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On the Admissibility of the Fixed Points Set of a Mapping with Respect to Another Mapping

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Abstract: Let (M, δ) be a metric space, $f : M \to M$, and $g : M \to [0, +\infty)$. In this paper, we obtain sufficient conditions under which the set of fixed points of f is g-admissible, i.e., $\mathrm{Fix}(f) \neq \emptyset$ and $\mathrm{Fix}(f) \subset g^{-1}(\{0\})$. Some special cases of our main results are discussed and some examples are given.

Keywords: fixed points set; g-admissible; μ-continuous; μ-regular; μ-Cauchy; μ-complete; picard sequence

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1. Introduction

Let (M, δ) be a metric space, $f : M \to M$, and $g : M \to [0, +\infty)$. In this paper, we are interested in obtaining sufficient conditions on (M, δ), f, and g, ensuring that

$$\begin{cases} \mathrm{Fix}(f) \neq \emptyset, \\ \mathrm{Fix}(f) \subset g^{-1}(\{0\}), \end{cases} \tag{1}$$

where $\mathrm{Fix}(f)$ denotes the set of fixed points of f. In this case, we say that $\mathrm{Fix}(f)$ is g-admissible.

The main motivation for studying problems of type (1) comes from the theory of partial metric spaces (see e.g., [1,2]) and essentially from Ioan A. Rus's paper [3]. Namely, let (M, ρ) be a complete partial metric space and $f : M \to M$ be a given mapping. Suppose that there exists $0 < \sigma < 1$ such that

$$\rho(f(y), f(z)) \leq \sigma \rho(y, z), \tag{2}$$

for all $y, z \in M$. Then (see [2]) f has a unique fixed point $z^* \in M$. Moreover, $\rho(z^*, z^*) = 0$. Consider now the mapping $\delta_\rho : M \times M \to [0, +\infty)$ defined by

$$\delta_\rho(y, z) = 2\rho(y, z) - \rho(y, y) - \rho(z, z), \quad y, z \in M.$$

Then (see [4]) (M, δ_ρ) is a complete metric space. Moreover, (2) is equivalent to

$$\delta_\rho(f(y), f(z)) + \rho(f(y), f(y)) + \rho(f(z), f(z)) \leq \sigma\big(\delta_\rho(y, z) + \rho(y, y) + \rho(z, z)\big),$$

for all $y, z \in M$. In [3], Ioan A. Rus proposed to study the class of mappings f satisfying the more general condition

$$\delta(f(y), f(z)) + g(f(y)) + g(f(z)) \leq \sigma(\delta(y, z) + g(y) + g(z)), \tag{3}$$

for all $y, z \in M$, where δ is a metric on M and $g : M \to [0, +\infty)$ is a given mapping. In [5], it was shown that, if (M, δ) is complete, g is lower semi-continuous, and f and g satisfy (3), then f admits a unique fixed point $z^* \in M$. Moreover, $g(z^*) = 0$. Some extensions and generalizations of the obtained result in [5] can be found in [6–12] (see also the references therein). Some recent results related to fixed point theory in partial metric spaces can be found in [13–16] (see also the references therein).

This work is concerned with the study of new classes of mappings $f : M \to M$ for which we derive g-admissibility results. The next section of this paper is devoted to some definitions and preliminaries. In Section 3, we present and prove our main results. In Section 4, we investigate some special cases and provide some examples.

2. Preliminaries

Let (M, δ) be a metric space and $\mu : M \times M \to \mathbb{R}$. Let H be the set of functions $h : [0, +\infty) \to [0, +\infty)$ satisfying:

(H1) h is nondecreasing,

(H2) $\sum_{i=0}^{+\infty} h^i(s) < +\infty$ for all $s > 0$, where h^i is the ith iterate of h.

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. We recall below some notions introduced in [17] (see also [18]).

Definition 1. *A sequence $\{z_n\} \subset M$ is μ-regular, if $\mu(z_n, z_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.*

Definition 2. *A sequence $\{z_n\} \subset M$ is μ-Cauchy, if $\{z_n\}$ is μ-regular and $\{z_n\}$ is a Cauchy sequence.*

Definition 3. *The metric space (M, δ) is μ-complete, if every μ-Cauchy sequence in M is convergent to an element of M.*

Definition 4. *A mapping $f : M \to M$ is μ-continuous, if for every μ-regular sequence $\{z_n\} \subset M$,*

$$\lim_{n\to+\infty} \delta(z_n, z) = 0,\ z \in M \implies \lim_{n\to+\infty} \delta(f(z_n), f(z)) = 0.$$

Definition 5 (see [8])**.** *Let $f : M \to M$ and $g : M \to [0, +\infty)$. The set* $\mathrm{Fix}(f)$ *is said to be g-admissible, if it satisfies* (1).

Some examples on the above notions are given below.

Example 1. *Let $M = \mathbb{R}$ and $\mu : M \times M \to \mathbb{R}$ be the mapping defined by*

$$\mu(t, s) = \sin^2\left(\frac{\pi t}{2}\right) + \sin^2\left(\frac{\pi s}{2}\right), \quad (t, s) \in M \times M.$$

Let $\{z_n\} \subset M$ be the sequence defined by

$$z_n = n, \quad n \in \mathbb{N}.$$

Then, for all $n \in \mathbb{N}$,

$$\begin{aligned} \mu(z_n, z_{n+1}) &= \sin^2\left(\frac{n\pi}{2}\right) + \sin^2\left(\frac{(n+1)\pi}{2}\right) \\ &= \sin^2\left(\frac{n\pi}{2}\right) + \cos^2\left(\frac{n\pi}{2}\right) \\ &= 1, \end{aligned}$$

which shows that $\{z_n\}$ is μ-regular.

Example 2. *Let $M = C([0, 1])$ and δ be the metric on M defined by*

$$\delta(f, g) = \max_{0\leq t\leq 1} |f(t) - g(t)|, \quad f, g \in M.$$

Let $\mu : M \times M \to \mathbb{R}$ be the mapping defined by

$$\mu(f,g) = \int_0^1 f(s)g(s)\,ds, \quad f,g \in M.$$

Consider the sequence $\{z_n\} \subset M$ defined by

$$z_n(t) = \exp\left(\frac{t}{n+1}\right), \quad 0 \le t \le 1.$$

Since $\{z_n\}$ is a convergent sequence (it converges uniformly to $z(t) = 1$), then $\{z_n\}$ is a Cauchy sequence. Moreover, for all $n \in \mathbb{N}$,

$$\mu(z_n, z_{n+1}) = \int_0^1 \exp\left(\frac{t}{n+1}\right)\exp\left(\frac{t}{n+2}\right)dt \ge 1.$$

Then $\{z_n\}$ is μ-regular. Consequently, $\{z_n\}$ is a μ-Cauchy sequence.

Example 3. *Let $M = (0,+\infty)$ and $\delta = |\cdot|$ (i.e., $\delta(y,z) = |y-z|$ for all $y,z \in M$). Obviously (M,δ) is not complete. Let $\mu : M \times M \to \mathbb{R}$ be the mapping defined by*

$$\mu(t,s) = \begin{cases} 1 & \text{if} \quad t,s \ge 1, \\ 0 & \text{if} \quad 0 < t < 1 \text{ or } 0 < s < 1. \end{cases}$$

Consider a μ-Cauchy sequence $\{z_n\} \subset M$. Due to the μ-regularity of $\{z_n\}$, by the definition of μ, we deduce that $\{z_n\} \subset \mathcal{M} = [1,\infty)$. Since $(\mathcal{M},\delta)$ is complete and $\{z_n\}$ is a Cauchy sequence, then $\{z_n\}$ converges to some $z \in \mathcal{M} \subset M$. This shows that (M,δ) is μ-complete. Consider now the function $f : M \to M$ defined by

$$f(t) = \begin{cases} -\ln t & \text{if} \quad 0 < t < 1, \\ t & \text{if} \quad t \ge 1. \end{cases}$$

Clearly f is not continuous at 1. However, for any μ-regular sequence $\{z_n\} \subset M$ (recall that in this case $z_n \ge 1$ for all n), if $z_n \to z \in M$, as $n \to +\infty$, then $z \ge 1$ and $f(z_n) = z_n \to f(z) = z$ as $n \to +\infty$. This shows that f is μ-continuous.

The proof of the following lemma can be found in [19].

Lemma 1. *Let $h \in H$. Then*

(i) *$h(s) < s$ for all $s > 0$,*
(ii) *$h(0) = 0$,*
(iii) *h is continuous at 0.*

3. Main Results

The first main result of this paper is the following.

Theorem 1. *Let (M,δ) be a metric space, $\mu : M \times M \to \mathbb{R}$, $f : M \to M$, and $g : M \to [0,+\infty)$. Assume that*

(i) *(M,δ) is μ-complete,*
(ii) *f is μ-continuous,*
(iii) *$\mu(z, f(z)) \ge 1$ for all $z \in M$.*

Suppose also that there exists $h \in H$ such that

$$\mu(y,z)\big(\delta(f(y),f(z)) + g(f(y)) + g(f(z))\big) \le h\big(\delta(y,z) + g(y) + g(z)\big), \tag{4}$$

for all $y,z \in M$. Then

(I) $\mathrm{Fix}(f)$ *is* g*-admissible,*
(II) *For all* $z_0 \in M$*, the Picard sequence* $\{f^n(z_0)\}$ *converges to a fixed point of* f*.*

Proof. We first show that

$$\mathrm{Fix}(f) \subset g^{-1}(\{0\}). \tag{5}$$

If $\mathrm{Fix}(f) = \emptyset$, then (5) is obvious. So, suppose that $\mathrm{Fix}(f) \neq \emptyset$ and let $z \in \mathrm{Fix}(f)$. Taking $y = z$ in (4), we obtain

$$\mu(z,z)\big(\delta(f(z),f(z)) + g(f(z)) + g(f(z))\big) \leq h\big(\delta(z,z) + g(z) + g(z)\big),$$

that is,

$$2g(z)\mu(z,f(z)) \leq h\big(2g(z)\big).$$

Due to (iii), we deduce that

$$2g(z) \leq h\big(2g(z)\big). \tag{6}$$

If $g(z) > 0$, then by the statement (i) of Lemma 1, we have

$$h\big(2g(z)\big) < 2g(z). \tag{7}$$

Hence, (6) and (7) lead to a contradiction. Consequently, $g(z) = 0$ and (5) follows.

The second step consists in showing that $\mathrm{Fix}(f)$ is a nonempty set. Fix $z_0 \in M$ and consider the Picard sequence $\{z_n\} \subset M$ defined by

$$z_{n+1} = f(z_n), \quad n \in \mathbb{N}. \tag{8}$$

Then, for $n \geq 1$, taking $(y,z) = (z_{n-1}, z_n)$ in (4), we obtain

$$\mu(z_{n-1},z_n)\big(\delta(f(z_{n-1}),f(z_n)) + g(f(z_{n-1})) + g(f(z_n))\big) \leq h\big(\delta(z_{n-1},z_n) + g(z_{n-1}) + g(z_n)\big),$$

that is,

$$\mu(z_{n-1},z_n)\big(\delta(z_n,z_{n+1}) + g(z_n) + g(z_{n+1})\big) \leq h\big(\delta(z_{n-1},z_n) + g(z_{n-1}) + g(z_n)\big).$$

Since by (iii) $\{z_n\}$ is μ-regular, the above inequality leads to

$$\tau_n \leq h(\tau_{n-1}), \quad n \geq 1,$$

where

$$\tau_n = \delta(z_n, z_{n+1}) + g(z_n) + g(z_{n+1}). \tag{9}$$

By (H1), the above inequality leads to

$$\tau_1 \leq h(\tau_0), \;\; \tau_2 \leq h(\tau_1) \leq h^2(\tau_0), \;\; \tau_3 \leq h(\tau_2) \leq h^3(\tau_0), \cdots,$$

that is,

$$\tau_n \leq h^n(\tau_0), \quad n \geq 1. \tag{10}$$

Since $\delta(z_n, z_{n+1}) \leq \tau_n$, we deduce that

$$\delta(z_n, z_{n+1}) \leq h^n(\tau_0), \quad n \geq 1. \tag{11}$$

Using (11) and the triangle inequality, for $n < m$, it holds that

$$\begin{aligned}\delta(z_n, z_m) &\leq \sum_{r=n}^{m-1} \delta(z_r, z_{r+1}) \\ &\leq \sum_{r=n}^{m-1} h^r(\tau_0) \\ &\leq \sum_{r=n}^{+\infty} h^r(\tau_0).\end{aligned}$$

Hence, due to (H2) and the statement (ii) of Lemma 1, we deduce that

$$\lim_{n,m\to+\infty} \delta(z_n, z_m) = 0,$$

which proves that $\{z_n\}$ is a Cauchy sequence, and then $\{z_n\}$ is μ-Cauchy. Therefore, by (i), we deduce that $\{z_n\}$ converges to some $z^* \in M$. Since $\{z_n\}$ is μ-regular and f is μ-continuous (by (ii)), it holds that $\{f(z_n)\} = \{z_{n+1}\}$ converges to $f(z^*)$. Then the uniqueness of the limit leads to $z^* = f(z^*)$, that is, $z^* \in \mathrm{Fix}(f)$. Consequently

$$\mathrm{Fix}(f) \neq \emptyset. \tag{12}$$

Hence, (5) and (12) show that $\mathrm{Fix}(f)$ is g-admissible, which proves (I). Finally, (II) follows from the the arbitrary choice of z_0 and the convergence of $\{z_n\}$ to an element of $\mathrm{Fix}(f)$. □

Our next result is the following.

Theorem 2. *Let (M, δ) be a metric space, $\mu : M \times M \to \mathbb{R}$, $f : M \to M$, and $g : M \to [0, +\infty)$. Assume that conditions* (i) *and* (iii) *of Theorem 1, and* (4) *hold. Moreover, suppose that*

(a) *g is lower semi-continuous,*
(b) *For any μ-regular sequence $\{u_n\} \subset M$,*

$$\lim_{n\to+\infty} \delta(u_n, u) = 0,\ u \in M \implies \exists\, k \in \mathbb{N} :\ \mu(u_n, u) \geq 1,\ n \geq k.$$

Then

(I) $\mathrm{Fix}(f)$ *is g-admissible,*
(II) *For all $z_0 \in M$, the Picard sequence $\{f^n(z_0)\}$ converges to a fixed point of f.*

Proof. From the proof of Theorem 1, (5) holds, and the Picard sequence $\{z_n\}$ defined by (8) is μ-regular and converges to some $z^* \in M$. By (b), we deduce that there exists $k \in \mathbb{N}$ such that

$$\mu(z_n, z^*) \geq 1, \quad n \geq k. \tag{13}$$

On the other hand, by (H2), the statement (ii) of Lemma 1, and (10), we deduce that

$$\lim_{n\to+\infty} \tau_n = 0.$$

Hence, from (9) (since $\{z_n\}$ is convergent), it follows that

$$\lim_{n\to+\infty} g(z_n) = 0, \tag{14}$$

which implies by (a) that

$$g(z^*) \leq \liminf_{n\to+\infty} g(z_n) = 0,$$

that is,

$$g(z^*) = 0. \tag{15}$$

Next, taking $(y,z) = (z_n, z^*)$ in (4), using (13) and (15), it holds that

$$\delta(z_{n+1}, f(z^*)) + g(z_{n+1}) + g(f(z^*)) \le h\big(\delta(z_n, z^*) + g(z_n)\big), \quad n \ge k,$$

which yields

$$\delta(z_{n+1}, f(z^*)) \le h\big(\delta(z_n, z^*) + g(z_n)\big), \quad n \ge k.$$

Passing to the limit as $n \to +\infty$ in the above inequality, using the statements (ii) and (iii) of Lemma 1, and (14), we obtain $\delta(z^*, f(z^*)) = 0$, that is, $z^* \in \mathrm{Fix}(f)$. Hence, by (5), $\mathrm{Fix}(f)$ is g-admissible. □

4. Special Cases

Some special cases of our main results are deduced in this section.

Corollary 1. *Let (M,δ) be a complete metric space, $f : M \to M$, and $g : M \to [0,+\infty)$. Suppose that there exist $h \in H$ and $F : [0,+\infty) \to \mathbb{R}$ such that*

$$\delta(f(y), f(z)) + g(f(y)) + g(f(z)) \le h(\delta(y,z) + g(y) + g(z)) + F(\delta(z, f(y))), \quad y,z \in M. \tag{16}$$

Suppose also that

(i) *g is lower semi-continuous,*
(ii) *F is continuous,*
(iii) *$F(0) = 0$.*

Then

(I) *$\mathrm{Fix}(f)$ is g-admissible,*
(II) *For all $z_0 \in M$, the Picard sequence $\{f^n(z_0)\}$ converges to a fixed point of f.*

Proof. The main idea of the proof consists in showing that f and g satisfy (4) for a judicious choice of the mapping μ. Namely, let us introduce the mapping $\mu : M \times M \to \mathbb{R}$ defined by

$$\mu(y,z) = \begin{cases} 1 & \text{if} \quad z = f(y), \\ \\ 0 & \text{if} \quad z \ne f(y). \end{cases} \tag{17}$$

Obviously, one has

$$\mu(y, f(y)) = 1, \quad y \in M.$$

Let us fix $(y,z) \in M$. If $z \ne f(y)$, then by the definition of μ, (4) follows immediately. If $z = f(y)$, it follows from (16) that

$$\delta(f(y), f(z)) + g(f(y)) + g(f(z)) \le h(\delta(y,z) + g(y) + g(z)) + F(0).$$

Since $F(0) = 0$ and $\mu(y,z) = 1$, it holds that

$$\mu(y,z)(\delta(f(y), f(z)) + g(f(y)) + g(f(z))) \le h(\delta(y,z) + g(y) + g(z)).$$

Then, in both cases, (4) is satisfied.

Next, let us show that f is μ-continuous. Let $\{z_n\}$ be a μ-regular sequence. By the definition of μ, it holds that $z_{n+1} = f(z_n)$ for all $n \in \mathbb{N}$. Let $u \in M$ be such that $\lim_{n\to+\infty} \delta(z_n, u) = 0$. Taking $(y,z) = (z_{n-1}, z_n)$ in (16), we obtain (since $F(0) = 0$)

$$\delta(z_n, z_{n+1}) + g(z_n) + g(z_{n+1}) \le h(\delta(z_{n-1}, z_n) + g(z_{n-1}) + g(z_n)), \quad n \ge 1.$$

Since $h \in H$, the above inequality leads to

$$\lim_{n \to +\infty} g(z_n) = 0.$$

Due to the lower semi-continuity of g, we deduce that

$$g(u) = 0.$$

Hence, taking $(y,z) = (z_n, u)$ in (16), it holds that

$$\delta(f(z_n), f(u)) \leq \delta(f(z_n), f(u)) + g(f(z_n)) + g(f(u)) \leq h(\delta(z_n, u) + g(z_n)) + F(\delta(u, z_{n+1})),$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \to +\infty$, using the statements (ii) and (iii) of Lemma 1, the continuity of F, and the fact that $F(0) = 0$, we obtain

$$\lim_{n \to +\infty} \delta(f(z_n), f(u)) = 0.$$

This shows that f is μ-continuous.

Notice that since (M, δ) is complete, then (M, δ) is μ-complete. Finally, the desired results follow from Theorem 1. $\square$

Remark 1. *The class of mappings $f : M \to M$ satisfying* (16) *with $g \equiv 0$ and $F(s) = Ls$, $L \geq 0$, has been introduced in* [20] *under the name of (h, L)-weak contractions. Namely, it was shown that, if f belongs to the class of (h, L)-weak contractions, then $\mathrm{Fix}(f) \neq \emptyset$ and for all $z_0 \in M$, the Picard sequence $\{f^n(z_0)\}$ converges to a fixed point of f. Clearly, taking $g \equiv 0$ and $F(s) = Ls$ in Corollary 1, we refind these results. For additional references related to (h, L)-weak contractions and their generalizations, see e.g.,* [17,19,21–25] *and the references therein.*

An example supporting Corollary 1 is given below.

Example 4. *Let $M = [0, 1]$ and δ be the standard metric on M given by*

$$\delta(y, z) = |y - z|, \quad y, z \in M. \tag{18}$$

Consider the mappings $f : M \to M$ and $g : M \to [0, +\infty)$ defined respectively by

$$f(y) = \begin{cases} \dfrac{1}{4} & \text{if} \quad y \in [0, 1), \\ 0 & \text{if} \quad y = 1, \end{cases} \tag{19}$$

and

$$g(y) = \left| 4y^2 - 3y + \frac{1}{2} \right|, \quad y \in M.$$

We can check easily that $\mathrm{Fix}(f) = \left\{ \frac{1}{4} \right\}$ *and* $g^{-1}(\{0\}) = \left\{ \frac{1}{2}, \frac{1}{4} \right\}$*. We claim that*

$$\delta(f(y), f(z)) + g(f(y)) + g(f(z)) \leq \frac{1}{2}(\delta(y, z) + g(y) + g(z)) + \delta(z, f(y)), \quad y, z \in M. \tag{20}$$

- *Case 1: $0 \leq y, z < 1$. In this case, we have*

$$\delta(f(y), f(z)) + g(f(y)) + g(f(z)) = \left| \frac{1}{4} - \frac{1}{4} \right| + g\left(\frac{1}{4}\right) + g\left(\frac{1}{4}\right) = 0.$$

Then (20) *holds.*

• *Case 2:* $0 \le y < 1$ *and* $z = 1$*. In this case,*

$$\begin{aligned}\delta(f(y), f(z)) + g(f(y)) + g(f(z)) &= \frac{1}{4} + g\left(\frac{1}{4}\right) + g(0) \\ &= \frac{3}{4} \\ &= \delta(z, f(y)) \\ &\le \frac{1}{2}(\delta(y,z) + g(y) + g(z)) + \delta(z, f(y)).\end{aligned}$$

• *Case 3:* $y = 1$ *and* $0 \le z < 1$*. In this case,*

$$\begin{aligned}\delta(f(y), f(z)) + g(f(y)) + g(f(z)) &= \frac{3}{4} \\ &= \frac{g(y)}{2} \\ &\le \frac{1}{2}(\delta(y,z) + g(y) + g(z)) + \delta(z, f(y)).\end{aligned}$$

• *Case 4:* $y = z = 1$*. In this case,*

$$\begin{aligned}\delta(f(y), f(z)) + g(f(y)) + g(f(z)) &= 2g(0) \\ &= 1 \\ &= \delta(z, f(y)) \\ &\le \frac{1}{2}(\delta(y,z) + g(y) + g(z)) + \delta(z, f(y)).\end{aligned}$$

Hence, (20) *is satisfied in all cases. Notice that* (20) *is a special case of* (16) *with*

$$h(s) = \frac{s}{2}, \quad s \ge 0$$

and

$$F(s) = s, \quad s \ge 0.$$

Then, by Corollary 1, $\mathrm{Fix}(f)$ *is* g*-admissible, and for all* $z_0 \in M$*, the Picard sequence* $\{f^n(z_0)\}$ *converges to a fixed point of* f*. Notice that*

$$\mathrm{Fix}(f) = \left\{\frac{1}{4}\right\} \subset \left\{\frac{1}{2}, \frac{1}{4}\right\} = g^{-1}(\{0\}),$$

which confirms the results given by Corollary 1.

Corollary 2. *Let* (M, δ) *be a complete metric space,* $f : M \to M$*, and* $g : M \to [0, +\infty)$*. Suppose that there exist* $\sigma > 0$ *and* $F : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *such that*

$$\begin{aligned}&\delta(f(y), f(z)) + g(f(y)) + g(f(z)) \\ &\le F\Big(\delta(y, f(y)) + g(y) + g(f(y)), \delta(y,z) + g(y) + g(z)\Big)(\delta(z, f(z)) + g(z) + g(f(z))) \\ &+ \sigma(\delta(y,z) + g(y) + g(z)),\end{aligned} \tag{21}$$

for all $y, z \in M$*. Suppose also that*

(i) g *is lower semi-continuous,*
(ii) F *is continuous,*
(iii) *There exists* $\lambda_F > 0$ *such that* $F(s,s) = \lambda_F$ *for all* $s \ge 0$*,*
(iv) $\sigma + \lambda_F < 1$.

Then

(I) $\mathrm{Fix}(f)$ *is* g*-admissible,*
(II) *For all* $z_0 \in M$*, the Picard sequence* $\{f^n(z_0)\}$ *converges to a fixed point of* f*.*

Proof. First, we shall prove that f and g satisfy (4) for some $h \in H$, where μ is defined by (17). Let us fix $(y,z) \in M$. If $z \neq f(y)$, then by the definition of μ, (4) follows immediately. If $z = f(y)$, it follows from (21) that

$$\begin{aligned}&\delta(f(y),f(z)) + g(f(y)) + g(f(z)) \\ &\leq F\Big(\delta(y,z) + g(y) + g(z), \delta(y,z) + g(y) + g(z)\Big)(\delta(f(y),f(z)) + g(f(y)) + g(f(z))) \\ &+ \sigma(\delta(y,z) + g(y) + g(z)).\end{aligned}$$

Using (iii), we deduce that

$$\begin{aligned}\delta(f(y),f(z)) + g(f(y)) + g(f(z)) \quad &\leq \quad \lambda_F(\delta(f(y),f(z)) + g(f(y)) + g(f(z))) \\ &\quad\ +\sigma(\delta(y,z) + g(y) + g(z)),\end{aligned}$$

which yields

$$(1-\lambda_F)(\delta(f(y),f(z)) + g(f(y)) + g(f(z))) \leq \sigma(\delta(y,z) + g(y) + g(z)).$$

Since $1-\lambda_F > 0$ and $\mu(y,z) = 1$, it holds that

$$\mu(y,z)(\delta(f(y),f(z)) + g(f(y)) + g(f(z))) \leq h(\delta(y,z) + g(y) + g(z)),$$

where

$$h(s) = \frac{\sigma s}{1-\lambda_F}, \quad s \geq 0.$$

Notice that form (iv), it follows that $h \in H$. Hence, f and g satisfy (4), where h is defined above and μ is given by (17). Moreover, proceeding as in the proof of Corollary 1, one can show that f is μ-continuous. Hence, applying Theorem 1, (I) and (II) follow. □

Corollary 3. *Under the assumptions of Corollary 2, the mapping* f *admits one and only one fixed point.*

Proof. Suppose that $u,v \in \mathrm{Fix}(f)$. Since by Corollary 2 we know that $\mathrm{Fix}(f)$ is g-admissible, then $g(u) = g(v) = 0$. Taking $(y,z) = (u,v)$ in (21), it holds that

$$\begin{aligned}&\delta(u,v) + g(u) + g(v) \\ &\leq F\Big(\delta(u,u) + g(u) + g(u), \delta(u,v) + g(u) + g(v)\Big)(\delta(v,v) + g(v) + g(v)) \\ &+ \sigma(\delta(u,v) + g(u) + g(v)),\end{aligned}$$

that is,

$$\delta(u,v) \leq \sigma\delta(u,v).$$

Since $\sigma < 1$, we deduce that $u = v$. □

Corollary 4. *Let (M, δ) be a complete metric space, $f : M \to M$, and $g : M \to [0, +\infty)$ be a lowe semi-continuous function. Suppose that*

$$\begin{aligned} &\delta(f(y), f(z)) + g(f(y)) + g(f(z)) \\ &\leq \lambda \left[\frac{\tau + \delta(y, f(y)) + g(y) + g(f(y))}{\tau + \delta(y, z) + g(y) + g(z)} \right] (\delta(z, f(z)) + g(z) + g(f(z))) \\ &+ \sigma(\delta(y, z) + g(y) + g(z)), \end{aligned} \tag{22}$$

for all $y, z \in M$, where $\lambda, \sigma, \tau > 0$ and $\lambda + \sigma < 1$. Then

(I) $\mathrm{Fix}(f)$ *is g-admissible,*
(II) *For all $z_0 \in M$, the Picard sequence $\{f^n(z_0)\}$ converges to a fixed point of f.*
(III) $\mathrm{card}(\mathrm{Fix}(f)) = 1$.

Proof. We have just to observe that (22) is a special case of (21) with

$$F(t, s) = \frac{\tau + t}{\tau + s}, \quad t, s \geq 0.$$

Next, the results follow from Corollaries 2 and 3. □

Remark 2. *The class of mappings $f : M \to M$ satisfying (22) with $g \equiv 0$ and $\tau = 1$ has been introduced in [26] under the name of rational-type contractions. Namely, it was shown that, if f belongs to this class, then $\mathrm{card}(\mathrm{Fix}(f)) = 1$ and for all $z_0 \in M$, the Picard sequence $\{f^n(z_0)\}$ converges to the fixed point of f. Clearly, taking $g \equiv 0$ and $\tau = 1$ in Corollary 4, we refind these results. For additional references related to rational-type contractions, see, e.g., [27–30] and the references therein.*

An example supporting Corollary 4 is given below.

Example 5. *Let $M = [0, 1]$ and δ be the metric on M given by (18). Let us consider the mappings $f : M \to M$ and $g : M \to [0, +\infty)$, where f is given by (19) and*

$$g(y) = \frac{5}{6} y \left| y - \frac{1}{4} \right|, \quad y \in M.$$

Obviously, $\mathrm{Fix}(f) = \left\{ \frac{1}{4} \right\}$ and $g^{-1}(\{0\}) = \left\{ 0, \frac{1}{4} \right\}$. We claim that f and g satisfy (22) with $\sigma = \frac{1}{2}$, $0 < \lambda < \frac{1}{2}$ and $\tau > 0$.

• *Case 1: $0 \leq y, z < 1$. In this case,*

$$\delta(f(y), f(z)) + g(f(y)) + g(f(z)) = 2g\left(\frac{1}{4}\right) = 0.$$

Then (22) *holds.*

- *Case 2:* $0 \le y < 1$ *and* $z = 1$*. In this case,*

$$\begin{aligned}
&\delta(f(y), f(z)) + g(f(y)) + g(f(z)) \\
&= \frac{1}{4} + g\left(\frac{1}{4}\right) + g(0) \\
&= \frac{1}{4} \\
&\le \frac{1}{2}\,\frac{5}{8} \\
&= \sigma g(z) \\
&\le \sigma(\delta(y, z) + g(y) + g(z)) \\
&+ \lambda\left[\frac{\tau + \delta(y, f(y)) + g(y) + g(f(y))}{\tau + \delta(y, z) + g(y) + g(z)}\right](\delta(z, f(z)) + g(z) + g(f(z))).
\end{aligned}$$

- *Case 3:* $0 \le z < 1$ *and* $y = 1$*. In this case*

$$\begin{aligned}
&\delta(f(y), f(z)) + g(f(y)) + g(f(z)) \\
&= \frac{1}{4} + g(0) + g\left(\frac{1}{4}\right) \\
&= \frac{1}{4} \\
&\le \sigma g(y) \\
&\le \sigma(\delta(y, z) + g(y) + g(z)) \\
&+ \lambda\left[\frac{\tau + \delta(y, f(y)) + g(y) + g(f(y))}{\tau + \delta(y, z) + g(y) + g(z)}\right](\delta(z, f(z)) + g(z) + g(f(z))).
\end{aligned}$$

- *Case 4:* $y = z = 1$*. In this case*

$$\delta(f(y), f(z)) + g(f(y)) + g(f(z)) = 2g(0) = 0.$$

Then (22) *holds. Hence,* (22) *is satisfied in all cases. Then, by Corollary* 4, $\mathrm{Fix}(f)$ *is* g*-admissible, and for all* $z_0 \in M$*, the Picard sequence* $\{f^n(z_0)\}$ *converges to the fixed point of* f*. Notice that*

$$\mathrm{Fix}(f) = \left\{\frac{1}{4}\right\} \subset \left\{0, \frac{1}{4}\right\} = g^{-1}(\{0\}),$$

which confirms the results given by Corollary 4.

5. Conclusions

We provided sufficient conditions under which the set of fixed points of a mapping $f : M \to M$ is included in the zero set of a mapping $g : M \to [0, +\infty)$ (see Theorems 1 and 2). Next, we discussed some special cases of our obtained results and provided some examples (see Section 4).

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