

Article

# Analytical Solution of Laterally Loaded Free-Head Long Piles in Elasto-Plastic Cohesive Soils

Ayman Abd-Elhamed <sup>1,2</sup>

<sup>1</sup> Physics and Engineering Mathematics Department, Faculty of Engineering-Mattaria, Helwan University, Cairo 11718, Egypt; Ayman.abdelhamed@ksiu.edu.eg

<sup>2</sup> Faculty of Engineering, King Salman International University, Ras-Sedr 46512, Egypt

**Abstract:** This research study presents a closed form solution of responses of laterally loaded long piles embedded on cohesive soils with a constant subgrade modulus. The surrounding soil medium is modelled as elastic-perfectly plastic. The closed form solution is derived by solving the governing differential equation of the pile–soil system. The most popular numerical computation software package MATLAB is utilized for the implementation of solutions. The provided analytical method reliably calculates the pile head deflection and bending moment required for engineering design purposes. Results are discussed and verified with solutions of an equivalent three-dimensional finite element (FE) model developed using ANSYS software. It was concluded that the proposed analytical model could efficiently provide the exact solution of embedded piles in elasto-plastic cohesive soil under lateral loads.

**Keywords:** analytical solution; laterally loaded long piles; cohesive soil; elasto-plastic soil; ANSYS software



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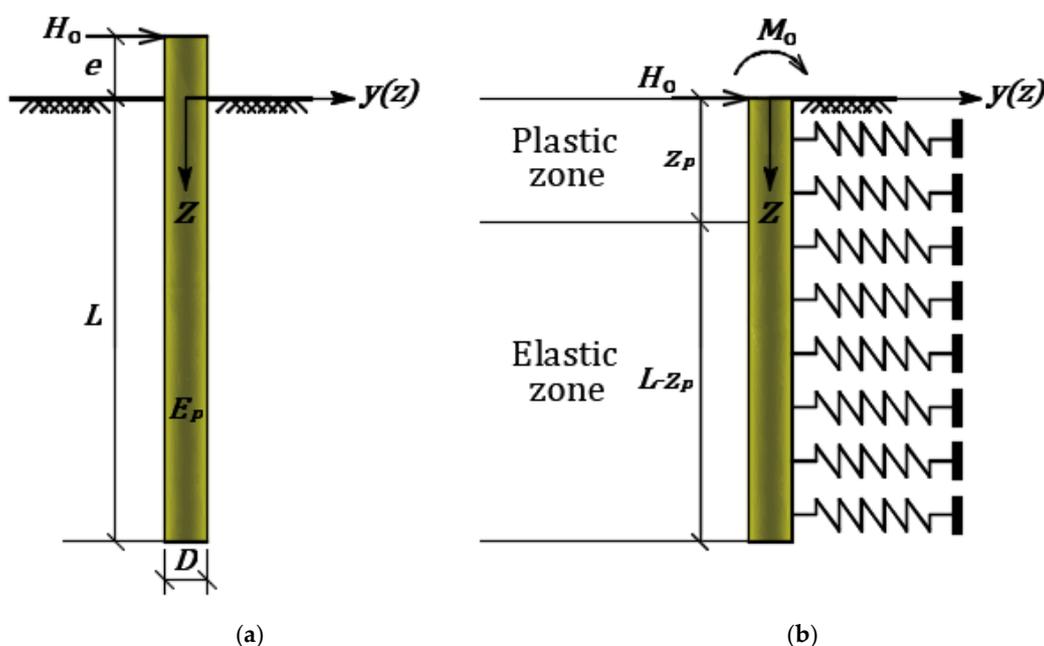
## 1. Introduction

When a shallow soil stratum is highly compressible and of low bearing capacity to support the load transmitted by the superstructure, piles are principally used to transmit the loads to the underlying stiffer soil strata of a higher bearing capacity. Various precious approaches have been proposed to analyze the laterally loaded piles, including the subgrade reaction approach [1,2] and p-y approach [3,4]. In the subgrade reaction approach, the supporting soil is idealized as a series of lateral elastic springs and the relationship between lateral applied force and lateral deflection pattern of a pile is treated as linear- elastic over the depth of the pile. Moreover, the subgrade reaction modulus is obtained through fitting with relevant numerical solutions [5,6]. In the p-y approach the soil is conveniently simulated by a series of independent non-linear springs varying with the depth, and the soil–pile interface responses are captured by representing the soil nonlinear springs with p-y curves along the pile length. Several researchers on the piles subjected to lateral loads continued to be unabated in which the soil is considered as a continuum-based model. In these studies, different approaches such as the finite element approach [7], boundary element approach [8], and finite difference approach [9], have been introduced to predict the response of laterally loaded piles. A large number of studies have examined the behaviour of the soil–pile interaction of laterally loaded piles considering that the relationship between soil lateral reaction and pile deflection is linear [10], and the lateral load behaviour of soil is non-linear. Therefore, the development of algebraic expressions via a closed form solution is proposed in this study to predict the non-linear responses of laterally loaded piles in clay strata. The proposed analytical solution in the current study provides a better approach for structural designers to simply solve for the lateral soil reaction, displacement, shear force, and bending moment responses of laterally loaded free-head long piles embedded in homogeneous cohesive soils with a constant lateral subgrade reaction with depth. Consequently, the techniques can be easily applied in

practice as an alternative approach to analyse and design laterally loaded long piles. In addition, the proposed solution incorporates the contribution of the Winkler foundation model with the linear elastic-perfect-plastic  $p$ - $y$  soil response. The proposed analytical solution is verified by comparing the results obtained for the pile's deflection and induced bending moment with numerical results using the finite element software ANSYS. The accuracy of the location at which the maximum bending moment occurs, and the magnitude of that maximum bending moment along the pile are further verified analytically by comparing them to those obtained from the FE analysis. In addition, a numerical example is introduced to investigate the effect of the horizontal loads on the depth of the plastic zone.

### 2. Problem Definition and Formulation

A free-head cylindrical pile of length  $L$ , diameter  $D$ , and Young's modulus  $E_p$  embedded in a homogeneous clay soil and subjected to lateral load  $H_0$  at an eccentricity,  $e$  above ground surface is schematically shown in Figure 1a. In order to simulate the pile–soil interaction, the pile shaft is assumed to be perfectly glued to the surrounding soils suggesting that there is no relative movement along the pile–soil interface. Furthermore, a series of springs distributed along the pile shaft are used to characterize that interaction for the pile subjected to the lateral load  $H_0$  and bending moment,  $M_0 = H_0e$  at the ground surface, as shown in Figure 1b.



**Figure 1.** (a) Schematic diagram of pile under lateral load, (b) illustration of lateral soil reaction distribution along the pile shaft.

Free-head and fixed-tip boundary conditions are assumed. Therefore, the pile head is unrestrained to rotate and move laterally while the pile base is fully fixed against translation and rotation. Unquestionably, the soil behaves non-linearly under higher lateral load. This nonlinearity is described by the well-known ideal elasto-plastic  $p$ - $y$  relationship at any depth. This model is shown in Figure 2a in which  $k$  is the gradient of the  $p$ - $y$  curve that offers the soil modulus at any point  $z$  below the surface along the pile,  $p_u$  is the limiting soil horizontal resistance per unit depth, and  $y_y$  is the threshold of soil displacement above which the soil is initiated to yield. On the other hand, the soil near the ground surface presumably yields firstly and propagates downward to a depth called plastic depth,  $z_p$  [11]. Below plastic depth, the soil reaction is linear and still in the elastic state. Accordingly, the lateral soil reaction along the pile shaft may be characterized by two zones: the plastic zone above which extends to a depth  $z_p$  depending on both the applied lateral force and its point

of application, and the bottom elastic zone below this depth (see Figures 1b and 2b) [12]. It is worth noting that the limiting lateral force for clay soils increases with depth from  $2c_uD$  to  $3c_uD$  at ground surface up to  $8c_uD$  to  $12c_uD$  at a depth of about  $3D$ , and beyond this depth it keeps constant [13]. Where  $c_u$  is the undrained shear strength of the soil,  $D$  is the pile diameter. Consequently, the limiting lateral force is proposed to equal  $9c_uD$  to the end of the plastic zone (see Figure 2b) [14].

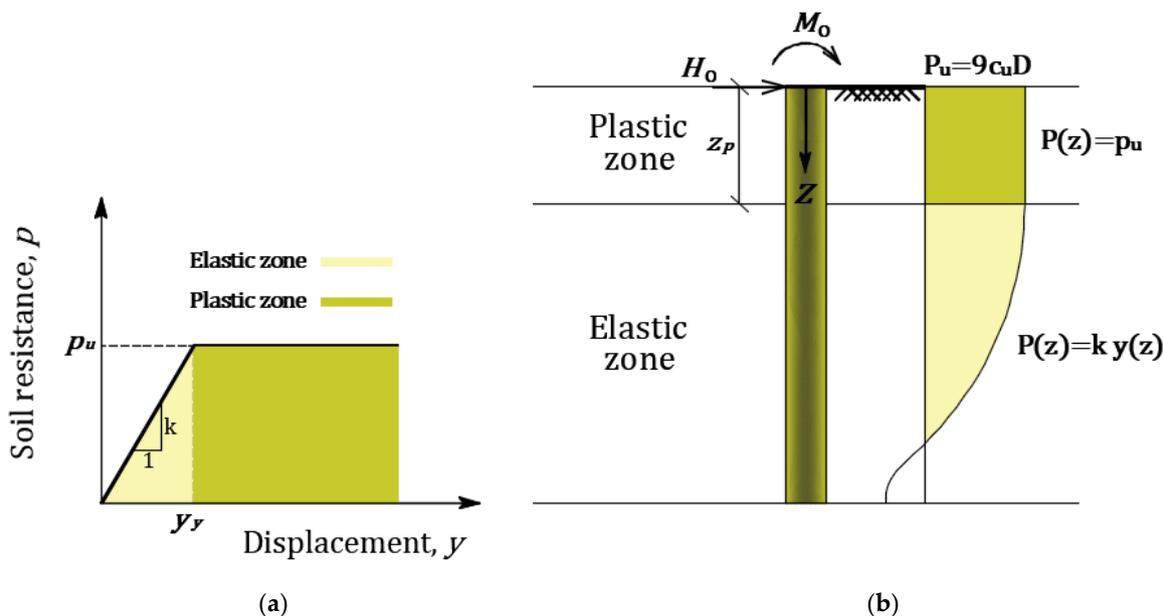


Figure 2. (a) Elasto-plastic behavior of the soil, (b) lateral soil reaction distribution [14].

### 3. Mathematical Formulation

In the case of the pile subjected to lateral forces, it tries to push the surrounding soil horizontally in the loading direction to its limit state. Subsequently, the surface soils tend to yield, and the cracks tend to be very shallow. The lateral soil reaction along the pile shaft may be characterized by two zones: the top plastic zone which extends downward to depth  $z_p$  depending on both applied lateral force and its point of application, and the bottom elastic zone below this depth (see Figure 2).

The governing flexural equation for the response of the soil–pile system according to Winkler’s foundation concept can be written as:

$$EI \frac{d^4 y}{dz^4} + P(z) = 0, \tag{1}$$

In the above,  $E$  is the modulus of elasticity of concrete,  $I$  is the effective moment of inertia,  $y$  is the lateral deflection of the pile, and  $p(z)$  is the soil reaction at any point at a depth  $z$  along the axis of the pile which may be described for the elasto-plastic model as:

$$P(z) = \begin{cases} k \times y & \rightarrow z \geq z_p \\ p_u & \rightarrow z \leq z_p \end{cases}, \tag{2}$$

where  $k$  is the subgrade reaction modulus of the soil,  $z_p$  is the depth of the plastic zone, and  $p_u$  is the lateral limit force below which the soil reaction has a linear relationship with lateral pile deflection.

The integration of Equation (1) considering the boundary and continuity conditions presented in Tables 1 and 2 leads to the analytical solution for piles subjected to lateral loads and embedded in linear-plastic soil.

**Table 1.** Boundary conditions used for elastic solution.

Boundary Condition	Equations
Shear force $V = H_0$ at $z = 0$	$EIY'''(0) = H_0$
Bending moment $M = M_0$ at $z = 0$	$EIY''(0) = M_0$

**Table 2.** Boundary conditions used for elasto-plastic solution.

Boundary Condition	Equations
Shear force $V = H_0$ at $z = 0$	$EIY'''(0) = H_0$
Bending moment $M = M_0$ at $z = 0$	$EIY''(0) = M_0$
Slope from plastic zone equal slope from elastic zone at $z = z_p$ (compatibility for slope)	$Y'(z_p)_{plastic} = y'(z_p)_{elastic}$
Deflection from plastic zone equal deflection from elastic zone at $z = z_p$ (compatibility for deflection)	$Y(z_p)_{plastic} = y(z_p)_{elastic}$
Bending moment from plastic zone equal bending moment from elastic zone at $z = z_p$	$y''(z_p) = \frac{M(z_p)}{EI}$
Shear force from plastic zone equal shear force from elastic zone at $z = z_p$	$y'''(z_p) = \frac{V(z_p)}{EI}$
For infinitely long piles, the pile is fixed at the bottom	$y(L) = 0$
	$y'(L) = 0$
	$y''(L) = 0$
	$y'''(L) = 0$

3.1. Elastic Soil Behaviour

Based on the assumption that the applied lateral force does not exceed the ultimate load  $p_u$ , the relationship between lateral soil resistance per unit pile length and pile lateral deflection under working loads is linear–elastic. Consequently, under this assumption the governing equation of motion can be expressed as:

$$EI \frac{d^4y}{dz^4} + k \times y = 0, \tag{3}$$

If one assumes that the modulus of subgrade reaction  $k$  in clay soil is independent of depth [15], thereby the general solution of Equation (3) is of the form:

$$y(z) = e^{-\beta z} \{g_1 \sin(\beta z) + g_2 \cos(\beta z)\} + e^{\beta z} \{g_3 \sin(\beta z) + g_4 \cos(\beta z)\}, \tag{4}$$

where  $\beta$  is the reciprocal of the characteristic length and  $g_i, i = 1, 2, 3, 4$  are four independent constant parameters.

Assuming that the pile is infinite and the deflection at the tip of the long piles is negligible, then the lateral deflections at working loads can be simplified as

$$y(z) = e^{-\beta z} \{g_1 \sin(\beta z) + g_2 \cos(\beta z)\}, \tag{5}$$

Further higher derivatives for Equation (5) yields

$$y'(z) = \beta e^{-\beta z} \{(g_1 - g_2) \cos(\beta z) + (g_1 + g_2) \sin(\beta z)\}, \tag{6}$$

$$y''(z) = -2\beta^2 e^{-\beta z} \{g_1 \cos(\beta z) - g_2 \sin(\beta z)\}, \tag{7}$$

$$y'''(z) = 2\beta^3 e^{-\beta z} \{(g_1 + g_2) \cos(\beta z) + (g_1 - g_2) \sin(\beta z)\}, \tag{8}$$

$$y''''(z) = -4\beta^4 e^{-\beta z} \{g_1 \sin(\beta z) + g_2 \cos(\beta z)\}, \tag{9}$$

By substituting Equations (5) and (9) into Equation (3),  $\beta$  for the free-load pile can be assigned as:

$$\beta = 1/L = \sqrt[4]{kD/4EI}, \tag{10}$$

where  $EI$  is stiffness of pile section and  $D$  is width or diameter of pile.

Two constant parameters  $g_1$  and  $g_2$  are obtained for piles subjected to applied horizontal load  $H_0$  and applied bending moment  $M_0$  by imposing the equilibrium conditions for both the bending moment and shear force at pile top using Table 1. Thereupon, by substituting the boundary conditions into Equations (7) and (8), two integration coefficients  $g_1$  and  $g_2$  can be determined as:

$$g_1 = \frac{1}{2\beta^2 EI} \{-M_0\}, \tag{11}$$

$$g_2 = \frac{1}{2\beta^2 EI} \left\{ M_0 + \frac{H_0}{\beta} \right\}, \tag{12}$$

The deflection along the laterally loaded pile can be obtained by substituting Equations (11) and (12) into Equation (5) as

$$y(z) = \frac{e^{-\beta z}}{2\beta^2 EI} \left\{ -M_0 \sin(\beta z) + \left( M_0 + \frac{H_0}{\beta} \right) \cos(\beta z) \right\}, \tag{13}$$

Consequently, the bending moment  $M$ , and the shear force  $V$  are found by successive differentiation of Equation (13) as:

$$M(z) = e^{-\beta z} \left\{ M_0 \cos(\beta z) + \left( M_0 + \frac{H_0}{\beta} \right) \sin(\beta z) \right\}, \tag{14}$$

$$V(z) = e^{-\beta z} \{ H_0 \cos(\beta z) - (2\beta M_0 + H_0) \sin(\beta z) \}, \tag{15}$$

By setting  $V(z) = 0$ , the point of zero shear  $z_0$  can be written as:

$$z_0 = \frac{1}{\beta} \tan^{-1} \frac{H_0}{2\beta M_0 + H_0}, \tag{16}$$

Therefore, the maximum bending moment can be obtained as follows:

$$M_{max} = e^{-\beta z_0} \left\{ M_0 \cos(\beta z_0) + \left( M_0 + \frac{H_0}{\beta} \right) \sin(\beta z_0) \right\}, \tag{17}$$

### 3.2. Elasto-Plastic Soil Behaviour

Admittedly, the soil behavior under higher lateral load levels is predominantly non-linear. A simple approach proposed by Madhav et al. [16] reasonably captures this nonlinearity through considering the elastic-plastic subgrade model. Consequently, the non-linear responses for long piles under lateral loading may be obtained through two zones as follows.

#### 3.2.1. Plastic Zone $z \leq z_p$

$$p(z) = -EI \frac{d^4 Y}{dz^4} = -p_u, \tag{18}$$

Through several integrations for Equation (18), the key responses of the pile in the plastic zone can be expressed as follows:

$$V(z) = EI \frac{d^3 Y}{dz^3} = -p_u z + C_1, \tag{19}$$

$$M(z) = EI \frac{d^2 Y}{dz^2} = -p_u \frac{z^2}{2} + C_1 z + C_2, \tag{20}$$

$$\varphi(z) = \frac{dY}{dz} = \frac{1}{EI} \left[ -p_u \frac{z^3}{6} + C_1 \frac{z^2}{2} + C_2 z + C_3 \right], \tag{21}$$

$$Y(z) = \frac{1}{EI} \left[ -p_u \frac{z^4}{24} + C_1 \frac{z^3}{6} + C_2 \frac{z^2}{2} + C_3 z + C_4 \right], \tag{22}$$

3.2.2. Elastic Zone  $z \geq z_p$

$$y(z) = e^{-\beta z} \{C_5 \sin(\beta z) + C_6 \cos(\beta z)\}, \tag{23}$$

$$y'(z) = \beta e^{-\beta z} \{(C_5 - C_6) \cos(\beta z) - (C_5 + C_6) \sin(\beta z)\}, \tag{24}$$

$$y''(z) = -2\beta^2 e^{-\beta z} \{C_5 \cos(\beta z) - C_6 \sin(\beta z)\}, \tag{25}$$

$$y'''(z) = 2\beta^3 e^{-\beta z} \{(C_5 + C_6) \cos(\beta z) + (C_5 - C_6) \sin(\beta z)\}, \tag{26}$$

$$y''''(z) = -4\beta^4 e^{-\beta z} \{C_5 \sin(\beta z) + C_6 \cos(\beta z)\}, \tag{27}$$

The solution of the above equations requires boundary conditions to obtain the unknown parameters  $C_i$ ,  $i = 1, 2, 3, 4, 5, 6$  and  $z = z_p$ . All boundary conditions associated with the above equations must be determined by recourse to the boundary conditions in Table 2. The equilibrium for the bending moment and shear force at  $z = 0$  and compatibility for displacement and slope at  $z = z_p$  is considered.

By considering equilibrium for shear at the pile head, the undetermined constants  $C_1$  can be determined as:

$$C_1 = H_0, \tag{28}$$

Subsequently, by imposing the equilibrium for the bending moment at the pile head, the undetermined constants  $C_2$  can be expressed as:

$$C_2 = M_0, \tag{29}$$

Then, the plastic zone develops at a depth  $z_p$  of the form:

$$z_p = \left( \frac{H_0^2}{p_u^2} + 2 \frac{M_0}{p_u} \right) 0.5 + \frac{H_0}{p_u} - \frac{1}{\beta}, \tag{30}$$

Applying the compatibility for slope at  $z = z_p$  by equating Equation (21) with Equation (24) yields

$$C_3 = -EI\beta \left\{ y(z_p) + \frac{M(z_p)}{2\beta^2 EI} \right\} + p_u \frac{z_p^3}{6} - H_0 \frac{z_p^2}{2} - M_0 z_p, \tag{31}$$

where

$$y(z_p) = \frac{p_u}{k}, \tag{32}$$

and

$$M(z_p) = -p_u \frac{z_p^2}{2} + H_0 z_p + M_0, \tag{33}$$

Similarly, applying the compatibility for deflection at  $z = z_p$  by equating Equation (22) with Equation (23) yields,

$$C_4 = EI y(z_p) + p_u \frac{z_p^4}{24} - H_0 \frac{z_p^3}{6} - M_0 \frac{z_p^2}{2} - C_3 z_p, \tag{34}$$

Based on Equations (21) and (22), the rotation  $\varphi_{top}$  and horizontal deflection  $Y_{top}$  of the pile at  $z = 0$  can be expressed as

$$\varphi_{top} = -\beta \left\{ y(z_p) + \frac{M(z_p)}{2\beta^2 EI} \right\} + p_u \frac{z_p^3}{6} - H_0 \frac{z_p^2}{2} - M_0 z_p, \tag{35}$$

$$Y_{top} = y(z_p) + \frac{1}{EI} \left\{ p_u \frac{z_p^4}{24} - H_0 \frac{z_p^3}{6} - M_0 \frac{z_p^2}{2} - C_3 z_p \right\}, \tag{36}$$

Furthermore, Equation (26) can be re-expressed as:

$$y'''(z) = \underbrace{2\beta^3 e^{-\beta z} \{C_5 \cos(\beta z) - C_6 \sin(\beta z) + C_5 \sin(\beta z) + C_6 \cos(\beta z)\}}_{(-\beta)y''(z)},$$

$$y'''(z) = \underbrace{(-\beta) \left( -2\beta^2 e^{-\beta z} \right) \{C_5 \cos(\beta z) - C_6 \sin(\beta z)\}}_{\left( \frac{1}{2\beta} \right) y''''(z)} - \underbrace{\left( \frac{1}{2\beta} \right) \left( -4\beta^4 e^{-\beta z} \right) \{C_5 \sin(\beta z) + C_6 \cos(\beta z)\}}_{\left( \frac{1}{2\beta} \right) y''''(z)} \tag{37}$$

$$y'''(z) = -\beta y''(z) + 2\beta^3 y(z), \tag{38}$$

Applying equilibrium principles at  $z = z_p$  yields

$$C_5 = e^{\beta z_p} \left\{ y(z_p) \sin(\beta z_p) - \frac{M(z_p)}{2\beta^2 EI} \cos(\beta z_p) \right\}, \tag{39}$$

and  $C_6$  is given by

$$C_6 = e^{\beta z_p} \left\{ y(z_p) \cos(\beta z_p) + \frac{M(z_p)}{2\beta^2 EI} \sin(\beta z_p) \right\}, \tag{40}$$

There are two cases that may arise to obtain the maximum value of the bending moment and the corresponding position.

**Case 1: the bending moment occurs in the elastic region.**

In this case, the largest bending moment value is assumed to occur in the elastic region. Substituting into Equation (26) with  $z = z_0$  and equating with zero yields

$$y'''(z_0) = 2\beta^3 EI e^{-\beta z_0} \left\{ e^{\beta z_p} y(z_p) [\sin(\beta z_p) \cos(\beta z_0) + \cos(\beta z_p) \cos(\beta z_0) + e^{\beta z_p} \frac{M(z_p)}{2\beta^2 EI} [\sin(\beta z_p) \cos(\beta z_0) - \cos(\beta z_p) \cos(\beta z_0)] + e^{\beta z_p} y(z_p) [\sin(\beta z_p) \sin(\beta z_0) - \cos(\beta z_p) \sin(\beta z_0)] - e^{\beta z_p} \frac{M(z_p)}{2\beta^2 EI} [\sin(\beta z_p) \sin(\beta z_0) + \cos(\beta z_p) \sin(\beta z_0)] \right\} = 0,$$

The depth of maximum bending moment which occurs in elastic zone can be expressed as

$$z_0 = \frac{1}{\beta} \tan^{-1} \left\{ \frac{\sin(\beta z_p) \left[ y(z_p) + \frac{M(z_p)}{2\beta^2 EI} \right] + \cos(\beta z_p) \left[ y(z_p) - \frac{M(z_p)}{2\beta^2 EI} \right]}{\cos(\beta z_p) \left[ y(z_p) + \frac{M(z_p)}{2\beta^2 EI} \right] - \sin(\beta z_p) \left[ y(z_p) - \frac{M(z_p)}{2\beta^2 EI} \right]} \right\}, \tag{41}$$

Furthermore,

$$M_{max} = -2\beta^2 EI e^{-\beta z_0} \{C_5 \cos(\beta z_0) - C_6 \sin(\beta z_0)\}, \tag{42}$$

**Case 2: the bending moment occurs in the plastic region.**

This case assumes the occurrence of largest bending moment value is in the plastic region. Proceeding similarly as Case 1, the depth of maximum bending moment which occurs in elastic zone can be expressed as

$$z_0 = \frac{H_0}{p_u}, \tag{43}$$

and its value is

$$M_{max} = \frac{H_0^2}{2p_u} + M_0, \tag{44}$$

#### 4. Validation of the Proposed Methods

The capability of the proposed analytical solutions presented in this research to predict the lateral behaviour of single piles embedded in a cohesive soil and the corresponding pile responses have been demonstrated and compared with the FE results carried out using ANSYS software.

For this purpose, a concrete pile with length  $L = 15$  m, diameter  $D = 0.4$  m, Poisson’s ratio  $\nu = 0.2$ , and Young’s modulus of elasticity  $E_p = 35,000$  MPa is embedded in a clay layer with lateral reaction modulus  $ks = 50,000$  kpa and shear strength  $Cu = 14.4$  kpa. The pile head is unrestrained to rotate and move laterally while the pile tip is completely fixed.

Figure 3 presents the computed pile head deflections versus applied lateral forces for both the analytical solution and FE solution with ANSYS software. The figure indicates that the results of the proposed analytical solution are closer to the results from the FE method. The lateral deflection obtained by the ANSYS workbench 18.2 for different lateral loads, 2, 75.5, and 82 kN are shown in Figure 4. The captured results obtained from Figure 3 are compared with those obtained by proposed analytical method as shown in Table 3. The presented results clearly indicate excellent agreement with the FE solution.

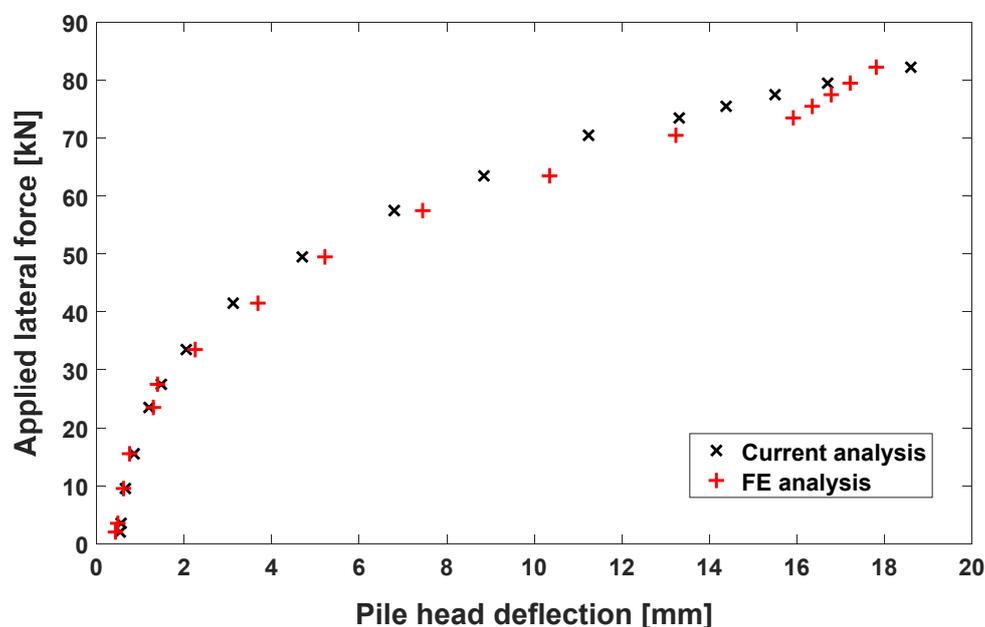
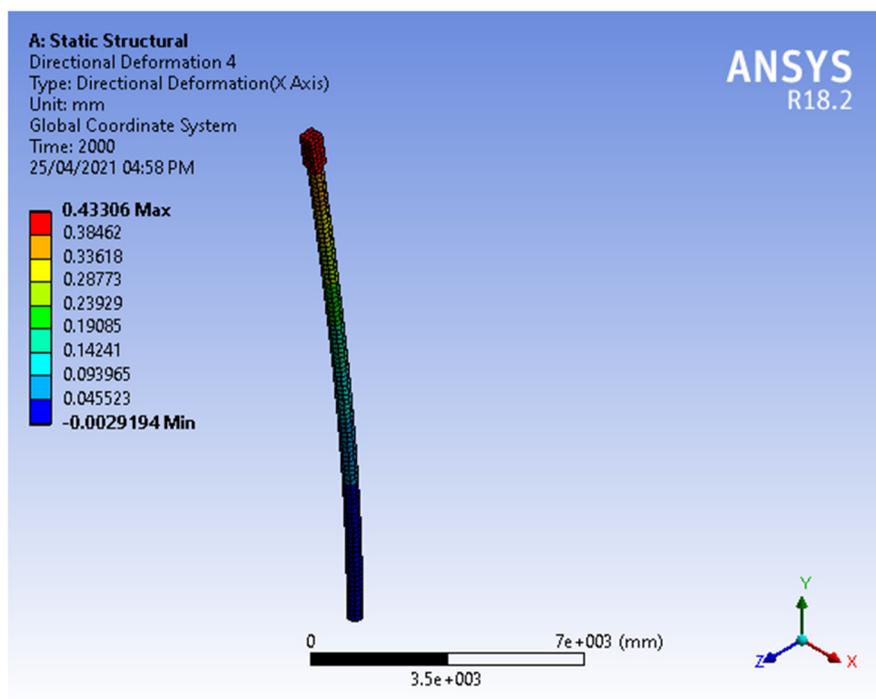
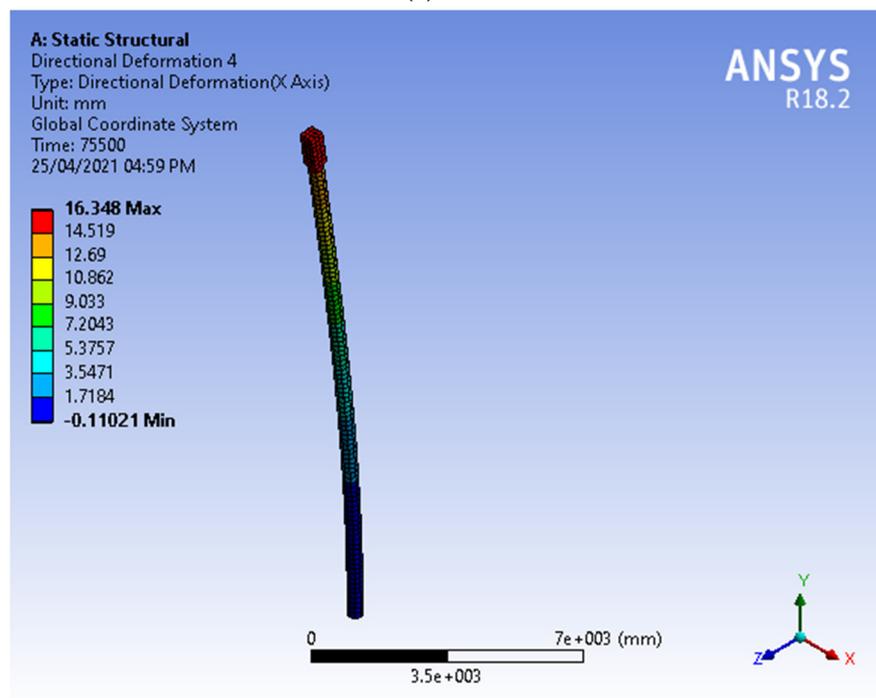


Figure 3. Comparison between the analytical and numerical lateral deflections of pile head versus applied lateral forces.

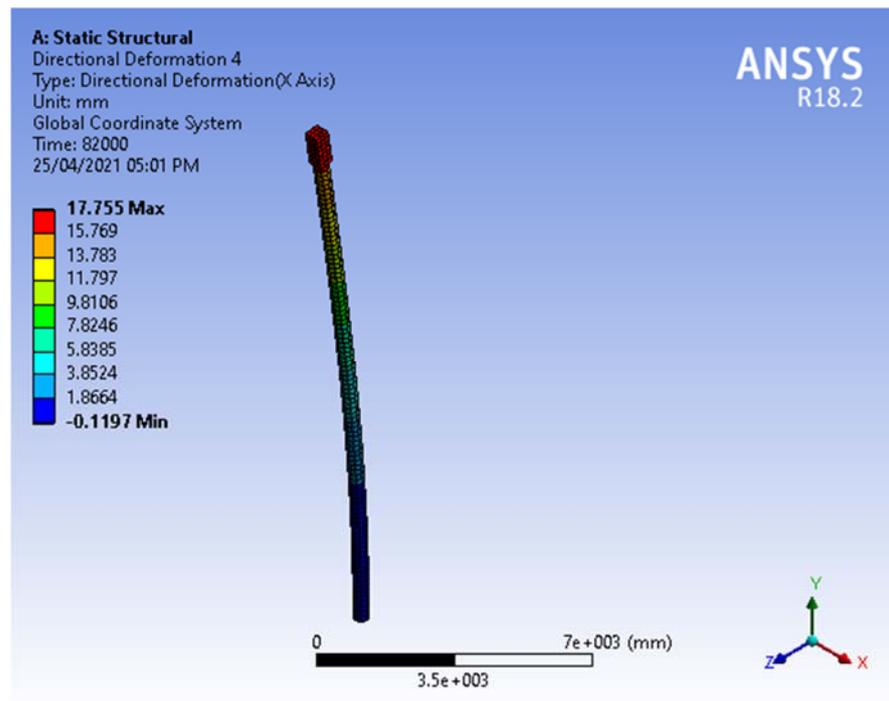


(a)



(b)

Figure 4. Cont.



(c)

**Figure 4.** The lateral deflection profiles of piles for different lateral loads (a) 2 kN, (b) 75.5 kN, and (c) 80 kN applied at a distance of 1 m above ground surface.

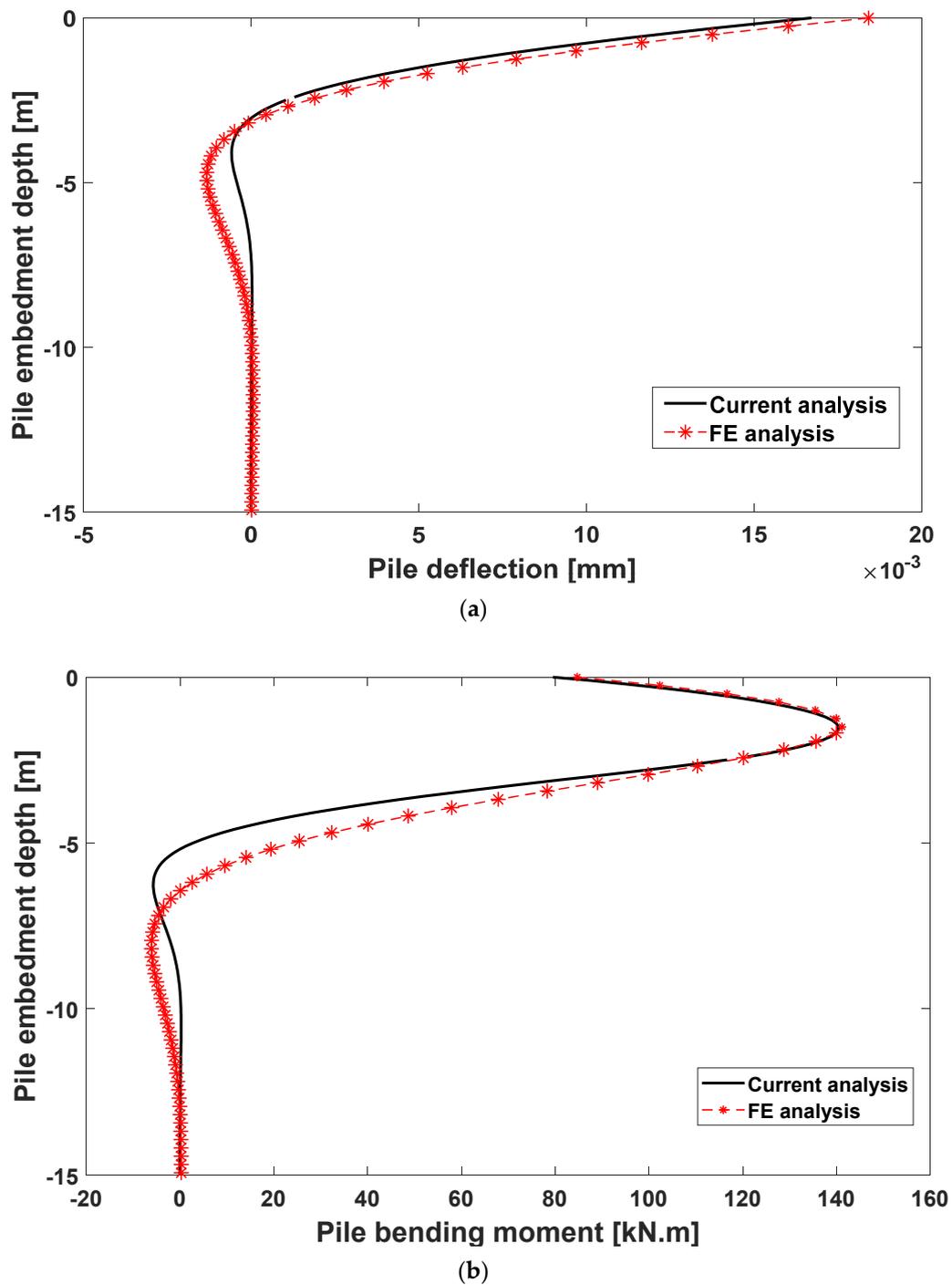
**Table 3.** Pile head deflections obtained from the analytical and FE methods.

Applied Lateral Load (kN)	Pile Head Deflections (mm)		%
	Analytical Method	FE, ANSYS Method	
2	0.54	0.43	19
75.5	14.38	16.34	13.7
82	18.3	17.75	3

Additionally, to ensure that the proposed solution produces a reliable pile deflection and bending moment not only at the pile head but also for the entire pile length, the lateral deflection and bending moment distribution along the pile shaft are presented and compared with the results of the FE analysis as shown in Figure 5.

The captured peak lateral deflection and bending moment obtained from the analytical solution are 16.710 mm and 140.459 kN·m, respectively, while the corresponding values obtained from the FE analysis are 17.809 mm and 141.053 kN·m, respectively. The results clearly indicate the agreement between the analytical and FE solutions in calculating peak value for design purposes.

Additionally, it is observed that the lateral deflection and bending moment profiles obtained using the proposed analytical method reach a good agreement with the FE solution.



**Figure 5.** Comparison of the results with the FE results. (a) The distribution of lateral deflection along the pile. (b) The distribution of the bending moment along the pile.

### 5. Conclusions

An analytical solution of the lateral response of an infinite pile embedded in homogeneous elasto-plastic soil deposits has been proposed in the current study. The solution is based on conceptual assumptions of the Winkler model which treat the pile as a flexible beam and soil restraint surrounding the pile as a series of non-linear independent springs. The flexural differential equation describing the response of the soil–pile system was solved analytically. Additionally, the location at which the maximum bending moment occurs, and the magnitude of that maximum bending moment along the pile were obtained from

the analysis. For verification of the proposed analytical procedure, the obtained results are compared with the FE results, carried out using ANSYS software, showing excellent agreements. The proposed exact solution could efficiently provide a better approach for structural designers to predict the non-linear responses of laterally loaded free-head long piles embedded in clay soil deposits with a uniform subgrade reaction modulus. Additionally, the proposed analytical solution has the advantage that it can provide a better approach for engineers to analyse and design the piles subjected to lateral loads without resorting to a 3D FE analysis, which is time consuming. Consequently, the mathematical forms of the theoretical solutions can be easily applied in practice as an alternative approach to analyse and design laterally loaded long piles.

It is emphasized that the simplified analytical solution obtained in this study is limited by the linearity of the soil–pile system, which often significantly affects the overall responses of the pile and the soil. Therefore, the proposed solution must be modified before applied to a non-linear case. Moreover, further investigation and analyses are needed to analyse the laterally loaded long piles in soil with variable subgrade reactions with depth, or for piles embedded in multilayered soil.

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**Conflicts of Interest:** The author declares no conflict of interest.

## Nomenclature

$D$	Pile diameter or width	$H_0$	Applied lateral load
$L$	Pile length	$M_0$	Applied moment
$I$	Moment of inertia of pile shaft	$y(z)$	Lateral deflection of pile
$E$	Young's modulus of pile shaft	$\varphi(z)$	Pile rotation
$k$	Coefficient of the subgrade reaction	$M(z)$	Pile bending moment
$c_u$	Undrained shear strength	$V(z)$	Pile shear force
$p_u$	Limit lateral force	$p(z)$	Load per unit length
$\beta$	Reciprocal of the characteristic length	$M_{ma}$	Maximum moment
$e$	Distance of pile above ground surface	$Y_{top}$	Top horizontal deflection
$g_i$	Integration coefficients	$\varphi_{top}$	Top pile rotation
$C_i$	Unknown parameters	$z_0$	Point of zero shear
$\nu$	Soil Poisson's ratio	$z_p$	Depth of the plastic zone

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