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Theoretical and Numerical Aspect of Fractional Differential Equations with Purely Integral Conditions

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Abstract: In this paper, we are interested in the study of a Caputo time fractional advection–diffusion equation with nonhomogeneous boundary conditions of integral types $\int_0^1 v(x,t)dx$ and $\int_0^1 x^n v(x,t)dx$. The existence and uniqueness of the given problem's solution is proved using the method of the energy inequalities known as the "a priori estimate" method relying on the range density of the operator generated by the considered problem. The approximate solution for this problem with these new kinds of boundary conditions is established by using a combination of the finite difference method and the numerical integration. Finally, we give some numerical tests to illustrate the usefulness of the obtained results.

Keywords: fractional derivatives; Caputo derivative; fractional advection–diffusion equation; finite difference schemes; integral conditions

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1. Introduction

Fractional Partial Differential Equations (FPDEs) have become very important in recent years due to their use in several mathematical models. FPDEs are considered as the generalization of a partial differential equation (PDE) of an integer order of an arbitrary order. These generalizations play an essential role in engineering, physics and applied mathematics. Due to the properties of Fractional Differential Equations (FDE), different models are created for complex phenomena using FPDEs, for example, in electroanalytical chemistry, viscoelasticity [1,2], porous environment, fluid flow, thermodynamic [3,4], diffusion transport, rheology [5–7], electromagnetism, signal processing [8], electrical network [9] and others [10–12]. Some relevant applications of fractional differential equations in the modeling of tribo-fatigue systems and new materials can be mentioned as methods for the experimental study of friction in an active system [13], the volumetric damage state of the tribofatigue system in [14], the tribo-fatigue damage transition and mapping for wheel material under rolling–sliding contact condition [15]; this study is based on construction of a tribo-fatigue damage map of high-speed railway wheel material under different tangential forces and contact pressure conditions through JD-1 testing equipment. Several problems have been mentioned in modern physics and technology using partial differential equations where the nonlocal conditions are described by integrals as $\int_0^1 v(x,t)dx$, and $\int_0^1 \varphi(x)v(x,t)dx$. These integral conditions are of great interest due to their applications in many fields: In population dynamics, heat diffusion–advection, models of blood circulation, chemical engineering thermoelasticity [16]. The existence and uniqueness of the solution of these problems have been studied by many authors [17–20]. Some results have been obtained by construction of a variational formulation which depends on the choice of spaces and their norms, Lax–Milgram theorem, Poincaré theorem, fixed point theory and Laplace transforms. For the numerical study of FPDE with classical boundary nonlocal conditions, we can cite the works of A. Alikhanov [21–23], Meerschaert, M.M, S. Shen and F Liu [24,25], El-Nabulsi, R.A [26–28] and others [29–33]. Among these authors we can cite Yuriy Povstenko [34] who studied the time fractional diffusion-wave equation with classic boundary conditions. Taki and Bouziani [35] have studied a problem of FPDE which has the boundary condition of integral type $\int_0^1 v(x,t)dx$ with respect to time derivative of order α $(0<\alpha<1)$. In this paper, we are interested in a new problem of FPDE with two boundary conditions of integrals types; $\int_0^1 v(x,t)dx$, and $\int_0^1 x^n v(x,t)dx$. We consider the time fractional advection–diffusion equation, obtained by replacing the second-order time derivative in standard wave equation with a fractional derivative of order α $(1<\alpha<2)$, and classical boundary conditions with integral boundary conditions. The physical interpretation of the fractional derivative is that it represents a degree of memory in the diffusing material. For the theoretical study, we use the energy inequalities method to prove the existence and the uniqueness. The numerical study is based on the application of a combination of the finite difference method with a numerical integration method to obtain an approximate solution of the proposed problem. We use a uniform space–time discretization. The Caputo fractional operator of order α $(1<\alpha<2)$ is approximated by a scheme called L_2, and the integer-order differential operators are approximated by central and advanced numerical schemes. Stability and convergence of the numerical scheme obtained show that the method used is conditionally stable and convergent. Numerical tests carried out give very satisfactory results; that is, the values of the approximate solution are very close to the exact solution. All numerical and graphical results obtained are produced using MATLAB software.

2. Notions and Preliminaries

First we need a definition of Caputo derivative to explain the problem that we shall study in this work: Let $\Gamma(.)$ denote the gamma function. For any positive non-integer value $1<\alpha<2$, the Caputo derivative is defined as follows:

Definition 1 (See [17]). *Let us denote by $C_0(0,1)$ the space of continuous functions with compact support in $(0,1)$, and its bilinear form is given by*

$$(u,w)=\int_0^1 \Im_x^m u.\Im_x^m w dx \qquad (m\in\mathbb{N}^*), \tag{1}$$

where

$$\Im_x^m u=\int_0^x \frac{(x-\xi)^{m-1}}{(m-1)!}u(\xi,t)d\xi \qquad (m\in\mathbb{N}^*).$$

For $m=1$, we have $\Im_x u=\int_0^x u(\xi,t)d\xi$ and $\Im_t u=\int_0^t u(x,\tau)d\tau$. The bilinear form (1) is considered as scalar product on $C_0(0,1)$ when it is not complete.

Definition 2 (See [18]). *We denote by*

$$B_2^m(0,1)=\begin{cases} L^2(0,1) & for\ m=0 \\ u\ mesurable\ /\Im_x^m u\in L^2(0,1) & for\ m\in\mathbb{N}^*,\end{cases}$$

the completion of $C_0(0,1)$ for the scalar product defined by (1).The norm associated to the scalar product is

$$\|u\|_{B_2^m(0,1)}=\|\Im_x^m u\|_{L^2(0,1)}=\left(\int_0^T(\Im_x^m u)^2dx\right)^{\frac{1}{2}}.$$

Lemma 1 (See [12]). *For all $m\in\mathbb{N}^*$, we obtain*

$$\|u\|_{B_2^m(0,1)}\le\left(\frac{1}{2}\right)^m\|u\|^2_{L^2(0,1)}. \tag{2}$$

Definition 3 (See [12]). *Let X be a Banach space with the norm $\|u\|_X$, and let $u:(0,T) \to X$ be an abstract function; by $\|u(.,t)\|_X$ we denote the norm of the element $u(.,t) \in X$ at a fixed t.*

We denote by $L^2(0,T;X)$ the set of all measurable abstract functions $u(.,t)$ from $(0,T)$ into X such that

$$\|u\|_{L^2(0,T;X)} = \left(\int_0^T \|u(.,t)\|_X dt\right)^{\frac{1}{2}} < \infty.$$

Lemma 2. (*Cauchy inequality with ε*) *(See [12]). For all $\varepsilon \succ 0$ and arbitrary variables $a,b \in \mathbb{R}$, we have the following inequality:*

$$|ab| \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2. \tag{3}$$

Definition 4 (See [2]). *The left Caputo derivative for $1 < \alpha < 2$ can be expressed as*

$${}_0^c\partial_t^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{f''(s)}{(t-s)^{\alpha-1}} ds;\ t > 0.$$

Definition 5 (See [2]). *The integral of order α of the function $f \in L^1[a,b]$ is defined by:*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds;\ t > 0.$$

Lemma 3 (See [12]). *For all real $1 < \alpha < 2$ we have the inequality*

$$\int_0^1 {}_0^c\partial_t^\alpha (\Im_x u)^2 dx \leq 2\int_0^1 ({}_0^c\partial_t^\alpha u)(\Im_x u) dx.$$

Lemma 4 (See [35]). *For all real $1 < \alpha < 2$ we have the inequality*

$$\int_Q ({}_0^c\partial_t^\alpha u)(\Im_x u) dxdt \leq \int_Q \left({}_0^c\partial_t^{\frac{\alpha}{2}} \Im_x u\right)^2 dxdt.$$

3. Theoretical Study

In this section, we prove the existence and uniqueness of the strong solution and its dependence on the data of the problem of fractional partial differential equations (FPDEs) with boundary conditions of integral type.

3.1. Position of the Problem

In the rectangular domain

$$Q = \left\{(x,t) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < t < T\right\},\ \text{where } T > 0,$$

we consider the fractional differential equation:

$$\pounds v = {}_0^c\partial_t^\alpha v + a(x,t)\frac{\partial^2 v}{\partial x^2} + b(x,t)\frac{\partial v}{\partial x} + c(x,t)v = g(x,t),\ \text{where } 1 < \alpha < 2. \tag{4}$$

With Equation (4), we associate the initial conditions:

$$\begin{cases} \ell v = v(x,0) = \Phi(x), & x \in (0,1), \\ qv = \frac{\partial v(x,0)}{\partial t} = \Psi(x), & x \in (0,1), \end{cases} \tag{5}$$

and the purely integral conditions

$$\begin{cases} \int_0^1 v(x,t)dx = \mu(t), & t \in (0,T), \\ \int_0^1 xv(x,t)dx = E(t), & t \in (0,T), \end{cases} \tag{6}$$

where $\Phi, \Psi, \mu, E, a, b, c$ et g are known continuous functions.

Hypothesis 1. *(1) For all $x, t \in \overline{Q}$, we assume that*

$$\sup_Q a(x,t) \leq 0, \sup_Q \frac{\partial^4 a(x,t)}{\partial x^4} \geq 0, \inf_Q \frac{\partial^3 b(x,t)}{\partial x^3} \leq 0, c(x,t) \geq 0, \sup_Q \frac{\partial^2 c(x,t)}{\partial x^2} \geq 0. \tag{7}$$

(2) For all $x, t \in \overline{Q}$ there exist $M > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned} 0 < M \leq 4\frac{\partial^2 a(x,t)}{\partial x^2} - 4\sup_Q a(x,t) - \frac{1}{2}\sup_Q \frac{\partial^4 a(x,t)}{\partial x^4} + \frac{1}{2}\inf_Q \frac{\partial^3 b(x,t)}{\partial x^3} \\ -\frac{1}{2}\sup_Q \frac{\partial^2 c(x,t)}{\partial x^2} - 3\frac{\partial b(x,t)}{\partial x} + 2c(x,t) - \frac{1}{2\varepsilon} \end{aligned} \tag{8}$$

this hypothesis is equivalent to the following one.

There exists $M > 0$ such that: Every positive number $M' > 0$ could be written as $M + \frac{1}{2\varepsilon}$ with $M > 0$ and $\varepsilon > 0$

$$\begin{aligned} 0 < M' \leq 4\frac{\partial^2 a(x,t)}{\partial x^2} - 4\sup_Q a(x,t) - \frac{1}{2}\sup_Q \frac{\partial^4 a(x,t)}{\partial x^4} + \frac{1}{2}\inf_Q \frac{\partial^3 b(x,t)}{\partial x^3} \\ -\frac{1}{2}\sup_Q \frac{\partial^2 c(x,t)}{\partial x^2} - 3\frac{\partial b(x,t)}{\partial x} + 2c(x,t). \end{aligned} \tag{9}$$

(3) The functions $\Phi(x)$ and $\Psi(x)$ satisfy the following compatibility conditions

$$\int_0^1 \Phi dx = \mu(0), \int_0^1 x\Phi dx = E(0), \int_0^1 \Psi dx = \mu'(0), \int_0^1 x\Psi dx = E'(0). \tag{10}$$

Our proof consists of three essential steps:

1. Reformulation of the problem into a problem with homogeneous conditions.
2. The uniqueness of the solution to the problem using the a priori estimate method.
3. The existence of the solution of the problem based on the density of the range of the operator generated by the abstract formulation of the problem.

3.2. Reformulation of the Problem

We transform a problem ((4)–(6)) with nonhomogeneous integral conditions to the equivalent problem with homogeneous integral conditions by introducing a new unknown function $\widetilde{u}$ defined by

$$v(x,t) = \widetilde{u}(x,t) + U(x,t), \tag{11}$$

where

$$U(x,t) = 2(2-3x)\mu(t) + 6(2x-1)E(t). \tag{12}$$

Now we study a new problem with homogeneous integral conditions

$$\begin{cases} \pounds\widetilde{u} = {}_0^c\partial_t^\alpha \widetilde{u} + a(x,t)\frac{\partial^2 \widetilde{u}}{\partial x^2} + b(x,t)\frac{\partial \widetilde{u}}{\partial x} + c(x,t)\widetilde{u} = h(x,t), \\ \ell v = \widetilde{u}(x,0) = \varphi(x), \quad x \in (0,1), \\ qv = \frac{\partial \widetilde{u}(x,0)}{\partial t} = \psi(x), \quad x \in (0,1), \\ \int_0^1 \widetilde{u}(x,t)dx = 0, \quad t \in (0,T), \\ \int_0^1 x\widetilde{u}(x,t)dx = 0, \quad t \in (0,T), \end{cases} \tag{13}$$

where

$$\begin{aligned} h(x,t) &= g(x,t) - \pounds U(x,t), \\ \varphi(x) &= \Phi(x) - \ell U, \\ \psi(x) &= \Psi(x) - qU. \end{aligned}$$

and

$$\int_0^1 \varphi(x)dx = 0, \int_0^1 x\varphi(x)dx = 0, \int_0^1 \psi(x) = 0, \int_0^1 x\psi(x) = 0.$$

Again we introduce a new function u defined by

$$u(x,t) = \widetilde{u}(x,t) - \psi(x)t - \varphi(x), \tag{14}$$

therefore problem (13) is given as follow

$$\begin{cases} \pounds u = {}_0^c\partial_t^\alpha u + a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u = f(x,t), \\ \ell u = u(x,0) = 0, \quad x \in (0,1), \\ qu = \frac{\partial u(x,0)}{\partial t} = 0, \quad x \in (0,1), \\ \int_0^1 u(x,t)dx = 0, \quad t \in (0,T), \\ \int_0^1 xu(x,t)dx = 0, \quad t \in (0,T). \end{cases} \tag{15}$$

Thus, instead of seeking for solution v of problems (4)–(6), we establish the existence and the uniqueness of the solution u of problem (15)

The solution v is simply given by:

$$v(x,t) = \widetilde{u}(x,t) + U(x,t). \tag{16}$$

3.3. Energy Inequality Method and Consequences

To prove the existence of the solution, we use the energy inequality method known also as the "a priori estimate" method, which consists mainly of reformulating problem (15) in an equivalent operational form:

$$Lu = \mathcal{F},$$

where the operator $L = (\pounds, \ell, q)$ acts from B to F, where B is a Banach space of functions $u \in L^2(Q)$ with the finite norm:

$$\|u\|_B = \left(\int_Q \left({}_0^c\partial_t^{\frac{\alpha}{2}} (\Im_x u) \right)^2 dxdt + \int_Q (\Im_x u)^2 dxdt \right)^{\frac{1}{2}}, \tag{17}$$

and F is a Hilbert space consisting of all the elements $F = (f,0,0)$ with the finite norm:

$$\|\mathsf{F}\|_F = \left(\int_Q f^2 dxdt \right)^{\frac{1}{2}}. \tag{18}$$

Let $D(L)$, the domain of the operator L, be the set of all functions u such that $\Im_x u$, $\Im_x \left({}_0^c\partial_t^\alpha u \right)$, $\Im_x \frac{\partial u}{\partial x}$, $\Im_x \frac{\partial^2 u}{\partial x^2} \in L^2(Q)$ and u satisfy the integral conditions (6).

Theorem 1. *Under assumptions (7) and (8), for all $u \in D(L)$ we have the estimate*

$$\|u\|_B \le C\|Lu\|_F, \tag{19}$$

where C is a positive constant independent of $u, u \in D(L)$.

Proof. Multiplying the fractional differential equation in problem (15) by $Mu = -2\Im_x^2 u$ and integrating it on Q we find

$$\begin{aligned}
&-2\int_Q ({}_0^c\partial_t^\alpha u)\Im_x^2 u dxdt - 2\int_Q a(x,t)\frac{\partial^2 u}{\partial x^2}\Im_x^2 u dxdt \\
&-2\int_Q b(x,t)\frac{\partial u}{\partial x}\Im_x^2 u dxdt - 2\int_Q c(x,t)u\,\Im_x^2 u dxdt \\
=\;& -2\int_Q f\;\Im_x^2 u dxdt.
\end{aligned} \tag{20}$$

Integrating by parts the four integrals on the left side of (20), we obtain

$$-2\int_Q ({}_0^c\partial_t^\alpha u)\Im_x^2 u dxdt = 2\int_Q ({}_0^c\partial_t^\alpha \Im_x u)(\Im_x u)dxdt, \tag{21}$$

$$\begin{aligned}
-2\int_Q a(x,t)\frac{\partial^2 u}{\partial x^2}\Im_x^2 u dxdt \;=\;& 4\int_Q \frac{\partial^2 a}{\partial x^2}(\Im_x u)^2 dx - 2\int_Q au^2 dxdt \\
&-\int_Q \frac{\partial a^4}{\partial x^4}\left(\Im_x^2 u\right)^2 dx,
\end{aligned} \tag{22}$$

$$-2\int_Q b(x,t)\frac{\partial u}{\partial x}\Im_x^2 u dx = \int_Q \frac{\partial^3 b}{\partial x^3}\left(\Im_x^2 u\right)^2 dx - 3\int_Q \frac{\partial b}{\partial x}(\Im_x u)^2 dx, \tag{23}$$

$$-2\int_Q c(x,t)u\,\Im_x^2 u dx = -\int_Q \frac{\partial^2 c}{\partial x^2}\left(\Im_x^2 u\right)^2 dx + 2\int_Q c(\Im_x u)^2 dx. \tag{24}$$

Substituting (21)–(24) into (20) yields

$$\begin{aligned}
&2\int_Q ({}_0^c\partial_t^\alpha \Im_x u)(\Im_x u)dx + 4\int_Q \frac{\partial^2 a}{\partial x^2}(\Im_x u)^2 dx - 2\int_Q au^2 dx \\
&-\int_Q \frac{\partial a^4}{\partial x^4}\left(\Im_x^2 u\right)^2 dx + \int_Q \frac{\partial^3 b}{\partial x^3}\left(\Im_x^2 u\right)^2 dx - 3\int_Q \frac{\partial b}{\partial x}(\Im_x u)^2 dx \\
&-\int_Q \frac{\partial^2 c}{\partial x^2}\left(\Im_x^2 u\right)^2 dx + 2\int_Q c(\Im_x u)^2 dx \\
=\;& -2\int_Q f\;\Im_x^2 u dx.
\end{aligned} \tag{25}$$

By the elementary inequalities in Lemmas 3 and 4 and assumptions (7) and (8) we obtain

$$\begin{aligned}
&2\int_Q \left({}_0^c\partial_t^{\frac{\alpha}{2}}(\Im_x u)\right)^2 dxdt + \int_Q (4\frac{\partial^2 a}{\partial x^2} - 4\sup au \\
&-\frac{1}{2}\frac{\partial a^4}{\partial x^4} + \frac{1}{2}\inf\frac{\partial^3 b}{\partial x^3} - 3\frac{\partial b}{\partial x} \\
&-\frac{1}{2}\sup\frac{\partial^2 c}{\partial x^2} + 2c)(\Im_x u)^2 dxdt \\
\le\;& -2\int_Q f\;\Im_x^2 u dxdt.
\end{aligned} \tag{26}$$

The estimate of the right side of (26) gives:

$$\begin{aligned}
& \int_Q \left({}_0^c\partial_t^{\frac{\alpha}{2}}(\Im_x u)\right)^2 dxdt + \int_Q (4\frac{\partial^2 a}{\partial x^2} - 4\sup au \\
& -\frac{1}{2}\frac{\partial a^4}{\partial x^4}dxdt + \frac{1}{2}\inf\frac{\partial^3 b}{\partial x^3} - 3\frac{\partial b}{\partial x} \\
& -2\sup\frac{\partial^2 c}{\partial x^2} + 2c - \frac{1}{2\varepsilon})(\Im_x u)^2 dxdt \\
\leq\ & \varepsilon\int_Q f^2 dxdt.
\end{aligned} \tag{27}$$

So, by using assumptions (7) and (8) we find

$$\begin{aligned}
& 2\int_Q \left({}_0^c\partial_t^{\frac{\alpha}{2}}(\Im_x u)\right)^2 dxdt + M\int_Q (\Im_x u)^2 dxdt \\
\leq\ & \varepsilon\int_Q f^2 dxdt.
\end{aligned} \tag{28}$$

Finally, we obtain a priori estimate

$$\|u\|_B \leq C\|Lu\|_F, \tag{29}$$

where

$$C = \left(\frac{\varepsilon}{\min(2,M)}\right)^{\frac{1}{2}}.$$

□

We proved the uniqueness of the solution in the case of existence, and we have

Corollary 1. *The operator L from B to F has a closure $\overline{L}$.*

Proof. See [18]. □

The a priori estimate (19) can be extended to cover the strong solution of problem (15) by passing to the limit.

Corollary 2. *The range of the operator $\overline{L}$ is closed in F and $R(\overline{L}) = \overline{R(L)}$.*

Consequently, the strong solution of the problem is unique if it exists, and depends continuously on $\mathcal{F} = (f, 0, 0)$.

3.4. Existence of the Solution

To prove the existence, it suffices to prove that $R(L)$ is dense in F; that is, its orthogonal is reduced to the singleton $\{0\}$.

Theorem 2. *Let us suppose that assumptions (7) and (8) and integral conditions (6) are filled, and for $\omega \in L^2(Q)$ and for all $u \in\ D(L)$, we have*

$$\int_Q \pounds u.\omega dxdt = 0, \tag{30}$$

then ω is almost everywhere in Q.

Proof. We can rewrite Equation (30) as follows

$$\int_Q ({}_0^c\partial_t^\alpha u\omega)dxdt = -\int_Q a(x,t)\frac{\partial^2 u}{\partial x^2}\omega dxdt - \int_Q b(x,t)\frac{\partial u}{\partial x}\omega dxdt - \int_Q c(x,t)u\omega dxdt. \quad (31)$$

We express the function ω in terms of u as follows:

$$\omega = -2\Im_x^2 u. \quad (32)$$

Substituting ω by its representation (32) into (31), integrating by parts and taking into account conditions (6) and assumptions (7) and (8), we obtain:

$$2\int_Q ({}_0^c\partial_t^\alpha \Im_x u)\Im_x u dxdt = -4\int_Q \frac{\partial^2 a}{\partial x^2}(\Im_x u)^2 dxdt + 2\int_Q au^2 dxdt + \int_Q \frac{\partial^4 a}{\partial x^4}(\Im_x u)^2 dxdt$$

$$-\int_Q \frac{\partial^3 b}{\partial x^3}(\Im_x u)^2 dxdt + 3\int_Q \frac{\partial b}{\partial x}(\Im_x u)^2 dxdt + \int_Q \frac{\partial^2 c}{\partial x^2}(\Im_x u)^2 dxdt - 2\int_Q c(\Im_x u)^2 dxdt.$$

□

Under assumptions (7) and (8) and conditions (6), we obtain

$$2\int_Q ({}_0^c\partial_t^\alpha \Im_x u)\Im_x u dxdt = -\int_Q (4\frac{\partial^2 a}{\partial x^2} + 4\sup au + \frac{1}{2}\frac{\partial^4 a}{\partial x^4} - \frac{1}{2}\inf\frac{\partial^3 b}{\partial x^3} + 3\frac{\partial b}{\partial x} + 2\sup\frac{\partial^2 c}{\partial x^2} - 2c)(\Im_x u)^2 dxdt.$$

Using condition (6) under assumptions, we find

$$2\int_Q ({}_0^c\partial_t^\alpha \Im_x u)\Im_x u dxdt \le -\left(\frac{1}{2\varepsilon} + M\right)\int_Q (\Im_x u)^2 dxdt.$$

By Lemmas 2–4 we obtain

$$2\int_Q \left({}_0^c\partial_t^{\frac{\alpha}{2}}(\Im_x u)\right)^2 dxdt \le -\left(\frac{1}{2\varepsilon} + M\right)\int_Q (\Im_x u)^2 dxdt.$$

Then

$$(\Im_x u)^2 = 0. \quad (33)$$

We obtain

$$u = 0.$$

So $u = 0$ in Ω which gives $\omega = 0$ in $L^2(Q)$.

4. Numerical Study

In this section we present the numerical technique that we will apply to the problem considered above, and we illustrate the schemes obtained with well-chosen applications.

4.1. Finite Difference Method

4.1.1. Discretization of the Problem

We consider a uniform subdivision of intervals $[0,1]$ and $[0,T]$ as follows

$$x_i = ih;\ i = 0,\dots,N \text{ and } t_k = kh_t;\ k = 0,\dots,M.$$

We denote by v_i^k the approximate solution of $v(x_i,t_k)$ at point (x_i,t_k), and L the operator defined by

$$L = a\frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x} + c,\ L(.)_i^k = a_i^k\frac{\partial^2(.)}{\partial x^2} + b_i^k\frac{\partial(.)}{\partial x} + c_i^k \tag{34}$$

where

$$a_i^k = a(x_i, t_k),\quad b_i^k = b(x_i, t_k),\quad c_i^k = c(x_i, t_k).$$

From the Taylor development of function v at the point (x_i, t_k) we have

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_i^k = \frac{1}{h^2}\left(v_{i-1}^k - 2v_i^k + v_{i+1}^k\right) + O\left(h^2\right),\ \left(\frac{\partial v}{\partial x}\right)_i^k = \frac{v_{i+1}^k - v_i^k}{h} + O(h). \tag{35}$$

Substituting (35) in the operator L_i^k expressed in (34) gives

$$Lv_i^{k+1} = \left(\frac{a_i^{k+1}}{h^2} + \frac{b_i^{k+1}}{h}\right)v_{i+1}^{k+1} + \left(c_i^{k+1} - 2\frac{a_i^{k+1}}{h^2} - \frac{b_i^{k+1}}{h}\right)v_i^{k+1} + \frac{a_i^{k+1}}{h^2}v_{i-1}^{k+1}. \tag{36}$$

The discretization of Caputo derivative fractional operator ${}_0^c\partial_t^\alpha v$ [25] with $1 < \alpha < 2$ is defined by

$$({}_0^c\partial_t^\alpha v)_i^{k+1} \simeq \gamma\sum_{j=0}^{k}\left(v_i^{k-j-1} - 2v_i^{k-j} + v_i^{k-j+1}\right)d_j \tag{37}$$

$$\text{where}\begin{cases} d_j = (j+1)^{2-\alpha} - j^{2-\alpha} \\ d_0 = 1; k = 1, \ldots, M \end{cases},\quad \gamma = \frac{h_t^{-\alpha}}{\Gamma(3-\alpha)}.$$

Writing fractional differential Equation (4) in point $(ih, (k+1)h_t)$, we find

$$\gamma\sum_{j=0}^{k}\left(v_i^{k-j-1} - 2v_i^{k-j} + v_i^{k-j+1}\right)d_j + Lv_i^{k+1} = g_i^{k+1},\ i = \overline{1, N-1}. \tag{38}$$

Then

$$\begin{aligned} F_i^{k+1}v_{i-1}^{k+1} + A_i^{k+1}v_i^{k+1} + B_i^{k+1}v_{i+1}^{k+1} - 2\gamma d_k v_i^k + \gamma d_k v_i^{k-1} + \gamma\sum_{j=1}^{k-1}S_j d_j & \\ +\gamma\left(v_i^{-1} - 2v_i^0 + v_i^1\right)d_k &= g_i^{k+1} \end{aligned} \tag{39}$$

where

$$\begin{aligned} A_i^{k+1} &= \gamma + c_i^{k+1} - 2\frac{a_i^{k+1}}{h^2} - \frac{b_i^{k+1}}{h},\quad B_i^{k+1} = \frac{a_i^{k+1}}{h^2} + \frac{b_i^{k+1}}{h}, \\ F_i^{k+1} &= \frac{a_i^{k+1}}{h^2},\quad S_j = v_i^{k-j-1} - 2v_i^{k-j} + v_i^{k-j+1}. \end{aligned}$$

To eliminate v_i^{-1}, we use initial condition (5); we find

$$\left(\frac{\partial v}{\partial t}\right)_i^n \simeq \frac{v_i^n - v_i^{n-1}}{h_t}.$$

Therefore,

$$v_i^{-1} \simeq \Phi_i - h_t\Psi_i = v_i^0 - h_t\Psi_i,\quad i = \overline{1, N-1}. \tag{40}$$

Substituting (40) into (39), we obtain

$$
\begin{aligned}
& F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} - 2\gamma d_k v_i^k + \gamma d_k v_i^{k-1} + \gamma \sum_{j=1}^{k-1} S_j d_j \\
= \quad & d_k \gamma v_i^0 + d_k \gamma h_t \Psi_i - d_k \gamma v_i^1 + g_i^{k+1}.
\end{aligned} \tag{41}
$$

For $k = 0$, relation (41) gives

$$
F_i^1 v_{i-1}^1 + A_i^1 v_i^1 + B_i^1 v_{i+1}^1 = \gamma v_i^0 + \gamma h_t \Psi_i + g_i^1 \quad \text{with} \quad i = \overline{1, N-1}. \tag{42}
$$

By conditions (6), and the trapeze method we obtain,

$$
v_0^1 = \frac{2\mu(h_t) - 2E(h_t)}{h} + 2\sum_{j=1}^{N-1} (jh-1) v_j^1,\ v_N^1 = \frac{2E(h_t)}{h} - 2\sum_{j=1}^{N-1} jh v_j^1.
$$

For $i = 1$

$$
\begin{aligned}
& \Big(A_1^1 + 2F_1^1(h-1)\Big) v_1^1 + \Big(B_1^1 + 2F_1^1(2h-1)\Big) v_2^1 + 2F_1^1 \sum_{j=3}^{N-1} (jh-1) v_j^1 \\
& = \gamma v_1^0 + \gamma h_t \Psi_1 + g_1^1 + \frac{2F_1^1}{h} (E(h_t) - \mu(h_t)).
\end{aligned} \tag{43}
$$

For $i = N-1$

$$
\begin{aligned}
& -2B_{N-1}^1 \sum_{j=1}^{N-3} jh v_j^1 + \Big(F_{N-1}^1 - 2B_{N-1}^1 (N-2)h\Big) v_{N-2}^1 + \Big(A_{N-1}^1 - 2B_{N-1}^1 (N-1)h\Big) v_{N-1}^1 \\
& = \gamma v_{N-1}^0 + \gamma h_t \Psi_{N-1} + g_{N-1}^1 - \frac{2B_{N-1}^1}{h} E(h_t).
\end{aligned} \tag{44}
$$

Matrix's form
We denote by

$$
w_i = \gamma v_i^0 + \gamma h_t \Psi_i + g_i^1, \quad y_1^1 = \frac{2F_1^1}{h}(E(h_t) - \mu(h_t)), \quad z_{N-1}^1 = -\frac{2B_{N-1}^1}{h} E(h_t),
$$

$$
\begin{aligned}
P^1 &= \left(l_{i,j}\right)_{N-1,N-1} \text{ is square matrix defined by} \\
l_{1,1} &= A_1^1 + 2F_1^1(h-1),\ l_{1,2} = B_1^1 + 2F_1^1(2h-1), \\
l_{N-1,N-2} &= F_{N-1}^1 - 2B_{N-1}^1 (N-2)h,\ l_{N-1,N-1} = A_{N-1}^1 - 2B_{N-1}^1 (N-1)h\,,
\end{aligned}
$$

$$
l_{i,j} = \begin{cases}
2F_1^1(jh-1) & \text{when} \quad i = 1,\ j = \overline{3, N-1} \\
0 & \text{when} \quad |i-j| \geq 2\,,\, i = \overline{2, N-2} \\
A_i^1 & \text{when} \quad i = j,\ i = \overline{2, N-2} \\
B_i^1 & \text{when} \quad i = j-1,\ i = \overline{1, N-2} \\
F_i^1 & \text{when} \quad i = j+1,\ i = \overline{2, N-1} \\
-2B_{N-1}^1 jh & \text{when} \quad i = N-1,\ j = \overline{1, N-3}.
\end{cases}
$$

Taking into account (42)–(44), we obtain the matricial system

$$
P^1.V^1 = H^1 \tag{45}
$$

where

$$
H^1 = W^1 + R^1,\ W^1 = \left(w_1^1, w_2^1, \ldots, w_{N-1}^1\right)^T,\ R^1 = \left(y_1^1, 0, 0, \ldots, 0, z_{N-1}^1\right)^T.
$$

To solve the system (45) we can apply one of the direct methods.

4.1.2. General Case

It is readily seen that, *for* $k \geq 1$

$$\sum_{j=1}^{k-1} S_j d_j = (d_2 - 2d_1) v_i^{k-1} + d_1 v_i^k + d_{k-1} v_i^0 + (d_{k-2} - 2d_{k-1}) v_i^1 + \sum_{m=2}^{k-2} \sigma_m v_i^{k-m} \tag{46}$$

where

$$\sigma_m = d_{m-1} - 2d_m + d_{m+1},\ m = \overline{2, k-2}.$$

Lemma 5. *If* $k \geq 1$; *the numerical scheme (41) is equivalent to*

$$F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} = -\gamma \sum_{m=1}^{k-1} \sigma_m v_i^{k-m} + \gamma(2 - d_1) v_i^k + \gamma(d_k - d_{k-1}) v_i^0$$
$$+ \gamma d_k h_t \Psi_i + g_i^{k+1}, \quad \text{for} \quad i = 1, \ldots, N-1. \tag{47}$$

Proof. From scheme (41), we have

$$F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} - 2\gamma d_k v_i^k + \gamma d_k v_i^{k-1} + \gamma \sum_{j=1}^{k-1} S_j d_j$$
$$= \quad d_k \gamma v_i^0 + d_k \gamma h_t \Psi_i - d_k \gamma v_i^1 + g_i^{k+1}.$$

□

So

$$F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} + \gamma \sum_{j=2}^{k-2} S_j d_i$$
$$+\gamma(v_i^{k+1} - 2v_i^k + v_i^{k-1})\, d_0 + \gamma(v_i^1 - 2v_i^0 + v_i^{-1}) d_k = g_i^{k+1}. \tag{48}$$

Using (46) we obtain

$$F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} = -\gamma \sum_{m=1}^{k-1} \sigma_m v_i^{k-m} + \gamma(2 - d_1) v_i^k + \gamma(d_k - d_{k-1}) v_i^0$$
$$+ \gamma d_k h_t \Psi_i + g_i^{k+1}, \text{ for } \quad i = 1, \ldots, N-1. \tag{49}$$

Using conditions (6), and by trapeze method we obtain,
For $i = 1$

$$\left(A_1^{k+1} + 2F_1^{k+1}(h-1)\right) v_1^{k+1} + \left(B_1^{k+1} + 2F_1^{k+1}(2h-1)\right) v_2^{k+1} + 2F_1^{k+1} \sum_{j=3}^{N-1} (jh - 1) v_j^{k+1}$$
$$= \frac{2F_1^{k+1}}{h}(E((k+1)h_t) - \mu((k+1)h_t)) - \gamma \sum_{m=1}^{k-1} \sigma_m v_1^{k-m} + \gamma(d_k - d_{k-1}) v_1^0 + \gamma d_k h_t \Psi_1 + g_1^{k+1}. \tag{50}$$

For $i = N-1$

$$-2B_{N-1}^{k+1} \sum_{j=1}^{N-3} jh v_j^{k+1} + \left(F_{N-1}^{k+1} - 2B_{N-1}^{k+1}(N-2)h\right) v_{N-2}^{k+1} + \left(A_{N-1}^{k+1} - 2B_{N-1}^{k+1}(N-1)h\right) v_{N-1}^{k+1}$$

$$= -\frac{2B_{N-1}^{k+1}}{h}E((k+1)h_t) - \gamma\sum_{m=1}^{k-1}\sigma_m v_{N-1}^{k-m} + \gamma(2-d_1)v_{N-1}^k$$
$$+\gamma(d_k - d_{k-1})v_{N-1}^0 + \gamma d_k h_t \Psi_{N-1} + g_{N-1}^{k+1}. \tag{51}$$

Matrix's form

We take expression (49) for $i = \overline{2, N-2}$ and Equations (50) and (51) to formulate the matrix systems:

$$\begin{cases} P^{k+1}V^{k+1} = H^{k+1}; \quad k \geq 1 \\ \\ V^0,\ V^1 \text{ are known} \end{cases} \tag{52}$$

where

$$P^{k+1} = \left(l_{i,j}^{k+1}\right)_{N-1,N-1}$$

is square matrix defined by

$$\begin{aligned} l_{1,1}^{k+1} &= A_1^{k+1} + 2F_1^{k+1}(h-1),\ l_{1,2}^{k+1} = B_1^{k+1} + 2F_1^{k+1}(2h-1), \\ l_{N-1,N-2}^{k+1} &= F_{N-1}^{k+1} - 2B_{N-1}^{k+1}(N-2)h,\ l_{N-1,N-1}^{k+1} = A_{N-1}^{k+1} - 2B_{N-1}^{k+1}(N-1)h\,, \end{aligned}$$

$$l_{i,j}^{k+1} = \begin{cases} 2F_1^{k+1}(jh-1) & \text{when} \quad i=1,\ j=\overline{3,N-1} \\ 0 & \text{when} \quad |i-j| \geq 2\,,\, i = \overline{2,N-2} \\ A_i^{k+1} & \text{when} \quad i=j,\ i=\overline{2,N-2} \\ B_i^{k+1} & \text{when} \quad i=j-1,\ i=\overline{1,N-2} \\ F_i^{k+1} & \text{when} \quad i=j+1,\ i=\overline{2,N-1} \\ -2B_{N-1}^{k+1}jh & \text{when} \quad i=N-1,\ j=\overline{1,N-3} \end{cases}$$

and

$$\begin{aligned} V^{\,k+1} &= \left(v_1^{k+1},\ldots,v_{N-1}^{k+1}\right)^T;\ H^{k+1} = -\gamma\sum_{m=1}^{k-1}\sigma_m V^{\,k-m} + W^{\,k+1} + R^{\,k+1}; k\geq 1 \\ W^{\,k+1} &= \left(w_1^{k+1}, w_2^{k+1},\ldots,w_{N-1}^{k+1}\right)^T,\ R^{k+1} = \left(y_1^{k+1},0,0,\ldots,0,z_{N-1}^{k+1}\right)^T, \\ w_i^{k+1} &= \gamma(2-d_1)v_i^k + \gamma(d_k - 2d_{k-1})v_i^0 + \gamma d_k h_t \Psi_i + g_i^{k+1}, \\ y_1^{k+1} &= \frac{2F_1^{k+1}}{h}(E((k+1)h_t) - \mu((k+1)h_t));\ z_{N-1}^{k+1} = -\frac{2B_{N-1}^{k+1}}{h}E((k+1)h_t). \end{aligned}$$

In order to prove system (52) has a unique solution we denote ρ as an eigenvalue of the matrix P^k, and $X = (x_1, x_2, \ldots, x_{N-1})^T$ is a nonzero eigenvector corresponding to ρ

We choose i such that

$$|x_i| = \max\{|x_j| : j = 1;\ldots;N-1\}.$$

Then

$$\sum_{j=1}^{N-1} l_{i,j}x_j = \rho x_i;\ i = \overline{1;N-1}.$$

Therefore

$$\rho = l_{i,i} + \sum_{\substack{j=1 \\ j\neq i}}^{N-1} l_{i,\,j}\frac{x_j}{x_i}. \tag{53}$$

Substituting the values of $l_{i,j}$ into (53), and taking into account that F_i^k, a_i^k are negative and $\left|\frac{x_j}{x_i}\right| \leq 1$ we get:

For $i = 1$

$$\begin{aligned}\rho &= \left(A_1^{k+1}+2F_1^{k+1}(h-1)\right)+\left(B_1^{k+1}+2F_1^{k+1}(2h-1)\right)\frac{x_2}{x_1}+2F_1^{k+1}\sum_{j=3}^{N-1}(jh-1)\frac{x_j}{x_1} \\ &= \gamma+c_i^{k+1}-F_1^{k+1}-B_1^{k+1}\left(1-\frac{x_2}{x_1}\right)+2F_1^{k+1}\sum_{j=2}^{N-1}(jh-1)\frac{x_j}{x_1}.\end{aligned} \tag{54}$$

For $i = N-1$

$$\begin{aligned}\rho &= l_{i,i}+\sum_{\substack{j=1\\ j\neq i}}^{N-1} l_{i,j}\frac{x_j}{x_i} \\ &= A_{N-1}^{k+1}-2B_{N-1}^{k+1}(N-1)h+\left(F_{N-1}^{k+1}-2B_{N-1}^{k+1}(N-2)h\right)\left(\frac{x_{N-2}}{x_{N-1}}\right)-2B_{N-1}^{k+1}\sum_{j=1}^{N-3}jh\frac{x_j}{x_{N-1}} \\ &= \gamma+c_{N-1}^{k+1}-B_{N-1}^{k+1}+F_{N-1}^{k+1}\left(\frac{x_{N-2}}{x_{N-1}}-1\right)-2B_{N-1}^{k+1}(N-1)h-2B_{N-1}^{k+1}\sum_{j=1}^{N-2}jh\frac{x_j}{x_{N-1}}.\end{aligned} \tag{55}$$

For $i = \overline{2, N-2}$

$$\begin{aligned}\rho &= l_{i,i}+\sum_{\substack{j=1\\ j\neq i}}^{N-1} l_{i,j}\frac{x_j}{x_i} \\ &= A_i^{k+1}+F_i^{k+1}\frac{x_{i-1}}{x_i}+B_i^{k+1}\frac{x_{i+1}}{x_i} \\ &= \gamma+c_i^{k+1}-B_i^{k+1}-F_i^{k+1}+F_i^{k+1}\frac{x_{i-1}}{x_i}+B_i^{k+1}\frac{x_{i+1}}{x_i} \\ &= \gamma+c_i^{k+1}+F_i^{k+1}\left(\frac{x_{i-1}}{x_i}-1\right)+\frac{a_i^{k+1}+hb_i^{k+1}}{h^2}\left(\frac{x_{i+1}}{x_i}-1\right).\end{aligned} \tag{56}$$

From the above we conclude for $i = \overline{1, N-1}$,
If $b_i^{k+1} < 0$, $B_{N-1}^{k+1} < 0$ then $\rho > 0$.
If $b_i^{k+1} > 0$ and $h \leq \min_{1\leq i\leq N-1}\left(\frac{-a_i^{k+1}}{b_i^{k+1}}\right)$, $\rho > 0$, then all eigenvalues of matrix P^{k+1} are strictly positive; therefore, P^{k+1} is invertible.

4.2. Stability and Convergence

4.2.1. Stability

We have

$$F_i^{k+1}+A_i^{k+1}+B_i^{k+1}=\gamma+c_i^{k+1},\ F_i^{k+1}\leq 0,\ A_i^{k+1}+B_i^{k+1}\geq 0.$$

Let u_i^{k+1}be the approximate solution of (49), and e_i^{k+1} the error at point (x_i, t_{k+1}) defined by

$$v_i^{k+1}-u_i^{k+1}=e_i^{k+1},\ \text{and}\ \left\|E^k\right\|=\max_{1\leq i\leq N-1}|e_i^k|,\ \ E^k=\left(e_1^k,\ldots,e_{N-1}^k\right)^T.$$

For $k = 0$ we apply (42) and we get

$$\begin{aligned}
\left\|E^1\right\| &\leq \left(\gamma + c_i^1\right)\left\|E^1\right\| = \left(F_i^1 + A_i^1 + B_i^1\right)\left\|E^1\right\| \\
&= \left(F_i^1\left\|E^1\right\| + \left(A_i^1 + B_i^1\right)\left\|E^1\right\|\right) \\
&\leq \left(\left(A_i^1 + B_i^1\right)\left\|E^1\right\| + F_i^1\left|e_{i-1}^1\right|\right) \\
&\leq \max_{1 \leq i \leq N-1}\left|F_i^1 e_{i-1}^1 + A_i^1 e_i^1 + B_i^1 e_{i+1}^1\right| = \gamma\left\|E^0\right\|.
\end{aligned}$$

So

$$\left\|E^1\right\| \preceq \frac{\gamma}{\gamma + c_i^1}\left\|E^0\right\| \preceq \left\|E^0\right\|. \tag{57}$$

Therefore the method is stable.

Lemma 6. *For $k \geq 1$ the scheme (48) is stable and we have*

$$\left\|E^{k+1}\right\| \leq C\left\|E^0\right\|,\ C > 0, \text{ for all } k \geq 1.$$

Proof. We use mathematical induction. □

We assume $\|E^j\| \leq c_j\|E^0\|$, and $C_{\max} = \max c_j$; where $c_j \succ 0$, $j = \overline{1, k}$.
From (49), where $i = \overline{1, N-1}$, we get

$$F_i^{k+1}e_{i-1}^{k+1} + A_i^{k+1}e_i^{k+1} + B_i^{k+1}e_{i+1}^{k+1} = -\gamma\sum_{m=1}^{k-1}\sigma_m e_i^{k-m} + \gamma(2 - d_1)e_i^k + \gamma(d_k - d_{k-1})e_i^0,$$

so

$$\begin{aligned}
\left(\gamma + c_i^{k+1}\right)\left\|E^{k+1}\right\| &\leq \left(\left(A_i^{k+1} + B_i^{k+1}\right)\left\|E^{k+1}\right\| + F_i^{k+1}\left|e_{i-1}^{k+1}\right|\right) \\
&\leq \left\|-\gamma\sum_{m=1}^{k-1}\sigma_m e_i^{k-m} + \gamma(2 - d_1)e_i^k + \gamma(d_k - d_{k-1})e_i^0\right\| \\
&\leq \gamma\left(\sum_{m=1}^{k-1}|\sigma_m|\left\|E_i^{k-m}\right\| + (2 - d_1)\left\|e_i^k\right\| + (d_{k-1} - d_k)\left\|e_i^0\right\|\right) \\
&\leq \gamma C_{\max}\left(\sum_{m=1}^{k-1}|\sigma_m| + 2 - d_1 + d_{k-1} - d_k\right)\left\|E^0\right\| \\
&\leq \gamma C_{\max}\left(5 - 2^{3-\alpha}\right)\left\|E^0\right\|
\end{aligned}$$

where

$$\sum_{m=1}^{k-1}|\sigma_m| + 2 - d_1 + d_{k-1} - d_k = 5 - 2^{3-\alpha},\ 0 < \sigma_m < 1,\ \ -1 < d_k - d_{k-1} < 0,\ 1 < 2 - d_1 < 2.$$

Then

$$\left\|E^{k+1}\right\| \leq C\left\|E^0\right\|;\ C = C_{\max}\left(5 - 2^{3-\alpha}\right). \tag{58}$$

Therefore, the method is stable.

4.2.2. Convergence

Let $v(x_i; t_{k+1})$ be the exact solution and v_i^{k+1} be the approximate solution of scheme (38); we put $v(x_i; t_{k+1}) - v_i^{k+1} = \epsilon_i^{k+1}$ for $i = \overline{1, N-1}$, $k = \overline{1, M-1}$.

The scheme L_2 defined in (37) verifed [25].

$$\left| \frac{\partial^\alpha v}{\partial t^\alpha} - \left(\frac{\partial^\alpha v}{\partial t^\alpha} \right)_{L_2} \right| \leq O(h_t). \tag{59}$$

Substituting into (38) and using (35) and (59) leads to

$$\gamma \sum_{j=0}^{k} \Big(v(x_i;t_{k-j-1}) - \epsilon_i^{k-j-1} - 2\Big(v(x_i;t_{k-j}) - \epsilon_i^{k-j} \Big) + \Big(v(x_i;t_{k-j+1}) - \epsilon_i^{k-j+1} \Big) \Big) d_j$$

$$+L\Big(v(x_i;t_{k+1}) - \epsilon_i^{k+1} \Big) = g_i^{k+1}.$$

Then

$$\gamma \sum_{j=0}^{k} \Big(v(x_i;t_{k-j-1}) - 2v(x_i;t_{k-j}) + \Big(v(x_i;t_{k-j+1}) \Big) \Big) d_j + Lv(x_i;t_{k+1})$$

$$-\gamma \sum_{j=0}^{k} \Big(\epsilon_i^{k-j-1} - 2\epsilon_i^{k-j} + \epsilon_i^{k-j+1} \Big) d_j - L\epsilon_i^{k+1} = g_i^{k+1}.$$

So

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + O(h_t) + Lv(x;t) + O(h) - \gamma \sum_{j=0}^{k} \Big(\epsilon_i^{k-j-1} - 2\epsilon_i^{k-j} + \epsilon_i^{k-j+1} \Big) d_j - L\epsilon_i^{k+1} = g_i^{k+1}.$$

Hence

$$\gamma \sum_{j=0}^{k} \Big(\epsilon_i^{k-j-1} - 2\epsilon_i^{k-j} + \epsilon_i^{k-j+1} \Big) d_j + L\epsilon_i^{k+1} = O(h+h_t). \tag{60}$$

Taking

$$\left| \epsilon_l^k \right| = \left\| \epsilon^k \right\| = \max_{1 \leq i \leq N-1} \left| \epsilon_i^k \right|; \epsilon^k = \Big(\epsilon_1^k, , \ldots, \epsilon_{N-1}^k \Big)^T; \; \left\| \epsilon_i^0 \right\| = 0$$

for $k = 0$, we get

$$F_i^1 \epsilon_{i-1}^1 + A_i^1 \epsilon_i^1 + B_i^1 \epsilon_{i+1}^1 = \gamma \epsilon_i^0 + O(h+h_t) \quad \text{with} \quad i = \overline{1, N-1}. \tag{61}$$

We have

$$\begin{aligned} \left\| \epsilon^1 \right\| &= \left| \epsilon_l^1 \right| \leq \Big(F_i^1 + A_i^1 + B_i^1 \Big) \left| \epsilon_l^1 \right| \\ &\leq \Big(\Big(A_i^1 + B_i^1 \Big) \left| \epsilon_l^1 \right| + F_i^1 \left| \epsilon_l^1 \right| \Big) \\ &\leq \max_{1 \leq i \leq N-1} \left| F_i^1 \epsilon_{i-1}^1 + A_i^1 \epsilon_l^1 + B_i^1 \epsilon_l^1 \right| = O(h+h_t). \end{aligned}$$

Hence

$$\left\| \epsilon^1 \right\| \leq O(h+h_t). \tag{62}$$

We assume : $\left| \epsilon_l^j \right| \leq O(h+h_t); j = \overline{1,k}$.
From (60), we get

$$F_i^{k+1} \epsilon_{i-1}^{k+1} + A_i^{k+1} \epsilon_i^{k+1} + B_i^{k+1} \epsilon_{i+1}^{k+1} = -\gamma \sum_{m=1}^{k-1} \sigma_m \epsilon_i^{k-m} + \gamma(2-d_1)\epsilon_i^k + O(h+h_t). \tag{63}$$

We have

$$\begin{aligned}
\left\|\epsilon^{k+1}\right\| &\leq \left(\gamma+c_i^{k+1}\right)\left|\epsilon_l^{k+1}\right| = \left(F_i^{k+1}+A_i^{k+1}+B_i^{k+1}\right)\left|\epsilon_l^{k+1}\right| \\
&\leq \left(F_i^{k+1}\left|\epsilon_{i-1}^{k+1}\right| + \left(A_i^{k+1}+B_i^{k+1}\right)\left|\epsilon_l^{k+1}\right|\right) \\
&\leq \max_{1\leq i\leq N-1}\left|-\gamma\sum_{m=1}^{k-1}\sigma_m\epsilon_i^{k-m}+\gamma(2-d_1)\epsilon_i^k+O(h+h_t)\right| \\
&\leq \gamma\sum_{m=1}^{k-1}\sigma_m\left\|\epsilon^{k-m}\right\|+\gamma(2-d_1)\left\|\epsilon^k\right\|+O(h+h_t) \\
&\leq \gamma\left(\sum_{m=1}^{k-1}\sigma_m+(2-d_1)\right)O(h+h_t)+O(h+h_t).
\end{aligned}$$

Hence

$$\left\|\epsilon^{k+1}\right\| \leq \frac{\gamma}{\gamma+c_i^{k+1}}O(h+h_t)+\frac{1}{\gamma+c_i^{k+1}}O(h+h_t)\leq O(h+h_t). \quad (64)$$

Therefore, the method is convergent.

4.3. Applications

In this section, we give some numerical investigation tests.

Example 1. *We consider problems (4)–(6) with*
$\alpha=\frac{3}{2},\quad a(x,t)=-x-t,$
$b(x,t)=x+t,\quad c(x)=2,\quad g(x,t)=(\frac{3}{4}\sqrt{\pi}+2t\sqrt{t})e^x,\quad \phi(x)=\psi(x)=0,$
$\mu(t)=(e-1)t^{\frac{3}{2}},\quad E(t)=t^{\frac{3}{2}}.$

The analytical solution is given by $v(x,t)=t^{\frac{3}{2}}e^x$.
The approximate solution $u(x,t)$ where A.E is the absolute error.
We see in Figures 1–3 and Tables 1–3 that the absolute error(A.E) gradually decreases when the step h_t takes small values and getting very close to zero. That is, for $h_t=0.01$, $h_t=0.001$, and $h_t=0.0001$ the absolute error(A.E) decreases towards zero and the approximate solution tends towards the exact solution with the convergence order of $O(h+h_t)$.

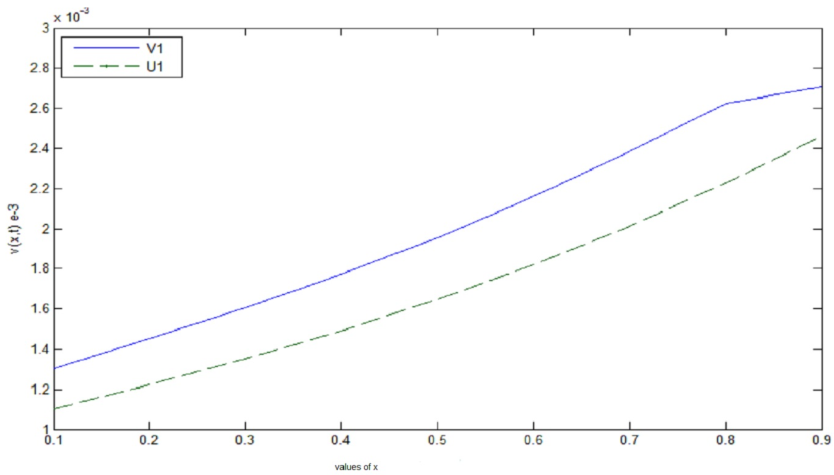

Figure 1. $h=0.1, h_t=0.01$.

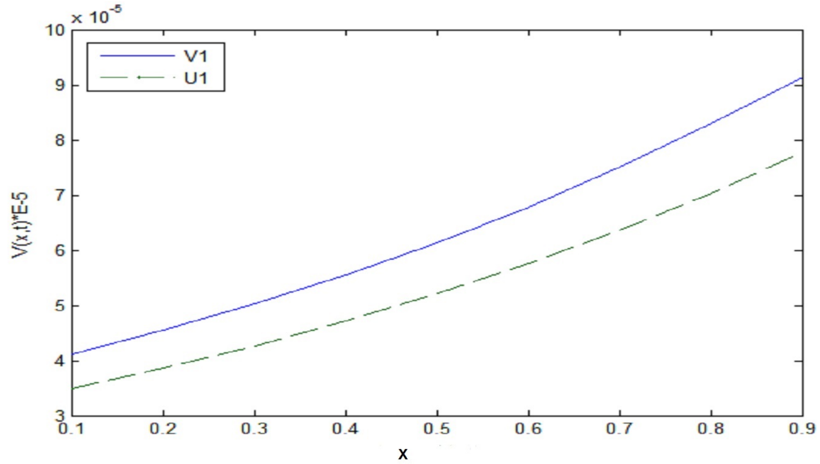

Figure 2. $h_t = 10^{-3}$.

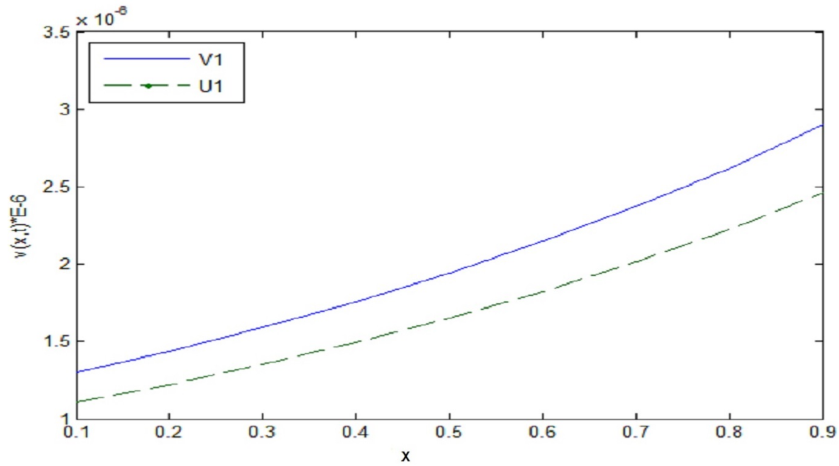

Figure 3. $h_t = 10^{-4}$.

For $k = 1$ (second iteration),

Table 4 shows the absolute error decrease to zero and Figures 4–6 show the approximate solution u^2 after two steps $2h_t$ tend towards the exact solution when h_t close to zero, with convergence order $O(h + h_t)$.

Table 1. $h = 0.1$; $h_t = 0.01$.

x_i	$u^1(x,t)$	$v^1(x,t)$	$\mathbf{A.E}$
0.1	1.1052×10^{-3}	1.3042×10^{-3}	1.99×10^{-4}
0.2	1.2214×10^{-3}	1.4523×10^{-3}	2.30×10^{-4}
0.3	1.3499×10^{-3}	1.6038×10^{-3}	2.53×10^{-4}
0.4	1.4918×10^{-3}	1.7710×10^{-3}	2.79×10^{-4}
0.5	1.6487×10^{-3}	1.9558×10^{-3}	3.07×10^{-4}
0.6	1.8221×10^{-3}	2.1600×10^{-3}	3.38×10^{-4}
0.7	2.0138×10^{-3}	2.3851×10^{-3}	3.71×10^{-4}
0.8	2.2255×10^{-3}	2.6235×10^{-3}	3.98×10^{-4}
0.9	2.4596×10^{-3}	2.7079×10^{-3}	2.48×10^{-4}

Table 2. $h = 0.1, h_t = 0.001$.

x_i	$u^1(x,t)$	$v^1(x,t)$	A.E
0.1	3.4949×10^{-5}	4.1175×10^{-5}	6.23×10^{-6}
0.2	3.8624×10^{-5}	4.5515×10^{-5}	6.89×10^{-6}
0.3	4.2686×10^{-5}	5.0301×10^{-5}	7.61×10^{-6}
0.4	4.7176×10^{-5}	5.5590×10^{-5}	8.41×10^{-6}
0.5	5.2137×10^{-5}	6.1435×10^{-5}	9.30×10^{-6}
0.6	5.7620×10^{-5}	6.7895×10^{-5}	1.03×10^{-5}
0.7	6.3680×10^{-5}	7.5034×10^{-5}	1.14×10^{-5}
0.8	7.0378×10^{-5}	8.2923×10^{-5}	1.25×10^{-5}
0.9	7.7802×10^{-5}	9.1415×10^{-5}	1.36×10^{-5}

Table 3. $h = 0.1, h_t = 0.0001$.

x_i	$u^1(x,t)$	$v^1(x,t)$	A.E
0.1	1.1052×10^{-6}	1.3020×10^{-6}	1×10^{-7}
0.2	1.2214×10^{-6}	1.4389×10^{-6}	2×10^{-7}
0.3	1.3499×10^{-6}	1.5903×10^{-6}	2×10^{-7}
0.4	1.4918×10^{-6}	1.7575×10^{-6}	2×10^{-7}
0.5	1.6487×10^{-6}	1.9424×10^{-6}	2×10^{-7}
0.6	1.8221×10^{-6}	2.1466×10^{-6}	3×10^{-7}
0.7	2.0138×10^{-6}	2.3724×10^{-6}	3×10^{-7}
0.8	2.2255×10^{-6}	2.6219×10^{-6}	3×10^{-7}
0.9	2.4596×10^{-6}	2.8974×10^{-6}	4×10^{-7}

Table 4. $h = 0.1, h_t = 10^{-2}, 10^{-3}, 10^{-5}$.

x_i	$h_t = 10^{-2}$	$h_t = 10^{-3}$	$h_t = 10^{-5}$
0.1	1.84×10^{-3}	5.80×10^{-5}	5.80×10^{-8}
0.2	1.74×10^{-3}	5.45×10^{-5}	5.45×10^{-8}
0.3	1.62×10^{-3}	5.06×10^{-5}	5.06×10^{-8}
0.4	1.49×10^{-3}	4.63×10^{-5}	4.63×10^{-8}
0.5	1.33×10^{-3}	4.15×10^{-5}	4.16×10^{-8}
0.6	1.17×10^{-3}	3.63×10^{-5}	3.63×10^{-8}
0.7	9.82×10^{-4}	3.05×10^{-5}	3.05×10^{-8}
0.8	7.40×10^{-4}	2.40×10^{-5}	2.41×10^{-8}
0.9	1.26×10^{-4}	1.64×10^{-5}	1.69×10^{-8}

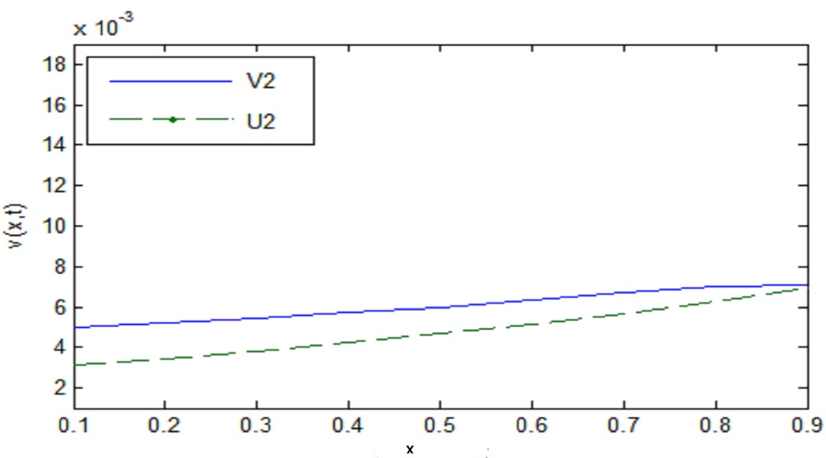

Figure 4. $h_t = 10^{-2}$.

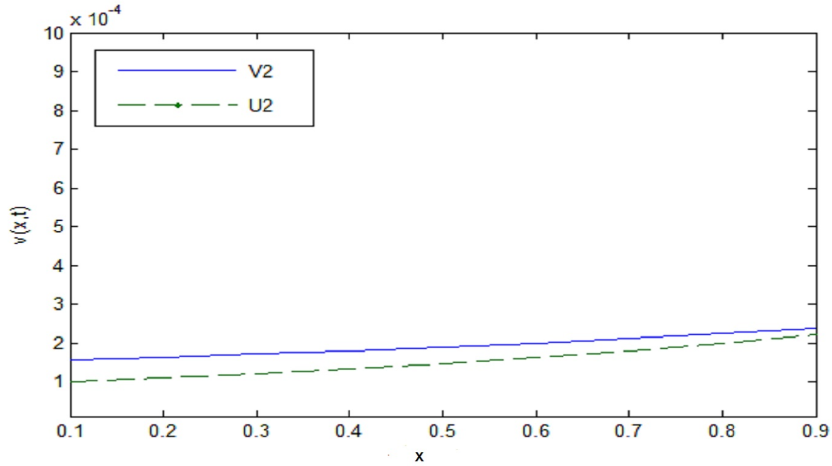

Figure 5. $h_t = 10^{-3}$.

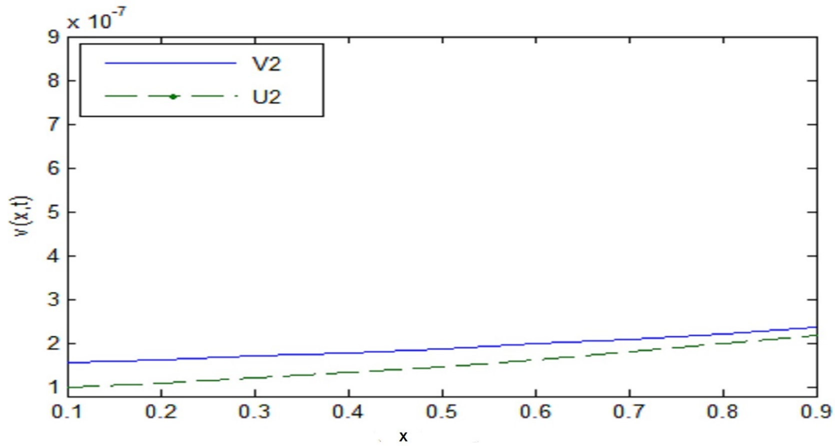

Figure 6. $h_t = 10^{-5}$.

Remark 1. *For space step $h = 0.01$ we find Figures A1 and A2 in Appendix A.*

Example 2. *We take:* $\alpha = \frac{3}{2}$, $\quad a(x,t) = -x^2 - t$, $\quad b(x,t) = x - t$,
$c(x) = x + 2t$, $\quad g(x,t) = (4\sqrt{t} + (t+1)^2(x^2 + 2t)e^x$, $\Phi(x) = e^x$;
$\psi(x) = 2e^x$, $\quad \mu(t) = (t+1)^2$, $\quad E(t) = (t+1)^2$.

The analytical solution of this problem is given by $v(x,t) = (t+1)^2 e^x$.

Tables 5–7 show the values of the absolute error.

In this example from Tables 5–7 and Figures 7–9 we see again for space step $h = 0.1$ the absolute error tends to zero when the time step h_t $(10^{-2}, 10^{-3}, 10^{-4})$ takes a value close to zero, with convergence order $O(h + h_t)$. In Figure 9 we take into account $x \in [0.8965, 0.9000]$ to see the variation of error because it is very close to zero when $x \in [0.1, 0.8]$.

Table 5. $h = 0.1,\ h_t = 10^{-2}$.

x_i	**A.E**
0.1	6.72×10^{-4}
0.2	2.47×10^{-3}
0.3	2.44×10^{-3}
0.4	2.41×10^{-3}
0.5	2.39×10^{-3}
0.6	2.38×10^{-3}
0.7	1.84×10^{-3}
0.8	1.40×10^{-2}
0.9	2.06×10^{-1}

Table 6. $h = 0.1,\ h_t = 10^{-3}$.

x_i	**A.E**
0.1	1.22×10^{-5}
0.2	4.28×10^{-5}
0.3	4.24×10^{-5}
0.4	4.20×10^{-5}
0.5	4.15×10^{-5}
0.6	4.10×10^{-5}
0.7	4.03×10^{-5}
0.8	4.83×10^{-5}
0.9	5.48×10^{-3}

Table 7. $h = 0.1, h_t = 10^{-4}$.

x_i	**A.E**
0.1	3.75×10^{-7}
0.2	1.26×10^{-6}
0.3	1.24×10^{-6}
0.4	1.25×10^{-6}
0.5	1.23×10^{-6}
0.6	1.22×10^{-6}
0.7	1.20×10^{-6}
0.8	1.19×10^{-6}
0.9	1.72×10^{-4}

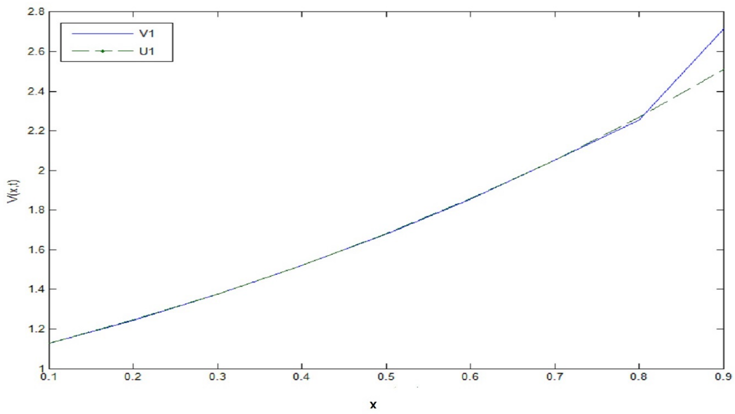

Figure 7. $h = 0.1, h_t = 10^{-2}$.

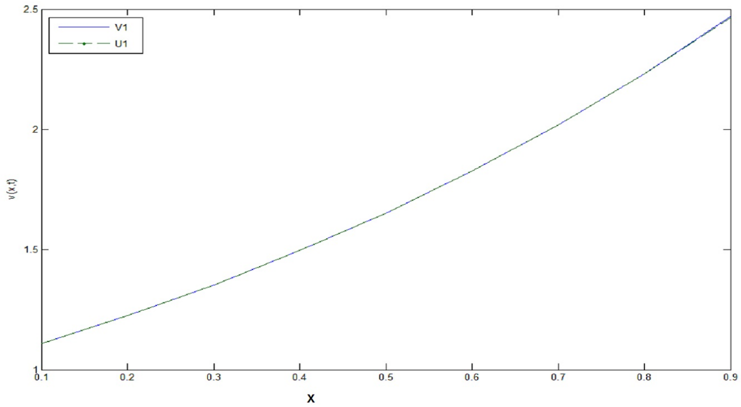

Figure 8. $h = 0.1, h_t = 10^{-3}$.

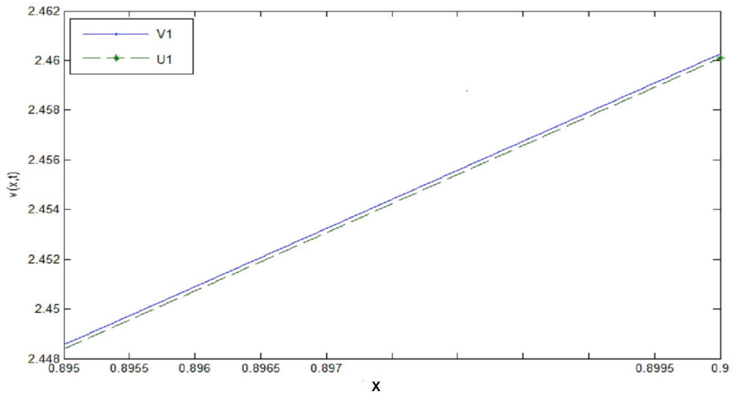

Figure 9. $h = 0.1, h_t = 10^{-4}, x \in [0.8965, 0.9000]$.

Remark 2. *For the space step $h = 0.01$ we find Figures A3–A5 in Appendix A.*

Table 8 shows the error norm $\left\|E^k\right\|_\infty$ for different values of α defined by

$$\left\|E^k\right\|_\infty = \max_{1 \leq i \leq N-1} |e_i|, \text{ where } E^k = V^{\,k} - U^{\,k} = \left(e_1^k, \dots, e_{N-1}^k\right)^T$$

Table 8. $h = 0.1$.

	Value of h_t	10^{-3}	10^{-5}	10^{-7}
$\|E^1\|_\infty$ for	$\alpha = 1.2$	9.5736×10^{-4}	1.3196×10^{-6}	5.2768×10^{-9}
	$\alpha = 1.4$	1.1294×10^{-4}	1.2671×10^{-7}	2.0154×10^{-10}
	$\alpha = 1.6$	2.3162×10^{-5}	1.2692×10^{-8}	7.9794×10^{-12}
	$\alpha = 1.8$	1.53×10^{-4}	1.4449×10^{-9}	3.4062×10^{-13}
	$\alpha = 1.9$	4.7963×10^{-6}	6.2306×10^{-10}	8.6153×10^{-14}

We see in Table 8, for the space step $h = 0.1$, and for the different values of α, the error keeps the same behavior; that is, the error norm tends towards zero when the time step h_t takes values close to zero, with an order of convergence $O(h + h_t)$. For $\alpha = 1.2$ (value close to 1) the error is greater compared to the case $\alpha = 1.9$ (value close to 2) due of the fractional operator being approximated by the formula called L_2.

5. Conclusions

In this paper, the existence and uniqueness of the solution of a Caputo time fractional problem with nonhomogeneous boundary integral conditions are established. We use the "energy inequality" method which is an efficient functional analysis method.

For the numerical study, these kinds of boundary conditions have been applied for the first time to such problems, where we combined the finite difference method with the numerical integration to obtain the approximate solution. The results obtained by this technique are very encouraging from the point of view that the numerical schemes are stable and convergent.

As further research directions, we aim at studying the same problem using the compact finite difference method in order to obtain more precise results, change the sense of fractional derivatives and/or extend the study to time–space fractional derivatives.

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Appendix A

In Example 1 for $h = 0.01$

From Tables A1 and A2 and Figures A1 and A2 with space step $h = 0.01$, we see that the approximate solution u^1 tends to the exact solution v^1 when h_t $(h_t = 10^{-3}, h_t = 10^{-5})$ takes values close to zero, with convergence order $O(h + h_t)$.

Table A1. The absolute error for $h = 0.01;\ h_t = 10^{-3}$.

$i = \overline{1,9}$	$\overline{10,18}$	$\overline{19,27}$	$\overline{28,36}$	$\overline{37,45}$	$\overline{46,54}$	$\overline{55,63}$	$\overline{64,72}$	$\overline{73,81}$	$\overline{82,89}$	$\overline{90,99}$
5×10^{-6}	6×10^{-6}	6×10^{-6}	7×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
5×10^{-6}	6×10^{-6}	7×10^{-6}	7×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
5×10^{-6}	6×10^{-6}	7×10^{-6}	7×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
6×10^{-6}	6×10^{-6}	7×10^{-6}	7×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
6×10^{-6}	6×10^{-6}	7×10^{-6}	7×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
6×10^{-6}	6×10^{-6}	7×10^{-6}	7×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
6×10^{-6}	6×10^{-6}	7×10^{-6}	8×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
6×10^{-6}	6×10^{-6}	7×10^{-6}	8×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
6×10^{-6}	6×10^{-6}	7×10^{-6}	8×10^{-6}	8×10^{-6}	9×10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}

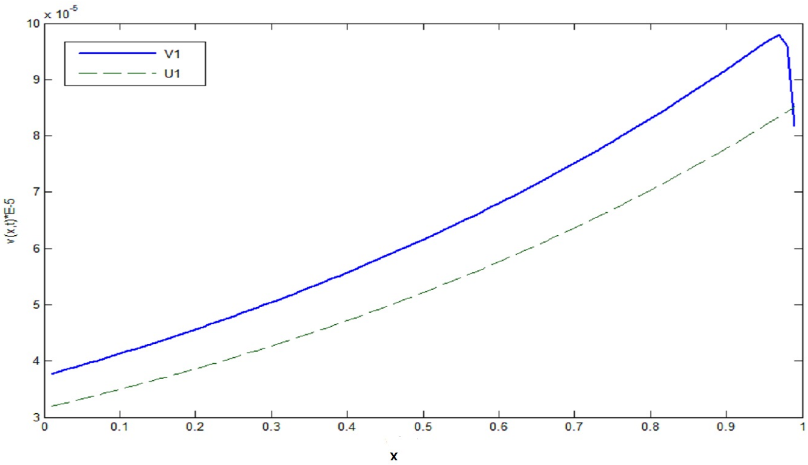

Figure A1. $h_t = 10^{-3}$.

Table A2. The absolute error for $h = 0.01, h_t = 10^{-5}$.

$i = \overline{1,9}$	$\overline{10,18}$	$\overline{19,27}$	$\overline{28,36}$	$\overline{37,45}$	$\overline{46,54}$	$\overline{55,63}$	$\overline{64,72}$	$\overline{73,81}$	$\overline{82,89}$	$\overline{90,99}$
5×10^{-9}	6×10^{-9}	6×10^{-9}	7×10^{-9}	8×10^{-9}	8×10^{-9}	9×10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
5×10^{-9}	6×10^{-9}	6×10^{-9}	7×10^{-9}	8×10^{-9}	9×10^{-9}	9×10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
5×10^{-9}	6×10^{-9}	6×10^{-9}	7×10^{-9}	8×10^{-9}	9×10^{-9}	9×10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
5×10^{-9}	6×10^{-9}	7×10^{-9}	7×10^{-9}	8×10^{-9}	9×10^{-9}	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
5×10^{-9}	6×10^{-9}	7×10^{-9}	7×10^{-9}	8×10^{-9}	9×10^{-9}	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
5×10^{-9}	6×10^{-9}	7×10^{-9}	7×10^{-9}	8×10^{-9}	9×10^{-9}	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
6×10^{-9}	6×10^{-9}	7×10^{-9}	7×10^{-9}	8×10^{-9}	9×10^{-9}	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
6×10^{-9}	6×10^{-9}	7×10^{-9}	7×10^{-9}	8×10^{-9}	9×10^{-9}	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
6×10^{-9}	6×10^{-9}	7×10^{-9}	8×10^{-9}	8×10^{-9}	9×10^{-9}	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}

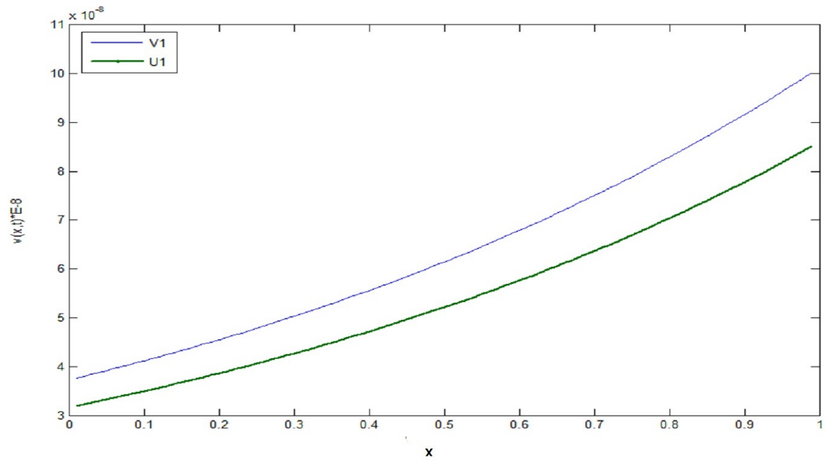

Figure A2. $h_t = 10^{-5}$.

In Example 2, for $h = 0.01$, $\alpha = 1.5$

The Figures A3–A5 show where the space step is fixed at $h = 0.01$ and the time step h_t decreases towards zero ($h_t = 0.001$, $h_t = 0.0001$, $h_t = 0.00001$), the approximate solution u^1 tends to the exact solution v^1, in the case where $h_t = 0.00001$ we see that the two curves of u^1 and v^1 are almost identical.

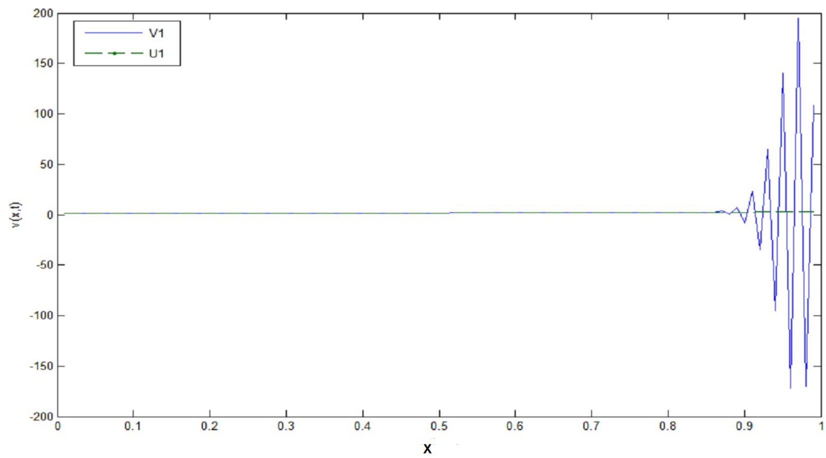

Figure A3. $h_t = 0.001$.

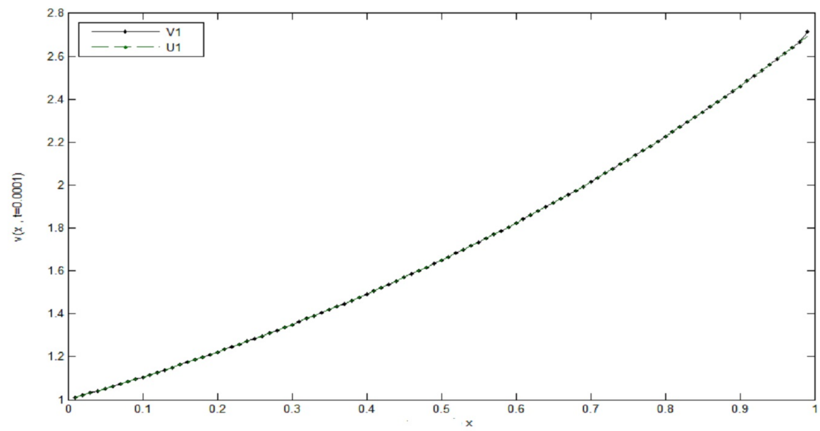

Figure A4. $h_t = 0.0001$.

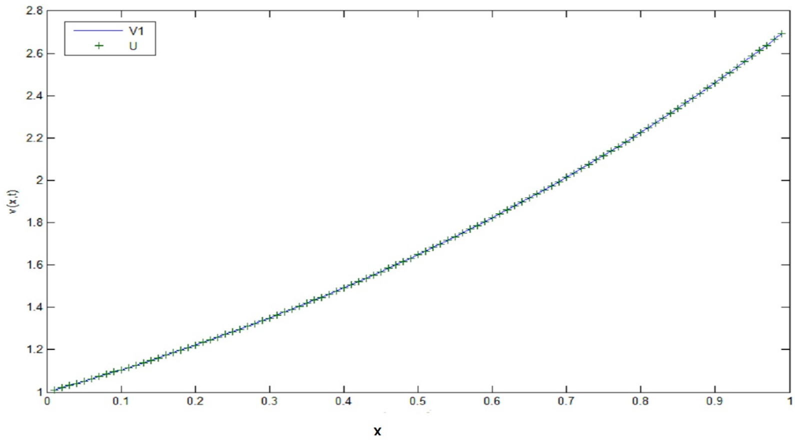

Figure A5. $h_t = 0.00001$.

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