Article

Infinite Sum of the Incomplete Gamma Function Expressed in Terms of the Hurwitz Zeta Function

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Abstract: We apply our simultaneous contour integral method to an infinite sum in Prudnikov et al. and use it to derive the infinite sum of the Incomplete gamma function in terms of the Hurwitz zeta function. We then evaluate this formula to derive new series in terms of special functions and fundamental constants. All the results in this work are new.

Keywords: incomplete gamma function; Hurwitz zeta function; Catalan's constant; Euler's constant; infinite sum; contour integral

MSC: Primary 30E20; 33-01; 33-03; 33-04; 33E20

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1. Significance Statement

The Hurwitz zeta function was first studied by Adolf Hurwitz in 1882 [1], where he investigated some properties of Dirichlet functions occurring in the determination of the class numbers of binary quadratic forms. In more recent work, the Hurwitz zeta function $\zeta(k, a)$ was again studied but from the perspective of evaluating this function in terms of the infinite sum of the incomplete gamma function $\Gamma(k, a)$ section (8) in [2].

Some examples of this type of evaluation are presented in the works by Kanemitsu et al. [3,4] and Bailey et al. [5]. In those publications, the authors produced another approach to evaluating the Hurwitz zeta function $\zeta(s, a)$ using the infinite sum of the incomplete gamma function for relatively small values of the parameters s and a . The evaluation of the Hurwitz zeta function was also published where the Stokes phenomenon associated with the Hurwitz zeta function was examined in [6] and the error bounds for the asymptotic expansion of the Hurwitz zeta $\zeta(s, a)$ function were evaluated for large a in [7].

In the book of Prudnikov et al. Section (5.2) in [8], one will find the infinite sum of the incomplete gamma function expressed in terms of definite integrals. However, with a closer look one sees these infinite sums are taken over the first parameter of the incomplete gamma function.

In this present paper, we will derive a new expression for the Hurwitz zeta function expressed in terms of the infinite sum of the incomplete gamma function and evaluate this expression in terms of other special functions and fundamental constants not previously published. This expression is new because the way we have written the infinite sum of the incomplete gamma function does not exist in current literature to the best of our knowledge. The current literature on this topic is either in the form of asymptotic series expansions or closed forms which do not have as wide a range of evaluation of the parameters as compared to this work. It is our aim and hope to add to current tables of the infinite sum of the incomplete gamma function expressed in terms of special functions with the goal of aiding researchers requiring such formulae.

2. Introduction

In this present work, we derive a new expression for the Hurwitz zeta function in terms of the infinite sum of the incomplete gamma function given by:

$$\sum_{n=1}^{\infty} (-1)^n n^{-2k-1} \left(e^{i\pi an} (-in)^k \Gamma(k+1, ian\pi) - e^{-i\pi an} (in)^k \Gamma(k+1, -ian\pi) \right) \\ = \frac{i\pi^{k+1} \left(a^{k+1} + 2^{k+1}(k+1)\zeta\left(-k, \frac{a+1}{2}\right) \right)}{k+1},$$

where the variables k, a are general complex numbers. This new expression is then used to derive special cases in terms of fundamental constants and special functions. The derivations follow the method used by us in [9]. This method involves using a form of the generalized Cauchy's integral formula is given by:

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw, \quad (1)$$

where $y, w \in \mathbb{C}$, and C is in general an open contour in the complex plane where the bilinear concomitant [9] has the same value at the end points of the contour. This method involves using a form of Equation (1) then multiplying both sides by a function, then taking the definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2.1. The Incomplete Gamma Function

The multivalued incomplete gamma functions [2], $\gamma(s, z)$ and $\Gamma(s, z)$, are defined by

$$\gamma(s, z) = \int_0^z t^{s-1} e^{-t} dt,$$

and

$$\Gamma(s, z) = \int_z^{\infty} t^{s-1} e^{-t} dt,$$

where $\operatorname{Re}(s) > 0$. The incomplete gamma function has a recurrence relation given by

$$\gamma(s, z) + \Gamma(s, z) = \Gamma(s),$$

where $s \neq 0, -1, -2, \dots$. The incomplete gamma function is continued analytically by

$$\gamma(a, ze^{2m\pi i}) = e^{2\pi mia} \gamma(a, z),$$

and

$$\Gamma(s, ze^{2m\pi i}) = e^{2\pi mis} \Gamma(s, z) + (1 - e^{2\pi mis}) \Gamma(s),$$

where $m \in \mathbb{Z}$. When $z \neq 0$, $\Gamma(s, z)$ is an entire function of s and $\gamma(s, z)$ is meromorphic with simple poles at $s = -n$ for $n = 0, 1, 2, \dots$ with residue $\frac{(-1)^n}{n!}$. These definitions are listed in Section 8.2(i) and (ii) in [2].

2.2. The Hurwitz Zeta Function

The Hurwitz zeta function (25.11)(i) in [2] is defined by the infinite sum,

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

where $\zeta(s, a)$ has a meromorphic continuation in the s -plane, its only singularity in $\mathbb{C}$ being a simple pole at $s = 1$ with residue 1. As a function of a , with $s (\neq 1)$ fixed, $\zeta(s, a)$ is analytic in the half-plane $\operatorname{Re}(a) > 0$.

The Hurwitz zeta function is continued analytically with a definite integral representation (25.11.25) in [2], given by

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} dx,$$

where $\operatorname{Re}(s) > 1, \operatorname{Re}(a) > 0$.

3. Hurwitz Zeta Function in Terms of the Contour Integral

We use the method in [9]. The cut and contour are in the first quadrant of the complex w -plane with $0 < \operatorname{Re}(w) < 1$. The cut approaches the origin from the interior of the first quadrant and goes to infinity vertically and the contour goes around the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula, we first replace $y \rightarrow \log(a), k \rightarrow k + 2$ to get

$$-\frac{\log^{k+2}(a)}{2(k+2)!} = -\frac{1}{2\pi i} \int_C \frac{1}{2} a^w w^{-k-3} dw. \quad (2)$$

Using a generalization of Cauchy's integral formula, we first replace $y \rightarrow \log(a) + \pi(2y + 1)$ followed by multiplying both sides by $-\pi$ then taking the infinite sum over $y \in [0, \infty)$ to get

$$-\frac{2^{k+1} \pi^{k+2} \zeta\left(-k-1, \frac{\log(a)+\pi}{2\pi}\right)}{(k+1)!} = \frac{1}{2\pi i} \int_C \frac{1}{2} \pi a^w w^{-k-2} \operatorname{csch}(\pi w) dw, \quad (3)$$

from Equation (5.1.25.6) in [10] where the multivalued complex logarithmic function on $\mathbb{C} \setminus \{0\}$ is defined in Section (3.3.3) in [11].

4. Incomplete Gamma Function in Terms of the Contour Integral

Using Equation (1), we replace $y \rightarrow y + \log(a)$ and multiply both sides by e^{iny} and take the definite integral over $y \in [0, \infty)$ to get

$$\frac{a^{-in} (-in)^{-k-1} \Gamma(k+1, -in \log(a))}{\Gamma(k+1)} = -\frac{1}{2\pi i} \int_C \frac{a^w w^{-k-1}}{w + in} dw. \quad (4)$$

Next, we form a second equation by replacing $n \rightarrow -n, k \rightarrow k + 1$ and adding both equations followed by taking the infinite sum over $n \in [1, \infty)$ to get

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(-1)^n a^{-in} (n^2)^{-k-1} \left(a^{2in} (-in)^k \Gamma(k+2, in \log(a)) + (in)^k \Gamma(k+2, -in \log(a)) \right)}{2\Gamma(k+2)} \\ &= \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_C \frac{(-1)^n a^w w^{-k-1}}{n^2 + w^2} dw \\ &= \frac{1}{2\pi i} \int_C \sum_{n=1}^{\infty} \frac{(-1)^n a^w w^{-k-1}}{n^2 + w^2} dw \\ &= \frac{1}{2\pi i} \int_C \frac{1}{2} a^w w^{-k-3} (\pi w \operatorname{csch}(\pi w) - 1) dw, \end{aligned} \quad (5)$$

from Equation (1.422.3) in [12], where $|\arg(n \log(a))| < \pi$.

5. Main Results

Infinite Sum of the Incomplete Gamma Function in Terms of the Hurwitz Zeta Function

Theorem 1. For $a, k \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \frac{1}{2} (-1)^{n-1} (n^2)^{-k-2} \left(a^{in} (-in)^{k+2} \Gamma(k+2, in \log(a)) + a^{-in} (in)^{k+2} \Gamma(k+2, -in \log(a)) \right) \\ = -2^{k+1} \pi^{k+2} \zeta \left(-k-1, \frac{\log(a) + \pi}{2\pi} \right) - \frac{\log^{k+2}(a)}{2(k+2)}, \quad (6)$$

or by replacing $a \rightarrow e^{a\pi}$ and replacing $k \rightarrow k-1$ we get

$$\sum_{n=1}^{\infty} (-1)^n n^{-2k-1} \left(e^{i\pi an} (-in)^k \Gamma(k+1, ian\pi) - e^{-i\pi an} (in)^k \Gamma(k+1, -ian\pi) \right) \\ = \frac{i\pi^{k+1} \left(a^{k+1} + 2^{k+1} (k+1) \zeta \left(-k, \frac{a+1}{2} \right) \right)}{k+1}. \quad (7)$$

Proof. Since the addition of the right-hand sides of Equations (2) and (3) is equal to the right-hand side of Equation (5) we can equate the left-hand sides and simplify the Gamma function to yield the stated result. The logarithmic function is defined in Section (4.1) in [13]. $\square$

6. Representation in Terms of π , Catalan's G and Glaisher's γ constants

Proposition 1. The infinite sum of the exponential integral function $E_n(z)$ in terms of fundamental constants,

$$\sum_{n=1}^{\infty} \frac{e^{\frac{i\pi n}{2}} \left(E_1 \left(-\frac{1}{2} in\pi \right) + e^{i\pi n} E_1 \left(\frac{in\pi}{2} \right) \right)}{n^2} = -\frac{1}{48} \pi \left(\pi \log \left(\frac{16}{e^9 A^{24}} \right) + 48G \right).$$

Proof. Use Equation (7) and take the first partial derivative with respect to k then set $k = 1, a = 1/2$ and simplify in terms of the exponential integral function $E_n(z)$, from Equation (5.1.45) in [13], Catalan's constant G , from Section (9.73) in [12], Glaisher's constant A , from Section (2.15) in [14] and using Equation (12.5.48) in [15]. The evaluation of the multivalued complex exponential integral function is given in [16]. $\square$

7. Representation in Terms of Euler's Constant γ

Proposition 2.

$$\sum_{n=1}^{\infty} \left(-\Gamma(0, -in\pi) - e^{2i\pi n} \Gamma(0, in\pi) \right) = \log(2) - \gamma.$$

Proof. Use Equation (7) and replace $k \rightarrow k-1, a \rightarrow a/(\pi i)$ to get

$$\sum_{n=1}^{\infty} (-1)^n e^{-an} n^{-k} \left((-1)^{1-k} e^{2an} \Gamma(k, an) - \Gamma(k, -an) \right) \\ = \frac{i^{2-k} \pi^k}{k} \left(2^k k \zeta \left(1-k, \frac{1}{2} \left(1 - \frac{ia}{\pi} \right) \right) + \pi^{-k} (-ia)^k \right), \quad (8)$$

then apply L'Hopital's rule to the right-hand side as $k \rightarrow 0, a \rightarrow \pi i$ and simplify in terms of Euler's constant γ , from Section (9.73) in [12] and using Equation (64:4:1) and entry (1) in Table below (44:7:1) in [17]. $\square$

8. Representation in Terms of the Log-Gamma Function

Lemma 1. For $a \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n (e^{ian} \Gamma(0, ian) - e^{-ian} \Gamma(0, -ian))}{n} = i \log \left(a^a e^{-a} (2\pi)^{\pi-a} \Gamma \left(\frac{a+\pi}{2\pi} \right)^{-2\pi} \right). \quad (9)$$

Proof. Use Equation (8) and take the first partial derivative with respect to k and set $k = 1$ and simplify using Equation (64:10:2) in [17]. $\square$

Proposition 3.

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(-1)^n (e^{in} \Gamma(0, in) - e^{-in} \Gamma(0, -in))}{n} \\ &= i \left(-1 + (\pi - 1) \log(2\pi) - 2\pi \log \left(\Gamma \left(\frac{1+\pi}{2\pi} \right) \right) \right). \end{aligned}$$

Proof. Use Equation (9) and set $a = 1$ and simplify. $\square$

9. Representation in Terms of Catalan's Constant G

Proposition 4.

$$\sum_{n=1}^{\infty} n \left(e^{\frac{3i\pi n}{2}} \Gamma \left(-1, \frac{in\pi}{2} \right) - e^{\frac{i\pi n}{2}} \Gamma \left(-1, -\frac{1}{2}in\pi \right) \right) = \frac{i(8G + 4 - \pi^2)}{2\pi}.$$

Proof. Use Equation (7) and set $k = -1, a = \pi i/2$ and simplify in terms Catalan's constant G , from Section (9.73) in [12], Equation (64:4:1) in [17] and Equation (5) in [18]. $\square$

10. Representation in Terms of π

Proposition 5.

$$\sum_{n=1}^{\infty} \left(-n \left(\Gamma(-1, -in\pi) - e^{2i\pi n} \Gamma(-1, in\pi) \right) \right) = -\frac{i(\pi^2 - 12)}{12\pi}.$$

Proof. Use Equation (8) and set $k = -1, a = \pi i$ and simplify using entry (1) in Table below (64:4:2) in [17]. $\square$

11. Representation in Terms of the Riemann Zeta-Function $\zeta(s)$

Proposition 6. For $k \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} n^{-k} \left(-\Gamma(k, -in\pi) - e^{-i\pi(k-2n)} \Gamma(k, in\pi) \right) = \frac{i^{2-k} \pi^k \left(2^k k \zeta(1-k) + 1 \right)}{k}.$$

Proof. Use Equation (7) and set $a = \pi i$ and simplify using entry (2) in the Table below (64:7) in [17]. The Zeta function $\zeta(k)$ is given in Section (25) in [2]. $\square$

12. Representation in Terms of π and e

Proposition 7.

$$\sum_{n=1}^{\infty} \left(-\frac{i(\Gamma(0, -in\pi) - \Gamma(0, in\pi))}{\pi n} \right) = \log \left(\frac{e}{\pi} \right).$$

Proof. Use Equation (7) and take the first partial derivative with respect to k and set $k = 0$ and simplify using Equation (64:10:2) in [17]. $\square$

13. Summary of Expression

In this section, we provide a summary including the Hurwitz zeta function $\zeta(k, a)$ and the Incomplete Gamma Trio or IGT, for easy reading.

$$\sum_{n=1}^{\infty} (-1)^n e^{-an} n^{-k} \left((-1)^{1-k} e^{2an} \Gamma(k, an) - \Gamma(k, -an) \right) \\ = \frac{i^{2-k} \pi^k}{k} \left(2^k k \zeta \left(1-k, \frac{1}{2} \left(1 - \frac{ia}{\pi} \right) \right) + \pi^{-k} (-ia)^k \right),$$

and

$$\begin{cases} \sum_{n=1}^{\infty} \left(\Gamma(0, -in\pi) + e^{2i\pi n} \Gamma(0, in\pi) \right) = \gamma - \log(2), \\ \sum_{n=1}^{\infty} n \left(e^{\frac{3i\pi n}{2}} \Gamma \left(-1, \frac{in\pi}{2} \right) - e^{\frac{i\pi n}{2}} \Gamma \left(-1, -\frac{1}{2} in\pi \right) \right) = \frac{i(8G + 4 - \pi^2)}{2\pi}, \\ \frac{1}{\pi i} \sum_{n=1}^{\infty} \left(\frac{\Gamma(0, -in\pi) - \Gamma(0, in\pi)}{n} \right) = \log \left(\frac{e}{\pi} \right). \end{cases}$$

14. Discussion

In this short note, the authors derived an expression for the Hurwitz zeta function $\zeta(s, a)$ in terms of the infinite sum of the incomplete gamma function $\Gamma(s, a)$. The infinite sum derived allowed for large values of the parameters and the derivations involved fundamental constants and special functions. We numerically verified the results for complex values of the parameters using Wolfram's Mathematica software. We will be applying our contour integral method to other infinite sums in future work.

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