) = (0, 0, 0) \ngeq \eta(\imath) = (0.3, 0.3, 0.4)$. So η is not 3-PFI (right) of P.

Lemma 11. *Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ and $m = (m_1, m_2, \ldots, m_m)$ be two m-PFI(right) and m-PFI(left) of P, respectively. Then $\eta \wedge m$ is m-PFQI of P.*

Proof. Let $n \in P$. If $n \neq bt$ for some $b, t \in P$. Then, $((\eta \wedge m) \circ \delta) \wedge (\delta \circ (\eta \wedge m)) \leq (\eta \wedge m)$. If $n = ab$ for some $a, b \in P$, then

$$\begin{aligned}
(((\eta_\kappa \wedge m_\kappa) \circ \delta_\kappa) \wedge (\delta_\kappa \circ (\eta_\kappa \wedge m_\kappa)))(n) &= ((\eta_\kappa \wedge m_\kappa) \circ \delta_\kappa)(n) \wedge (\delta_\kappa \circ (\eta_\kappa \wedge m_\kappa))(n) \\
&= \bigvee_{n=ab} \{(\eta_\kappa \wedge m_\kappa)(a) \wedge \delta_\kappa(b)\} \wedge \bigvee_{n=ab} \{\delta_\kappa(a) \wedge (\eta_\kappa \wedge m_\kappa)(b)\} \\
&= \bigvee_{n=ab} \{(\eta_\kappa \wedge m_\kappa)(a)\} \wedge \bigvee_{n=ab} \{(\eta_\kappa \wedge m_\kappa)(b)\} \\
&= \bigvee_{n=ab} \{(\eta_\kappa \wedge m_\kappa)(a) \wedge (\eta_\kappa \wedge m_\kappa)(b)\} \\
&= \bigvee_{n=ab} \{(\eta_\kappa(a) \wedge m_\kappa(a)) \wedge (\eta_\kappa(b) \wedge m_\kappa(b))\} \\
&\leq \bigvee_{n=ab} \{\eta_\kappa(a) \wedge m_\kappa(b)\} \\
&\leq \bigvee_{n=ab} \{\eta_\kappa(ab) \wedge m_\kappa(ab)\} \\
&= \bigvee_{n=ab} \{(\eta_\kappa \wedge m_\kappa)(ab)\} \\
&= (\eta_\kappa \wedge m_\kappa)(n) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Hence, $((\eta \wedge m) \circ \delta) \wedge (\delta \circ (\eta \wedge m)) \leq (\eta \wedge m)$, that is, $\eta \wedge m$ is m-PFQI of P. □

Now, we define the m-PFII of a semigroup.

Definition 8. *An m-PFSS $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ of P is called an m-PFII of P if for all $a, b, c \in P$, $\eta(abc) \geq \eta(b)$, that is, $\eta_\kappa(abc) \geq \eta_\kappa(b)$ for all $\kappa \in \{1, 2, \ldots, m\}$.*

Lemma 12. *A subset H of P is an interior ideal of P if, and only if, C_H is an m-PFII of P.*

Proof. Let H be any interior ideal of P. From Lemma 2, C_H is an m-PFSS of P. Now we show that $C_H(abc) \geq C_H(b)$ for every $a, b, c \in P$. We consider the following two cases:

Case 1: Let $b \in H$ and $a, c \in P$. Then $C_H(b) = (1, 1, \ldots, 1)$. Since H is an interior ideal of P, then $abc \in H$. Then, $C_H(abc) = (1, 1, \ldots, 1)$. Hence, $C_H(abc) \geq C_H(b)$.

Case 2: Let $b \notin H$ and $a, c \in P$. Then, $C_H(b) = (0, 0, \ldots, 0)$. Clearly, $C_H(abc) \geq C_H(b)$. Hence, C_H of H is an m-PFII of P.

Conversely, consider C_H of H is an m-PFII of P. Then by Lemma 2, H is a sub-semigroup of P. Let $b \in H$ and $a, c \in P$. Then $C_H(b) = (1, 1, \ldots, 1)$. By hypothesis, $C_H(abc) \geq C_H(b) = (1, 1, \ldots, 1)$. Hence $C_H(abc) = (1, 1, \ldots, 1)$. This implies that $abc \in H$, that is H is an interior ideal of P. □

Lemma 13. *An m-PFSS η of P is an m-PFII of P if, and only if, $\delta \circ \eta \circ \delta \leq \eta$.*

Proof. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ be m-PFII of P. We show that $\delta \circ \eta \circ \delta \leq \eta$. Let $n \in P$. Then, for all $\kappa \in \{1, 2, \ldots, m\}$.

$$\begin{aligned}
(\delta_\kappa \circ \eta_\kappa \circ \delta_\kappa)(n) &= \bigvee_{n=uv} \{(\delta_\kappa \circ \eta_\kappa)(u) \wedge \delta_\kappa(v)\} \\
&= \bigvee_{n=uv} \{(\delta_\kappa \circ \eta_\kappa)(u)\} \\
&= \bigvee_{n=uv} \{ \bigvee_{u=ab} (\delta_\kappa(a) \wedge \eta_\kappa(b)\} \\
&= \bigvee_{n=(ab)v} \{\eta_\kappa(b)\} \\
&\leq \bigvee_{n=(ab)v} \{\eta_\kappa((ab)v)\} \text{ as } \eta \text{ is an } m\text{-PFII of } P. \\
&= \eta_\kappa(n) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Therefore $\delta \circ \eta \circ \delta \leq \eta$.

Conversely, let $\delta \circ \eta \circ \delta \leq \eta$. We only show that $\eta_\kappa(abc) \geq \eta_\kappa(b)$ for every $a, b, c \in P$ and for all $\kappa \in \{1, 2, \ldots, m\}$. Let $n = abc$. Now, for all $\kappa \in \{1, 2, \ldots, m\}$.

$$\begin{aligned}
\eta_\kappa(abc) &\geq ((\delta_\kappa \circ \eta_\kappa) \circ \delta_\kappa)((ab)c) \\
&= \bigvee_{(ab)c=uv} \{((\delta_\kappa \circ \eta_\kappa)(u) \wedge \delta_\kappa(v)\} \\
&\geq (\delta_\kappa \circ \eta_\kappa)(ab) \wedge \delta_\kappa(c) \\
&= (\delta_\kappa \circ \eta_\kappa)(ab) \\
&= \bigvee_{ab=pq} \{\delta_\kappa(p) \wedge \eta_\kappa(q)\} \\
&\geq \delta_\kappa(a) \wedge \eta_\kappa(b) \\
&= \eta_\kappa(b) \text{ for all } \kappa \in \{1, 2, \ldots, m\}.
\end{aligned}$$

Therefore, $\eta_\kappa(abc) \geq \eta_\kappa(b)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Hence, η is m-PFII of P. □

Proposition 5. *A subset $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ of P is m-PFII of P if, and only if, $\eta_t = \{a \in P | \eta(a) \geq t\} \neq \phi$ is an interior ideal of P for all $t = (t_1, t_2, \ldots, t_m) \in (0, 1]^m$.*

Proof. This is the same as the proof of Propositions 1 and 2. □

4. Characterization of Regular and Intra-Regular Semigroups by m-Polar Fuzzy Ideals

A semigroup P is called regular if for all $x \in P$, there exists an element $a \in P$ such that $x = xax$. A semigroup P is called an intra-regular semigroup if for all $x \in P$, there exists elements $b, c \in P$ such that $x = bx^2c$. Regular and intra-regular semigroups have been studied by several authors, see [24,28]. The characterizations of the regular and intra-regular semigroups in terms of m-PF ideals and m-PFBI are discussed with the help of many theorems in this section.

Theorem 1 ([28])**.** *The following results are equivalent in P.*

1. P is regular;
2. $H \cap I = HI$, for every right ideal H and left ideal I of P;
3. $J = JPJ$, for every quasi ideal J of P.

Theorem 2. *Each m-PFQI η of P is an m-PFBI of P.*

Proof. Suppose that $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ be m-PFQI of P. Let $a, b \in P$. Then,

$$\begin{aligned} \eta_\kappa(ab) &\geq ((\eta_\kappa \circ \delta_\kappa) \wedge (\delta_\kappa \circ \eta_\kappa))(ab) \\ &= (\eta_\kappa \circ \delta_\kappa)(ab) \wedge (\delta_\kappa \circ \eta_\kappa)(ab) \\ &= [\bigvee_{ab=op} \{(\eta_\kappa(o) \wedge \delta_\kappa)(p)\}] \wedge [\bigvee_{ab=uv} \{(\delta_\kappa(u) \wedge \eta_\kappa(v)\}] \\ &\geq \{(\eta_\kappa(a) \wedge \delta_\kappa)(b)\} \wedge \{(\delta_\kappa(a) \wedge \eta_\kappa(b)\} \\ &= \{(\eta_\kappa(a) \wedge 1\} \wedge \{1 \wedge \eta_\kappa(b)\} \\ &= \eta_\kappa(a) \wedge \eta_\kappa(b) \text{ for all } \kappa \in \{1, 2, \ldots, m\}. \end{aligned}$$

So, $\eta_\kappa(ab) \geq \eta_\kappa(a) \wedge \eta_\kappa(b)$. Now, let $a, b, c \in P$. Then,

$$\begin{aligned} (\delta_\kappa \circ \eta_\kappa))((ab)c) &= \bigvee_{(ab)c=uv} \{(\delta_\kappa(u) \wedge \eta_\kappa(v)\} \\ &\geq \delta_\kappa(ab) \wedge \eta_\kappa(c) \\ &= 1 \wedge \eta_\kappa(c) \\ &= \eta_\kappa(c). \end{aligned}$$

Therefore, $(\delta_\kappa \circ \eta_\kappa)(abc) \geq \eta_\kappa(c)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Since $(ab)c = a(bc) \in aP$, so $(ab)c = ap$ for some $p \in P$. Therefore,

$$\begin{aligned} (\eta_\kappa \circ \delta_\kappa)(abc) &= \bigvee_{(ab)c=ob} \{(\eta_\kappa(o) \wedge \delta_\kappa)(b)\} \\ &\geq \eta_\kappa(a) \wedge \delta_\kappa(p) \text{ since } (ab)c = ap \\ &= \eta_\kappa(a) \wedge 1 \\ &= \eta_\kappa(a). \end{aligned}$$

Therefore, $(\eta_\kappa \circ \delta_\kappa)(abc) \geq \eta_\kappa(a)$ for all $\kappa \in \{1, 2, \ldots, m\}$. Now, by our supposition

$$\begin{aligned} \eta_\kappa(abc) &\geq ((\eta_\kappa \circ \delta_\kappa) \wedge (\delta_\kappa \circ \eta_\kappa))(abc) \\ &= (\eta_\kappa \circ \delta_\kappa)(abc) \wedge (\delta_\kappa \circ \eta_\kappa)(abc) \\ &\geq \eta_\kappa(a) \wedge \eta_\kappa(c) \text{ for all } \kappa \in \{1, 2, \ldots, m\}. \end{aligned}$$

Therefore, $\eta(abc) \geq \eta(a) \wedge \eta(c)$. Hence, η is m-PFBI of P. □

Theorem 3. *The given statements are equivalent in P.*

1. *P is regular;*

2. *$\eta \wedge m' = \eta \circ m'$ for every m-PFI(right) η and m-PFI(left) m' of P.*

Proof. $(1) \implies (2)$: Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ and $m' = (m_1, m_2, \ldots, m_m)$ be two m-PFI(right) and m-PFI(left) of P. Let $o \in P$, we have

$$\begin{aligned} (\eta_\kappa \circ m'_\kappa)(o) &= \bigvee_{o=bc} \{(\eta_\kappa(b) \wedge m'_\kappa(c)\} \\ &\leq \bigvee_{o=bc} \{(\eta_\kappa(bc) \wedge m'_\kappa(bc)\} \\ &= \eta_\kappa(o) \wedge m'_\kappa(o) \\ &= (\eta_\kappa \wedge m'_\kappa)(o) \text{ for all } \kappa \in \{1, 2, \ldots, m\}. \end{aligned}$$

Therefore, $\eta \circ m' \leq \eta \wedge m'$. As P is regular, then for every, $o \in P$, there exists $a \in P$, such that $o = (oa)o$.

$$\begin{aligned} (\eta_\kappa \wedge m'_\kappa)(o) &= \eta_\kappa(o) \wedge m'_\kappa(o) \\ &\leq \eta_\kappa(oa) \wedge m'_\kappa(o) \text{ as } \eta \text{ is } m\text{-PFRI of } P. \\ &\leq \bigvee_{o=bc} \{(\eta_\kappa(b) \wedge m'_\kappa(c)\} \\ &= (\eta_\kappa \circ m'_\kappa)(o) \text{ for all } \kappa \in \{1, 2, \ldots, m\}. \end{aligned}$$

So, $\eta \wedge m' \leq \eta \circ m'$. Therefore, $\eta \wedge m' = \eta \circ m'$.

$(2) \implies (1)$: Let $o \in P$. Then, $\eta = oP$ is a left ideal of P and $m' = oP \cup Po$ is a right ideal of P generated by o. Then, by using Lemma 2, C_η and $C_{m'}$ the m-polar fuzzy characteristic fuctions of η and m' are m-PFI(left) and m-PFI(right) of P, respectively. Then, we have

$$\begin{aligned} C_{m'\eta} &= (C_{m'} \circ C_\eta) \text{ by Lemma 1} \\ &= (C_{m'} \wedge C_\eta) \text{ by 2} \\ &= C_{m' \cap \eta} \text{ by Lemma 1.} \end{aligned}$$

Therefore, $m' \cap \eta = m'\eta$. As a result, Theorem 1 shows that P is regular. □

Theorem 4. *The following statements are equivalent in P.*

1. *P is regular;*
2. *$\eta = \eta \circ \delta \circ \eta$ for every m-PFGBI η of P;*
3. *$\eta = \eta \circ \delta \circ \eta$ for every m-PFQI η of P.*

Proof. $(1) \implies (2)$: Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ be an m-PFGBI of P and $o \in P$. Since P is regular, there exists $a \in P$ *such that* $o = (oa)o$. Therefore, we have

$$\begin{aligned} (\eta_\kappa \circ \delta_\kappa \circ \eta_\kappa)(o) &= \bigvee_{o=bc} \{(\eta_\kappa \circ \delta_\kappa)(b) \wedge \eta_\kappa(c)\} \text{ for some } b, c \in P \\ &\geq (\eta_\kappa \circ \delta_\kappa)(oa) \wedge \eta_\kappa(o) \text{ since } o = (oa)o \\ &= \bigvee_{oa=pq} \{\eta_\kappa(p) \wedge \delta_\kappa)(q)\} \wedge \eta_\kappa(o) \\ &\geq \{\eta_\kappa(o) \wedge \delta_\kappa)(a)\} \wedge \eta_\kappa(o) \\ &= \eta_\kappa(o) \text{ for all } \kappa \in \{1, 2, \ldots, m\}. \end{aligned}$$

Hence, $\eta \circ \delta \circ \eta \geq \eta$. Since η is an m-PFGBI of P. Therefore, we have

$$\begin{aligned}
(\eta_\kappa \circ \delta_\kappa \circ \eta_\kappa)(o) &= \bigvee_{o=rs} \{(\eta_\kappa \circ \delta_\kappa)(r) \wedge \eta_\kappa(s)\} \text{ for some } r,s \in P \\
&= \bigvee_{o=rs} \{ \bigvee_{r=uv} \{\eta_\kappa(u) \wedge \delta_\kappa(v)\} \wedge \eta_\kappa(s)\}\} \text{ for some } r,s \in P \\
&= \bigvee_{o=rs} \{ \bigvee_{r=uv} \{\eta_\kappa(u) \wedge \eta_\kappa(s)\}\} \\
&\leq \bigvee_{o=rs} \{ \bigvee_{r=uv} \{\eta_\kappa((uv)(s)\}\} \\
&= \bigvee_{o=rs} \eta_\kappa(rs) \\
&= \eta_\kappa(o) \text{ for all } \kappa \in \{1,2,\ldots,m\}.
\end{aligned}$$

So, $\eta \circ \delta \circ \eta \leq \eta$. Therefore, $\eta = \eta \circ \delta \circ \eta$.

$(2) \Longrightarrow (3)$: It is obvious.

$(3) \Longrightarrow (1)$: Let η, ρ be m-PFI (right) and m-PFI (left) of P, respectively. Then $\eta \wedge \rho$ is an m-PFQI of P. According to hypothesis

$$\begin{aligned}
\eta_\kappa \wedge \rho_\kappa &\leq (\eta_\kappa \wedge \rho_\kappa) \circ \delta_\kappa \circ (\eta_\kappa \wedge \rho_\kappa) \\
&\leq \eta_\kappa \circ \delta_\kappa \circ \rho_\kappa \\
&\leq \eta_\kappa \circ \rho_\kappa.
\end{aligned}$$

However, $\eta_\kappa \circ \rho_\kappa \leq \eta_\kappa \wedge \rho_\kappa$ always hold. Hence, $\eta_\kappa \circ \rho_\kappa = \eta_\kappa \wedge \rho_\kappa$, that is $\eta \circ \rho \leq \eta \wedge \rho$. Therefore by Theorem 3, P is a regular semigroup. Hence, proved. □

Theorem 5. *The following statements are equivalent in P.*

1. *P is regular;*
2. *$\rho \wedge m' \wedge \eta \leq \rho \circ m' \circ \eta$ for every m-PFI(right) ρ, every m-PFGBI m' and every m-PFI(left) η of P;*
3. *$\rho \wedge m' \wedge \eta \leq \rho \circ m' \circ \eta$ for every m-PFI(right) ρ, every m-PFBI m' and every m-PFI(left) η of P;*
4. *$\rho \wedge m' \wedge \eta \leq \rho \circ m' \circ \eta$ for every m-PFI(right) ρ, every m-PFQI m' and every m-PFI(left) η of P.*

Proof. $(1) \Longrightarrow (2)$: Consider b is any element of P. As P is regular, there exists $a \in P$ *such that* $b = bab$. It follows that $b = (ba)b = b(ab)$ for each $a \in P$ and P is semigroup. Hence, we have

$$\begin{aligned}
(\rho \circ m' \circ \eta)(b) &= \bigvee_{b=ac} \{(\rho \circ m')(a) \wedge \eta(c)\} \\
&\geq (\rho \circ m')(b) \wedge \eta(ab) \text{ since } b = b(ab) \\
&\geq \bigvee_{b=pq} \{\rho(p) \wedge m'(q)\} \wedge \eta(b) \text{ as } \eta \text{ is an } m\text{-PFI(left) of } P. \\
&\geq (\rho(ba) \wedge m'(b)) \wedge \eta(b) \text{ since } b = (ba)b \\
&\geq (\rho(b) \wedge m'(b)) \wedge \eta(b) \\
&= ((\rho \wedge m')(b)) \wedge \eta(b) \\
&= (\rho \wedge m' \wedge \eta)(b).
\end{aligned}$$

Therefore, $\rho \wedge m' \wedge \eta \leq \rho \circ m' \circ \eta$. So (1) implies (2).

$(2) \Longrightarrow (3) \Longrightarrow (4)$: Straightforward.

$(4) \Longrightarrow (1)$: As δ is an m-PFQI of P, by the supposition, we have

$$
\begin{aligned}
(\rho \wedge \eta)(b) &= ((\rho \wedge \delta) \wedge \eta)(b) \\
&\leq ((\rho \circ \delta) \circ \eta(b) \\
&= \bigvee_{b=rs} \{(\rho \circ \delta)(r) \circ \eta(s)\} \\
&= \bigvee_{b=rs} \{(\bigvee_{r=cd} \{\rho(c) \wedge \delta(d)\}) \wedge \eta(s)\} \\
&= \bigvee_{b=rs} \{(\bigvee_{r=cd} \{\rho(c) \wedge 1\}) \wedge \eta(s)\} \\
&= \bigvee_{b=rs} \{(\bigvee_{r=cd} \rho(c)) \wedge \eta(s)\} \\
&\leq \bigvee_{b=rs} \{(\bigvee_{r=cd} \rho(cd)\}) \wedge \eta(s)\} \\
&= \bigvee_{b=rs} \{\rho(r) \wedge \eta(s)\} \\
&= (\rho \circ \eta)(b).
\end{aligned}
$$

Therefore $\rho \wedge \eta \leq \rho \circ \eta$. But $\rho \circ \eta \leq \rho \wedge \eta$ always. So, $\rho \circ \eta = \rho \wedge \eta$. Hence, by using Theorem 3, P is regular. □

Theorem 6 ([28]). *The following conditions are equivalent in P.*

1. P is intra-regular;
2. $H \cap I \subseteq HI$ for every right ideal H and every left ideal I of P.

Definition 9 ([24]). *A semigroup P is both regular and intra-regular if and only if $H = H^2$ for every bi-ideal H of P.*

Theorem 7. *A semigroup P is intra-regular if, and only if $\eta \wedge \rho \leq \eta \circ \rho$ for every m-PFI(left) η and, for every, m-PFI(right) ρ of P.*

Proof. Consider a is any element of P. As P is intra-regular, there exists $x, y \in P$ *such that* $a = xa^2y$. Hence, we have

$$
\begin{aligned}
(\eta \circ \rho)(a) &= \bigvee_{a=bc} \{\eta(b) \wedge \rho(c)\} \\
&\geq \eta(xa) \wedge \rho(ay) \\
&\geq \eta(a) \wedge \rho(a) \\
&= (\eta \wedge \rho)(a).
\end{aligned}
$$

This implies $\eta \circ \rho \geq \eta \wedge \rho$.

Conversely, assume that $\eta \wedge \rho \leq \eta \circ \rho$ for all m-PFI(left) η and m-PFI(right) ρ of P. Let H be a right ideal and I be a left ideal of P, then C_H is an m-PFI(right) and C_I is an m-PFI(left) of P. By Lemma 1, $C_{H \cap I} = C_H \wedge C_I \leq C_H \circ C_I = C_{HI}$ which implies that $H \cap I \subseteq HI$. Therefore, by Theorem 6, P is intra-regular. □

Theorem 8. *For every m-PFBI η of P, $\eta \circ \eta = \eta$ if and only if, P is both regular and intra-regular.*

Proof. Let P be both regular and intra-regular semigroup. Let η be an m-PFBI of P. Thus, for $x \in P$, there exists $a, b, c \in P$ such that $x = xax$ and $x = bx^2c$. Therefore, $x = xax = xaxax = xa(bx^2c)ax = (xabx)(xcax)$. Hence, we have

$$
\begin{aligned}
(\eta \circ \eta)(x) &= \bigvee_{x=bc} \{\eta(b) \wedge \eta(c)\} \\
&\geq \eta(xabx) \wedge \eta(xcax) \\
&\geq \eta(x) \wedge \eta(x) \\
&= \eta(x).
\end{aligned}
$$

This implies $\eta \circ \eta \geq \eta$. By Lemma 3, $\eta \circ \eta \leq \eta$ holds always. Therefore, $\eta \circ \eta = \eta$.

Conversely, let H be a bi-ideal of P. Since every m-PFBI is m-PFSS of P. Then, Lemma 2, implies that C_H is m-PFBI of P. Hence, by our supposition, $C_H = C_H \circ C_H$. Thus, $H = H^2$. Therefore, by Theorem 9, P is both regular and intra-regular. □

5. Comparative Study and Discussion

This section explains how this paper and the previous one are related to Shabir et al. [27]. Shabir et al. [24] studied regular and intra-regular semiring in terms of BFIs. Shabir et al. extended the work of [24] and initiated the concept of m-PFIs in LA-semigroups and characterized the regular LA-semigroups by the properties of these m-PFIs [27]. By extending the work of [24,27], the concept of m-PFIs in semigroups is introduced, and characterizations of regular and intra-regular semigroups by the properties of m-PFIs are given in this paper. Our approach is superior to that of Shabir et al. [27] because the associative property in LA-semigroups does not hold. There are also numerous structures that are handled by semigroups but not by LA-semigroups. If we take any non-empty set and define the operation on it as $a * b = a$, then it is a semigroup, but not an LA-semigroup. To overcome this problem, we used a semigroup to generalize the whole results of Shabir et al. [27] and, as a result, our methodology offers a broader variety of applications than Shabir et al. [27].

6. Conclusions

When data for real world complex situations come from m factors ($m \geq 2$), then m-PFS is used to deal such problems. The structure of semigroups is investigated using the idea of m-PFS in this research paper. Shabir et al. [27] used LA-semigroups as the basis for their algebraic structure, which we converted into semigroups. Most importantly, we proved some results related to fuzzy ideals in semigroups in terms of m-PFIs in semigroups. This paper presents a significant number of m-PFS theory applications. We also studied the characterization of regular and intra-regular semigroups by m-PFIs (left) (resp. m-PFIs right) and m-PFBI.

Our future plans are to study the m-PFIs in terms of semirings, ternary semigroups, ternary semirings, near rings and hyperstructures.

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