Article

Fixed Point Results via Least Upper Bound Property and Its Applications to Fuzzy Caputo Fractional Volterra–Fredholm Integro-Differential Equations

Humaira ¹ , Muhammad Sarwar ^{1,*} , Thabet Abdeljawad ^{2,*} and Nabil Mlaiki ³¹ Department of Mathematics, University of Malakand, Chakdara Dir(L) 18800, Pakistan; humaira.swatpk@gmail.com² Department of Medical Research, China Medical University, Taichung 40402, Taiwan³ Department Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia; Nmlaki@psu.edu.sa

* Correspondence: sarwar@uom.edu.pk (M.S.); tabdeljawad@psu.edu.sa (T.A.)

Abstract: In recent years, complex-valued fuzzy metric spaces (in short CVFMS) were introduced by Shukla et al. (Fixed Point Theory 32 (2018)). This setting is a valuable extension of fuzzy metric spaces with the complex grade of membership function. They also established fixed-point results under contractive condition in the aforementioned spaces and generalized some essential existence results in fixed-point theory. The purpose of this manuscript is to derive some fixed-point results for multivalued mappings enjoying the least upper bound property in CVFMS. Furthermore, we studied the existence theorem for a unique solution to the Fuzzy fractional Volterra–Fredholm integro-differential equations (FCFVFIDEs) as an application to our derived result involving the Caputo derivative.

Keywords: complex-valued fuzzy metric space; fuzzy mappings; fixed-point; cauchy sequence and contractive condition; least upper bound property

MSC: Primary 47H10; Secondary 54H25

Citation: Humaira; Sarwar, M.; Abdeljawad, T.; Mlaiki, N. Fixed Point Results via Least Upper Bound Property and Its Applications to Fuzzy Caputo Fractional Volterra–Fredholm Integro-Differential Equations. *Mathematics* **2021**, *9*, 1969. <https://doi.org/10.3390/math9161969>

Academic Editors: Michal Fečkan and Marius-F. Danca

Received: 22 June 2021

Accepted: 6 August 2021

Published: 17 August 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

It is a well-known fact that metric fixed-point theory is developed by Banach fixed-point theorem. This result is widely applied in nonlinear functional analysis. Indeed, it is the abstract setting of the successive approximation method to investigate the solution of differential equations. Additionally, the advances made in fixed-point theory are applied to differential equations and integral equations. Specifically, fixed-point theory has applications in nonlinear fractional differential equations.

Mathematical tools such as mathematical logics and mathematical arithmetic etc. are used to model many natural phenomena. However, it is not easy to obtain the deterministic models of mathematical problems using the above-mentioned tools. Such models also have some vagueness and errors. To obtain or reduce the errors and vagueness, it is essential to introduce another way of modeling and investigating solutions. In 1965, Zadeh introduced the fuzzy sets concept [1]. In recent years, fuzzy sets were applied in many applied branches of science and engineering. This concept has clear advantages over deterministic-stochastic problems. Observing these applications, the mathematical models are converted to fuzzy fields, which form a natural association with between crisp and fuzzy problems, as well as having a natural association between fuzzy fractional and fuzzy non-fractional problems.

Agarwal et al. solved fuzzy fractional differential equations in the sense of the Riemann–Liouville derivative. Following this, several authors have extended the definitions of generalized gH-differentiability, Caputo derivative and different types of integral

equations in fuzzy field. For example Ahmad et al. [2] performed an analysis of fuzzy fractional order Volterra–Fredholm integro-differentials. In [3], the authors studied fuzzy fractional differential equations under a generalized Caputo derivative. Hoa [4] studied a fuzzy fractional functional integral and differential equations. Moreover, fuzzy fractional functional differential equations under Caputo gH-differentiability were investigated in [5]. Using the Caputo–Katagampola fractional derivative approach, Hoa et al. [6] studied fuzzy fractional differential equations. In 2020, using the concept of kernel ψ -functions, Vu and Hoa [7] investigated the applications of contractive-like mapping principal to fuzzy fractional integral equations. A variety of fuzzy fractional differential and integral equation applications, in different fields of the sciences, such as electrochemistry, physics, economy, chemistry, electromagnetic, viscoelasticity and control theory, are present in the literature—for example, [8–13].

Classically, a fuzzy set is associated with a membership function, which assigns a numerical value ranging between zero and one to each of its elements. In other words, that fuzzy set is the generalization of the traditional set. Ramot et al. proposed complex fuzzy sets, which are characterized by complex valued membership functions [14]. This extension looks like an extension from real numbers to complex numbers. After this, complex fuzzy sets and logics were systematically reviewed by some authors [15]. Nadler introduced the concept of multivalued contraction mappings and obtained the fixed-point results [16]. Heilpern established the idea of fuzzy contractions, which represents the fuzzy generalization of Banach’s contraction principle [17]. Continuing this, Weiss and Butnairu also obtained fixed points of fuzzy mappings [18,19]. Kramosil and Michalek established the notion of fuzzy metric space [20]. Grabiec followed the work of Kramosil, Michalek and obtained the fuzzy version of the Banach contraction principle [21]. George and Veeramani modified the setting of fuzzy metric spaces due to Kramosil, and defined the Hausdorff topology of fuzzy metric space [22]. Following this, many authors have studied different fixed-point results in fuzzy metric spaces [23]. Furthermore, there are many extensions of metric space terms, including fuzzy metric spaces.

Very recently, Shukla et al. have initiated a new approach to complex valued fuzzy metric space, viewing it as a generalization of fuzzy matrices by replacing $[0, 1]$ for the grade of membership with the complex unit closed interval [24]. They obtained some significant fixed-point results with valid illustrated examples. This work is quite new and interesting, so researchers are interested in generalizing more results in this setting and discussing its applications.

Due to important applications of rational type contractions in complex valued metric spaces, and the the work carried out in [24], using Dass and Gupta’s [25] rational type expression, some fixed-point results are established in the context of CVFMS. For the authenticity of the presented results, an example and existence theorem for the solution of fuzzy fractional Volterra–Fredholm integro-differential equation under a generalized fuzzy Caputo derivative is also discussed.

2. Preliminaries

In this section, we present some basic definitions and lemmas of CVFMS and prove some properties for multi-valued mappings in this setting. In This manuscript is labeled

- (I) The set of complex numbers by $\mathfrak{C}$,
- (II) $\wp = \{(\lambda, \chi) : 0 \leq \lambda < \infty, 0 \leq \chi < \infty\} \subset \mathfrak{C}$ where $(0, 0) = \theta, (1, 1) = \ell$,
- (III) $\mathbb{T} = \{(\lambda, \chi) : 0 \leq \lambda \leq 1, 0 \leq \chi \leq 1\}$,
- (IV) $\mathbb{T}_0 = \{(\lambda, \chi) : 0 \leq \lambda < 1, 0 \leq \chi < 1\}$,
- (V) $\mathbb{T}^+ = \{(\lambda, \chi) : 0 < \lambda \leq 1, 0 \leq \chi \leq 1\}$,
- (VII) $\wp_\theta = \{(\lambda, \chi) : 0 < \lambda < \infty, 0 < \chi < \infty\}$.

Define a partial ordering $\preceq$ on $\mathfrak{C}$ by $c_1 \preceq c_2$ iff $c_2 - c_1 \in \wp$. The relations $c_1 \preceq c_2$ and $c_1 \prec c_2$ indicate that

- i. $\operatorname{Re}(c_1) \leq \operatorname{Re}(c_2), \operatorname{Im}(c_1) \leq \operatorname{Im}(c_2)$,
- ii. $\operatorname{Re}(c_1) < \operatorname{Re}(c_2), \operatorname{Im}(c_1) < \operatorname{Im}(c_2)$.

For $c, \lambda \in \mathcal{C}$, $\lambda \preceq c$ iff $\lambda - c \in \wp_\theta$. Suppose $\mathfrak{G} \subset \mathcal{C}$. Let the $\inf \mathfrak{G}$ exists and it is the lower bound of $\mathfrak{G}$, that is $\inf \mathfrak{G} \preceq c \quad \forall c \in \mathfrak{G}$ and $\mathbf{v} \preceq \inf \mathfrak{G}$ for each lower bound $\mathbf{v}$ of $\mathfrak{G}$, then $\inf \mathfrak{G}$ is called the greatest lower bound (glb) of $\mathfrak{G}$. In the same fashion, one can define $\sup \mathfrak{G}$, the least upper bound (lub) of $\mathfrak{G}$.

Definition 1 ([24]). A sequence $\{c_b\}$ is monotonic with respect to $\preceq$ if either $c_b \preceq c_{b+1}$ or $c_{b+1} \preceq c_b \quad \forall b \in \mathbb{N}$.

Definition 2 ([24]). A binary relation $\diamond : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ is called a complex valued $\mathfrak{t}$ -norm if the conditions given below hold:

- (1) $\mathfrak{h}_1 \diamond \mathfrak{h}_2 = \mathfrak{h}_2 \diamond \mathfrak{h}_1$;
- (2) $\mathfrak{h}_1 \diamond \mathfrak{h}_2 \preceq \mathfrak{h}_3 \diamond \mathfrak{h}_4$ whenever $\mathfrak{h}_1 \preceq \mathfrak{h}_3, \mathfrak{h}_2 \preceq \mathfrak{h}_4$;
- (3) $\mathfrak{h}_1 \diamond (\mathfrak{h}_2 \diamond \mathfrak{h}_3) = (\mathfrak{h}_1 \diamond \mathfrak{h}_2) \diamond \mathfrak{h}_3$;
- (4) $\mathfrak{h} \diamond \theta = \theta, \mathfrak{h} \diamond \ell = \mathfrak{h}$;

for all $\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4 \in \mathcal{T}$

Definition 3 ([24]). If $\mathcal{S}$ is a non-empty set and $\diamond$ is continuous complex-valued $\mathfrak{t}$ -norm, $\mathfrak{U}$ a complex fuzzy set on $\mathcal{S} \times \mathcal{S} \times \wp_\theta \rightarrow \mathcal{T}$, observing the following conditions:

- (1) $0 \preceq \mathfrak{U}(\mathfrak{h}, \lambda, \mathbf{r})$;
- (2) $\mathfrak{U}(\mathfrak{h}, \lambda, \mathbf{r}) = \ell$ for every $\mathbf{r} \in \wp_\theta$ if and only if $\mathfrak{h} = \lambda$;
- (3) $\mathfrak{U}(\mathfrak{h}, \lambda, \mathbf{r}) = \mathfrak{U}(\lambda, \mathfrak{h}, \mathbf{r})$;
- (4) $\mathfrak{U}(\mathfrak{h}, \lambda, \mathbf{r}) \diamond \mathfrak{U}(\lambda, \mathbf{y}, \mathbf{r}') \preceq \mathfrak{U}(\mathfrak{h}, \mathbf{y}, \mathbf{r} + \mathbf{r}')$;
- (5) $\mathfrak{U}(\mathfrak{h}, \lambda, \diamond) : \wp_\theta \rightarrow \mathcal{T}$ is continuous for all $\mathfrak{h}, \lambda, \mathbf{y} \in \mathcal{S}$ and $\mathbf{r}, \mathbf{r}' \in \wp_\theta$.

Then $(\mathcal{S}, \mathfrak{U}, \diamond)$ is known to be a CVFMS. The function $\mathfrak{U}(\mathfrak{h}, \lambda, \mathbf{r})$ represents the degree of nearness and non-nearness between $\mathfrak{h}$ and λ with respect to the complex parameter $\mathbf{r}$, respectively.

Example 1. Let $X = \mathbb{N}$ (set of natural numbers). Define $\diamond$ by $c' \diamond c'' = (a'a'', b'b'')$ for all $c' = (a', b'), c'' = (a'', b'') \in \mathcal{T}$. Define complex fuzzy set $\mathfrak{U}$ as

$$\mathfrak{U}(\lambda, \chi, \mathbf{r}) = \begin{cases} \frac{\lambda}{\chi} \ell & \text{if } \lambda \leq \chi \\ \frac{\chi}{\lambda} \ell & \text{if } \chi \leq \lambda, \end{cases}$$

for each $\lambda, \chi \in X, c \in \wp_\theta$. Then $(X, \mathfrak{U}, \diamond)$ is CVFMS.

Definition 4 ([24]). Let $\mathcal{S}$ be a non-empty set. A complex fuzzy set A is characterized by a mapping defined on $\mathcal{S}$ and ranging closed unit complex interval $\mathcal{T}$.

Definition 5 ([24]). Suppose $(\mathcal{S}, \mathfrak{U}, \diamond)$ is CVFMS. A sequence $\{\lambda_b\}$ in $\mathcal{S}$ is called a Cauchy sequence if

$$\lim_{b \rightarrow \infty} \inf_{d > b} \mathfrak{U}(\lambda_b, \lambda_d, \mathbf{r}) = \ell \quad \forall \mathbf{r} \in \wp_\theta.$$

The CVFMS $(\mathcal{S}, \mathfrak{U}, \diamond)$ is said to be complete if every Cauchy sequence converges to an element of $\mathcal{S}$.

Definition 6 ([24]). For assumed $\mathbf{t} \in \mathcal{T}_\theta, \mathbf{r} \in \wp_\theta$ and $\mathbf{u}_\theta \in \mathcal{S}$, we fixed $B[\mathbf{u}_\theta, \mathbf{t}, \mathbf{r}] \{z \in \mathcal{S} : \ell - \mathbf{t} \preceq \mathfrak{U}(\mathbf{u}_\theta, z, \mathbf{r})\}$.

Lemma 1 ([24]). If $(\mathcal{S}, \mathfrak{U}, \diamond)$ is a CVFMS. If $\mathbf{r}, \mathbf{r}' \in \wp_\theta$ and $\mathbf{r} \preceq \mathbf{r}'$, then $\mathfrak{U}(\lambda, \xi, \mathbf{r}) \preceq \mathfrak{U}(\lambda, \xi, \mathbf{r}') \quad \forall \lambda, \xi \in \mathcal{S}$.

Lemma 2 ([24]). Let $(\mathcal{S}, \mathfrak{U}, \diamond)$ be CVFMS. A sequence $\{\lambda_b\}$ in $\mathcal{S}$ converges to $\mathbf{v} \in \mathcal{S}$ iff $\lim_{b \rightarrow \infty} \mathfrak{U}(\lambda_b, \mathbf{v}, \mathbf{r}) = \ell$ holds $\forall \mathbf{r} \in \wp_\theta$.

Lemma 3 ([24]). Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be CVFMS. If $\mathbf{r}, \mathbf{r}' \in \wp_\theta$ and $\mathbf{r} \preceq \mathbf{r}'$, then $\mathcal{U}(\lambda, \xi, \mathbf{r}) \preceq \mathcal{U}(\lambda, \xi, \mathbf{r}') \forall \lambda, \xi \in \mathcal{S}$.

Lemma 4 ([24]). Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be CVFMS. A sequence $\{\lambda_q\}$ in $\mathcal{S}$ converges to $\mathbf{v} \in \mathcal{S}$ iff $\lim_{q \rightarrow \infty} \mathcal{U}(\lambda_q, \mathbf{v}, \mathbf{r}) = \ell$ holds $\forall t \in \wp_\theta$.

Remark 1 ([24]). Suppose $\lambda_b \in \wp \forall b \in \mathbb{N}$ and $\preceq$ are in partial order, then:

- (a) If the sequence $\{\lambda_b\}$ is monotonic and there exists $\gamma, \eta \in \wp$ with $\gamma \preceq \lambda_b \preceq \eta, \forall b \in \mathbb{N}$, then there exists $\lambda \in \wp$ such that $\lim_{b \rightarrow \infty} \lambda_b = \lambda$.
- (b) Although the partial ordering $\preceq$ is not a linear order on $\mathfrak{C}$, the pair $(\mathfrak{C}, \preceq)$ is a lattice.
- (c) If $\mathcal{S} \subset \mathfrak{C}$, then $\inf \mathcal{S}$ and $\sup \mathcal{S}$ both exist for $\gamma, \eta \in \mathfrak{C}$ with $\gamma \preceq s \preceq \eta \forall s \in \mathcal{S}$.

Remark 2 ([24]). Let $\lambda_b, \lambda'_b, \hbar \in \wp, \forall b \in \mathbb{N}$, then

- (a) If $\lambda_b \preceq \lambda'_b \preceq \ell \forall b \in \mathbb{N}$ and $\lim_{b \rightarrow \infty} \lambda_b = \ell$, then $\lim_{b \rightarrow \infty} \lambda'_b = \ell$.
- (b) If $\lambda_b \preceq \hbar \forall b \in \mathbb{N}$ and $\lim_{b \rightarrow \infty} \lambda_b = \lambda$, then $\lambda \preceq \mathbf{w}$.
- (c) If $\hbar \preceq \lambda_b \forall b \in \mathbb{N}$ and $\lim_{b \rightarrow \infty} \lambda_b = \lambda$, then $\mathbf{w} \preceq \lambda$.

A relatively important notion in complex fuzzy set theory is σ -level set. Let A be a complex fuzzy set in $\mathcal{S}$. Then, the function values of $A(\lambda)$ are said to be the grade of membership of $\lambda \in A$. The collection of all those elements in $\mathcal{S}$ belonging to A have at least a degree $\sigma \in \mathbb{T}^+$, which is called the σ -level set and denoted by $[A]_\sigma$. That is,

$$[A]_\sigma = \{\lambda : A(\lambda) \succeq \sigma\} \quad \text{if } \sigma \in \mathbb{T} = [A_\sigma^-, A_\sigma^+].$$

Please note that the σ -level representation of fuzzy valued function T is expressed by $T_\sigma(t) = [T_\sigma^-(t), A_\sigma^+(t)], \sigma \in [0, 1]$.

Definition 7. Let $T : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ be a fuzzy mapping. An element $\mathbf{u} \in \mathcal{S}$ is known to be a fuzzy fixed point of T if there exists an $\sigma \in \mathbb{T}^+$ such that $\mathbf{u} \in [Tu]_\sigma$, where $\mathfrak{F}(\mathcal{S})$ is a collection of complex fuzzy sets.

Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be a CVFMS. We denote the family of all nonempty, closed and bounded subsets of a complex valued fuzzy metric space by $\text{CB}(\mathcal{S})$. From now on we denote for $\bar{w} \in \mathcal{S}$, for $\mathbf{r} \in \wp_\theta, c \in \mathcal{S}$ and $\mathfrak{G} \in \text{CB}(\mathcal{S})$:

$$s(\bar{w}, \mathbf{r}) = \{(\bar{z}, \mathbf{r}) \in \mathbb{T} : (\bar{z}, \mathbf{r}) \preceq (\bar{w}, \mathbf{r})\}$$

and

$$s(c, \mathfrak{G}, \mathbf{r}) = \bigcup_{d \in \mathfrak{G}} s(\mathcal{U}(c, d, \mathbf{r})) = \bigcup_{d \in \mathfrak{G}} \{\bar{z} \in \mathcal{S} : \bar{z} \preceq \mathcal{U}(c, d, \mathbf{r})\}$$

For $C, \mathfrak{G} \in \text{CB}(\mathcal{S})$, we denote

$$s(C, \mathfrak{G}, \mathbf{r}) = \left(\bigcap_{c \in C} s(\mathcal{U}(c, \mathfrak{G}, \mathbf{r})) \right) \cap \left(\bigcap_{d \in \mathfrak{G}} s(\mathcal{U}(d, C, \mathbf{r})) \right)$$

Let T be a multivalued mapping from $\mathcal{S}$ into $\text{CB}(\mathcal{S})$, for $z \in \mathcal{S}$ and $\mathcal{A} \in \text{CB}(\mathcal{S})$, we define

$$\mathcal{W}_z(\mathcal{A}, \mathbf{r}) = \{\mathcal{U}(z, a, \mathbf{r}) : a \in \mathcal{A}\}.$$

Thus for $z, w \in \mathcal{S}$

$$\mathcal{W}_z(Tw, \mathbf{r}) = \{\mathcal{U}(z, u, \mathbf{r}) : u \in Tw\}.$$

Definition 8. In a $(\mathcal{S}, \mathcal{U}, \diamond)$ CVFMS a subset K of $\mathcal{S}$ is said to be bounded from above if there exists some $\mathbf{w} \in \mathcal{S}$, such that $k \preceq \mathbf{w}$ for all $k \in K$.

Definition 9. In a CVFMS, a multivalued mapping $\mathcal{T} : \mathcal{S} \rightarrow 2^\wp$ is said to be bounded from above if and only if, for each $z \in \mathcal{S}$, there exists $x_z \in \wp$, such that

$$w \preceq x_z$$

for all $w \in \mathcal{T}z$.

Definition 10. A fuzzy mapping $F : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ is supposed to have an upper bound property on $(\mathcal{S}, \mathcal{U}, \diamond)$, if, for any $z \in \mathcal{S}$ related with some $\sigma \in \mathbb{T}$, the multivalued mapping $\mathcal{T} : \mathcal{S} \rightarrow 2^\wp$ defined by

$$\mathcal{T}_z(w) = \mathcal{W}_z([Fw]_\sigma)$$

is bounded from above, i.e., for $z, w \in \mathcal{S}$ there is an element $l_z([Fw]_\sigma) \in \wp$ with

$$v \preceq l_z([Fw]_\sigma)$$

for each $v \in \mathcal{W}_z([Fw]_\sigma)$, where $l_z([Fw]_\sigma)$ is known as the upper bound of F .

Lemma 5. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be CVFMS.

- (i) Let $(a, f), (b, f) \in \wp$. If $(a, f) \preceq (b, f)$ then $s(a, r) \subseteq s(b, r)$
- (ii) Let $(a, f) \in \wp, \mathfrak{P}, \mathfrak{G} \in \text{CB}(\mathcal{S})$ and $c \in \mathfrak{P}$. If $(a, f) \in s(\mathfrak{P}, \mathfrak{G}, r)$ then $a \in s(c, \mathfrak{G}, r)$ for all $c \in \mathfrak{P}$ or $a \in s(\mathfrak{P}, d, r)$ for all $d \in \mathfrak{G}$.

Proof.

- (i) Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be a CVFMS. Suppose $(x, f) \in s(a, r)$ then $(x, f) \preceq (a, f)$. But $(a, f) \preceq (b, f)$, therefore $(x, f) \preceq (b, f)$. Consequently $(x, f) \in s(b, r)$. Hence $s(a, r) \subseteq s(b, r)$.
- (ii) Suppose $c \in \mathfrak{P}$ and $(a, f) \in s(\mathfrak{P}, \mathfrak{G}, r)$

$$(a, f) \in \left(\bigcap_{c \in \mathfrak{P}} s(c, \mathfrak{G}, r) \right) \cap \left(\bigcap_{d \in \mathfrak{G}} s(d, \mathfrak{P}, r) \right),$$

yields that

$$(a, f) \in \left(\bigcap_{c \in \mathfrak{P}} s(c, \mathfrak{G}, r) \right) \text{ and } (a, f) \in \left(\bigcap_{d \in \mathfrak{G}} s(d, \mathfrak{P}, r) \right).$$

Since $(a, f) \in \bigcap_{c \in \mathfrak{P}} s(c, \mathfrak{G}, r)$ implies that $(a, f) \in s(c, \mathfrak{G}, r)$ for all $c \in \mathfrak{P}$. Similarly $b \in s(d, \mathfrak{P}, r)$ for all $d \in \mathfrak{G}$.

□

Remark 3. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be CVFMS. If $\mathbb{T} = [0, 1]$, then $(\mathcal{S}, \mathcal{U}, \diamond)$ is a fuzzy metric space. Moreover, for $\mathfrak{P}, \mathfrak{G} \in \text{CB}(\mathcal{S})$, then $H(\mathfrak{P}, \mathfrak{G}, r) = \sup s(\mathfrak{P}, \mathfrak{G}, r)$ is the Hausdorff distance induced by $\mathcal{U}$.

Definition 11. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be CVFMS and let G be fuzzy mappings from $\mathcal{S}$ into $\mathfrak{F}(\mathcal{S})$. A point $\lambda \in G$ is called a fuzzy fixed point of G if $\lambda \in [G\lambda]_\sigma$, for some $\sigma \in \mathbb{T}$.

Definition 12. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be CVFMS and the fuzzy mapping $U : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ satisfies the least upper bound property (lub) on $(\mathcal{S}, \mathcal{U}, \diamond)$, if for any $h \in \mathcal{S}$ and $\sigma \in (0, 1]$, the least upper bound (lub) of $h_\omega([U\chi]_\sigma, r)$ exists in $\mathfrak{C}$ for all $h, \chi \in \mathcal{S}$ and $r \in \wp_\theta$. If $\mathcal{U}(h, [U\chi]_\sigma, \diamond)$ be the lub of $h_\omega([U\chi]_\sigma, r)$. Then,

$$\mathcal{U}(h, [U\chi]_\sigma, r) = \sup \{ \mathcal{U}(h, u, r) : u \in [U\chi]_\sigma, r \}.$$

Definition 13. The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathfrak{F}(\mathcal{S})$ is defined as follows

$$u \ominus_{\text{gH}} v = w \Leftrightarrow \begin{cases} (i) u = v + w \\ \text{or } (ii) v = u + (-1)w. \end{cases}$$

Definition 14 ([2]). The generalized Hukuhara derivative of a fuzzy-valued function $T : (a, b) \rightarrow \mathfrak{F}(\mathcal{S})$ at t_0 is defined as

$$T'_{\text{gH}}(t_0) = \lim_{h \rightarrow 0} \frac{T(t_0 + \delta) \ominus_{\text{gH}} T(t_0)}{\delta},$$

if $(T'_{\text{gH}}(t_0) \rightarrow \mathfrak{F}(\mathcal{S}))$, we say that T is generalized Hukuhara differentiable (gH-differentiable) at t_0 .

Additionally, we say that T is $[(i) - \text{gH}]$ -differentiable at t_0 if

$$(T'_{\text{gH}})_\sigma(t_0) = [(T'_\sigma)^-(t_0), (T'_\sigma)^+(t_0)], \quad 0 \leq \sigma \leq 1,$$

and that f is $[(ii) - \text{gH}]$ -differentiable at t_0 if

$$(T'_{\text{gH}})_\sigma(t_0) = [(T'_\sigma)^+(t_0), (T'_\sigma)^-(t_0)], \quad 0 \leq \sigma \leq 1.$$

Definition 15. Consider $f : [a, b] \rightarrow \mathbb{R}$, fractional derivative of $f(t)$ in the Caputo sense is defined as

$$(D_*^q f)(t) = (I^{m-q} D^m f)(t) = \frac{1}{\Gamma(m-q)} \int_a^t (t-s)^{(q-m-1)} f^{(m)}(s) ds \quad m-1 < q \leq m, m \in \mathbb{N}, t > a$$

where D stands for classic derivative.

We denote $C^F[a, b]$ as the space of all continuous fuzzy-valued functions on $[a, b]$. Additionally, we denote the space of all Lebesgue integrable fuzzy-valued functions on the bounded interval $[a, b] \subset \mathbb{R}$ by $L^F[a, b]$.

Definition 16. Let $f' \in C^F[a, b] \cap L^F[a, b]$. The fractional generalized Hukuhara Caputo derivative of fuzzy-valued function f is defined as follows:

$$({}_{\text{gH}}D_*^q f)(t) = I_a^{1-q} (f'_{\text{gH}})(t) = \frac{1}{\Gamma(1-q)} \int_a^t \frac{(f'_{\text{gH}})(s) ds}{(t-s)^q}, \quad a < s < t, \quad 0 < q < 1.$$

Additionally, we say that f is ${}^{\text{cf}}[(i) - \text{gH}]$ -differentiable at t_0 if

$$({}_{\text{gH}}D_*^q f)_\sigma(t_0) = [(D_*^q f_\sigma)^-(t_0), (D_*^q f_\sigma)^+(t_0)], \quad 0 \leq \sigma \leq 1,$$

and that f is ${}^{\text{cf}}[(ii) - \text{gH}]$ -differentiable at t_0 if

$$({}_{\text{gH}}D_*^q f)_\sigma(t_0) = [(D_*^q f_\sigma)^+(t_0), (D_*^q f_\sigma)^-(t_0)], \quad 0 \leq \sigma \leq 1.$$

3. Main Results

Theorem 1. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ CVFMS such that, for any sequence $\{\mathbf{r}_n\}$ in $\wp_\theta$ with $\lim_{n \rightarrow \infty} \mathbf{r}_n = \infty$, we have

$$\lim_{r \rightarrow \infty} \mathcal{U}(\mathbf{w}, \mathbf{z}, \mathbf{r}_n) = \ell, \quad \text{for all } \mathbf{w}, \mathbf{z} \in \mathcal{S}, \mathbf{q} + \mathbf{e} = \mathbf{z} \in (0, 1) \text{ and } \mathbf{r} > 0.$$

Assume that there exists some $\sigma \in (0, 1]$, such that, for each $z \in \mathcal{S}$, such that $[F\mathbf{w}]_\sigma$ is a nonempty compact subset of $\mathcal{S}$. Let $F : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ be a fuzzy mapping with the least upper bound property, such that

$$\frac{[1 + \mathfrak{U}(z, [Fz]_\sigma, \mathbf{qr})]\mathfrak{U}(\mathbf{w}, [F\mathbf{w}]_\sigma, \mathbf{er})}{1 + \mathfrak{U}(\mathbf{w}, z, \mathbf{r})} + \mathfrak{U}(z, \mathbf{w}, \mathbf{r}) \in s([F\mathbf{w}]_\sigma, [Fz]_\sigma, \mathfrak{z}\mathbf{r}). \quad (1)$$

Then F has a unique σ -fuzzy fixed point.

Proof. Let $\mathbf{c}_0$ be any arbitrary point in $\mathcal{S}$. Define a sequence $\{\mathbf{c}_n\}$ in $\mathcal{S}$ by

$$\mathbf{c}_n \in [F\mathbf{c}_{n-1}]_\sigma \quad \text{for all } n \in \{1, 2, \dots\}.$$

First of all, we have to show that $\{\mathbf{c}_n\}$ is a Cauchy sequence. For this, we define

$$\Lambda_b = \{\mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}) : m > n\} \subseteq \mathbb{T},$$

for $n \in \{1, 2, \dots\}$ and fixed $\mathbf{r} \in \wp_\theta$. Since $\theta \prec (\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}) \preceq \ell$ for all $b \in \{1, 2, \dots\}$. Using Remark 1, we obtain that, for all $b \in 1, 2, \dots$, the infimum, $\inf \Lambda_b = \varrho_b$ (say) exists. For $\mathbf{r} \in \wp_\theta$, $n, m \in \mathbb{N}$ with $m > n$, from (1) by setting $z = \mathbf{c}_n$ and $\mathbf{w} = \mathbf{c}_m$, we obtain

$$\frac{[1 + \mathfrak{U}(\mathbf{c}_n, [F\mathbf{c}_n]_\sigma, \mathbf{qr})]\mathfrak{U}(\mathbf{c}_m, [F\mathbf{c}_m]_\sigma, \mathbf{er})}{1 + \mathfrak{U}(\mathbf{c}_m, \mathbf{c}_n, \mathbf{r})} + \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}) \in s([F\mathbf{c}_n]_\sigma, [F\mathbf{c}_m]_\sigma, \mathfrak{z}\mathbf{r}).$$

Using Lemma 5 (ii), we obtain

$$\frac{[1 + \mathfrak{U}(\mathbf{c}_n, [F\mathbf{c}_n]_\sigma, \mathbf{qr})]\mathfrak{U}(\mathbf{c}_m, [F\mathbf{c}_m]_\sigma, \mathbf{er})}{1 + \mathfrak{U}(\mathbf{c}_m, \mathbf{c}_n, \mathbf{r})} + \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}) \in s(\mathbf{c}_{n+1}, [F\mathbf{c}_m]_\sigma, \mathfrak{z}\mathbf{r}).$$

Since $[F\mathbf{c}_m]_\sigma$ is nonempty subset of $\mathcal{S}$, there exists some $\mathbf{c}_{m+1} \in [F\mathbf{c}_m]_\sigma$ such that

$$\frac{[1 + \mathfrak{U}(\mathbf{c}_n, [F\mathbf{c}_n]_\sigma, \mathbf{qr})]\mathfrak{U}(\mathbf{c}_m, [F\mathbf{c}_m]_\sigma, \mathbf{er})}{1 + \mathfrak{U}(\mathbf{c}_m, \mathbf{c}_n, \mathbf{r})} + \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}) \in s(\mathfrak{U}(\mathbf{c}_{n+1}, \mathbf{c}_{m+1}, \mathfrak{z}\mathbf{r})).$$

Using Definition 12, we obtain

$$\mathfrak{U}(\mathbf{c}_{n+1}, \mathbf{c}_{m+1}, \mathfrak{z}\mathbf{r}) \succeq \frac{[1 + \mathfrak{U}(\mathbf{c}_n, [F\mathbf{c}_n]_\sigma, \mathbf{qr})]\mathfrak{U}(\mathbf{c}_m, [F\mathbf{c}_m]_\sigma, \mathbf{er})}{1 + \mathfrak{U}(\mathbf{c}_m, \mathbf{c}_n, \mathbf{r})} + \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}).$$

Applying the least upper bound property of F

$$\begin{aligned} \mathfrak{U}(\mathbf{c}_{n+1}, \mathbf{c}_{m+1}, \mathfrak{z}\mathbf{r}) &\succeq \frac{[1 + \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_{n+1}, \mathbf{qr})]\mathfrak{U}(\mathbf{c}_m, \mathbf{c}_{m+1}, \mathbf{er})}{1 + \mathfrak{U}(\mathbf{c}_m, \mathbf{c}_n, \mathbf{r})} + \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}) \\ &\succeq \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}). \end{aligned}$$

Utilizing Lemma 3, this yields

$$\mathfrak{U}(\mathbf{c}_{n+1}, \mathbf{c}_{m+1}, \mathbf{r}) \succeq \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \frac{\mathbf{r}}{3}) \succeq \mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}), \quad (2)$$

which implies that

$$\mathfrak{U}(\mathbf{c}_n, \mathbf{c}_m, \mathbf{r}) \preceq \mathfrak{U}(\mathbf{c}_{n+1}, \mathbf{c}_{m+1}, \mathbf{r}) \quad \text{for all } n, m \in \mathbb{N} \text{ with } m > n.$$

Therefore, by definition, we have

$$\theta \preceq \varrho_n \preceq \varrho_{n+1} \preceq \ell. \quad (3)$$

Thus, $\{\varrho_n\}$ is a monotonic sequence in $\wp$, and by the use of Remark 1 and (3), there exists $\ell_a \in \wp$, such that

$$\lim_{n \rightarrow \infty} \varrho_n = \ell_a. \quad (4)$$

Again, from (2), we have, for $\mathbf{c} \in \wp$,

$$\varrho_{n+1} = \inf_{m > n} \mathcal{U}(\mathbf{c}_{n+1}, \mathbf{c}_{m+1}, \mathbf{r}) \succeq \inf_{m > n} \mathcal{U}(\mathbf{c}_n, \mathbf{c}_m, \frac{\mathbf{r}}{3}).$$

Similarly, for $\mathbf{c} \in \wp_\theta$, we obtain

$$\begin{aligned} \mathcal{U}(\mathbf{c}_{n+1}, \mathbf{c}_{m+1}, \mathbf{r}) &\succeq \mathcal{U}(\mathbf{c}_n, \mathbf{c}_m, \frac{\mathbf{r}}{3}) \\ &\succeq \mathcal{U}(F\mathbf{c}_{n-1}, F\mathbf{c}_{m-1}, \frac{\mathbf{r}}{3}) \\ &\succeq \mathcal{U}(\mathbf{c}_{n-1}, \mathbf{c}_{m-1}, \frac{\mathbf{r}}{3^2}) = \mathcal{U}(F\mathbf{c}_{n-2}, F\mathbf{c}_{m-2}, \frac{\mathbf{r}}{3^2}) \\ &\succeq \mathcal{U}(\mathbf{c}_{n-2}, \mathbf{c}_{m-2}, \frac{\mathbf{r}}{3^3}) \succeq \cdots \succeq \mathcal{U}(\mathbf{c}_0, \mathbf{c}_{m-n}, \frac{\mathbf{r}}{3^{n+1}}), \end{aligned}$$

hence, for all $\mathbf{c} \in \wp_\theta$,

$$\varrho_{n+1} = \inf_{m > n} \mathcal{U}(\mathbf{c}_{n+1}, \mathbf{c}_{m+1}, \mathbf{r}) \succeq \inf_{m > n} \mathcal{U}(\mathbf{c}_0, \mathbf{c}_{m-n}, \frac{\mathbf{r}}{3^{n+1}}) \succeq \inf_{\mathbf{w} \in \mathcal{S}} \mathcal{U}(\mathbf{c}_0, \mathbf{w}, \frac{\mathbf{r}}{3^{n+1}}).$$

Since $\lim_{n \rightarrow \infty} \frac{\mathbf{r}}{3^{n+1}} = \infty$, using (4) and from hypothesis, we have

$$\ell_a \succeq \inf_{\mathbf{w} \in \mathcal{S}} \mathcal{U}(\mathbf{c}_0, \mathbf{w}, \frac{\mathbf{r}}{3^{n+1}}) = \ell. \quad (5)$$

From (4) and (5), we obtain

$$\lim_{n \rightarrow \infty} \varrho_n = \ell.$$

Hence, $\{\mathbf{c}_n\}$ is a Cauchy sequence in $\mathcal{S}$. Since $\mathcal{S}$ is complete and from Lemma 4, there exists $\mathbf{x} \in \mathcal{S}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{U}(\mathbf{c}_n, \mathbf{x}, \mathbf{r}) = \ell \quad \text{for all } \mathbf{c} \in \wp_\theta. \quad (6)$$

Considering (1), for any $\mathbf{c} \in \wp_\theta$, we obtain

$$\frac{[1 + \mathcal{U}(\mathbf{c}_n, [F\mathbf{c}_n]_\sigma, \mathbf{q}\mathbf{r})]\mathcal{U}(\mathbf{x}, [F\mathbf{x}]_\sigma, \mathbf{e}\mathbf{r})}{1 + \mathcal{U}(\mathbf{x}, \mathbf{c}_n, \mathbf{r})} + \mathcal{U}(\mathbf{c}_n, \mathbf{x}, \mathbf{r}) \in s([F\mathbf{c}_n]_\sigma, [F\mathbf{x}]_\sigma, \mathfrak{z}\mathbf{r}).$$

Using Lemma 5 (ii), we obtain

$$\frac{[1 + \mathcal{U}(\mathbf{c}_n, [F\mathbf{c}_n]_\sigma, \mathbf{q}\mathbf{r})]\mathcal{U}(\mathbf{x}, [F\mathbf{x}]_\sigma, \mathbf{e}\mathbf{r})}{1 + \mathcal{U}(\mathbf{x}, \mathbf{c}_n, \mathbf{r})} + \mathcal{U}(\mathbf{c}_n, \mathbf{x}, \mathbf{r}) \in s(\mathbf{c}_{n+1}, [F\mathbf{x}]_\sigma, \mathbf{r}).$$

By definition, we obtain

$$\mathcal{U}(\mathbf{c}_{n+1}, [F\mathbf{x}]_\sigma, \mathfrak{z}\mathbf{r}) \succeq \frac{[1 + \mathcal{U}(\mathbf{c}_n, [F\mathbf{c}_n]_\sigma, \mathbf{q}\mathbf{r})]\mathcal{U}(\mathbf{x}, [F\mathbf{x}]_\sigma, \mathbf{e}\mathbf{r})}{1 + \mathcal{U}(\mathbf{x}, \mathbf{c}_n, \mathbf{r})} + \mathcal{U}(\mathbf{c}_n, \mathbf{x}, \mathbf{r}).$$

Applying the least upper bound property of F

$$\begin{aligned} \mathcal{U}(\mathbf{c}_{n+1}, [F\mathbf{x}]_\sigma, \mathfrak{z}\mathbf{r}) &\succeq \frac{[1 + \mathcal{U}(\mathbf{c}_n, \mathbf{c}_{n+1}, \mathbf{q}\mathbf{r})]\mathcal{U}(\mathbf{x}, \mathbf{x}_n, \mathbf{e}\mathbf{r})}{1 + \mathcal{U}(\mathbf{x}, \mathbf{c}_n, \mathbf{r})} + \mathcal{U}(\mathbf{c}_n, \mathbf{x}, \mathbf{r}) \\ &\succeq \mathcal{U}(\mathbf{c}_n, \mathbf{x}, \mathbf{r}). \end{aligned} \quad (7)$$

From the definition and using (7), we obtain

$$\begin{aligned}\mathcal{U}(x, [Fx]_{\sigma}, \mathbf{r}) &\succeq \mathcal{U}(x, \mathbf{c}_{n+1}, \frac{\mathbf{r}}{2}) * \mathcal{U}(\mathbf{c}_{n+1}, [Fx]_{\sigma}, \frac{\mathbf{r}}{2}) \\ &\succeq \mathcal{U}(x, \mathbf{c}_{n+1}, \frac{\mathbf{r}}{2}) * \mathcal{U}(\mathbf{c}_n, x, \frac{\mathbf{r}}{2}).\end{aligned}$$

Taking $\lim_{n \rightarrow \infty}$, and using (6) and Remark 2, we can see that

$$\mathcal{U}(x, [Fx]_{\sigma}, \mathbf{r}) = \ell \quad \text{for all } \mathbf{r} \in \wp_{\theta}.$$

i.e., $x \in [Fx]_{\sigma}$. Let x_1 be another fixed point of F , and there exists $\mathbf{r} \in \wp_{\theta}$ such that $\mathcal{U}(x, x_1, \mathbf{r}) \neq \ell$ then it yields from (1) that

$$\frac{[1 + \mathcal{U}(x, [Fx]_{\sigma}, \mathbf{qr})]\mathcal{U}(x_1, [Fx_1]_{\sigma}, \mathbf{er})}{1 + \mathcal{U}(x_1, x, \mathbf{r})} + \mathcal{U}(x, x_1, \mathbf{r}) \in s([Fx]_{\sigma}, [Fx_1]_{\sigma}, \mathfrak{z}\mathbf{r}).$$

Using Lemma 5 (ii), we get

$$\frac{[1 + \mathcal{U}(x, [Fx]_{\sigma}, \mathbf{qr})]\mathcal{U}(x_1, [Fx_1]_{\sigma}, \mathbf{er})}{1 + \mathcal{U}(x_1, x, \mathbf{r})} + \mathcal{U}(x, x_1, \mathbf{r}) \in s(x, [Fx_1]_{\sigma}, \mathfrak{z}\mathbf{r}).$$

Since $[Fx_1]_{\sigma}$ is nonempty subset of $\mathcal{S}$, there exists some $x_1 \in [Fx_1]_{\sigma}$ such that

$$\frac{[1 + \mathcal{U}(x, [Fx]_{\sigma}, \mathbf{qr})]\mathcal{U}(x_1, [Fx_1]_{\sigma}, \mathbf{er})}{1 + \mathcal{U}(x_1, x, \mathbf{r})} + \mathcal{U}(x, x_1, \mathbf{r}) \in s(\mathcal{U}(x, x_1, \mathfrak{z}\mathbf{r})).$$

Using Definition 12, we obtain

$$\mathcal{U}(x, x_1, \mathfrak{z}\mathbf{r}) \succeq \frac{[1 + \mathcal{U}(x, [Fx]_{\sigma}, \mathbf{qr})]\mathcal{U}(x_1, [Fx_1]_{\sigma}, \mathbf{er})}{1 + \mathcal{U}(x_1, x, \mathbf{r})} + \mathcal{U}(x, x_1, \mathbf{r}).$$

Applying the least upper bound property of F

$$\begin{aligned}\mathcal{U}(x, x_1, \mathfrak{z}\mathbf{r}) &\succeq \frac{[1 + \mathcal{U}(x, x, \mathbf{qr})]\mathcal{U}(x_1, x_1, \mathbf{er})}{1 + \mathcal{U}(x_1, x, \mathbf{r})} + \mathcal{U}(x, x_1, \mathbf{r}) \\ &\succeq \mathcal{U}(x, x_1, \mathbf{r}).\end{aligned}$$

On simplification, we get

$$\mathcal{U}(x, x_1, \mathbf{r}) \succeq \mathcal{U}(x, x_1, \frac{\mathbf{r}}{3}) \succeq \mathcal{U}(x, x_1, \frac{\mathbf{r}}{3^2}) \succeq \cdots \succeq \mathcal{U}(x, x_1, \frac{\mathbf{r}}{3^n}).$$

Using $\lim_{n \rightarrow \infty} \frac{\mathbf{r}}{3^n} = \infty$ and $\mathcal{U}(x, x_1, \frac{\mathbf{r}}{3^n}) \succeq \inf_{\mathbf{w} \in \mathcal{S}} \mathcal{U}(x, x_1, \frac{\mathbf{r}}{3^n})$. From this we get $\mathcal{U}(x, x_1, \mathbf{r}) \succeq \ell$, which is a contradiction. Thus, $\mathcal{U}(x, x_1, \mathbf{r}) = \ell$, for all $\mathbf{r} \in \wp_{\theta}$. i.e., $x = x_1$, which follows the uniqueness. $\square$

In the succeeding theorem, we use Definition 6 to demonstrate the existence of fixed-point for a mapping enjoying a restricted condition.

Theorem 2. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ CVFMS and $F : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ be a fuzzy mapping where the least upper bound property enjoys:

- (1) There exists $u_{\theta} \in \mathcal{S}$ and $\mathbf{r} \in \mathfrak{T}_{\theta}$ with $\ell - \mathbf{t} \preceq \mathcal{U}(u_{\theta}, F[u_{\theta}]_{\sigma}, \mathbf{r})$ for all $\mathbf{r} \in \wp_{\theta}$.
- (2)

$$\frac{[1 + \mathcal{U}(z, [Fz]_{\sigma}, \mathbf{qr})]\mathcal{U}(\mathbf{w}, [F\mathbf{w}]_{\sigma}, \mathbf{er})}{1 + \mathcal{U}(\mathbf{w}, z, \mathbf{r})} + \mathcal{U}(z, \mathbf{w}, \mathbf{r}) \in s([F\mathbf{w}]_{\sigma}, [Fz]_{\sigma}, \mathfrak{z}\mathbf{r}). \quad (8)$$

for all $z, \mathbf{w} \in B[u_\theta, F u_\theta, \mathbf{r}]$ and for each $z \in \mathcal{S}$, there exist some $\sigma \in (0, 1]$, such that $[F \mathbf{w}]_\sigma$ be a nonempty closed and bounded subset of $\mathcal{S}$, while $\mathbf{q} + \mathbf{e} = \mathbf{z} \in [0, 1]$. Then, F has a unique σ -fuzzy fixed point in $B[u_\theta, u_\theta, \mathbf{r}]$.

Proof. To prove this, it is enough to show that $B[u_\theta, \mathbf{t}, \mathbf{r}]$ is complete and $[F z]_\sigma \in B[u_\theta, \mathbf{t}, \mathbf{r}]$ for all $z \in B[u_\theta, F u_\theta, \mathbf{r}]$.

Let $\{c_n\}$ be a Cauchy sequence in $B[u_\theta, \mathbf{t}, \mathbf{r}]$. Thus, from the completeness of the ground set $\mathcal{S}$ and Lemma 4, there is an $v \in \mathcal{S}$ with

$$\lim_{n \rightarrow \infty} \mathcal{U}(c_n, v, \mathbf{r}) = \ell \quad \text{for all } \mathbf{r} \in \wp_\theta,$$

at this instant, for all $m, n \in \mathbb{N}$,

$$\mathcal{U}(u_\theta, v, \mathbf{r} + \frac{\mathbf{r}}{m}) \succeq \mathcal{U}(u_\theta, v, \mathbf{r}) * \mathcal{U}(u_\theta, v, \frac{\mathbf{r}}{m}).$$

Since $c_n \in B[u_\theta, \mathbf{c}, \mathbf{r}]$, and $\lim_{n \rightarrow \infty} \mathcal{U}(u_\theta, v, \mathbf{r}) = \ell$, so, by utilizing Remark 2 and using the properties of t-norm, we obtain

$$\mathcal{U}(u_\theta, v, \mathbf{r} + \frac{\mathbf{r}}{m}) \succeq (\ell - \mathbf{c}) \diamond \ell = \ell - \mathbf{c}.$$

Setting alimit such that $m \rightarrow \infty$ and using Remark 2, we have $\mathcal{U}(u_\theta, v, \mathbf{r}) \succeq \ell - \mathbf{c}$. Consequently, $v \in B[u_\theta, \mathbf{t}, \mathbf{r}]$.

For each $z \in B[u_\theta, \mathbf{c}, \mathbf{r}]$, it can be seen (8) that

$$\frac{[1 + \mathcal{U}(u_\theta, [F u_\theta]_\sigma, \mathbf{r}\mathbf{q})] \mathcal{U}(z, [F z]_\sigma, \mathbf{r}\mathbf{e})}{1 + \mathcal{U}(z, u_\theta, \mathbf{r})} + \mathcal{U}(u_\theta, z, \mathbf{r}) \in s([F u_\theta]_\sigma, [F z]_\sigma, \mathbf{z}\mathbf{r}).$$

By Definition, we obtain

$$\begin{aligned} \mathcal{U}([F u_\theta]_\sigma, [F z]_\sigma, \mathbf{z}\mathbf{r}) &\succeq \frac{[1 + \mathcal{U}(u_\theta, [F u_\theta]_\sigma, \mathbf{r}\mathbf{q})] \mathcal{U}(z, [F z]_\sigma, \mathbf{r}\mathbf{e})}{1 + \mathcal{U}(z, u_\theta, \mathbf{r})} + \mathcal{U}(u_\theta, z, \mathbf{r}) \\ &\succeq \mathcal{U}(u_\theta, z, \mathbf{r}). \end{aligned}$$

This yields

$$\begin{aligned} \mathcal{U}(u_\theta, [F z]_\sigma, \mathbf{r} + \frac{\mathbf{r}}{m}) &\succeq \mathcal{U}(u_\theta, F u_\theta, \frac{\mathbf{r}}{m}) \diamond \mathcal{U}(F u_\theta, [F z]_\sigma, \mathbf{r}) \\ &\succeq (\ell - \mathbf{t}) \diamond \mathcal{U}(u_\theta, z, \frac{\mathbf{r}}{3}) \\ &\succeq (\ell - \mathbf{t}) \diamond \mathcal{U}(u_\theta, z, \frac{\mathbf{r}}{3^2}) \succeq \cdots \succeq (\ell - \mathbf{t}) \diamond \mathcal{U}(u_\theta, z, \frac{\mathbf{r}}{3^n}), \end{aligned}$$

for all $n \in \mathbb{N}$. Using $\lim_{n \rightarrow \infty} \frac{\mathbf{r}}{3^n}$ and $\mathcal{U}(u_\theta, z, \frac{\mathbf{r}}{3^n}) \succeq \inf_{y \in \mathcal{S}} \mathcal{U}(u_\theta, z, \frac{\mathbf{r}}{3^n})$. It yields from above inequality

$$\begin{aligned} \mathcal{U}(u_\theta, [F z]_\sigma, \mathbf{r} + \frac{\mathbf{r}}{m}) &\succeq (\ell - \mathbf{t}) \diamond \ell \\ &\succeq (\ell - \mathbf{t}) \end{aligned}$$

Taking $\lim_{m \rightarrow \infty}$ and utilizing Remark 2, we have

$$\mathcal{U}(u_\theta, [F z]_\sigma, \mathbf{r}) \succeq (\ell - \mathbf{t})$$

Thus, $[F z] \in B[u_\theta, \mathbf{t}, \mathbf{r}]$. $\square$

In Theorem 1 the contractive condition (1) for F can be replaced by the following, analogous proof:

Corollary 1.

$$\frac{[1 + \mathcal{U}(z, [Fz]_{\sigma}, \mathbf{q}(\mathbf{r})\mathbf{r})]\mathcal{U}(\mathbf{w}, [F\mathbf{w}]_{\sigma}, \mathbf{e}(\mathbf{r})\mathbf{r})}{1 + \mathcal{U}(\mathbf{w}, z, \mathbf{r})} + \mathcal{U}(z, \mathbf{w}, \mathbf{r}) \in s([F\mathbf{w}]_{\sigma}, [Fz]_{\sigma}, \mathfrak{z}(\mathbf{r})\mathbf{r}),$$

for each $z, \mathbf{w} \in \wp_{\theta}$. Where $\mathbf{q}, \mathbf{e}, \mathfrak{z} : \wp_{\theta} \rightarrow (0, 1)$.

By setting $\mathcal{U}(z, \mathbf{w}, \mathbf{r}) = \theta$ in Theorem 1, we get the following corollary.

Corollary 2. Let $(\mathcal{S}, \mathcal{U}, *)$ be a complete complex valued fuzzy metric space, such that, for any sequence $\{\mathbf{r}_n\}$ in $\wp_{\theta}$ with $\lim_{n \rightarrow \infty} \mathbf{r}_n = \infty$, we have

$$\lim_{r \rightarrow \infty} M(\mathbf{w}, z, \mathbf{r}_n) = \ell, \text{ for all } \mathbf{w}, z \in \mathcal{S}, \mathbf{q} \in (0, 1) \text{ and } \mathbf{r} > 0.$$

Assume that there exists some $\sigma \in (0, 1]$, such that, for each $z \in \mathcal{S}$ such that $[F\mathbf{w}]_{\sigma}$ is a nonempty compact subset of $\mathcal{S}$ for all $\mathbf{w} \in \mathcal{S}$. Let $F : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ be a fuzzy mapping with least upper bound property, such that

$$[1 + \mathcal{U}(z, [Fz]_{\sigma}, \mathbf{q}\mathbf{r})]\mathcal{U}(\mathbf{w}, [F\mathbf{w}]_{\sigma}, \mathbf{e}\mathbf{r}) \in s([F\mathbf{w}]_{\sigma}, [Fz]_{\sigma}, \mathfrak{z}\mathbf{r}),$$

where $\mathbf{q} + \mathbf{e} = \mathfrak{z} < 1$. Then, F has a unique σ -fuzzy fixed point.

By setting $\mathcal{U}(\mathbf{w}, [F\mathbf{w}]_{\sigma}, \mathbf{e}\mathbf{r}) = \theta$ in Theorem 1, we get the following corollary.

Corollary 3. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be a complete complex valued fuzzy metric space such that, for any sequence, $\{\mathbf{r}_n\}$ in $\wp_{\theta}$ with $\lim_{n \rightarrow \infty} \mathbf{r}_n = \infty$, we have

$$\lim_{r \rightarrow \infty} M(\mathbf{w}, z, \mathbf{r}_n) = \ell, \text{ for all } \mathbf{w}, z \in \mathcal{S}, \mathbf{q} \in (0, 1) \text{ and } \mathbf{r} > 0.$$

Assume that there exists some $\sigma \in (0, 1]$, such that, for each $z \in \mathcal{S}$ such that $[F\mathbf{w}]_{\sigma}$ is a nonempty compact subset of $\mathcal{S}$ for all $\mathbf{w} \in \mathcal{S}$. Let $F : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ be a fuzzy mapping with the least upper bound property, such that

$$\mathcal{U}(z, \mathbf{w}, \mathbf{r}) \in s([F\mathbf{w}]_{\sigma}, [Fz]_{\sigma}, \mathfrak{z}\mathbf{r}),$$

where $\mathfrak{z} < 1$. Then F has a unique σ -fuzzy fixed point.

The task of Theorem 1, can also be obtained for self mapping while relaxing the least upper bound property, with analogous proof:

Corollary 4. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ CVFMS such that, for any sequence, $\{\mathbf{r}_n\}$ in $\wp_{\theta}$ with $\lim_{n \rightarrow \infty} \mathbf{r}_n = \infty$, we have

$$\lim_{r \rightarrow \infty} \mathcal{U}(\mathbf{w}, z, \mathbf{r}_n) = \ell, \text{ for all } \mathbf{w}, z \in \mathcal{S}, \mathbf{q} + \mathbf{e} = \mathfrak{z} \in (0, 1) \text{ and } \mathbf{r} > 0.$$

Let $F : \mathcal{S} \rightarrow \mathcal{S}$ enjoy

$$\mathcal{U}(F\mathbf{w}, Fz, \mathfrak{z}\mathbf{r}) \succeq \frac{[1 + \mathcal{U}(z, Fz, \mathbf{q}\mathbf{r})]\mathcal{U}(\mathbf{w}, F\mathbf{w}, \mathbf{e}\mathbf{r})}{1 + \mathcal{U}(\mathbf{w}, z, \mathbf{r})} + \mathcal{U}(z, \mathbf{w}, \mathbf{r}).$$

Then, F has a unique σ -fuzzy fixed point.

Remark 4. To obtain a unique fixed-point in the the above Corollary, it is sufficient that, to some extent, sequence $\{c_n\} \in \wp_\theta$ such that $\lim_{n \rightarrow \infty} c_n = \infty$, we get $\lim_{n \rightarrow \infty} \mathcal{U}(z, w, c_n) = \ell \quad \forall \quad z, w \in \mathcal{S}$. This state is obtained from the suppositions of Corollary 4 as, for any sequence, $\{c_n\} = \infty$ also for each $z, w \in \mathcal{S}$.

$$\lim_{n \rightarrow \infty} \mathcal{U}(z, w, c_n) \geq \lim_{n \rightarrow \infty} \inf_{v \in \mathcal{S}} \mathcal{U}(z, v, c_n) = \ell.$$

By the use of the above remark, and the rest of the proof of Corollary 4, we can use a more general statement for our main theorem, as follows.

Corollary 5. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be CVFMS. Suppose for any sequence $\{c_n\} \in \wp_\theta$ such that $\lim_{n \rightarrow \infty} c_n = \infty$, we obtained $\lim_{n \rightarrow \infty} \mathcal{U}(z, w, c_n) = \ell \quad \forall \quad z, w \in \mathcal{S}$. Moreover, let for any sequence in $\wp_\theta$ there exists $c_0 \in \mathcal{S}$ with $\lim_{n \rightarrow \infty} c_n = \infty$, we obtained

$$\lim_{n \rightarrow \infty} \inf_{y \in \Xi_{c_0}} \mathcal{U}(c_0, y, c_n) = \ell, \quad (9)$$

where Ξ_{c_0} represents the collection of F -iterates of c_0 . If $F : \mathcal{S} \rightarrow \mathcal{S}$ with:

$$\mathcal{U}(Fw, Fz, \mathfrak{z}r) \succeq \frac{[1 + \mathcal{U}(z, Fz, qr)] \mathcal{U}(w, Fw, er)}{1 + \mathcal{U}(w, z, r)} + \mathcal{U}(z, w, r),$$

where $q + e = \mathfrak{z} \in [0, 1)$. Then F has a unique σ -fuzzy fixed point in $\mathcal{S}$.

Proof. Define a sequence $\{c_n\}$ as $c_n = Fc_{n-1}$, for all $n \in \mathbb{N}$. Thanks to (9), which guarantee that $\{c_n\}$ is a Cauchy sequence, as for $r \in \wp_\theta$,

$$\begin{aligned} q_{n+1} &= \inf_{m > n} \mathcal{U}(c_n + 1, c_m + 1, r) \geq \inf_{m > n} \mathcal{U}(c_0, c_m + 1, \frac{r}{\mathfrak{z}^{n+1}}) \\ &= \inf_{m > n} \mathcal{U}(c_0, F^{m-n}c_0, \frac{r}{\mathfrak{z}^{n+1}}) = \inf_{y \in \Xi_{c_0} > n} \mathcal{U}(c_0, y, \frac{r}{\mathfrak{z}^{n+1}}). \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} q_{n+1} \succeq \lim_{n \rightarrow \infty} \inf_{y \in \Xi_{c_0} > n} \mathcal{U}(c_0, y, \frac{r}{\mathfrak{z}^{n+1}}) = \ell.$$

The proof is in the same fashion of Theorem 1. $\square$

Corollary 6. Let $(\mathcal{S}, \mathcal{U}, \diamond)$ be CVFMS such that, for any sequence $\{r_n\}$ in $\wp_\theta$ with $\lim_{n \rightarrow \infty} r_n = \infty$, we have

$$\lim_{r \rightarrow \infty} M(w, z, r_n) = \ell, \quad \text{for all } w, z \in \mathcal{S}, q \in (0, 1) \text{ and } r > 0.$$

Suppose $L : \mathcal{S} \rightarrow \text{CB}(\mathcal{S})$ be a multivalued mapping with least upper bound property, such that

$$\frac{[1 + \mathcal{U}(z, Lz, qr)] \mathcal{U}(w, Lw, er)}{1 + \mathcal{U}(w, z, r)} + \mathcal{U}(z, w, r) \in s(Lw, Lz, \mathfrak{z}r). \quad (10)$$

Then, L has a unique fixed point.

Proof. Consider the fuzzy mapping $F : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ defined by

$$F(x)(t) = \begin{cases} \sigma & \text{if } t \in Lx \\ \theta & \text{if } t \notin Lx, \end{cases}$$

where $\sigma \in \mathbb{T}^+$. Then,

$$[Fx]_\sigma = \{t : F(x)(t) \geq \sigma\} = Lx.$$

Thus, Theorem 1 can be applied to obtain a fixed point, i.e., there exists $v \in \mathcal{S}$ such that $v \in Fv$ $\square$

Example 2. Let $\mathbb{T}_{\mathbb{R}} = [0, 1]$ and $\mathcal{S} = \mathbb{T}_{\mathbb{R}} \times 0 \cup 0 \times \mathbb{T}_{\mathbb{R}}$. Let $\diamond$ be defined by

$$c_1 \diamond c_2 = (\max\{a' + a'' - 1, 0\}, \max\{b' + b'' - 1, 0\}),$$

for all $c_1 = (a', b'), c_2 = (a'', b'') \in \mathbb{T}$. Define $D : \mathcal{S} \times \mathcal{S} \rightarrow \mathfrak{C}$ by

$$D((z, 0), (w, 0)) = |z - w|(2, 1), D((0, z), (0, w)) = |z - w|(1, \frac{3}{5})$$

and

$$D((z, 0), (0, w)) = D((0, w), (z, 0)) = (2z + w, z + \frac{3}{5}w).$$

Clearly, $(\mathcal{S}, D)$ is a complex valued metric space. Let $\mathcal{U}_D$ be defined by

$$\mathcal{U}_D(u, v, c_1) = \ell - \frac{5D(u, v)}{18 + 5ab} \text{ for all } u, v \in \mathcal{S}, c_1 = (a, b) \in \wp_{\theta}.$$

Then, $(\mathcal{S}, \mathcal{U}_D, \diamond)$ is a complete CVFMS. Let $\sigma \in (0, 1]$ and $G : \mathcal{S} \rightarrow \mathfrak{F}(\mathcal{S})$ be a fuzzy mapping defined by:

$$G(\theta)(t) = \begin{cases} \ell & \text{if } t = \theta \\ \frac{1}{2}\ell & \text{if } \theta < t \leq \frac{w}{50} \\ \theta & \text{if } \frac{w}{50} < t \leq \ell, \end{cases}$$

if $(w, 0) \neq \theta$,

$$G(w)(t) = \begin{cases} \sigma & \text{if } \theta \leq t \leq \frac{w}{75} \\ \frac{\sigma}{3} & \text{if } \frac{w}{75} < t \leq \frac{w}{10} \\ \frac{\sigma}{4} & \text{if } \frac{w}{10} < t \leq \ell, \end{cases}$$

Then, for $w = \theta$, $[G\theta]_{\ell} = \{\theta\}$ and $\forall w, z \neq \theta$, $[Gw]_{\sigma} = [\theta, \frac{w}{75}]$. Thus,

$$\mathcal{W}_w([Gw]_{\sigma}, c_1) = \{\mathcal{U}_D(z, p, \mathbf{r}) : p \in [\theta, \frac{w}{75}]\}.$$

Let $\mathcal{U}_D(z, [Gw]_{\sigma}, \mathbf{r})$ be the least upper bound of $\mathcal{W}_w([Gw]_{\sigma}, \mathbf{r})$. Moreover, if $\omega_{wz} \in \mathbb{T}$ such that

$$\omega_{wz} = \ell - \frac{5D([Gw]_{\sigma}, [Gz]_{\sigma})}{18 + 5ab},$$

then,

$$s(\mathcal{U}_D([Gw]_{\sigma}, [Gz]_{\sigma}, \mathbf{r})) = \{\omega \in \mathbb{T} : \omega_{yz} \succeq \omega\}.$$

Consider

$$\begin{aligned} \omega_{wz} &= \ell - \frac{5D([Gw]_{\sigma}, [Gz]_{\sigma})}{18 + 5ab} \\ &= \ell - \frac{5|\frac{w}{75} - \frac{z}{75}|}{18 + 5ab} \\ &\succeq \ell - \frac{5|w - z|}{18 + 5ab} \\ &= \ell - \frac{5D(w, z)}{18 + 5ab} \\ &= \mathcal{U}_D(w, z, \mathbf{r}) \end{aligned}$$

Therefore, we have

$$\mathcal{U}_D(z, \mathbf{w}, \mathbf{r}) \in s([F\mathbf{w}]_\sigma, [Fz]_\sigma, \mathfrak{z}\mathbf{r}).$$

Hence, all conditions of Corollary 3 are satisfied by G ; therefore, there exists $(0, 0) \in \mathcal{S}$, such that $(0, 0) \in [G(0, 0)]_\sigma$.

4. Applications Fuzzy Caputo Fractional Volterra–Fredholm Integro-Differential Equations

Consider initial value problem

$$\begin{cases} ({}_{gH}D_*^\gamma u)(t) = f(t, u(t), \mathcal{K}u(t), \mathcal{H}u(t)), t \in J = [t_0, T] \\ u(t_0) = u_0 \in \mathbb{R}_{\mathcal{F}}, \end{cases} \quad (11)$$

where $0 < \gamma < 1$ is a real number and ${}_{gH}D_*^\gamma$ denote the Caputo fractional generalized derivative of order γ , $f: J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ is continuous in t , which satisfies some assumptions that will be specified later, and

$$\mathcal{K}u(t) = \int_{t_0}^t \mathcal{K}(t, ru(r))dr, \quad \mathcal{H}u(t) = \int_{t_0}^T \mathcal{H}(t, ru(r))dr, \quad (12)$$

This problem is equivalent to the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\gamma)} \int_{t_0}^t (t-r)^{\gamma-1} f(r, u(r), \mathcal{K}u(r), \mathcal{H}u(r))dr, \quad (13)$$

where u is a fuzzy valued ${}^{cf}[(i) - gH]$ -differentiable on J .

For a detailed study of problem (11), we recommend that the readers look at [8].

To study our results for the existence of a fixed point, we define the integral operator $\mathcal{T}$ as

$$\mathcal{T}u(t) = u_0 + \frac{1}{\Gamma(\gamma)} \int_{t_0}^t (t-r)^{\gamma-1} f(r, u(r), \mathcal{K}u(r), \mathcal{H}u(r))dr. \quad (14)$$

For the sake of simplicity, we mentioned $\frac{1}{\Gamma(\gamma)} \int_{t_0}^t (t-r)^{\gamma-1} f(r, u(r), \mathcal{K}u(r), \mathcal{H}u(r))dr = \mathcal{F}_u$. Now, we study the existence and uniqueness of solutions to problem (11). To proceed, we use the following hypotheses:

Hypothesis 1 (H1). $f: J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ is continuous such that

$$\mathcal{H}_{(\mathcal{T}u\mathcal{T}v)}(t) \preceq \frac{(1 + \mathcal{A}_{(uv)}(t))\mathcal{B}_{(uv)}(t)}{1 + \mathcal{G}_{(uv)}(t)} + \mathcal{G}_{(uv)}(t), \quad (15)$$

where

$$\begin{aligned} \mathcal{H}_{(\mathcal{T}u\mathcal{T}v)}(t) &= \frac{c}{c + \|\mathcal{F}_u - \mathcal{F}_v\|_\infty} \ell \\ \mathcal{A}_{(uv)}(t) &= \frac{c}{\|u - u_0 + \mathcal{F}_u\|_\infty} \ell \\ \mathcal{B}_{(uv)}(t) &= \frac{c}{\|v - v_0 + \mathcal{F}_v\|_\infty} \ell \\ \mathcal{G}_{(uv)}(t) &= \frac{c}{\|u - v\|_\infty} \ell \end{aligned}$$

Then, the initial value problem (11) has only one solution.

Proof. Consider $\mathcal{S} = ([0, T], \mathbb{R}_{\mathcal{F}})$ with the metric

$$\mathcal{D}(\hat{x}, \hat{y}) = \max_{t_0 \leq t \leq T} (\hat{x}(t) - \hat{y}(t))_\infty.$$

Let $\mathcal{U} : \mathcal{S} \times \mathcal{S} \times \wp_{\theta} \rightarrow \mathbb{T}$ defined by

$$\mathcal{U}(\hat{x}, \hat{y}, c) = \frac{c}{c + \mathcal{D}(\hat{x}, \hat{y})} \ell$$

for $c = (a, b) \in \wp_{\theta}$. It is obvious that $(\mathcal{S}, \mathcal{U}, *)$ is CVFMS. Consider (14), define the integral operator $\mathcal{T} : \mathcal{S} \rightarrow \mathcal{S}$. For $\hat{x}, \hat{y} \in \mathcal{S}$, we have

$$[\mathcal{T}\hat{x}]_{\sigma} = \{u \in [t, t_0] : u(j) = u_0 + \mathcal{F}_u \succeq \sigma, j \in [t, T], \mathcal{T}\hat{x}(u) \succeq \sigma\},$$

$$\begin{cases} \mathcal{U}(u, v, c) = \frac{c}{c + \max_{t_0 \leq t} \|u - v\|_{\infty}} \ell \\ \mathcal{U}(u, [\mathcal{T}u]_{\sigma}, qc) = \frac{c}{c + \max_{t_0 \leq t} \|u - u_0 + \mathcal{F}_u\|_{\infty}} \ell \\ \mathcal{U}(v, [\mathcal{T}v]_{\sigma}, ec) = \frac{c}{c + \max_{t_0 \leq t} \|v - v_0 + \mathcal{F}_v\|_{\infty}} \ell \\ \mathcal{U}([\mathcal{T}u]_{\sigma}, [\mathcal{T}v]_{\sigma}, 3c) = \frac{3c}{3c + \max_{t_0 \leq t} \|u_0 + \mathcal{F}_u - v_0 + \mathcal{F}_v\|_{\infty}} \ell \end{cases} \quad (16)$$

From assumption (15), we obtain, for each $t \in [t, t_0]$

$$\begin{aligned} \mathcal{H}_{(\mathcal{T}u\mathcal{T}v)}(t) &\succeq \frac{(1 + \mathcal{A}_{(uv)}(t))\mathcal{B}_{(uv)}(t)}{1 + \mathcal{G}_{(uv)}(t)} + \mathcal{G}_{(uv)}(t) \\ &= \frac{(1 + \frac{c}{\|u - u_0 + \mathcal{F}_u\|_{\infty}} \ell) \frac{c}{\|v - v_0 + \mathcal{F}_v\|_{\infty}} \ell}{1 + \frac{c}{\|u - v\|_{\infty}} \ell} + \frac{c}{\|u - v\|_{\infty} \ell} \\ &\succeq \frac{(1 + \frac{c}{c + \max_{t_0 \leq t} \|u - u_0 + \mathcal{F}_u\|_{\infty}} \ell) \frac{c}{c + \max_{t_0 \leq t} \|v - v_0 + \mathcal{F}_v\|_{\infty}} \ell}{1 + \frac{c}{c + \max_{t_0 \leq t} \|u - v\|_{\infty}} \ell} + \frac{c}{c + \max_{t_0 \leq t} \|u - v\|_{\infty} \ell}. \end{aligned}$$

This yields

$$\begin{aligned} \frac{3c}{3c + \max_{t_0 \leq t} \|\mathcal{F}_u - \mathcal{F}_v\|_{\infty}} \ell &\succeq \frac{(1 + \frac{c}{c + \max_{t_0 \leq t} \|u - u_0 + \mathcal{F}_u\|_{\infty}} \ell) \frac{c}{c + \max_{t_0 \leq t} \|v - v_0 + \mathcal{F}_v\|_{\infty}} \ell}{1 + \frac{c}{c + \max_{t_0 \leq t} \|u - v\|_{\infty}} \ell} \\ &+ \frac{c}{c + \max_{t_0 \leq t} \|u - v\|_{\infty} \ell}. \end{aligned}$$

By using (16), we get

$$\mathcal{U}([\mathcal{T}u]_{\sigma}, [\mathcal{T}v]_{\sigma}, 3c) \succeq \left(\frac{\mathcal{U}(u, [\mathcal{T}u]_{\sigma}, c) \mathcal{U}(v, [\mathcal{T}v]_{\sigma}, c)}{1 + \mathcal{U}(u, v, c)} + \mathcal{U}(u, v, c) \right).$$

Thus, all conditions of Theorem 1 hold. Therefore, there exists only one fixed point of $\mathcal{T}$ in $\mathcal{S}$, and so there exists a unique solution to the system (11). $\square$

5. Discussion and Conclusions

In many situations, classical models fail to describe the features of natural phenomena such as the dynamics of viscoelastic materials such as polymers, the atmospheric diffusion of pollution, and signal transmissions through strong magnetic fields. In such situations, fuzzy concepts are the best solution. This concept has the ability to model difficult uncertainties with ease. In our research work, we considered a complex fuzzy set in fuzzy metric spaces, which is more general than classical fuzzy metric fixed-point theory. We obtained complex fuzzy versions of rational type contractions via the least upper bound property in the new approach (complex valued fuzzy metric spaces). We also discussed its applications in multivalued mappings. Then, we proposed an existence theorem for a unique solution to fractional Volterra–Fredholm integro-differential equations under generalized fuzzy

Caputo Hukuhara differentiability using the technique of a fixed point. As an application, we provided an illustrative example, which shows the applicability and validity of the approach we used in this article.

Our results will open doors for researchers working on rational type contraction in complex valued fuzzy spaces. The studied results and their applications can be extended to functional, differential and integral equations via numerical experiment.

Author Contributions: Conceptualization, writing—original draft preparation, writing—review and editing, supervision, investigation, H. and M.S.; methodology, formal analysis, funding acquisition, visualization T.A. and N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We are very grateful to the editor and unbiased arbitrator for his/her prudent interpretation and proposition which refined the excellency of this manuscript. The last two authors would like to thank Prince Sultan University for funding this work through the research lab TAS.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control.* **1965**, *8*, 338–353. [\[CrossRef\]](#)
2. Ahmad, N.; Ullah, A.; Ullah, A.; Ahmad, S.; Shah, K.; Amtiaz, A. On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation. *Alex. Eng. J.* **2021**, *60*, 1827–1838. [\[CrossRef\]](#)
3. Allahviranloo, T.; Armand, A.; Gouyandeh, Z. Fuzzy fractional differential equations under generalized fuzzy Caputo derivative. *J. Intell. Fuzzy Syst.* **2014**, *26*, 1481–1490. [\[CrossRef\]](#)
4. Hoa, N.V. Fuzzy fractional functional integral and differential equations. *Fuzzy Sets Syst.* **2015**, *280*, 58–90. [\[CrossRef\]](#)
5. Hoa, N.V. Fuzzy fractional functional differential equations under Caputo gH-differentiability. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *22*, 1134–1157. [\[CrossRef\]](#)
6. Hoa, N.V.; Vu, H.; Duc, T.M. Fuzzy fractional differential equations under Caputo–Katugampola fractional derivative approach. *Fuzzy Sets Syst.* **2019**, *375*, 70–99. [\[CrossRef\]](#)
7. Vu, H.; Hoa, N.V. Applications of contractive-like mapping principles to fuzzy fractional integral equations with the kernel ψ -functions. *Soft Comput.* **2020**, *24*, 18841–18855. [\[CrossRef\]](#)
8. Allahviranloo, T.; Armand, A.; Gouyandeh, Z.; Ghadiri, H. Existence and uniqueness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equations. *J. Fuzzy Set Valued Anal.* **2013**, *2013*, 1–9. [\[CrossRef\]](#)
9. Alkan, S.; Hatipoglu, V. Approximate solution of Volterra-Fredholm integro-differential equations of fractional order. *Tablasi Math. J.* **2017**, *10*, 1–13. [\[CrossRef\]](#)
10. Hamoud, A.D.; Ghadle, K.P. The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations. *Korean J. Math.* **2017**, *3*, 323–334. [\[CrossRef\]](#)
11. Hamoud, A.D.; Ghadle, K.P. The combined modified of Laplace with Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro-differential equations. *J. Korean Soc. Ind. Appl. Math.* **2017**, *21*, 17–28. [\[CrossRef\]](#)
12. Ma, X.; Haung, C. Numerical solution of fractional integro-differential equations by hybrid collection method. *Appl. Math. Comput.* **2013**, *12*, 6750–6760. [\[CrossRef\]](#)
13. Mital, R.; Nigam, R. Solution of fractional integro-differential equations by Adomian decomposition method. *Int. J. Appl. Math. Mech.* **2008**, *2*, 87–94. [\[CrossRef\]](#)
14. Ramot, D.; Milo, F.; Friedman, M.; Kandel, A. Complex fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2002**, *2*, 171–186. [\[CrossRef\]](#)
15. Yazdanbakhsh, O.; Dick, S. A systematic review of complex fuzzy sets and logic. *Fuzzy Sets Syst.* **2018**, *338*, 1–22. [\[CrossRef\]](#)
16. Nadler, S.B. Multivalued contraction mappings. *Pac. J. Appl. Math.* **1969**, *30*, 475–488.
17. Heilpern, S. Fuzzy fixed point theorems. *J. Math. Anal.* **1981**, *83*, 566–569.
18. Butnariu, D. Fixed points for fuzzy mappings. *Fuzzy Sets Syst.* **1982**, *7*, 191–207.
19. Weiss, M.D. Fixed points, separation and induced topologies for fuzzy sets. *J. Math. Anal. Appl.* **1975**, *1*, 142–150.
20. Kramosil, I.; Michalek, J. Fuzzy metric and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344. [\[CrossRef\]](#)
21. Garbiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1988**, *27*, 385–399. [\[CrossRef\]](#)
22. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [\[CrossRef\]](#)

-
23. Dosenović, T.; Rakić, D.; Carić, B.; Radenović, S. Multivalued generalizations of fixed point results in fuzzy metric spaces. *Nonlinear Anal. Model. Control* **2016**, *2*, 211–222. [[CrossRef](#)]
 24. Shukla, S.; Rodríguez-López, R.; Abbas, M. Fixed point results for contractive mappings in complex valued fuzzy metric spaces. *Fixed Point Theory* **2018**, *19*, 751–774. [[CrossRef](#)]
 25. Dass, B.K.; Gupta, S. An extension of Banach contraction principle through rational expression. *Indian J. Pure Appl. Math.* **1975**, *6*, 1455–1458. [[CrossRef](#)]