mathematics

MDPI

Article

Applications to Boundary Value Problems and Homotopy Theory via Tripled Fixed Point Techniques in Partially Metric Spaces

Hasanen A. Hammad [1], Praveen Agarwal [2,3,4,5,6,*] and Juan L. G. Guirao [7,8]

[1] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt; hassanein_hamad@science.sohag.edu.eg
[2] Department of Mathematics, Anand International College of Engineering, Jaipur 302012, India
[3] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman AE 346, United Arab Emirates
[4] Institute of Mathematical Modeling, Almaty 050010, Kazakhstan
[5] International Center for Basic and Applied Sciences, Jaipur 302029, India
[6] Harish-Chandra Research Institute (HRI), Allahabad 211019, India
[7] Departamento de Matematica Aplicada y Estadistica, Universidad Politecnica de Cartagena, Hospital de Marina, 30203 Murcia, Spain; juan.garcia@upct.es or jlgarcia@kau.edu.sa
[8] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
* Correspondence: goyal.praveen2011@gmail.com or praveen.agarwal@anandice.ac.in

Abstract: In this manuscript, some tripled fixed point results were derived under (φ, ρ, ℓ)-contraction in the framework of ordered partially metric spaces. Moreover, we furnish an example which supports our theorem. Furthermore, some results about a homotopy results are obtained. Finally, theoretical results are involved in some applications, such as finding the unique solution to the boundary value problems and homotopy theory.

Keywords: tripled fixed point; boundary value problem; homotopy theory; partially ordered metric space

MSC: 54H25; 47H10; 34B24

check for updates

Citation: Hammad, H.A.; Agarwal, P.; Guirao, J.L.G. Applications to Boundary Value Problems and Homotopy Theory via Tripled Fixed Point Techniques in Partially Metric Spaces. *Mathematics* **2021**, *9*, 2012. https://doi.org/10.3390/math9162012

Academic Editors: Christopher Goodrich and Hsien-Chung Wu

Received: 9 June 2021
Accepted: 14 August 2021
Published: 23 August 2021

1. Introduction

Fixed point theory is one of the important and indispensable branches of non-linear analysis due to the proliferation of its applications in many disciplines such as engineering, computer science, physics, economics, biology, chemistry, etc. In mathematics, this technique is credited with clarifying and studying the behavior of dynamical systems, statistical methods, game theory models, differential equations, and many others. Specifically, this technique studies the existence and uniqueness of the solution to many integral and fractional equations, which facilitates the way to find numerical solutions to such problems, see [1–11].

Homotopy theory is a fundamental branch of algebraic topology where topological objects are studied up to homotopy equivalence. In the last century, strong links have emerged between this theory and many branches of mathematics. For example, this trend plays a prominent role in strengthening ties between homotopy theory and category theory (higher-dimensional), which have received considerable attention in recent years, see [12–15]. Furthermore, it is useful in quantum mechanics for dealing with Hamiltonian manifolds.

The idea of a partially metric space (PMS) was presented by Matthews [16] as a part of the study of denotational semantics of data flow networks. In fact, it is widely shared that PMSs play a prominent role in model building in computational and field theory in computer science, see [17–22].

Fixed point results for a single mapping in partially ordered metric space (POMSs) were introduced by Matthews [16,23], Oltra and Valero [24] and Altun et al. [25]. For more papers in common fixed point consequences in abstract spaces, we mention [26–32].

In [33], the notion of a coupled fixed point was presented and some pivotal results from it in the setting of PMSs are studied. Later, coupled fixed and coupled common fixed point theorems are proved by [34–37].

In 2011, Berinde and Borcut [38] generalized the idea of a coupled fixed point to a tripled fixed point (TFP) in the setting of POMSs. Under this space, Borcut [39,40], Karapnar et al. [41], Radenović [42] and Aydi et al. [43] introduced some circular theorems concerning TFP theorems. Moreover, more applications in this line are presented by Mustafa et al. [44] and Hammad and De la Sen [45,46].

The purpose of this manuscript is to prove some TFP consequences via (φ,ρ,ℓ)-contraction in the setting of ordered PMSs. In addition, to support our theoretical results, we give an example. Moreover, as applications, the existence and uniqueness of the solution to an initial value problem (IVP) and a homotopy theory are discussed.

2. Preliminaries

This part is devoted to recall the standard definition of a homotopy and some elementary properties for PMSs.

A homotopy between two functions is defined as follows: consider two continuous functions from a topological space to another; the two functions are considered homotopic if one can be continuously deformed into the other. This deformity is called a homotopy between the two functions. It can be formulated as follows:

Definition 1 ([12])**.** *Assume that $U, V : D \to E$ are continuous mappings defined on topological spaces D and E. Then the continuous function $H : D \times [0,1] \to E$ so that $H(\vartheta, 0) = U\vartheta$ and $H(\vartheta, 1) = V\vartheta$ is called a homotopy from U to V. Furthermore, U and V are called homotopic mappings.*

Definition 2 ([16])**.** *Let $\daleth \neq \emptyset$. The function $\Lambda : \daleth \times \daleth \to \mathbb{R}^+$ is called a partial metric on $\daleth$ if for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \daleth$, the hypotheses below hold:*

(Λ_1) $\vartheta_1 = \vartheta_2$ *iff* $\Lambda(\vartheta_1, \vartheta_1) = \Lambda(\vartheta_1, \vartheta_2) = \Lambda(\vartheta_2, \vartheta_2)$;
(Λ_2) $\Lambda(\vartheta_1, \vartheta_1) \leq \Lambda(\vartheta_1, \vartheta_2)$ *and* $\Lambda(\vartheta_2, \vartheta_2) \leq \Lambda(\vartheta_1, \vartheta_2)$;
(Λ_3) $\Lambda(\vartheta_1, \vartheta_2) = \Lambda(\vartheta_2, \vartheta_1)$;
(Λ_4) $\Lambda(\vartheta_1, \vartheta_2) \leq \Lambda(\vartheta_1, \vartheta_3) + \Lambda(\vartheta_3, \vartheta_2) - \Lambda(\vartheta_3, \vartheta_3)$.

We say that $(\daleth, \Lambda)$ is a PMS.

It should be noted that if Λ is partial metric on $\daleth$, then the function $\Omega_\Lambda : \daleth \times \daleth \to \mathbb{R}^+$, described by

$$\Omega_\Lambda(\vartheta_1, \vartheta_2) \leq 2\Lambda(\vartheta_1, \vartheta_2) - \Lambda(\vartheta_1, \vartheta_1) - \Lambda(\vartheta_2, \vartheta_2),$$

is a metric on $\daleth$.

Example 1 ([23])**.** *Let $\daleth = [0, \infty)$. The pair $(\daleth, \Lambda)$ is a PMS under the distance $\Lambda(\vartheta_1, \vartheta_2) = \max\{\vartheta_1, \vartheta_2\}$. It is obvious that Λ is not a usual metric and in this case $\Omega_\Lambda(\vartheta_1, \vartheta_2) = |\vartheta_1 - \vartheta_2|$.*

Example 2. *Assume $\daleth = \{[\vartheta_1, \vartheta_2] : \vartheta_1, \vartheta_2 \in \mathbb{R}, \vartheta_1 \leq \vartheta_2\}$ and define*

$$\Lambda([\vartheta_1, \vartheta_3], [\vartheta_2, \vartheta_4]) = \max\{\vartheta_3, \vartheta_4\} - \min\{\vartheta_1, \vartheta_2\}.$$

Then $(\daleth, \Lambda)$ is a PMS.

Every partial metric Λ on $\daleth$ produces a Y_0 topology χ_Λ on $\daleth$. The base of a topology χ_Λ is a family of open Λ-balls $\{O_\Lambda(z, \epsilon), z \in \daleth, \epsilon > 0\}$, where $O_\Lambda(z, \epsilon) = \{\vartheta \in \daleth : \Lambda(z, \vartheta) < \Lambda(z, z) + \epsilon\}$, $\forall z \in \daleth$ and $\epsilon > 0$.

Now, we state some topological properties on PMSs.

Definition 3 ([16]). *A sequence $\{\vartheta^{\omega}\}$ in a PMS $(\daleth, \Lambda)$ is called:*

(1) converges to the limit ϑ iff $\Lambda(\vartheta, \vartheta) = \lim_{\omega\to\infty} \Lambda(\vartheta, \vartheta^{\omega})$.
(2) a Cauchy sequence if $\lim_{\nu,\omega\to\infty} \Lambda(\vartheta^{\nu}, \vartheta^{\omega})$ exists and is finite.

Definition 4 ([16]). *(i) A PMS $(\daleth, \Lambda)$ is called complete if every Cauchy sequence $\{\vartheta^{\omega}\}$ in $\daleth$ converges (with respect to χ_{Λ}) to a point $\vartheta \in \daleth$ so that $\Lambda(\vartheta, \vartheta) = \lim_{\nu,\omega\to\infty} \Lambda(\vartheta^{\nu}, \vartheta^{\omega})$.*
(ii) A mapping $\xi : \daleth \to \daleth$ is said to be continuous at $\vartheta^0 \in \daleth$, if $\forall \epsilon > 0$, $\exists \delta > 0$ so that $\xi\big(O_{\Lambda}(\vartheta^0, \delta)\big) \subseteq O_{\Lambda}(\xi\vartheta^0, \epsilon)$.

The following lemmas are very important in the following.

Lemma 1 ([16]). *(i) We say that $\{\vartheta^{\omega}\}$ is a Cauchy sequence in the PMS $(\daleth, \Lambda)$ if it is a Cauchy sequence in the metric space $(\daleth, \Omega_{\Lambda})$.*
(ii) If the metric space $(\daleth, \Omega_{\Lambda})$ is complete, the PMS $(\daleth, \Lambda)$ is too. Furthermore,

$$\lim_{\omega\to\infty} \Omega_{\Lambda}(\vartheta, \vartheta^{\omega}) = 0 \Leftrightarrow \Lambda(\vartheta, \vartheta) = \lim_{\omega\to\infty} \Omega_{\Lambda}(\vartheta, \vartheta^{\omega}) = \lim_{\omega,\upsilon\to\infty} \Omega_{\Lambda}(\vartheta^{\upsilon}, \vartheta^{\omega}).$$

Lemma 2 ([47]). *Suppose that $(\daleth, \Lambda)$ is a PMS. If $\{\vartheta^{\omega}\} \in \daleth$ so that $\lim_{\omega\to\infty} \vartheta^{\omega} = r$ and $\Lambda(r, r) = 0$. Then $\lim_{\omega\to\infty} \Lambda(\vartheta^{\omega}, s) = \lim_{\omega\to\infty} \Lambda(r, s)$ for every $r, s \in \daleth$.*

Lemma 3 ([47]). *Assume that $(\daleth, \Lambda)$ is a PMS. Then*

(a) If $\Lambda(\vartheta_1, \vartheta_2) = 0$, then $\vartheta_1 = \vartheta_2$;
(b) If $\vartheta_1 \neq \vartheta_2$, then $\Lambda(\vartheta_1, \vartheta_2) > 0$.

Remark 1 ([47]). *If $\vartheta_1 = \vartheta_2$, then $\Lambda(\vartheta_1, \vartheta_2)$ may not be 0.*

Definition 5 ([46]). *A mapping $\xi : \daleth^3 \to \daleth$ (where $\daleth \times \daleth \times \daleth = \daleth^3$) on a partially ordered set (POS) $(\daleth, \preceq)$, has a mixed-monotone property, if for any $\vartheta, \theta, \omega \in \daleth$,*

$$\begin{aligned}
\vartheta_1, \vartheta_2 &\in \daleth,\ \vartheta_1 \preceq \vartheta_2 \Rightarrow \xi(\vartheta_1, \theta, \omega) \preceq \xi(\vartheta_2, \theta, \omega),\\
\theta_1, \theta_2 &\in \daleth,\ \theta_1 \preceq \theta_2 \Rightarrow \xi(\vartheta, \theta_1, \omega) \succeq \xi(\vartheta, \theta_2, \omega),\\
\omega_1, \omega_2 &\in \daleth,\ \omega_1 \preceq \omega_2 \Rightarrow \xi(\vartheta, \theta, \omega_1) \preceq \xi(\vartheta, \theta, \omega_2).
\end{aligned}$$

Definition 6 ([43]). *A mapping $\xi : \daleth^3 \to \daleth$ on a POS $(\daleth, \preceq)$ has a mixed ξ^*-monotone property where $\xi^* : \daleth \to \daleth$, if for any $\vartheta, \theta, \omega \in \daleth$,*

$$\begin{aligned}
\vartheta_1, \vartheta_2 &\in \daleth,\ \xi^*\vartheta_1 \preceq \xi^*\vartheta_2 \Rightarrow \xi(\vartheta_1, \theta, \omega) \preceq \xi(\vartheta_2, \theta, \omega),\\
\theta_1, \theta_2 &\in \daleth,\ \xi^*\theta_1 \preceq \xi^*\theta_2 \Rightarrow \xi(\vartheta, \theta_1, \omega) \succeq \xi(\vartheta, \theta_2, \omega),\\
\omega_1, \omega_2 &\in \daleth,\ \xi^*\omega_1 \preceq \xi^*\omega_2 \Rightarrow \xi(\vartheta, \theta, \omega_1) \preceq \xi(\vartheta, \theta, \omega_2).
\end{aligned}$$

Definition 7 ([39]). *We say that a trio $(\vartheta, \theta, \omega) \in \daleth^3$ is a TFP of the self-mapping $\xi : \daleth^3 \to \daleth$ if $\vartheta = \xi(\vartheta, \theta, \omega)$, $\theta = \xi(\theta, \omega, \vartheta)$ and $\omega = \xi(\omega, \vartheta, \theta)$.*

Definition 8 ([40]). *A trio $(\vartheta, \theta, \omega) \in \daleth^3$ is called a tripled coincidence point (TCP) of the two self-mappings $\xi : \daleth^3 \to \daleth$ and $\xi^* : \daleth \to \daleth$ if $\xi^*\vartheta = \xi(\vartheta, \theta, \omega)$, $\xi^*\theta = \xi(\theta, \omega, \vartheta)$ and $\xi^*\omega = \xi(\omega, \vartheta, \theta)$.*

Definition 9 ([40]). *Assume that $\daleth \neq \emptyset$ is a set, a trio $(\vartheta, \theta, \omega) \in \daleth^3$ is called a common TFP of $\xi : \daleth^3 \to \daleth$ and $\xi^* : \daleth \to \daleth$, if $\vartheta = \xi^*\vartheta = \xi(\vartheta, \theta, \omega)$, $\theta = \xi^*\theta = \xi(\theta, \omega, \vartheta)$ and $\omega = \xi^*\omega = \xi(\omega, \vartheta, \theta)$.*

Definition 10 ([48])**.** *The mappings $\xi : \daleth^3 \to \daleth$ and $\xi^* : \daleth \to \daleth$ are called $w-$compatible if $\xi^*(\xi(\vartheta,\theta,\omega)) = \xi(\xi^*\vartheta,\xi^*\theta,\xi^*\omega)$, $\xi^*(\xi(\theta,\omega,\vartheta)) = \xi(\xi^*\theta,\xi^*\omega,\xi^*\vartheta)$ and $\xi^*(\xi(\omega,\vartheta,\theta)) = \xi(\xi^*\omega,\xi^*\vartheta,\xi^*\theta)$, whenever $\xi^*\vartheta = \xi(\vartheta,\theta,\omega)$, $\xi^*\theta = \xi(\theta,\omega,\vartheta)$ and $\xi^*\omega = \xi(\omega,\vartheta,\theta)$.*

3. Theorems and Discussion

We begin this part with the following definition:

Definition 11. *Assume that $(\daleth,\Lambda)$ is a PMS, $\xi : \daleth^3 \to \daleth$ and $\xi^* : \daleth \to \daleth$ are mappings. We say that ξ verifies a $(\varphi,\rho,\ell)-$contraction w.r.t ξ^* if there are $\varphi,\rho,\ell : [0,\infty) \to [0,\infty)$ verifying the assertions below:*

- (a_1) *φ is continuous and monotonically non-decreasing, ρ is continuous and ℓ is lower semi continuous (LSC);*
- (a_2) *$\varphi(\nu) = 0$ iff $\nu = 0$, $\rho(0) = \ell(0) = 0$;*
- (a_3) *$\varphi(\nu) - \rho(\nu) + \ell(\nu) > 0$;*
- (a_4) *for all $\theta_1,\theta_2,\theta_3,\vartheta_1,\vartheta_2,\vartheta_3 \in \daleth$ with $\xi^*\theta_1 \preceq \xi^*\vartheta_1$, $\xi^*\theta_2 \succeq \xi^*\vartheta_2$ and $\xi^*\theta_3 \preceq \xi^*\vartheta_3$, we have*

$$\varphi(\Lambda(\xi(\theta_1,\theta_2,\theta_3),\xi(\vartheta_1,\vartheta_2,\vartheta_3))) \leq \rho(\sqsupset(\theta_1,\theta_2,\theta_3,\vartheta_1,\vartheta_2,\vartheta_3)) - \ell(\sqsupset(\theta_1,\theta_2,\theta_3,\vartheta_1,\vartheta_2,\vartheta_3)),$$

where

$$\begin{aligned} &\sqsupset(\theta_1,\theta_2,\theta_3,\vartheta_1,\vartheta_2,\vartheta_3) \\ =\ &\max\left\{\begin{array}{c} \Lambda(\xi^*\theta_1,\xi^*\vartheta_1),\Lambda(\xi^*\theta_2,\xi^*\vartheta_2),\Lambda(\xi^*\theta_3,\xi^*\vartheta_3), \\ \Lambda(\xi^*\theta_1,\xi(\theta_1,\theta_2,\theta_3)),\Lambda(\xi^*\theta_2,\xi(\theta_2,\theta_3,\theta_1)),\Lambda(\xi^*\theta_3,\xi(\theta_3,\theta_1,\theta_2)), \\ \Lambda(\xi^*\vartheta_1,\xi(\vartheta_1,\vartheta_2,\vartheta_3)),\Lambda(\xi^*\vartheta_2,\xi(\vartheta_2,\vartheta_3,\vartheta_1)),\Lambda(\xi^*\vartheta_3,\xi(\vartheta_3,\vartheta_1,\vartheta_2)), \\ \frac{\Lambda(\xi^*\theta_1,\xi(\theta_1,\theta_2,\theta_3))\Lambda(\xi^*\theta_2,\xi(\theta_2,\theta_3,\theta_1))}{1+\Lambda(\xi^*\theta_1,\xi^*\vartheta_1)+\Lambda(\xi^*\theta_2,\xi^*\vartheta_2)+\Lambda(\xi(\theta_1,\theta_2,\theta_3),\xi(\vartheta_1,\vartheta_2,\vartheta_3))}, \\ \frac{\Lambda(\xi^*\theta_1,\xi(\theta_1,\theta_2,\theta_3))\Lambda(\xi^*\theta_3,\xi(\theta_3,\theta_1,\theta_2))}{1+\Lambda(\xi^*\theta_1,\xi^*\vartheta_1)\Lambda(\xi^*\theta_3,\xi^*\vartheta_3)+\Lambda(\xi(\theta_1,\theta_2,\theta_3),\xi(\vartheta_1,\vartheta_2,\vartheta_3))}, \\ \frac{\Lambda(\xi^*\vartheta_1,\xi(\vartheta_1,\vartheta_2,\vartheta_3))\Lambda(\xi^*\vartheta_2,\xi(\vartheta_2,\vartheta_3,\vartheta_1))}{1+\Lambda(\xi^*\theta_1,\xi^*\vartheta_1)+\Lambda(\xi^*\theta_2,\xi^*\vartheta_2)+\Lambda(\xi(\theta_1,\theta_2,\theta_3),\xi(\vartheta_1,\vartheta_2,\vartheta_3))}, \\ \frac{\Lambda(\xi^*\vartheta_1,\xi(\vartheta_1,\vartheta_2,\vartheta_3))\Lambda(\xi^*\vartheta_3,\xi(\vartheta_3,\vartheta_1,\vartheta_2))}{1+\Lambda(\xi^*\theta_1,\xi^*\vartheta_1)+\Lambda(\xi^*\theta_2,\xi^*\vartheta_2)+\Lambda(\xi(\theta_1,\theta_2,\theta_3),\xi(\vartheta_1,\vartheta_2,\vartheta_3))} \end{array}\right\}. \end{aligned}$$

Theorem 1. *Let $(\daleth,\preceq)$ be a POS and Λ be a partial metric so that $(\daleth,\Lambda)$ is a PMS. Assume that $\xi : \daleth^3 \to \daleth$ and $\xi^* : \daleth \to \daleth$ are mappings which satisfy*

- *$(\rhd_i)$ ξ verifies a $(\varphi,\rho,\ell)-$contraction with respect to ξ^*;*
- *$(\rhd_{ii})$ $\xi(\daleth^3) \subseteq \xi^*(\daleth)$ and $\xi^*(\daleth)$ is a complete subspace of $\daleth$;*
- *$(\rhd_{iii})$ ξ has a mixed ξ^*-monotone property;*
- *$(\rhd_{iv})$ (1) if non-decreasing sequences $\{\theta_1^n\} \to \theta_1$ and $\{\theta_3^n\} \to \theta_3$, then $\theta_1^n \preceq \theta_1$ and $\theta_3^n \preceq \theta_3$ for all n;*
  (2) if a non-increasing sequence $\{\theta_2^n\} \to \theta_2$, then $\theta_2 \preceq \theta_2^n$, for all n.

If there are $\theta_1^0,\theta_2^0,\theta_3^0 \in \daleth$ so that $\xi^*\theta_1^0 \preceq \xi(\theta_1^0,\theta_2^0,\theta_3^0)$, $\xi^*\theta_2^0 \succeq \xi(\theta_2^0,\theta_3^0,\theta_1^0)$ and $\xi^*\theta_3^0 \preceq \xi(\theta_3^0,\theta_1^0,\theta_2^0)$, then ξ and ξ^* have a TCP in $\daleth^3$.

Proof. Assume that $\theta_1^0,\theta_2^0,\theta_3^0 \in \daleth^3$ such that $\xi^*\theta_1^0 \preceq \xi(\theta_1^0,\theta_2^0,\theta_3^0)$, $\xi^*\theta_2^0 \succeq \xi(\theta_2^0,\theta_3^0,\theta_1^0)$ and $\xi^*\theta_3^0 \preceq \xi(\theta_3^0,\theta_1^0,\theta_2^0)$. Because $\xi(\daleth^3) \subseteq \xi^*(\daleth)$, we select $\theta_1^1,\theta_2^1,\theta_3^1 \in \daleth$ so that

$$\xi^*\theta_1^0 \preceq \xi\left(\theta_1^0,\theta_2^0,\theta_3^0\right) = \xi^*\theta_1^1,\ \xi^*\theta_2^0 \succeq \xi\left(\theta_2^0,\theta_3^0,\theta_1^0\right) = \xi^*\theta_2^1 \text{ and } \xi^*\theta_3^0 \preceq \xi\left(\theta_3^0,\theta_1^0,\theta_2^0\right) = \xi^*\theta_3^1,$$

and select $\theta_1^2,\theta_2^2,\theta_3^2 \in \daleth$ so that

$$\xi\left(\theta_1^1,\theta_2^1,\theta_3^1\right) = \xi^*\theta_1^2,\ \xi\left(\theta_2^1,\theta_3^1,\theta_1^1\right) = \xi^*\theta_2^2 \text{ and } \xi\left(\theta_3^1,\theta_1^1,\theta_2^1\right) = \xi^*\theta_3^2.$$

Because ξ has the mixed ξ^*-monotone property, we obtain

$$\xi^*\theta_1^0 \preceq \xi^*\theta_1^1 \preceq \xi^*\theta_1^2,\ \xi^*\theta_2^0 \succeq \xi^*\theta_2^1 \succeq \xi^*\theta_2^2 \text{ and } \xi^*\theta_3^0 \preceq \xi^*\theta_3^1 \preceq \xi^*\theta_3^3.$$

Continuing with the same scenario, we build the sequences $\{\theta_1^\omega\}$, $\{\theta_2^\omega\}$ and $\{\theta_3^\omega\}$ in ⅂ so that

$$\xi^*\theta_1^{\omega+1} = \xi(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega),\ \xi^*\theta_2^{\omega+1} = \xi(\theta_2^\omega, \theta_3^\omega, \theta_1^\omega) \text{ and } \xi^*\theta_3^{\omega+1} = \xi(\theta_3^\omega, \theta_1^\omega, \theta_2^\omega),\ \omega = 0, 1, 2, \ldots$$

with

$$\begin{aligned} \xi^*\theta_1^0 &\preceq \xi^*\theta_1^1 \preceq \xi^*\theta_1^2 \preceq \cdots, \\ \xi^*\theta_2^0 &\succeq \xi^*\theta_2^1 \succeq \xi^*\theta_2^2 \succeq \cdots, \\ \text{and } \xi^*\theta_3^0 &\preceq \xi^*\theta_3^1 \preceq \xi^*\theta_3^3 \preceq \cdots. \end{aligned}$$

Now, if $\xi^*\theta_1^\omega = \xi^*\theta_1^{\omega+1}$, $\xi^*\theta_2^\omega = \xi^*\theta_2^{\omega+1}$ and $\xi^*\theta_3^\omega = \xi^*\theta_3^{\omega+1}$, for some ω, then a trio $(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega)$ is a TCP in ⅂^{3} and nothing proof. So, we assume that $\xi^*\theta_1^\omega = \xi^*\theta_1^{\omega+1}$ or $\xi^*\theta_2^\omega = \xi^*\theta_2^{\omega+1}$ or $\xi^*\theta_3^\omega = \xi^*\theta_3^{\omega+1}$, for all ω. Because $\xi^*\theta_1^\omega \preceq \xi^*\theta_1^{\omega+1}$, $\xi^*\theta_2^\omega \succeq \xi^*\theta_2^{\omega+1}$ and $\xi^*\theta_3^\omega \preceq \xi^*\theta_3^{\omega+1}$, from assertion (a_4), we can write

$$\begin{aligned} \varphi\left(\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right)\right) &= \varphi\left(\Lambda\left(\xi\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}\right), \xi(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega)\right)\right) \\ &\le \rho\left(\sqsupset\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}, \theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right)\right) - \ell\left(\sqsupset\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}, \theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right)\right), \end{aligned}$$

where

$$\begin{aligned} &\sqsupset\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}, \theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right) \\ &= \max\left\{\begin{array}{c} \Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right), \Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right), \Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right), \\ \Lambda\left(\xi^*\theta_1^{\omega-1}, \xi\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}\right)\right), \Lambda\left(\xi^*\theta_2^{\omega-1}, \xi\left(\theta_2^{\omega-1}, \theta_3^{\omega-1}, \theta_1^{\omega-1}\right)\right), \\ \Lambda\left(\xi^*\theta_3^{\omega-1}, \xi\left(\theta_3^{\omega-1}, \theta_1^{\omega-1}, \theta_2^{\omega-1}\right)\right), \Lambda\left(\xi^*\theta_1^\omega, \xi(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega)\right), \\ \Lambda\left(\xi^*\theta_2^\omega, \xi(\theta_2^\omega, \theta_3^\omega, \theta_1^\omega)\right), \Lambda\left(\xi^*\theta_3^\omega, \xi(\theta_3^\omega, \theta_1^\omega, \theta_2^\omega)\right), \\ \frac{\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}\right)\right)\Lambda\left(\xi^*\theta_2^{\omega-1}, \xi\left(\theta_2^{\omega-1}, \theta_3^{\omega-1}, \theta_1^{\omega-1}\right)\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)+\Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right)+\Lambda\left(\xi\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}\right), \xi\left(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right)\right)}, \\ \frac{\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}\right)\right)\Lambda\left(\xi^*\theta_3^{\omega-1}, \xi\left(\theta_3^{\omega-1}, \theta_1^{\omega-1}, \theta_2^{\omega-1}\right)\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)+\Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right)+\Lambda\left(\xi\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}\right), \xi\left(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right)\right)}, \\ \frac{\Lambda\left(\xi^*\theta_1^\omega, \xi\left(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right)\right)\Lambda\left(\xi^*\theta_2^\omega, \xi\left(\theta_2^\omega, \theta_3^\omega, \theta_1^\omega\right)\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)+\Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right)+\Lambda\left(\xi\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}\right), \xi\left(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right)\right)}, \\ \frac{\Lambda\left(\xi^*\theta_1^\omega, \xi\left(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right)\right)\Lambda\left(\xi^*\theta_3^\omega, \xi\left(\theta_3^\omega, \theta_1^\omega, \theta_2^\omega\right)\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)+\Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right)+\Lambda\left(\xi\left(\theta_1^{\omega-1}, \theta_2^{\omega-1}, \theta_3^{\omega-1}\right), \xi\left(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right)\right)} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} \Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right), \Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right), \Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right), \\ \Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right), \Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right), \Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right), \\ \Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right), \Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right), \Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right), \\ \frac{\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)\Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)+\Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right)+\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right)}, \\ \frac{\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)\Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)+\Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right)+\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right)}, \\ \frac{\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right)\Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)+\Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right)+\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right)}, \\ \frac{\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right)\Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right)+\Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right)+\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right)} \end{array}\right\}. \end{aligned}$$

However,

$$\begin{aligned}
\frac{\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right)\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right)+\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right)+\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right)} &\leq \max\{\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right)\},\\
\frac{\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right)\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right)+\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right)+\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right)} &\leq \max\{\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right)\},\\
\frac{\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right)\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right)+\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right)+\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right)} &\leq \Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right),
\end{aligned}$$

and

$$\frac{\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right)\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)}{1+\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right)+\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right)+\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right)} \leq \Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right).$$

Then

$$\sqsupset\left(\theta_1^{\omega-1},\theta_2^{\omega-1},\theta_3^{\omega-1},\theta_1^{\omega},\theta_2^{\omega},\theta_3^{\omega}\right)=\max\left\{\begin{array}{c}\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right),\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right),\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right),\\ \Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)\end{array}\right\}.$$

Hence

$$\begin{aligned}
\varphi\left(\Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right)\right) \leq\; & \rho\left(\max\left\{\begin{array}{c}\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right),\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right),\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right),\\ \Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)\end{array}\right\}\right)\\
& -\ell\left(\max\left\{\begin{array}{c}\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right),\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right),\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right),\\ \Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)\end{array}\right\}\right).
\end{aligned}$$

Analogously, for the second and third components, we can write

$$\begin{aligned}
\varphi\left(\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right)\right) \leq\; & \rho\left(\max\left\{\begin{array}{c}\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right),\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right),\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right),\\ \Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)\end{array}\right\}\right)\\
& -\ell\left(\max\left\{\begin{array}{c}\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right),\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right),\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right),\\ \Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)\end{array}\right\}\right),
\end{aligned}$$

and

$$\begin{aligned}
\varphi\left(\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)\right) \leq\; & \rho\left(\max\left\{\begin{array}{c}\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right),\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right),\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right),\\ \Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)\end{array}\right\}\right)\\
& -\ell\left(\max\left\{\begin{array}{c}\Lambda\left(\xi^*\theta_1^{\omega-1},\xi^*\theta_1^{\omega}\right),\Lambda\left(\xi^*\theta_2^{\omega-1},\xi^*\theta_2^{\omega}\right),\Lambda\left(\xi^*\theta_3^{\omega-1},\xi^*\theta_3^{\omega}\right),\\ \Lambda\left(\xi^*\theta_1^{\omega},\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_2^{\omega},\xi^*\theta_2^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^{\omega},\xi^*\theta_3^{\omega+1}\right)\end{array}\right\}\right).
\end{aligned}$$

Set $\nabla_\omega = \max\{\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right), \Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right), \Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right)\}$. Let us consider for all $\omega \geq 1$, $\nabla_\omega \neq 0$. Moreover, let, if possible for some ω, $\nabla_{\omega-1} < \nabla_\omega$. Now

$$\begin{aligned}
\varphi(\nabla_\omega) &= \varphi\left(\max\{\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right), \Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right), \Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right)\}\right) \\
&= \max\{\varphi\left(\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right)\right), \varphi\left(\Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right)\right), \varphi\left(\Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right)\right)\} \\
&\leq \rho\left(\max\left\{\begin{array}{l} \Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right), \Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right), \Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right), \\ \Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right), \Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right), \Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right) \end{array}\right\}\right) \\
&\quad -\ell\left(\max\left\{\begin{array}{l} \Lambda\left(\xi^*\theta_1^{\omega-1}, \xi^*\theta_1^\omega\right), \Lambda\left(\xi^*\theta_2^{\omega-1}, \xi^*\theta_2^\omega\right), \Lambda\left(\xi^*\theta_3^{\omega-1}, \xi^*\theta_3^\omega\right), \\ \Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right), \Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right), \Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right) \end{array}\right\}\right) \\
&= \rho(\max\{\nabla_{\omega-1}, \nabla_\omega\}) - \ell(\max\{\nabla_{\omega-1}, \nabla_\omega\}) \\
&= \rho(\nabla_\omega) - \ell(\nabla_\omega).
\end{aligned}$$

It follows from (a_2) and (a_3) that $\nabla_\omega = 0$, a contradiction. Hence $\nabla_{\omega-1} \leq \nabla_\omega$. This proves that the sequence $\{\nabla_\omega\}$ is a non-increasing and must converge to a real number ϖ(say)≥ 0. Furthermore,

$$\varphi(\nabla_\omega) \leq \rho(\nabla_{\omega-1}) - \ell(\nabla_{\omega-1}).$$

Passing $\omega \to \infty$, we have

$$\varphi(\varpi) \leq \rho(\varpi) - \ell(\varpi).$$

Based on assertions (a_2) and (a_3), we have $\varpi = 0$. Thus

$$\lim_{\omega\to\infty} \max\{\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right), \Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right), \Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right)\} = 0,$$

this leads to

$$\lim_{\omega\to\infty} \Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right) = 0,\ \lim_{\omega\to\infty} \Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right) = 0 \text{ and } \lim_{\omega\to\infty} \Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right) = 0. \tag{1}$$

By (Λ_2), one can write

$$\lim_{\omega\to\infty} \Lambda(\xi^*\theta_1^\omega, \xi^*\theta_1^\omega) = 0,\ \lim_{\omega\to\infty} \Lambda(\xi^*\theta_2^\omega, \xi^*\theta_2^\omega) = 0 \text{ and } \lim_{\omega\to\infty} \Lambda(\xi^*\theta_3^\omega, \xi^*\theta_3^\omega) = 0. \tag{2}$$

It follows from (1), (2) and the definition of Ω_Λ that

$$\lim_{\omega\to\infty} \Omega_\Lambda\left(\xi^*\theta_1^\omega, \xi^*\theta_1^{\omega+1}\right) = 0,\ \lim_{\omega\to\infty} \Omega_\Lambda\left(\xi^*\theta_2^\omega, \xi^*\theta_2^{\omega+1}\right) = 0 \text{ and } \lim_{\omega\to\infty} \Omega_\Lambda\left(\xi^*\theta_3^\omega, \xi^*\theta_3^{\omega+1}\right) = 0. \tag{3}$$

Now, using the contradiction method, we're going to prove that $\{\xi^*\theta_1^\omega\}$, $\{\xi^*\theta_2^\omega\}$ and $\{\xi^*\theta_3^\omega\}$ are Cauchy sequences. Assume that $\{\xi^*\theta_1^\omega\}$ or $\{\xi^*\theta_2^\omega\}$ or $\{\xi^*\theta_3^\omega\}$ is not Cauchy. This means $\Omega_\Lambda(\xi^*\theta_1^\upsilon, \xi^*\theta_1^\omega) \nrightarrow 0$ or $\Omega_\Lambda(\xi^*\theta_2^\upsilon, \xi^*\theta_2^\omega) \nrightarrow 0$ or $\Omega_\Lambda(\xi^*\theta_3^\upsilon, \xi^*\theta_3^\omega) \nrightarrow 0$ as $\omega, \upsilon \to \infty$. Consequently,

$$\max\{\Omega_\Lambda(\xi^*\theta_1^\upsilon, \xi^*\theta_1^\omega), \Omega_\Lambda(\xi^*\theta_2^\upsilon, \xi^*\theta_2^\omega), \Omega_\Lambda(\xi^*\theta_3^\upsilon, \xi^*\theta_3^\omega)\} \nrightarrow 0, \text{ as } \omega, \upsilon \to \infty.$$

Then there is an $\epsilon > 0$ and monotone increasing sequences $\{\upsilon_j\}$ and $\{\omega_j\}$ so that $\omega_j > \upsilon_j > j$,

$$\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{\upsilon_j}, \xi^*\theta_1^{\omega_j}\right), \Omega_\Lambda\left(\xi^*\theta_2^{\upsilon_j}, \xi^*\theta_2^{\omega_j}\right), \Omega_\Lambda\left(\xi^*\theta_3^{\upsilon_j}, \xi^*\theta_3^{\omega_j}\right)\right\} \geq \epsilon, \tag{4}$$

and

$$\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j-1}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j-1}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j-1}\right)\right\}<\epsilon. \tag{5}$$

From (4) and (5), we obtain that

$$\begin{aligned}\epsilon &\leq \max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j}\right)\right\}\\ &\leq \max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j-1}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j-1}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j-1}\right)\right\}\\ &\quad+\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{\omega_j-1},\xi^*\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_2^{\omega_j-1},\xi^*\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_3^{\omega_j-1},\xi^*\theta_3^{\omega_j}\right)\right\}\\ &< \epsilon+\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{\omega_j-1},\xi^*\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_2^{\omega_j-1},\xi^*\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_3^{\omega_j-1},\xi^*\theta_3^{\omega_j}\right)\right\}.\end{aligned}$$

As $j\to\infty$ and applying (3), one sees that

$$\lim_{j\to\infty}\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j}\right)\right\}=\epsilon. \tag{6}$$

According to the definition of Ω_Λ and (2), one can obtain

$$\lim_{j\to\infty}\max\left\{\Lambda\left(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j}\right),\Lambda\left(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j}\right),\Lambda\left(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j}\right)\right\}=\frac{\epsilon}{2}. \tag{7}$$

From (4), we find that

$$\begin{aligned}\epsilon &\leq \max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j}\right)\right\}\\ &\leq \max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j},\xi^*\theta_1^{v_j-1}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j},\xi^*\theta_2^{v_j-1}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j},\xi^*\theta_3^{v_j-1}\right)\right\}\\ &\quad+\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j-1},\xi^*\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j-1},\xi^*\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j-1},\xi^*\theta_3^{\omega_j}\right)\right\}\\ &\leq 2\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j},\xi^*\theta_1^{v_j-1}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j},\xi^*\theta_2^{v_j-1}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j},\xi^*\theta_3^{v_j-1}\right)\right\}\\ &\quad+\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j-1},\xi^*\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j-1},\xi^*\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j-1},\xi^*\theta_3^{\omega_j}\right)\right\}.\end{aligned} \tag{8}$$

Taking $j\to\infty$ and applying (3), (6) and (8), we have

$$\lim_{j\to\infty}\max\left\{\Omega_\Lambda\left(\xi^*\theta_1^{v_j-1},\xi^*\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_2^{v_j-1},\xi^*\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\xi^*\theta_3^{v_j-1},\xi^*\theta_3^{\omega_j}\right)\right\}=\epsilon. \tag{9}$$

Hence, we obtain

$$\lim_{j\to\infty}\max\left\{\Lambda\left(\xi^*\theta_1^{v_j-1},\xi^*\theta_1^{\omega_j}\right),\Lambda\left(\xi^*\theta_2^{v_j-1},\xi^*\theta_2^{\omega_j}\right),\Lambda\left(\xi^*\theta_3^{v_j-1},\xi^*\theta_3^{\omega_j}\right)\right\}=\frac{\epsilon}{2}. \tag{10}$$

From (5), we obtain that

$$\begin{aligned}
\epsilon &\le \max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j}\Big)\Big\}\\
&\le \max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{v_j-1}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{v_j-1}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{v_j-1}\Big)\Big\}\\
&\quad+\max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j-1},\xi^*\theta_1^{\omega_j+1}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j-1},\xi^*\theta_2^{\omega_j+1}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j-1},\xi^*\theta_3^{\omega_j+1}\Big)\Big\}\\
&\quad+\max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{\omega_j+1},\xi^*\theta_1^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{\omega_j+1},\xi^*\theta_2^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{\omega_j+1},\xi^*\theta_3^{\omega_j}\Big)\Big\}\\
&\le 2\max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{v_j-1}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{v_j-1}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{v_j-1}\Big)\Big\}\\
&\quad+\max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j}\Big)\Big\}\\
&\quad+2\max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{\omega_j+1},\xi^*\theta_1^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{\omega_j+1},\xi^*\theta_2^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{\omega_j+1},\xi^*\theta_3^{\omega_j}\Big)\Big\}. \qquad (11)
\end{aligned}$$

Letting $j\to\infty$ in (11), using (3) and (9), we have

$$\lim_{j\to\infty}\max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j-1},\xi^*\theta_1^{\omega_j+1}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j-1},\xi^*\theta_2^{\omega_j+1}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j-1},\xi^*\theta_3^{\omega_j+1}\Big)\Big\}=\epsilon.$$

Hence, we obtain

$$\lim_{j\to\infty}\max\Big\{\Lambda\Big(\xi^*\theta_1^{v_j-1},\xi^*\theta_1^{\omega_j+1}\Big),\Lambda\Big(\xi^*\theta_2^{v_j-1},\xi^*\theta_2^{\omega_j+1}\Big),\Lambda\Big(\xi^*\theta_3^{v_j-1},\xi^*\theta_3^{\omega_j+1}\Big)\Big\}=\frac{\epsilon}{2}.$$

Again, from (4), one can obtain

$$\begin{aligned}
\epsilon &\le \max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j}\Big)\Big\}\\
&\le \max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j+1}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j+1}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j+1}\Big)\Big\}\\
&\quad+\max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{\omega_j+1},\xi^*\theta_1^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{\omega_j+1},\xi^*\theta_2^{\omega_j}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{\omega_j+1},\xi^*\theta_3^{\omega_j}\Big)\Big\}
\end{aligned}$$

Setting $j\to\infty$, from(2) and (3), one sees that

$$\begin{aligned}
\epsilon &\le \lim_{j\to\infty}\max\Big\{\Omega_\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j+1}\Big),\Omega_\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j+1}\Big),\Omega_\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j+1}\Big)\Big\}+0\\
&\le \lim_{j\to\infty}\max\left\{\begin{array}{c}
2\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j+1}\Big)-\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{v_j}\Big)-\Lambda\Big(\xi^*\theta_1^{\omega_j+1},\xi^*\theta_1^{\omega_j+1}\Big),\\
2\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j+1}\Big)-\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{v_j}\Big)-\Lambda\Big(\xi^*\theta_2^{\omega_j+1},\xi^*\theta_2^{\omega_j+1}\Big),\\
2\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j+1}\Big)-\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{v_j}\Big)-\Lambda\Big(\xi^*\theta_3^{\omega_j+1},\xi^*\theta_3^{\omega_j+1}\Big)
\end{array}\right\}\\
&= 2\lim_{j\to\infty}\max\Big\{\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j+1}\Big),\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j+1}\Big),\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j+1}\Big)\Big\}.
\end{aligned}$$

Thus

$$\frac{\epsilon}{2}\le\lim_{j\to\infty}\max\Big\{\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j+1}\Big),\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j+1}\Big),\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j+1}\Big)\Big\}.$$

By the definition of φ,

$$\begin{aligned}
\varphi\Big(\frac{\epsilon}{2}\Big) &\le \lim_{j\to\infty}\varphi\Big(\max\Big\{\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j+1}\Big),\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j+1}\Big),\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j+1}\Big)\Big\}\Big) \qquad (12)\\
&= \lim_{j\to\infty}\max\Big\{\varphi\Big(\Lambda\Big(\xi^*\theta_1^{v_j},\xi^*\theta_1^{\omega_j+1}\Big)\Big),\varphi\Big(\Lambda\Big(\xi^*\theta_2^{v_j},\xi^*\theta_2^{\omega_j+1}\Big)\Big),\varphi\Big(\Lambda\Big(\xi^*\theta_3^{v_j},\xi^*\theta_3^{\omega_j+1}\Big)\Big)\Big\}.
\end{aligned}$$

Consider

$$
\begin{aligned}
\varphi\left(\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)\right) &= \varphi\left(\Lambda\left(\xi\left(\theta_1^{\upsilon_j-1},\theta_2^{\upsilon_j-1},\theta_3^{\upsilon_j-1}\right),\xi\left(\theta_1^{\omega_j},\theta_2^{\omega_j},\theta_3^{\omega_j}\right)\right)\right) \\
&\leq \rho\left(\sqsupset\left(\theta_1^{\upsilon_j-1},\theta_2^{\upsilon_j-1},\theta_3^{\upsilon_j-1},\theta_1^{\omega_j},\theta_2^{\omega_j},\theta_3^{\omega_j}\right)\right)-\ell\left(\sqsupset\left(\theta_1^{\upsilon_j-1},\theta_2^{\upsilon_j-1},\theta_3^{\upsilon_j-1},\theta_1^{\omega_j},\theta_2^{\omega_j},\theta_3^{\omega_j}\right)\right) \\
&= \rho\left(\max\left\{\begin{array}{c}
\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right),\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\omega_j}\right),\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\omega_j}\right), \\
\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\upsilon_j}\right),\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\upsilon_j}\right),\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\upsilon_j}\right), \\
\Lambda\left(\xi^*\theta_1^{\omega_j},\xi^*\theta_1^{\omega_j+1}\right),\Lambda\left(\xi^*\theta_2^{\omega_j},\xi^*\theta_2^{\omega_j+1}\right),\Lambda\left(\xi^*\theta_3^{\omega_j},\xi^*\theta_3^{\omega_j+1}\right), \\
\frac{\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\upsilon_j}\right)\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\upsilon_j}\right)}{1+\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right)+\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\omega_j}\right)+\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)}, \\
\frac{\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\upsilon_j}\right)\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\upsilon_j}\right)}{1+\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right)+\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\omega_j}\right)+\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)}, \\
\frac{\Lambda\left(\xi^*\theta_1^{\omega_j},\xi^*\theta_1^{\omega_j+1}\right)\Lambda\left(\xi^*\theta_2^{\omega_j},\xi^*\theta_2^{\omega_j+1}\right)}{1+\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right)+\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\omega_j}\right)+\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)}, \\
\frac{\Lambda\left(\xi^*\theta_1^{\omega_j},\xi^*\theta_1^{\omega_j+1}\right)\Lambda\left(\xi^*\theta_3^{\omega_j},\xi^*\theta_3^{\omega_j+1}\right)}{1+\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right)+\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\omega_j}\right)+\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)}
\end{array}\right\}\right) \\
&\quad -\ell\left(\max\left\{\begin{array}{c}
\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right),\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\omega_j}\right),\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\omega_j}\right), \\
\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\upsilon_j}\right),\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\upsilon_j}\right),\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\upsilon_j}\right), \\
\Lambda\left(\xi^*\theta_1^{\omega_j},\xi^*\theta_1^{\omega_j+1}\right),\Lambda\left(\xi^*\theta_2^{\omega_j},\xi^*\theta_2^{\omega_j+1}\right),\Lambda\left(\xi^*\theta_3^{\omega_j},\xi^*\theta_3^{\omega_j+1}\right), \\
\frac{\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\upsilon_j}\right)\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\upsilon_j}\right)}{1+\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right)+\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\omega_j}\right)+\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)}, \\
\frac{\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\upsilon_j}\right)\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\upsilon_j}\right)}{1+\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right)+\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\omega_j}\right)+\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)}, \\
\frac{\Lambda\left(\xi^*\theta_1^{\omega_j},\xi^*\theta_1^{\omega_j+1}\right)\Lambda\left(\xi^*\theta_2^{\omega_j},\xi^*\theta_2^{\omega_j+1}\right)}{1+\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right)+\Lambda\left(\xi^*\theta_2^{\upsilon_j-1},\xi^*\theta_2^{\omega_j}\right)+\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)}, \\
\frac{\Lambda\left(\xi^*\theta_1^{\omega_j},\xi^*\theta_1^{\omega_j+1}\right)\Lambda\left(\xi^*\theta_3^{\omega_j},\xi^*\theta_3^{\omega_j+1}\right)}{1+\Lambda\left(\xi^*\theta_1^{\upsilon_j-1},\xi^*\theta_1^{\omega_j}\right)+\Lambda\left(\xi^*\theta_3^{\upsilon_j-1},\xi^*\theta_3^{\omega_j}\right)+\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)}
\end{array}\right\}\right).
\end{aligned}
$$

Passing $j \to \infty$, we can write

$$\lim_{j\to\infty}\varphi\left(\Lambda\left(\xi^*\theta_1^{\upsilon_j},\xi^*\theta_1^{\omega_j+1}\right)\right) \leq \rho\left(\frac{\epsilon}{2}\right)-\ell\left(\frac{\epsilon}{2}\right).$$

Analogously, we obtain

$$\lim_{j\to\infty}\varphi\left(\Lambda\left(\xi^*\theta_2^{\upsilon_j},\xi^*\theta_2^{\omega_j+1}\right)\right) \leq \rho\left(\frac{\epsilon}{2}\right)-\ell\left(\frac{\epsilon}{2}\right),$$

and

$$\lim_{j\to\infty}\varphi\left(\Lambda\left(\xi^*\theta_3^{\upsilon_j},\xi^*\theta_3^{\omega_j+1}\right)\right) \leq \rho\left(\frac{\epsilon}{2}\right)-\ell\left(\frac{\epsilon}{2}\right).$$

So, from (12), one can write

$$\rho\left(\frac{\epsilon}{2}\right) \leq \rho\left(\frac{\epsilon}{2}\right)-\ell\left(\frac{\epsilon}{2}\right).$$

It follows from hypotheses (a_2) and (a_3) that $\frac{\epsilon}{2} = 0$, a contradiction. Hence, $\{\xi^*\theta_1^\omega\}$, $\{\xi^*\theta_2^\omega\}$ and $\{\xi^*\theta_3^\omega\}$ are Cauchy sequences in the metric space $(\daleth, \Omega_\Lambda)$. Therefore, $\Omega_\Lambda\left(\xi^*\theta_1^\upsilon, \xi^*\theta_1^\omega\right) \to 0$, $\Omega_\Lambda\left(\xi^*\theta_2^\upsilon, \xi^*\theta_2^\omega\right) \to 0$ and $\Omega_\Lambda\left(\xi^*\theta_3^\upsilon, \xi^*\theta_3^\omega\right) \to 0$ as $\upsilon, \omega \to \infty$. Thus, by (2) and the definition of Ω_Λ, we have

$$\lim_{\upsilon,\omega\to\infty} \Lambda(\xi^*\theta_1^\upsilon, \xi^*\theta_1^\omega) = 0, \lim_{\upsilon,\omega\to\infty} \Lambda(\xi^*\theta_2^\upsilon, \xi^*\theta_2^\omega) = 0 \text{ and } \lim_{\upsilon,\omega\to\infty} \Lambda(\xi^*\theta_3^\upsilon, \xi^*\theta_3^\omega) = 0. \quad (13)$$

Because $\xi^*(\daleth)$ is a complete subspace of $\daleth$ and $\{\xi^*\theta_1^\omega\}$, $\{\xi^*\theta_2^\omega\}$ and $\{\xi^*\theta_3^\omega\}$ are Cauchy sequences in a complete metric space $(\xi^*(\daleth), \Omega_\Lambda)$, then $\{\xi^*\theta_1^\omega\}$, $\{\xi^*\theta_2^\omega\}$ and $\{\xi^*\theta_3^\omega\}$ converges to some ϱ_1, ϱ_2 and ϱ_3 in $\xi^*(\daleth)$, respectively. Thus,

$$\lim_{\omega\to\infty} \Omega_\Lambda(\xi^*\theta_1^\omega, \varrho_1) = 0, \lim_{\omega\to\infty} \Omega_\Lambda(\xi^*\theta_2^\omega, \varrho_2) = 0 \text{ and } \lim_{\omega\to\infty} \Omega_\Lambda(\xi^*\theta_3^\omega, \varrho_3) = 0.$$

for some $\varrho_1, \varrho_2, \varrho_3 \in \xi^*(\daleth)$. Since $\varrho_1, \varrho_2, \varrho_3 \in \xi^*(\daleth)$, there are $\theta_1, \theta_2, \theta_3 \in \daleth$ so that $\varrho_1 = \xi^*\theta_1$, $\varrho_2 = \xi^*\theta_2$ and $\varrho_3 = \xi^*\theta_3$. Because $\{\xi^*\theta_1^\omega\}$, $\{\xi^*\theta_2^\omega\}$ and $\{\xi^*\theta_3^\omega\}$ are Cauchy sequences, then $\{\xi^*\theta_1^\omega\} \to \varrho_1$, $\{\xi^*\theta_2^\omega\} \to \varrho_2$, $\{\xi^*\theta_3^\omega\} \to \varrho_3$, $\{\xi^*\theta_1^{\omega+1}\} \to \varrho_1$, $\{\xi^*\theta_2^{\omega+1}\} \to \varrho_2$ and $\{\xi^*\theta_3^{\omega+1}\} \to \varrho_3$. Applying Lemma 1 (ii) and (13), we obtain

$$\begin{aligned}\Lambda(\varrho_1, \varrho_1) &= \lim_{\omega\to\infty} \Lambda(\xi^*\theta_1^\omega, \varrho_1) = \Lambda(\varrho_2, \varrho_2) \\ &= \lim_{\omega\to\infty} \Lambda(\xi^*\theta_2^\omega, \varrho_2) = \Lambda(\varrho_3, \varrho_3) = \lim_{\omega\to\infty} \Lambda(\xi^*\theta_3^\omega, \varrho_3) = 0. \end{aligned} \quad (14)$$

Next, we want to show that

$$\lim_{\omega\to\infty} \Lambda(\xi(\theta_1, \theta_2, \theta_3), \xi^*\theta_1^\omega) = \Lambda(\xi(\theta_1, \theta_2, \theta_3), \varrho_1).$$

Based on definition of Ω_Λ, we obtain

$$\begin{aligned}\Omega_\Lambda(\xi(\theta_1, \theta_2, \theta_3), \xi^*\theta_1^\omega) &= 2\Lambda(\xi(\theta_1, \theta_2, \theta_3), \xi^*\theta_1^\omega) \\ &\quad -\Lambda(\xi(\theta_1, \theta_2, \theta_3), \xi(\theta_1, \theta_2, \theta_3)) - \Lambda(\xi^*\theta_1^\omega, \xi^*\theta_1^\omega),\end{aligned}$$

Passing $\omega \to \infty$ and using (2),

$$\Omega_\Lambda(\xi(\theta_1, \theta_2, \theta_3), \varrho_1) = 2\lim_{\omega\to\infty} \Lambda(\xi(\theta_1, \theta_2, \theta_3), \xi^*\theta_1^\omega) - \Lambda(\xi(\theta_1, \theta_2, \theta_3), \xi(\theta_1, \theta_2, \theta_3)) - 0$$

According to definition of Ω_Λ and (13), one can write

$$\lim_{\omega\to\infty} \Lambda(\xi(\theta_1, \theta_2, \theta_3), \xi^*\theta_1^\omega) = \Lambda(\xi(\theta_1, \theta_2, \theta_3), \varrho_1),$$

Similarly

$$\lim_{\omega\to\infty} \Lambda(\xi(\theta_2, \theta_3, \theta_1), \xi^*\theta_2^\omega) = \Lambda(\xi(\theta_2, \theta_3, \theta_1), \varrho_2),$$

and

$$\lim_{\omega\to\infty} \Lambda(\xi(\theta_3, \theta_1, \theta_2), \xi^*\theta_3^\omega) = \Lambda(\xi(\theta_3, \theta_1, \theta_2), \varrho_3).$$

From (Λ_4), we obtain

$$\Lambda(\varrho_1, \xi(\theta_1, \theta_2, \theta_3)) \le \Lambda\left(\varrho_1, \xi^*\theta_1^{\omega+1}\right) + \Lambda\left(\xi^*\theta_1^{\omega+1}, \xi(\theta_1, \theta_2, \theta_3)\right).$$

Setting $\omega \to \infty$, we have

$$\Lambda(\varrho_1, \xi(\theta_1, \theta_2, \theta_3)) \le 0 + \lim_{\omega\to\infty} \Lambda(\xi(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega), \xi(\theta_1, \theta_2, \theta_3)). \quad (15)$$

From (a_1), (a_2) and (15), one can obtain

$$\begin{aligned}
\varphi(\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3))) &\leq \lim_{\omega\to\infty}\varphi(\Lambda(\xi(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega),\xi(\theta_1,\theta_2,\theta_3))) \\
&\leq \lim_{\omega\to\infty}[\rho(\sqsupset(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\theta_1,\theta_2,\theta_3)) - \ell(\sqsupset(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\theta_1,\theta_2,\theta_3))],
\end{aligned}$$

where

$$\begin{aligned}
&\sqsupset(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\theta_1,\theta_2,\theta_3) \\
=\;& \max\left\{\begin{array}{c}
\Lambda\left(\xi^*\theta_1^\omega,\varrho_1\right),\Lambda(\xi^*\theta_2^\omega,\varrho_2),\Lambda\left(\xi^*\theta_3^\omega,\varrho_3\right), \\
\Lambda\left(\xi^*\theta_1^\omega,\xi^*\theta_1^{\omega+1}\right),\Lambda\left(\xi^*\theta_2^\omega,\xi^*\theta_2^{\omega+1}\right),\Lambda\left(\xi^*\theta_3^\omega,\xi^*\theta_3^{\omega+1}\right), \\
\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2)), \\
\frac{\Lambda\left(\xi^*\theta_1^\omega,\xi^*\theta_1^{\omega+1}\right)\Lambda\left(\xi^*\theta_2^\omega,\xi^*\theta_2^{\omega+1}\right)}{1+\Lambda\left(\xi^*\theta_1^\omega,\varrho_1\right)+\Lambda\left(\xi^*\theta_2^\omega,\varrho_2\right)+\Lambda\left(\xi^*\theta_1^{\omega+1},\xi(\theta_1,\theta_2,\theta_3)\right)}, \\
\frac{\Lambda\left(\xi^*\theta_1^\omega,\xi^*\theta_1^{\omega+1}\right)\Lambda\left(\xi^*\theta_3^\omega,\xi^*\theta_3^{\omega+1}\right)}{1+\Lambda\left(\xi^*\theta_1^\omega,\varrho_1\right)+\Lambda\left(\xi^*\theta_3^\omega,\varrho_3\right)+\Lambda\left(\xi^*\theta_1^{\omega+1},\xi(\theta_1,\theta_2,\theta_3)\right)}, \\
\frac{\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3))\Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1))}{1+\Lambda\left(\xi^*\theta_1^\omega,\varrho_1\right)+\Lambda\left(\xi^*\theta_2^\omega,\varrho_2\right)+\Lambda\left(\xi^*\theta_1^{\omega+1},\xi(\theta_1,\theta_2,\theta_3)\right)}, \\
\frac{\Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1))\Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))}{1+\Lambda\left(\xi^*\theta_1^\omega,\varrho_1\right)+\Lambda\left(\xi^*\theta_3^\omega,\varrho_3\right)+\Lambda\left(\xi^*\theta_1^{\omega+1},\xi(\theta_1,\theta_2,\theta_3)\right)}
\end{array}\right\} \\
\to\;& \max\{\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\} \text{ as } \omega\to\infty.
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi(\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3))) \leq\;& \rho\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\end{array}\right\}\right) \\
&-\ell\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\end{array}\right\}\right),
\end{aligned}$$

Analogously,

$$\begin{aligned}
\varphi(\Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1))) \leq\;& \rho\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\end{array}\right\}\right) \\
&-\ell\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\end{array}\right\}\right),
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\Lambda(\varrho_3,\xi(\theta_3,\theta_2,\theta_1))) \leq\;& \rho\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\end{array}\right\}\right) \\
&-\ell\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\end{array}\right\}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\varphi\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_2,\theta_1))\end{array}\right\}\right) &= \max\left\{\begin{array}{c}\varphi(\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3))),\\ \varphi(\Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1))),\\ \varphi(\Lambda(\varrho_3,\xi(\theta_3,\theta_2,\theta_1)))\end{array}\right\}\\
&\leq \rho\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\end{array}\right\}\right)\\
&\quad -\ell\left(\max\left\{\begin{array}{c}\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\\ \Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\\ \Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\end{array}\right\}\right).
\end{aligned}$$

This implies that

$$\max\{\Lambda(\varrho_1,\xi(\theta_1,\theta_2,\theta_3)),\Lambda(\varrho_2,\xi(\theta_2,\theta_3,\theta_1)),\Lambda(\varrho_3,\xi(\theta_3,\theta_1,\theta_2))\}=0.$$

So $\varrho_1=\xi(\theta_1,\theta_2,\theta_3)$, $\varrho_2=\xi(\theta_2,\theta_3,\theta_1)$ and $\varrho_3=\xi(\theta_3,\theta_1,\theta_2)$. This leads to $\xi(\theta_1,\theta_2,\theta_3)=\xi^*\theta_1=\varrho_1$, $\xi(\theta_2,\theta_3,\theta_1)=\xi^*\theta_2=\varrho_2$ and $\xi(\theta_3,\theta_1,\theta_2)=\xi^*\theta_3=\varrho_3$. Therefore, ξ and ξ^* have a coincidence point in $\daleth^3$. □

The following theorem gives the uniqueness of a TFP:

Theorem 2. *Adding to hypotheses of Theorem 1 the following hypothesis:*

Let for each $(\theta_1,\theta_2,\theta_3),(\vartheta_1,\vartheta_2,\vartheta_3)\in\daleth^3$ there is a trio $(\varrho_1,\varrho_2,\varrho_3)\in\daleth^3$ so that a trio $(\xi(\varrho_1,\varrho_2,\varrho_3),\xi(\varrho_2,\varrho_3,\varrho_1),\ \xi(\varrho_3,\varrho_1,\varrho_2))$ is comparable to $(\xi(\theta_1,\theta_2,\theta_3),\xi(\theta_2,\theta_3,\theta_1),\xi(\theta_3,\theta_1,\theta_2))$ and $(\xi(\vartheta_1,\vartheta_2,\vartheta_3),\xi(\vartheta_2,\vartheta_3,\vartheta_1),\xi(\vartheta_3,\vartheta_1,\vartheta_2))$. If $(\theta_1,\theta_2,\theta_3)$ and $(\vartheta_1,\vartheta_2,\vartheta_3)$ are TCPs of ξ and ξ^* then

$$\begin{aligned}
\xi(\theta_1,\theta_2,\theta_3) &= \xi^*\theta_1=\xi^*\vartheta_1=\xi(\vartheta_1,\vartheta_2,\vartheta_3),\\
\xi(\theta_2,\theta_3,\theta_1) &= \xi^*\theta_2=\xi^*\vartheta_2=\xi(\vartheta_2,\vartheta_3,\vartheta_1),\\
\text{and } \xi(\theta_3,\theta_1,\theta_2) &= \xi^*\theta_3=\xi^*\vartheta_3=\xi(\vartheta_3,\vartheta_1,\vartheta_2).
\end{aligned}$$

Furthermore, if ξ and ξ^* are $w-$compatible, then there is a unique common TFP of ξ and ξ^* in $\daleth^3$.

Proof. The proof follows immediately from Theorem 1 and the concept of comparability. □

The result below follows from Theorem 1 and it is important in the next section.

Corollary 1. *Let $(\daleth,\preceq)$ be a POS and Λ be a partial metric so that $(\daleth,\Lambda)$ is a PMS. Assume that $\xi:\daleth^3\to\daleth$ is a mapping so that*

$$\begin{aligned}
\varphi(\Lambda(\xi(\theta_1,\theta_2,\theta_3),\xi(\vartheta_1,\vartheta_2,\vartheta_3))) &\leq \rho(\max\{\Lambda(\theta_1,\vartheta_1),\Lambda(\theta_2,\vartheta_2),\Lambda(\theta_3,\vartheta_3)\})\\
&\quad -\ell(\max\{\Lambda(\theta_1,\vartheta_1),\Lambda(\theta_2,\vartheta_2),\Lambda(\theta_3,\vartheta_3)\}), \quad (16)
\end{aligned}$$

for all $\theta_1,\theta_2,\theta_3,\vartheta_1,\vartheta_2,\vartheta_3\in\daleth$ with $\theta_1\preceq\vartheta_1$, $\theta_2\succeq\vartheta_2$ and $\theta_3\preceq\vartheta_3$, where φ,ρ and ℓ are described in Definition 11 and

(i) If non-decreasing sequences $\{\theta_1^n\}\to\theta_1$ and $\{\theta_3^n\}\to\theta_3$, then $\theta_1^n\preceq\theta_1$ and $\theta_3^n\preceq\theta_3$ for all n;

(ii) If a non-increasing sequence $\{\theta_2^n\}\to\theta_2$, then $\theta_2\preceq\theta_2^n$, for all n.

If there are $\theta_1^0,\theta_2^0,\theta_3^0\in\daleth$ so that $\theta_1^0\preceq\xi(\theta_1^0,\theta_2^0,\theta_3^0)$, $\theta_2^0\succeq\xi(\theta_2^0,\theta_3^0,\theta_1^0)$ and $\theta_3^0\preceq\xi(\theta_3^0,\theta_1^0,\theta_2^0)$, then ξ has a TCP in $\daleth^3$.

Example 3. *Suppose that $\daleth=[0,1]$. Describe a partially ordered $\preceq$ on $\daleth$ as*

$$\theta_1\preceq\theta_2\Longleftrightarrow\theta_1\leq\theta_2.$$

Define the mapping $\xi : \daleth^3 \to \daleth$ *by* $\xi(\theta_1, \theta_2, \theta_3) = \frac{\theta_1^2+\theta_2^2+\theta_3^2}{8(\theta_1+\theta_2+\theta_3+1)}$ *and* $\Lambda : \daleth \times \daleth \to [0, \infty)$ *by* $\Lambda(\theta_1, \theta_2) = \max\{\theta_1, \theta_2\}$. *It is clear that* $(\daleth, \Lambda)$ *is a PMS. Define* $\varphi, \rho, \ell : [0, \infty) \to [0, \infty)$ *by* $\varphi(\nu) = \nu$, $\rho(\nu) = \frac{5\nu}{8}$ *and* $\ell(\nu) = \frac{\nu}{4}$.

Consider

$$
\begin{aligned}
& \Lambda(\xi(\theta_1, \theta_2, \theta_3), \xi(\vartheta_1, \vartheta_2, \vartheta_3)) \\
= \;& \max\left\{ \frac{\theta_1^2 + \theta_2^2 + \theta_3^2}{8(\theta_1 + \theta_2 + \theta_3 + 1)}, \frac{\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2}{8(\vartheta_1 + \vartheta_2 + \vartheta_3 + 1)} \right\} \\
= \;& \frac{1}{8}\left(\max\left\{ \frac{\theta_1^2}{\theta_1 + \theta_2 + \theta_3 + 1}, \frac{\vartheta_1^2}{\vartheta_1 + \vartheta_2 + \vartheta_3 + 1} \right\} \right. \\
& + \max\left\{ \frac{\theta_2^2}{\theta_1 + \theta_2 + \theta_3 + 1}, \frac{\vartheta_2^2}{\vartheta_1 + \vartheta_2 + \vartheta_3 + 1} \right\} \\
& \left. + \max\left\{ \frac{\theta_3^2}{\theta_1 + \theta_2 + \theta_3 + 1}, \frac{\vartheta_3^2}{\vartheta_1 + \vartheta_2 + \vartheta_3 + 1} \right\} \right) \\
\leq \;& \frac{1}{8}\left(\max\left\{ \frac{\theta_1^2}{\theta_1 + 1}, \frac{\vartheta_1^2}{\vartheta_1 + 1} \right\} + \max\left\{ \frac{\theta_2^2}{\theta_2 + 1}, \frac{\vartheta_2^2}{\vartheta_2 + 1} \right\} + \max\left\{ \frac{\theta_3^2}{\theta_3 + 1}, \frac{\vartheta_3^2}{\vartheta_3 + 1} \right\} \right) \\
\leq \;& \frac{1}{8}\left(\max\left\{ \frac{\theta_1}{\theta_1 + 1}, \frac{\vartheta_1}{\vartheta_1 + 1} \right\} + \max\left\{ \frac{\theta_2}{\theta_2 + 1}, \frac{\vartheta_2}{\vartheta_2 + 1} \right\} + \max\left\{ \frac{\theta_3}{\theta_3 + 1}, \frac{\vartheta_3}{\vartheta_3 + 1} \right\} \right) \\
\leq \;& \frac{1}{8}(\max\{\theta_1, \vartheta_1\} + \max\{\theta_2, \vartheta_2\} + \max\{\theta_3, \vartheta_3\}) \\
= \;& \frac{1}{8}(\Lambda(\theta_1, \vartheta_1) + \Lambda(\theta_2, \vartheta_2) + \Lambda(\theta_3, \vartheta_3)) \\
\leq \;& \frac{3}{8}\max\{\Lambda(\theta_1, \vartheta_1), \Lambda(\theta_2, \vartheta_2), \Lambda(\theta_3, \vartheta_3)\} \\
= \;& \frac{5}{8}\max\{\Lambda(\theta_1, \vartheta_1), \Lambda(\theta_2, \vartheta_2), \Lambda(\theta_3, \vartheta_3)\} - \frac{1}{4}\max\{\Lambda(\theta_1, \vartheta_1), \Lambda(\theta_2, \vartheta_2), \Lambda(\theta_3, \vartheta_3)\} \\
= \;& \rho(\max\{\Lambda(\theta_1, \vartheta_1), \Lambda(\theta_2, \vartheta_2), \Lambda(\theta_3, \vartheta_3)\}) - \ell(\max\{\Lambda(\theta_1, \vartheta_1), \Lambda(\theta_2, \vartheta_2), \Lambda(\theta_3, \vartheta_3)\}).
\end{aligned}
$$

Therefore, all assertions of Corollary 1 are fulfilled and $(0, 0, 0)$ *is a unique TFP of* ξ *on* $\daleth^3$.

4. Application to IVPs

In the setting of PMSs, this part is devoted to discussing the existence of a uniqueness solution to the IVP below:

$$
\begin{cases} \theta_1'(\nu) = \Xi(\nu, \theta_1(\nu), \theta_1(\nu), \theta_1(\nu)), \; \nu \in \chi = [0, 1], \\ \theta_1(0) = \theta_1^0, \end{cases} \tag{17}
$$

where $\Xi : \chi \times [\frac{\theta_1^0}{5}, \infty) \times [\frac{\theta_1^0}{5}, \infty) \times [\frac{\theta_1^0}{5}, \infty) \to [\frac{\theta_1^0}{5}, \infty)$ is a continuous function for $\theta_1^0 \in \mathbb{R}$.

Now, we state and prove our main theorem in this part.

Theorem 3. *Assume that the IVP (17) with* $\Xi \in C\left(\chi \times [\frac{\theta_1^0}{5}, \infty) \times [\frac{\theta_1^0}{5}, \infty) \times [\frac{\theta_1^0}{5}, \infty)\right)$ *and*

$$
\int_0^\nu \Xi(\gamma, \theta_1(\gamma), \theta_2(\gamma), \theta_3(\gamma))d\gamma \leq \max\left\{ \begin{array}{l} \frac{1}{5}\int_0^\nu \Xi(\gamma, \theta_1(\gamma), \theta_1(\gamma), \theta_1(\gamma))d\gamma - \frac{16\theta_1^0}{25}, \\ \frac{1}{5}\int_0^\nu \Xi(\gamma, \theta_2(\gamma), \theta_2(\gamma), \theta_2(\gamma))d\gamma - \frac{16\theta_1^0}{25}, \\ \frac{1}{5}\int_0^\nu \Xi(\gamma, \theta_3(\gamma), \theta_3(\gamma), \theta_3(\gamma))d\gamma - \frac{16\theta_1^0}{25} \end{array} \right\}.
$$

Then the IVP (17) has a unique solution in $C\left(\chi,[\frac{\theta_1^0}{4},\infty)\right)$.

Proof. The IVP (17) is equivalent to the following integral equation:

$$\theta_1(\nu)=\theta_1^0+\int_0^\nu \Xi(\gamma,\theta_1(\gamma),\theta_2(\gamma),\theta_3(\gamma))d\gamma. \tag{18}$$

Assume that $\daleth=C\left(\chi,[\frac{\theta_1^0}{5},\infty)\right)$ and $\Lambda(\theta_1,\theta_2)=\max\left\{\theta_1-\frac{\theta_1^0}{5},\theta_2-\frac{\theta_1^0}{5},\theta_3-\frac{\theta_1^0}{5}\right\}$ for $\theta_1,\theta_2\in\daleth$. Define $\varphi,\rho,\ell:[0,\infty)\to[0,\infty)$ by $\varphi(\nu)=\nu$, $\rho(\nu)=\frac{4\nu}{5}$ and $\ell(\nu)=\frac{3\nu}{5}$. Describe the mapping $\xi:\daleth^3\to\daleth$ as

$$\xi(\theta_1,\theta_2,\theta_3)(\nu)=\theta_1^0+\int_0^\nu \Xi(\gamma,\theta_1(\gamma),\theta_2(\gamma),\theta_3(\gamma))d\gamma.$$

Now

$$\begin{aligned}
& \Lambda(\xi(\theta_1,\theta_2,\theta_3),\xi(\vartheta_1,\vartheta_2,\vartheta_3)) \\
= & \max\left\{\xi(\theta_1,\theta_2,\theta_3)-\frac{\theta_1^0}{5},\xi(\vartheta_1,\vartheta_2,\vartheta_3)-\frac{\theta_1^0}{5}\right\} \\
= & \max\left\{\frac{4\theta_1^0}{5}+\int_0^\nu \Xi(\gamma,\theta_1(\gamma),\theta_2(\gamma),\theta_3(\gamma))d\gamma,\frac{4\theta_1^0}{5}+\int_0^\nu \Xi(\gamma,\vartheta_1(\gamma),\vartheta_2(\gamma),\vartheta_3(\gamma))d\gamma\right\} \\
\leq & \max\left\{\begin{array}{c}
\frac{4\theta_1^0}{5}+\max\left\{\begin{array}{c}
\frac{1}{5}\int_0^\nu \Xi(\gamma,\theta_1(\gamma),\theta_1(\gamma),\theta_1(\gamma))d\gamma-\frac{16\theta_1^0}{25}, \\
\frac{1}{5}\int_0^\nu \Xi(\gamma,\theta_2(\gamma),\theta_2(\gamma),\theta_2(\gamma))d\gamma-\frac{16\theta_1^0}{25}, \\
\frac{1}{5}\int_0^\nu \Xi(\gamma,\theta_3(\gamma),\theta_3(\gamma),\theta_3(\gamma))d\gamma-\frac{16\theta_1^0}{25}
\end{array}\right\}, \\
\frac{4\theta_1^0}{5}+\max\left\{\begin{array}{c}
\frac{1}{5}\int_0^\nu \Xi(\gamma,\vartheta_1(\gamma),\vartheta_1(\gamma),\vartheta_1(\gamma))d\gamma-\frac{16\theta_1^0}{25}, \\
\frac{1}{5}\int_0^\nu \Xi(\gamma,\vartheta_2(\gamma),\vartheta_2(\gamma),\vartheta_2(\gamma))d\gamma-\frac{16\theta_1^0}{25}, \\
\frac{1}{5}\int_0^\nu \Xi(\gamma,\vartheta_3(\gamma),\vartheta_3(\gamma),\vartheta_3(\gamma))d\gamma-\frac{16\theta_1^0}{25}
\end{array}\right\}
\end{array}\right\} \\
= & \max\left\{\begin{array}{c}
\max\left\{\frac{\theta_1(\nu)}{5}-\frac{\theta_1^0}{25},\frac{\vartheta_1(\nu)}{5}-\frac{\theta_1^0}{25}\right\}, \\
\max\left\{\frac{\theta_2(\nu)}{5}-\frac{\theta_1^0}{25},\frac{\vartheta_2(\nu)}{5}-\frac{\theta_1^0}{25}\right\}, \\
\max\left\{\frac{\theta_3(\nu)}{5}-\frac{\theta_1^0}{25},\frac{\vartheta_3(\nu)}{5}-\frac{\theta_1^0}{25}\right\}
\end{array}\right\} \\
= & \frac{1}{5}\max\left\{\begin{array}{c}
\max\left\{\theta_1(\nu)-\frac{\theta_1^0}{5},\vartheta_1(\nu)-\frac{\theta_1^0}{5}\right\}, \\
\max\left\{\theta_2(\nu)-\frac{\theta_1^0}{5},\vartheta_2(\nu)-\frac{\theta_1^0}{5}\right\}, \\
\max\left\{\theta_3(\nu)-\frac{\theta_1^0}{5},\vartheta_3(\nu)-\frac{\theta_1^0}{5}\right\}
\end{array}\right\} \\
= & \frac{1}{5}\max\{\Lambda(\theta_1,\vartheta_1),\Lambda(\theta_2,\vartheta_2),\Lambda(\theta_3,\vartheta_3)\} \\
= & \frac{4}{5}\max\{\Lambda(\theta_1,\vartheta_1),\Lambda(\theta_2,\vartheta_2),\Lambda(\theta_3,\vartheta_3)\}-\frac{3}{5}\max\{\Lambda(\theta_1,\vartheta_1),\Lambda(\theta_2,\vartheta_2),\Lambda(\theta_3,\vartheta_3)\} \\
= & \rho(\max\{\Lambda(\theta_1,\vartheta_1),\Lambda(\theta_2,\vartheta_2),\Lambda(\theta_3,\vartheta_3)\})-\ell(\max\{\Lambda(\theta_1,\vartheta_1),\Lambda(\theta_2,\vartheta_2),\Lambda(\theta_3,\vartheta_3)\}).
\end{aligned}$$

Therefore, ξ verifies the stipulation (16) of Corollary 1. Thus, ξ has a unique TFP $(\theta_1,\theta_2,\theta_3)$ with $\theta_1=\theta_2=\theta_3$, which is a unique solution of the IVP (17). □

5. Application to a Homotopy

Here, we discuss a unique solution to homotopy theory.

Theorem 4. *Suppose that $(\daleth, \Lambda)$ is a complete PMS, A is an open subset of $\daleth$ and $\overline{A}$ is a closed subset of $\daleth$ so that $A \subseteq \overline{A}$. Assume that $H : \overline{A} \times \overline{A} \times \overline{A} \times [0,1] \to \daleth$ is an operator satisfies the hypotheses below:*

$(\aleph_1)$ $\theta_1 \neq H(\theta_1, \theta_2, \theta_3, \sigma)$, $\theta_2 \neq H(\theta_2, \theta_3, \theta_1, \sigma)$ *and* $\theta_3 \neq H(\theta_3, \theta_1, \theta_2, \sigma)$, *for each* $\theta_1, \theta_2, \theta_3 \in \partial A$ *(here ∂A refer to the boundary of A in $\daleth$) and $\sigma \in [0,1]$;*

$(\aleph_2)$

$$\begin{aligned}\varphi(\Lambda(H(\theta_1, \theta_2, \theta_3, \sigma), H(\vartheta_1, \vartheta_2, \vartheta_3, \sigma))) \quad \leq \quad & \rho(\max\{\Lambda(\theta_1, \vartheta_1), \Lambda(\theta_2, \vartheta_2), \Lambda(\theta_3, \vartheta_3)\}) \\ & -\ell(\max\{\Lambda(\theta_1, \vartheta_1), \Lambda(\theta_2, \vartheta_2), \Lambda(\theta_3, \vartheta_3)\}),\end{aligned}$$

for all $\theta_1, \theta_2, \theta_3, \vartheta_1, \vartheta_2, \vartheta_3 \in \overline{A}$ and $\sigma \in [0,1]$, where $\varphi, \rho : [0,\infty) \to [0,\infty)$ are continuous and non-decreasing and $\ell : [0,\infty) \to [0,\infty)$ is an LSC with $\varphi(\nu) - \rho(\nu) + \ell(\nu) > 0$, for $\nu > 0$;

$(\aleph_3)$ *There exists $C \geq 0$ so that*

$$\Lambda(H(\theta_1, \theta_2, \theta_3, \sigma), H(\theta_1, \theta_2, \theta_3, \sigma^*)) \leq C|\sigma - \sigma^*|,$$

for each $\theta_1, \theta_2, \theta_3 \in \overline{A}$ and $\sigma, \sigma^* \in [0,1]$.

Then $H(.,0)$ has a TFP, whenever $H(.,1)$ has a TFP.

Proof. Define the set

$$\mho = \{\sigma \in [0,1] : (\theta_1, \theta_2, \theta_3) = H(\theta_1, \theta_2, \theta_3, \sigma) \text{ for some } \theta_1, \theta_2, \theta_3 \in A\}.$$

Because $H(.,0)$ has a TFP in ∇, we obtain $0 \in \mho$, this proves that $\mho \neq \emptyset$.

We claim that $\mho$ is open and closed in $[0,1]$ so by the connectedness, we obtain $\mho = [0,1]$. Consequently, $H(.,1)$ has a TFP in ∇. Initially, we shall show that $\mho$ is open and closed in $[0,1]$. To do this, assume $\{\sigma^\omega\}_{\omega=1}^{\infty} \subseteq \mho$ with $\sigma^\omega \to \sigma \in [0,1]$ as $\omega \to \infty$. It must be shown that $\sigma \in \mho$.

Because $\sigma^\omega \in \mho$, for $\omega \geq 1$, there are $\theta_1^\omega, \theta_2^\omega, \theta_3^\omega \in A$ with $\varrho^\omega = \left(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega\right) = H\left(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega, \sigma^\omega\right)$. Consider

$$\begin{aligned}\Lambda\left(\theta_1^\omega, \theta_1^{\omega+1}\right) &= \Lambda\left(H(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega, \sigma^\omega), H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^{\omega+1}\right)\right) \\ &\leq \Lambda\left(H(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega, \sigma^\omega), H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^\omega\right)\right) \\ &\quad +\Lambda\left(H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^\omega\right), H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^{\omega+1}\right)\right) \\ &\quad -\Lambda\left(H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^\omega\right), H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^\omega\right)\right) \\ &\leq \Lambda\left(H(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega, \sigma^\omega), H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^\omega\right)\right) + C\left|\sigma^\omega - \sigma^{\omega+1}\right|.\end{aligned}$$

As $\omega \to \infty$ in the above inequality, we have

$$\lim_{\omega\to\infty} \Lambda\left(\theta_1^\omega, \theta_1^{\omega+1}\right) \leq \lim_{\omega\to\infty} \Lambda\left(H(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega, \sigma^\omega), H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^\omega\right)\right) + 0,$$

Since φ is non-decreasing and continuous, we obtain

$$\begin{aligned}\lim_{\omega\to\infty} \varphi\left(\Lambda\left(\theta_1^\omega, \theta_1^{\omega+1}\right)\right) &\leq \lim_{\omega\to\infty} \varphi\left[\Lambda\left(H(\theta_1^\omega, \theta_2^\omega, \theta_3^\omega, \sigma^\omega), H\left(\theta_1^{\omega+1}, \theta_2^{\omega+1}, \theta_3^{\omega+1}, \sigma^\omega\right)\right)\right] \\ &\leq \lim_{\omega\to\infty} \left[\begin{array}{c} \rho\left(\max\left\{\Lambda\left(\theta_1^\omega, \theta_1^{\omega+1}\right), \Lambda\left(\theta_2^\omega, \theta_2^{\omega+1}\right), \Lambda\left(\theta_3^\omega, \theta_3^{\omega+1}\right)\right\}\right) \\ -\ell\left(\max\left\{\Lambda\left(\theta_1^\omega, \theta_1^{\omega+1}\right), \Lambda\left(\theta_2^\omega, \theta_2^{\omega+1}\right), \Lambda\left(\theta_3^\omega, \theta_3^{\omega+1}\right)\right\}\right) \end{array}\right].\end{aligned}$$

Analogously,

$$\lim_{\omega\to\infty}\varphi\left(\Lambda\left(\theta_2^{\omega},\theta_2^{\omega+1}\right)\right)\leq\lim_{\omega\to\infty}\left[\begin{array}{c}\rho\left(\max\left\{\Lambda\left(\theta_1^{\omega},\theta_1^{\omega+1}\right),\Lambda\left(\theta_2^{\omega},\theta_2^{\omega+1}\right),\Lambda\left(\theta_3^{\omega},\theta_3^{\omega+1}\right)\right\}\right)\\-\ell\left(\max\left\{\Lambda\left(\theta_1^{\omega},\theta_1^{\omega+1}\right),\Lambda\left(\theta_2^{\omega},\theta_2^{\omega+1}\right),\Lambda\left(\theta_3^{\omega},\theta_3^{\omega+1}\right)\right\}\right)\end{array}\right],$$

and

$$\lim_{\omega\to\infty}\varphi\left(\Lambda\left(\theta_3^{\omega},\theta_3^{\omega+1}\right)\right)\leq\lim_{\omega\to\infty}\left[\begin{array}{c}\rho\left(\max\left\{\Lambda\left(\theta_1^{\omega},\theta_1^{\omega+1}\right),\Lambda\left(\theta_2^{\omega},\theta_2^{\omega+1}\right),\Lambda\left(\theta_3^{\omega},\theta_3^{\omega+1}\right)\right\}\right)\\-\ell\left(\max\left\{\Lambda\left(\theta_1^{\omega},\theta_1^{\omega+1}\right),\Lambda\left(\theta_2^{\omega},\theta_2^{\omega+1}\right),\Lambda\left(\theta_3^{\omega},\theta_3^{\omega+1}\right)\right\}\right)\end{array}\right].$$

This implies that

$$\lim_{\omega\to\infty}\Lambda\left(\theta_1^{\omega},\theta_1^{\omega+1}\right)=0,\ \lim_{\omega\to\infty}\Lambda\left(\theta_2^{\omega},\theta_2^{\omega+1}\right)=0\text{ and }\lim_{\omega\to\infty}\Lambda\left(\theta_3^{\omega},\theta_3^{\omega+1}\right)=0. \tag{19}$$

It follows from (Λ_2) that

$$\lim_{\omega\to\infty}\Lambda(\theta_1^{\omega},\theta_1^{\omega})=0,\ \lim_{\omega\to\infty}\Lambda(\theta_2^{\omega},\theta_2^{\omega})=0\text{ and }\lim_{\omega\to\infty}\Lambda(\theta_3^{\omega},\theta_3^{\omega})=0. \tag{20}$$

From the definition of Ω_Λ, we can write

$$\lim_{\omega\to\infty}\Omega_\Lambda\left(\theta_1^{\omega},\theta_1^{\omega+1}\right)=0,\ \lim_{\omega\to\infty}\Omega_\Lambda\left(\theta_2^{\omega},\theta_2^{\omega+1}\right)=0\text{ and }\lim_{\omega\to\infty}\Omega_\Lambda\left(\theta_3^{\omega},\theta_3^{\omega+1}\right)=0. \tag{21}$$

In order to prove that $\{\theta_1^{\omega}\}$, $\{\theta_2^{\omega}\}$ and $\{\theta_3^{\omega}\}$ are Cauchy sequences, assume that $\{\theta_1^{\omega}\}$ or $\{\theta_2^{\omega}\}$ or $\{\theta_3^{\omega}\}$ is not a Cauchy. Then there is an $\epsilon>0$ and monotone increasing sequences $\{v_j\}$ and $\{\omega_j\}$ so that $\omega_j>v_j>j$,

$$\max\left\{\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j}\right)\right\}\geq\epsilon, \tag{22}$$

and

$$\max\left\{\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j-1}\right),\Omega_\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j-1}\right),\Omega_\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j-1}\right)\right\}<\epsilon. \tag{23}$$

Using (22) and (23), we obtain

$$\begin{aligned}\epsilon &\leq \max\left\{\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j}\right)\right\}\\ &\leq \max\left\{\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j-1}\right),\Omega_\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j-1}\right),\Omega_\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j-1}\right)\right\}\\ &\quad+\max\left\{\Omega_\Lambda\left(\theta_1^{\omega_j-1},\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\theta_2^{\omega_j-1},\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\theta_3^{\omega_j-1},\theta_3^{\omega_j}\right)\right\}\\ &< \epsilon+\max\left\{\Omega_\Lambda\left(\theta_1^{\omega_j-1},\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\theta_2^{\omega_j-1},\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\theta_3^{\omega_j-1},\theta_3^{\omega_j}\right)\right\}.\end{aligned}$$

Letting $j\to\infty$ and using (21), we have

$$\lim_{j\to\infty}\max\left\{\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j}\right),\Omega_\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j}\right),\Omega_\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j}\right)\right\}=\epsilon. \tag{24}$$

Based on the definition of Ω_Λ and by (2), one can obtain

$$\lim_{j\to\infty}\max\left\{\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j}\right),\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j}\right),\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j}\right)\right\}=\frac{\epsilon}{2}. \tag{25}$$

Taking $j\to\infty$ and applying (24) and (21) in

$$\left|\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j+1}\right)-\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j}\right)\right|\leq\Omega_\Lambda\left(\theta_1^{\omega_j},\theta_1^{\omega_j+1}\right),$$

we have

$$\lim_{j\to\infty}\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j+1}\right)=\epsilon,$$

hence, we obtain

$$\lim_{j\to\infty}\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j+1}\right)=\frac{\epsilon}{2}.$$

By the same manner, one can obtain

$$\lim_{j\to\infty}\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j+1}\right)=\frac{\epsilon}{2},$$

and

$$\lim_{j\to\infty}\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j+1}\right)=\frac{\epsilon}{2}.$$

Let

$$\begin{aligned}\Omega_\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j+1}\right) &= \Lambda\left(H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_{13}^{v_j},\sigma^{v_j}\right),H\left(\theta_1^{\omega_j+1},\theta_2^{\omega_j+1},\theta_3^{\omega_j+1},\sigma^{\omega_j+1}\right)\right)\\ &\le \Lambda\left(H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_3^{v_j},\sigma^{v_j}\right),H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_3^{v_j},\sigma^{\omega_j+1}\right)\right)\\ &\quad+\Lambda\left(H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_3^{v_j},\sigma^{\omega_j+1}\right),H\left(\theta_1^{\omega_j+1},\theta_2^{\omega_j+1},\theta_3^{\omega_j+1},\sigma^{\omega_j+1}\right)\right)\\ &\quad-\Lambda\left(H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_3^{v_j},\sigma^{\omega_j+1}\right),H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_3^{v_j},\sigma^{\omega_j+1}\right)\right)\\ &\le C\left|\sigma^{v_j}-\sigma^{\omega_j+1}\right|+\Lambda\left(H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_3^{v_j},\sigma^{\omega_j+1}\right),H\left(\theta_1^{\omega_j+1},\theta_2^{\omega_j+1},\theta_3^{\omega_j+1},\sigma^{\omega_j+1}\right)\right).\end{aligned}$$

Letting $j\to\infty$ in the above and since $\{\sigma^{\omega_j}\}$ is Cauchy, we obtain

$$\frac{\epsilon}{2}\le\lim_{j\to\infty}\Lambda\left(H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_3^{v_j},\sigma^{\omega_j+1}\right),H\left(\theta_1^{\omega_j+1},\theta_2^{\omega_j+1},\theta_3^{\omega_j+1},\sigma^{\omega_j+1}\right)\right)$$

Because φ is non-decreasing and continuous, we have

$$\begin{aligned}\varphi\left(\frac{\epsilon}{2}\right) &\le \lim_{j\to\infty}\varphi\left[\Lambda\left(H\left(\theta_1^{v_j},\theta_2^{v_j},\theta_3^{v_j},\sigma^{\omega_j+1}\right),H\left(\theta_1^{\omega_j+1},\theta_2^{\omega_j+1},\theta_3^{\omega_j+1},\sigma^{\omega_j+1}\right)\right)\right]\\ &\le \lim_{\omega\to\infty}\left[\begin{array}{c}\rho\left(\max\left\{\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j+1}\right),\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j+1}\right),\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j+1}\right)\right\}\right)\\ -\ell\left(\Lambda\left(\theta_1^{v_j},\theta_1^{\omega_j+1}\right),\Lambda\left(\theta_2^{v_j},\theta_2^{\omega_j+1}\right),\Lambda\left(\theta_3^{v_j},\theta_3^{\omega_j+1}\right)\right)\end{array}\right]\\ &\le \rho\left(\frac{\epsilon}{2}\right)-\ell\left(\frac{\epsilon}{2}\right),\end{aligned}$$

this implies that $\epsilon\le 0$, which is a contradiction. Hence, $\{\theta_1^\omega\}$ is Cauchy sequence. Similarly, $\{\theta_2^\omega\}$ and $\{\theta_3^\omega\}$ are too in $(\urcorner,\Omega_\Lambda)$ and $\Omega_\Lambda\left(\theta_1^v,\theta_1^\omega\right)\to 0$, $\Omega_\Lambda(\theta_2^v,\theta_2^\omega)\to 0$ and $\Omega_\Lambda\left(\theta_3^v,\theta_3^\omega\right)\to 0$ as $v,\omega\to\infty$. Thus, by (20) and the definition of Ω_Λ, we have

$$\lim_{v,\omega\to\infty}\Lambda(\theta_1^v,\theta_1^\omega)=0,\ \lim_{v,\omega\to\infty}\Lambda(\theta_2^v,\theta_2^\omega)=0\text{ and }\lim_{v,\omega\to\infty}\Lambda(\theta_3^v,\theta_3^\omega)=0.$$

It follows from Lemma 1 (i) that $\{\theta_1^\omega\}$, $\{\theta_2^\omega\}$ and $\{\theta_3^\omega\}$ are Cauchy sequences in $(\urcorner,\Lambda)$. Because $(\urcorner,\Lambda)$ is a complete, from Lemma 1 (ii), there exist $\varrho_1,\varrho_2,\varrho_3\in\urcorner$ with

$$\begin{aligned}\Lambda(\varrho_1,\varrho_1) &= \lim_{\omega\to\infty}\Lambda(\theta_1^\omega,\varrho_1)=\lim_{\omega\to\infty}\Lambda\left(\theta_1^{\omega+1},\varrho_1\right)=\lim_{\omega,v\to\infty}\Lambda(\theta_1^\omega,\theta_1^v),\\ \Lambda(\varrho_2,\varrho_2) &= \lim_{\omega\to\infty}\Lambda(\theta_2^\omega,\varrho_2)=\lim_{\omega\to\infty}\Lambda\left(\theta_2^{\omega+1},\varrho_2\right)=\lim_{\omega,v\to\infty}\Lambda(\theta_2^\omega,\theta_2^v),\\ \Lambda(\varrho_3,\varrho_3) &= \lim_{\omega\to\infty}\Lambda(\theta_3^\omega,\varrho_3)=\lim_{\omega\to\infty}\Lambda\left(\theta_3^{\omega+1},\varrho_3\right)=\lim_{\omega,v\to\infty}\Lambda(\theta_3^\omega,\theta_3^v),\end{aligned}$$

Using Lemma 2, we have

$$\lim_{\omega\to\infty}\Lambda(\theta_1^\omega,H(\varrho_1,\varrho_2,\varrho_3,\sigma))=\Lambda(\varrho_1,H(\varrho_1,\varrho_2,\varrho_3,\sigma)).$$

Now

$$\begin{aligned}\Lambda(\theta_1^\omega, H(\varrho_1,\varrho_2,\varrho_3,\sigma)) &= \Lambda(H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma^\omega), H(\varrho_1,\varrho_2,\varrho_3,\sigma)) \\ &\le \Lambda(H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma^\omega), H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma)) \\ &\quad +\Lambda(H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma), H(\varrho_1,\varrho_2,\varrho_3,\sigma)) \\ &\quad -\Lambda(H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma), H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma)) \\ &= C|\sigma^\omega-\sigma| + \Lambda(H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma), H(\varrho_1,\varrho_2,\varrho_3,\sigma)).\end{aligned}$$

Passing $\omega \to \infty$, we obtain

$$\Lambda(\varrho_1, H(\varrho_1,\varrho_2,\varrho_3,\sigma)) \le \lim_{\omega\to\infty} \Lambda(H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma), H(\varrho_1,\varrho_2,\varrho_3,\sigma)).$$

Because φ is non-decreasing and continuous, we obtain

$$\begin{aligned}\varphi(\Lambda(\varrho_1, H(\varrho_1,\varrho_2,\varrho_3,\sigma))) &\le \lim_{\omega\to\infty} \varphi[\Lambda(H(\theta_1^\omega,\theta_2^\omega,\theta_3^\omega,\sigma), H(\varrho_1,\varrho_2,\varrho_3,\sigma))] \\ &\le \lim_{\omega\to\infty}\left[\begin{array}{c}\rho\big(\Lambda\big(\theta_1^\omega,\varrho_1\big),\Lambda\big(\theta_2^\omega,\varrho_2\big),\Lambda\big(\theta_3^\omega,\varrho_3\big)\big) \\ -\ell\big(\Lambda\big(\theta_1^\omega,\varrho_1\big),\Lambda\big(\theta_2^\omega,\varrho_2\big),\Lambda\big(\theta_3^\omega,\varrho_3\big)\big)\end{array}\right] \\ &= 0.\end{aligned}$$

This implies that $\Lambda(\varrho_1, H(\varrho_1,\varrho_2,\varrho_3,\sigma)) = 0$. Thus, $\varrho_1 = H(\varrho_1,\varrho_2,\varrho_3,\sigma)$. Analogously, $\varrho_2 = H(\varrho_2,\varrho_3,\varrho_1,\sigma)$ and $\varrho_3 = H(\varrho_3,\varrho_1,\varrho_2,\sigma)$. Hence, $\sigma \in \mho$, this proves that $\mho$ is closed in $[0,1]$.

Assume that $\sigma^0 \in \mho$. Then there are $\theta_1^0,\theta_2^0,\theta_3^0 \in A$ with $\theta_1^0 = H(\theta_1^0,\theta_2^0,\theta_3^0,\sigma^0)$. Because $\mho$ is open, then there exists $z > 0$ so that $O_\Lambda(\theta_1^0, z) \subseteq \mho$. Select $\sigma \in (\sigma^0-\epsilon, \sigma^0+\epsilon)$ so that $\left|\sigma-\sigma^0\right| \le \frac{1}{C^\omega} < \epsilon$.

Then $\theta_1 \in \overline{O_\Lambda(\theta_1^0, z)} = \{\theta_1 \in \daleth / \Lambda(\theta_1,\theta_1^0) \le z + \Lambda(\theta_1^0,\theta_1^0)$. We obtain

$$\begin{aligned}\Lambda(H(\theta_1,\theta_2,\theta_3,\sigma),\theta_1^0) &= \Lambda(H(\theta_1,\theta_2,\theta_3,\sigma), H(\theta_1^0,\theta_2^0,\theta_3^0,\sigma^0)) \\ &\le \Lambda(H(\theta_1,\theta_2,\theta_3,\sigma), H(\theta_1,\theta_2,\theta_3,\sigma^0)) \\ &\quad +\Lambda(H\Big(\theta_1,\theta_2,\theta_3,\sigma^0\Big), H(\theta_1^0,\theta_2^0,\theta_3^0,\sigma^0)) \\ &\quad -\Lambda(H\Big(\theta_1,\theta_2,\theta_3,\sigma^0\Big), H(\theta_1,\theta_2,\theta_3,\sigma^0)) \\ &\le C\Big|\sigma-\sigma^0\Big| + \Lambda(H\Big(\theta_1,\theta_2,\theta_3,\sigma^0\Big), H(\theta_1^0,\theta_2^0,\theta_3^0,\sigma^0)) \\ &\le \frac{1}{C^{\omega-1}} + \Lambda(H\Big(\theta_1,\theta_2,\theta_3,\sigma^0\Big), H(\theta_1^0,\theta_2^0,\theta_3^0,\sigma^0)).\end{aligned}$$

Letting $\omega \to \infty$, we have

$$\Lambda(H(\theta_1,\theta_2,\theta_3,\sigma),\theta_1^0) \le \Lambda(H\Big(\theta_1,\theta_2,\theta_3,\sigma^0\Big), H(\theta_1^0,\theta_2^0,\theta_3^0,\sigma^0)).$$

Because φ is non-decreasing and continuous, we obtain

$$\begin{aligned}\varphi\Big(\Lambda(H(\theta_1,\theta_2,\theta_3,\sigma),\theta_1^0)\Big) &\le \varphi\Big[\Lambda\Big(H\Big(\theta_1,\theta_2,\theta_3,\sigma^0\Big), H(\theta_1^0,\theta_2^0,\theta_3^0,\sigma^0)\Big)\Big] \\ &\le \rho\Big(\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big) - \ell\Big(\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big).\end{aligned}$$

By the same scenario, one can write

$$\varphi\Big(\Lambda(H(\theta_2,\theta_3\theta_1,\sigma),\theta_2^0)\Big) \le \rho\Big(\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big) - \ell\Big(\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big),$$

and

$$\varphi\Big(\Lambda(H(\theta_3,\theta_1,\theta_2,\sigma),\theta_3^0)\Big) \le \rho\Big(\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big) - \ell\Big(\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big).$$

Hence

$$\begin{aligned}
& \varphi\Big(\max\Big\{\Lambda(H(\theta_1,\theta_2,\theta_3,\sigma),\theta_1^0),\Lambda(H(\theta_2,\theta_3\theta_1,\sigma),\theta_2^0),\Lambda(H(\theta_3,\theta_1,\theta_2,\sigma),\theta_3^0)\Big\}\Big) \\
\leq\ & \rho\Big(\max\Big\{\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big\}\Big)-\ell\Big(\max\Big\{\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big\}\Big) \\
\leq\ & \varphi\Big(\max\Big\{\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big\}\Big).
\end{aligned}$$

Since φ is non-decreasing, we obtain

$$\begin{aligned}
& \max\Big\{\Lambda(H(\theta_1,\theta_2,\theta_3,\sigma),\theta_1^0),\Lambda(H(\theta_2,\theta_3\theta_1,\sigma),\theta_2^0),\Lambda(H(\theta_3,\theta_1,\theta_2,\sigma),\theta_3^0)\Big\} \\
\leq\ & \max\Big\{\Lambda\Big(\theta_1,\theta_1^0\Big),\Lambda\Big(\theta_2,\theta_2^0\Big),\Lambda\Big(\theta_3,\theta_3^0\Big)\Big\} \\
\leq\ & \max\Big\{z+\Lambda\Big(\theta_1^0,\theta_1^0\Big),z+\Lambda\Big(\theta_2^0,\theta_2^0\Big),z+\Lambda\Big(\theta_3^0,\theta_3^0\Big)\Big\}.
\end{aligned}$$

Therefore, for each $\sigma\in(\sigma^0-\epsilon,\sigma^0+\epsilon)$, we have $H:\overline{O_\Lambda(\theta_1^0,z)}\to\overline{O_\Lambda(\theta_1^0,z)}$.

Moreover, because assertion $(\aleph_2)$ holds and $\varphi,\rho:[0,\infty)\to[0,\infty)$ are continuous and non-decreasing and $\ell:[0,\infty)\to[0,\infty)$ is LSC with $\varphi(\nu)-\rho(\nu)+\ell(\nu)>0$, for $\nu>0$. Then, all hypotheses of Corollary 1 are fulfilled. Hence, we conclude that $H(.,\sigma)$ has a TFP in $\overline{A}$. Since this TFP must be contained in A since $(\aleph_1)$ is satisfied. Thus, $\sigma\in\mho$ for any $\sigma\in(\sigma^0-\epsilon,\sigma^0+\epsilon)$. Hence, $(\sigma^0-\epsilon,\sigma^0+\epsilon)\subseteq\mho$ and therefore $\mho$ is open in $[0,1]$. For the reverse implication, we use the same strategy. This finishes the proof. □

Corollary 2. *Suppose that $(\daleth,\Lambda)$ is a complete PMS, A is an open subset of $\daleth$ and $H:\overline{A}\times\overline{A}\times\overline{A}\times[0,1]\to\daleth$ with hypotheses below:*

(i) $\theta_1\neq H(\theta_1,\theta_2,\theta_3,\sigma)$, $\theta_2\neq H(\theta_2,\theta_3,\theta_1,\sigma)$ *and* $\theta_3\neq H(\theta_3,\theta_1,\theta_2,\sigma)$, *for each* $\theta_1,\theta_2,\theta_3\in\partial A$ *(here ∂A refer to the boundary of A in $\daleth$) and* $\sigma\in[0,1]$;

(ii) *There are* $\theta_1,\theta_2,\theta_3,\vartheta_1,\vartheta_2,\vartheta_3\in\overline{A}$ *and* $\sigma\in[0,1]$, $K\in[0,1)$ *so that*

$$\Lambda(H(\theta_1,\theta_2,\theta_3,\sigma),H(\vartheta_1,\vartheta_2,\vartheta_3,\sigma))\leq K\max\{\Lambda(\theta_1,\vartheta_1),\Lambda(\theta_2,\vartheta_2),\Lambda(\theta_3,\vartheta_3)\},$$

(iii) *There exist* $C\geq 0$ *so that*

$$\Lambda(H(\theta_1,\theta_2,\theta_3,\sigma),H(\theta_1,\theta_2,\theta_3,\sigma^*))\leq C|\sigma-\sigma^*|,$$

for each $\theta_1,\theta_2,\theta_3\in\overline{A}$ *and* $\sigma,\sigma^*\in[0,1]$.

Then $H(.,0)$ has a TFP, whenever $H(.,1)$ has a TFP.

Proof. The proof follows immediately from Theorem 4 by putting $\varphi(\nu)=\nu$, $\rho(\nu)=K\nu-\nu$ with $K\in[0,1)$ and $\ell(\nu)=\nu$, for $\nu>0$. □

Author Contributions: H.A.H.: Writing–original draft; J.L.G.G.: Methodology; P.A. Writing–review and editing. All authors contributed equally to this article. All authors have read and agreed to the published version of the manuscript.

Funding: This paper has been partially supported by Ministerio de Ciencia, Innovacion y Universidades grant number PGC2018-0971-B-100 and Fundacion Seneca de la Region de Murcia grant number 20783/PI/18.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data used to support the findings of this study are included within the article.

Acknowledgments: Juan L.G. Guirao is thankful to the Ministerio de Ciencia, Innovacion y Universidades grant number PGC2018-0971-B-100 and Fundacion Seneca de la Region de Murcia grant number 20783/PI/18 for parrtially support this research. Praveen Agarwal was very thankful to the SERB (project TAR/2018/000001), DST (project DST/INT/DAAD/P-21/2019, INT/RUS/RFBR/308) and NBHM (project 02011/12/ 2020NBHM(R.P)/R&D II/7867) for their necessary support.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Fredholm, E.I. Sur une classe d'équations fonctionnelles. *Acta Math.* **1903**, *27*, 365–390. [CrossRef]
2. Rus, M.D. A note on the existence of positive solution of Fredholm integral equations. *Fixed Point Theory* **2004**, *5*, 369–377.
3. Berenguer, M.I.; Munoz, M.V.F.; Guillem, A.I.G.; Galan, M.R. Numerical treatment of fixed point applied to the nonlinear fredholm integral equation. *Fixed Point Theory Appl.* **2009**, *1*, 638–735. [CrossRef]
4. Hammad, H.A.; De la Sen, M. A Solution of Fredholm integral equation by using the cyclic η_s^q-rational contractive mappings technique in *b*-metric-like spaces. *Symmetry* **2019**, *11*, 1184. [CrossRef]
5. Hammad, H.A.; De la Sen, M. Solution of nonlinear integral equation via fixed point of cyclic α_s^q-rational contraction mappings in metric-like spaces. *Bull. Braz. Math. Soc.* **2020**, *51*, 81–105. [CrossRef]
6. Aksoy, N.; Yildirim. The solvability of first type boundary value problem for a Schrödinger equation. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 211–220. [CrossRef]
7. Busovikov, V.M.; Sakbaev, V.Z. Dirichlet problem for poisson equation on the rectangle in infinite dimensional Hilbert space. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 329–344. [CrossRef]
8. Kaur, D.; Agarwal, P.; Rakshit, M.; Chand, M. fractional calculus involving (p,q)-Mathieu type series. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 15–34. [CrossRef]
9. Modanli, M.; Akgül, A. On solutions of fractional order telegraph partial differential equation by crank-nicholson finite difference method. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 163–170. [CrossRef]
10. Hammad, H.A.; Aydi, H.; De la Sen, M. Solutions of fractional differential type equations by fixed point techniques for multi-valued contractions. *Complixty* **2021**, *2021*, 5730853.
11. Hammad, H.A.; Aydi, H.; Mlaiki, N. Contributions of the fixed point technique to solve the 2D Volterra integral equations, Riemann-Liouville fractional integrals, and Atangana-Baleanu integral operators. *Adv. Differ. Equations* **2021**, *97*, 1–20. [CrossRef]
12. The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*; Institute for Advanced Study: 2013. Available online: https://arxiv.org/pdf/1308.0729.pdf (accessed on 5 May 2021).
13. Ege, M.E.; Alaca, C. Fixed point results and an application to homotopy in modular metric spaces. *J. Nonlinear Sci. Appl.* **2015**, *8*, 900–908). [CrossRef]
14. Agarwal, R.P.; J. Dshalalow, D. O'Regan, Fixed point and homotopy results for generalized contractive maps of Reich type. *Appl. Anal.* **2010**, *82*, 329–350. [CrossRef]
15. Vetro, C.; Vetro, F. A homotopy fixed point theorem in 0-complete partial metric space. *Filomat* **2015**, *29*, 2037–2048. [CrossRef]
16. Matthews, S.G. *Partial Metric Topology*; Research Report 212; Department of Computer Science, University of Warwick: Coventry, UK, 1992.
17. Kopperman, R.; Matthews, S.G.; Pajoohesh, H. What do partial metrics represent? In *Spatial Representation: Discrete vs. Continuous Computational Models*; Dagstuhl Seminar Proceedings, No. 04351; Internationales Begegnungs-undForschungszentrum für Informatik (IBFI): Schloss Dagstuhl, Germany, 2005.
18. Künzi, H.P.A.; Pajoohesh, H.; Schellekens, M.P. Partial quasi-metrics. *Theor. Comput. Sci.* **2006**, *365*, 237–246. [CrossRef]
19. O'Neill, S.J. *Two Topologies Are Better Than One*; Technical Report; University of Warwick: Coventry, UK, 1995. Available online: http://www.dcs.warwick.ac.uk/reports/283.html (accessed on 5 May 2021).
20. Schellekens, M. The Smyth completion: A common foundation for denotational semantics and complexity analysis. *Electron. Notes Theor. Comput. Sci.* **1995**, *1*, 535–556. [CrossRef]
21. Schellekens, M.P. A characterization of partial metrizability: domains are quantifiable. *Theor. Comput. Sci.* **2003**, *305*, 409–432. [CrossRef]
22. Waszkiewicz, P. Partial metrizability of continuous posets. *Math. Struct. Comput. Sci.* **2006**, *16*, 359–372. [CrossRef]
23. Matthews, S.G. Partial metric topology. In Proceedings of the 8th Summer Conference on General Topology and Applications. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197. [CrossRef]
24. Oltra, S.; Valero, O. Banach's fixed point theorem for partial metric spaces. *Rend. Ist. Mat. Univ. Trieste* **2004**, *36*, 17–26.
25. Altun, I.; Sola, F.; Simsek, H. Generalized contractions on partial metric spaces. *Topol. Appl.* **2010**, *157*, 2778–2785. [CrossRef]
26. Vetro, F.; Radenović, S. Nonlinear quasi-contractions of Ciric type in partial metric spaces. *Appl. Math. Comput.* **2012**, *219*, 1594–1600. [CrossRef]
27. Rao, K.P.R.; Kishore, G.N.V. A unique common fixed point theorem for four maps under $\psi - \phi$ contractive condition in partial metric spaces. *Bull. Math. Anal. Appl.* **2011**, *3*, 56–63.
28. Karapınar, E. Generalizations of Caristi Kirk's theorem on partial metric spaces. *Fixed Point Theory Appl.* **2011**, *4*. [CrossRef]
29. Aydi, H. Some coupled fixed point results on partial metric spaces. *Int. J. Math. Math. Sci.* **2011**, *2011*, 647091. [CrossRef]
30. Shukla, S.; Radenović, S. Some common fixed point theorems for *F*-contraction type mappings in 0-complete partial metric spaces. *J. Math.* **2013**, *2013*, 878730. [CrossRef]
31. Rao, N.S.; Kalyani, K. Unique fixed point theorems in partially ordered metric spaces. *Heliyon* **2020**, *6*, e05563.

32. Gupta, V.; Jungck, G.; Mani, N. Some novel fixed point theorems in partially ordered metric spaces. *AIMS Math.* **2020**, *5*, 4444–4452. [CrossRef]
33. Bhaskar, T.G.; Lakshmikantham, V. Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal.* **2006**, *65*, 1379–1393. [CrossRef]
34. Lakshmikantham, V.; Cirić, L. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal.* **2009**, *70*, 4341–4349. [CrossRef]
35. Hammad, H.A.; De la Sen, M. A coupled fixed point technique for solving coupled systems of functional and nonlinear integral equations. *Mathematics* **2019**, *7*, 634. [CrossRef]
36. Shatanawi, W.; Samet, B.; Abbas, M. Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces. *Math. Comput. Model.* **2012**, *55*, 680–687. [CrossRef]
37. Abbas, M.; Alikhan, M.; Radenović, S. Common coupled fixed point theorems in cone metric spaces for w-compatible mappings. *Appl. Math. Comput.* **2010**, *217*, 195–202. [CrossRef]
38. Berinde, V.; Borcut, M. Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Anal.* **2011**, *74*, 4889–4897. [CrossRef]
39. Borcut, M. Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. *Appl. Math. Comput.* **2012**, *218*, 7339–7346.
40. Borcut, M.; Berinde, V. Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. *Appl. Math. Comput.* **2012**, *218*, 5929–5936.
41. Choudhury, B.S.; Karapınar, E.; Kundu, A. Tripled coincidence point theorems for nonlinear contractions in partially ordered metric spaces. *Int. J. Math. Math. Sci.* **2012**, *2012*, 329298. [CrossRef]
42. Radenović, S. A note on tripled coincidence and tripled common fixed point theorems in partially ordered metric spaces. *Appl. Math. Comput.* **2014**, *236*, 367–372. [CrossRef]
43. Aydi, H.; Karapinar, E.; Postolache, M. Tripled coincidence point theorems for weak φ-contractions in partially ordered metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 44. [CrossRef]
44. Mustafa, Z.; Roshan, J.R.; Parvaneh, V. Existence of a tripled coincidence point in ordered Gb-metric spaces and applications to a system of integral equations. *J. Inequal. Appl.* **2013**, *1*, 453. [CrossRef]
45. Hammad, H.A.; De la Sen, M. A technique of tripled coincidence points for solving a system of nonlinear integral equations in POCML spaces. *J. Inequal. Appl.* **2020**, *211*. [CrossRef]
46. Hammad, H.A.; De la Sen, M. A tripled fixed point technique for solving a tripled-system of integral equations and Markov process in CCbMS. *Adv. Diff. Equ.* **2020**, *1*, 567. [CrossRef]
47. Abdeljawad, T.; Karapınar, E.; Tas, K. Existence and uniqueness of a common fixed point on partial metric spaces. *Appl. Math. Lett.* **2011**, *24*, 1894–1899. [CrossRef]
48. Hammad, H.A.; De la Sen, M. Tripled fixed point techniques for solving system of tripled fractional differential equations. *AIMS Math.* **2020**, *63*, 2330–2343. [CrossRef]