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A New Birnbaum–Saunders Distribution and Its Mathematical Features Applied to Bimodal Real-World Data from Environment and Medicine

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Abstract: In this paper, we propose and derive a Birnbaum–Saunders distribution to model bimodal data. This new distribution is obtained using the product of the standard Birnbaum–Saunders distribution and a polynomial function of the fourth degree. We study the mathematical and statistical properties of the bimodal Birnbaum–Saunders distribution, including probabilistic features and moments. Inference on its parameters is conducted using the estimation methods of moments and maximum likelihood. Based on the acceptance–rejection criterion, an algorithm is proposed to generate values of a random variable that follows the new bimodal Birnbaum–Saunders distribution. We carry out a simulation study using the Monte Carlo method to assess the statistical performance of the parameter estimators. Illustrations with real-world data sets from environmental and medical sciences are provided to show applications that can be of potential use in real problems.

Keywords: Birnbaum–Saunders distribution; data science; estimation of moments and maximum likelihood; Monte Carlo method; polynomial functions; proportionate-effect law; R software

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1. Introduction

Many phenomena may be modeled by continuous statistical distributions [1]. These distributions may be divided into symmetrical and asymmetrical with a wide variety of shapes (symmetry, skewed to the right, skewed to the left, unimodality, multimodality, platykurtosis, leptokurtosis, etc.) and supports (real numbers, positive reals, etc.). The most popular continuous distribution is the Gaussian or normal model, which has been the reference distribution for many years in statistics. However, other continuous distributions have gained space in the landscape of statistics due mainly to the important development of computation. For more details of theory, methodology, and applications of diverse continuous distributions, see [1].

A model that has received considerable attention recently is the Birnbaum–Saunders (BS) distribution, which is unimodal, asymmetric (skewed to the right), and has two parameters that modify its shape and scale. The BS distribution has good properties. For example, it is closely related to the normal distribution; it is closed under scalar multiplication (proportionality) and under reciprocation; its median coincides with the BS scale parameter; it has different shapes for its probability density function (PDF), which cover high, medium, and low asymmetry levels; it belongs to the class of log-symmetric distributions; it is a mixture model; it has mathematical arguments to be considered as a cumulative damage distribution; for more details on these and other properties, see [2–7].

After its origins in physics of materials and engineering [3], the BS distribution has been widely studied from theoretical and methodological perspectives by teams from prac-

tically all over the world [8–17]. In addition, this model has been applied in diverse areas other than engineering by international multidisciplinary groups [7,18–25]. Among these areas, we can highlight a number of applications in environmental and medical phenomena due to the mathematical justification stated in [7,18] when employing the BS distribution to model such phenomena.

Despite the wide applicability of the BS distribution, its shapes fail to model bimodal data. Note that phenomena associated with this type of data is more frequent than thought so that, under the scenario of a suitable application of the BS distribution, new bimodal models of this kind are needed to form part of the pool of statistical tools of diverse practitioners. Some BS distributions that included a bimodality parameter were introduced in [26,27]. However, these distributions showed little flexibility in the modeling of bimodal data. Therefore, a bimodal BS distribution that permits real-world phenomena to be modeled with high flexibility to gain precision in the information obtained from the data analysis based on such a model is needed.

The main objective of this investigation is to propose and derive a new bimodal BS distribution. Its derivation is followed by the corresponding mathematical expressions that allow the computation of probabilistic features and moments. These features enable us to obtain location, scale, variability, and shape measures, such as the mean, variance, and coefficients of variation (CV), skewness/asymmetry (CS), and kurtosis (CK). In addition, a simulation algorithm is presented, permitting us to generate values for a random variable that has the new proposed distribution. Such an algorithm allows us to simulate bimodal BS distributed data for assessing the performance of the parameter estimators of this new model. A computational implementation in the R software is provided, which helps to exemplify the obtained results with two real-world data sets from environmental and medical sciences and to show potential applications.

The remainder of the paper is structured in the following form: Section 2 presents background on the BS distribution and its use to model environmental and medical phenomena. In Section 3, we propose and derive the new model with its mathematical properties. In Section 4, the estimation of parameters of the bimodal BS distribution is performed. Section 5 proposes an acceptance–rejection simulation algorithm that allows us to generate bimodal BS distributed values. This section also provides the numerical results of our investigation with simulated and real data sets. At last, in Section 6, we give the conclusions, limitations, and ideas for further work that have emerged from the present study.

2. The Birnbaum–Saunders Distribution

In this section, we provide preliminary elements of the BS distribution. In addition, we illustrate how this distribution has mathematical arguments to describe environmental and medical data.

2.1. Definition

The results of this section can be seen in detail in [5]. If a random variable Y has a BS distribution with parameters α and β , the notation $Y \sim \text{BS}(\alpha, \beta)$ is employed. The BS distributed random variable Y is related to the standard normal distributed random variable V by means of

$$Y = \beta \left(\frac{\alpha}{2} V + \sqrt{\left(\frac{\alpha}{2} V \right)^2 + 1} \right)^2,$$

where $V \sim N(0, 1)$, $\alpha > 0$, and $\beta > 0$. The BS PDF is given by

$$f_Y(y; \alpha, \beta) = \frac{y^{-\frac{3}{2}}(y + \beta)}{2\alpha\beta^{\frac{1}{2}}} \phi(a(y)), \quad y > 0, \quad (1)$$

with $a(y) = (1/\alpha)(\sqrt{y/\beta} - \sqrt{\beta/y})$, where ϕ is the standard normal PDF.

2.2. Justifying the BS Distribution in Environmental and Medical Contexts

The BS distribution can be adequate to model data from certain fields based on an empirical fit using goodness-of-fit techniques. However, this empirical argument by means of a simple fit can be unsatisfactory. Indeed, if we show that a distribution is suitable to model a phenomenon, employing a theoretical justification may be a better argument. For more details about the results presented below, see [5] (fatigue), [18] (environment), [7] (medicine), and [28] (neural activity).

Assume a random variable associated with a generic metric. For instance, the tumor size can result in the death of a patient in a medical setting, or in an environmental context, the accumulation of pollutants in the atmosphere leads to a critical environmental situation. A generic metric can be stated considering the effect of several independent causes that interact, which work together to increase the value of the metric over time.

When the effects of such causes are accumulated randomly, this accumulation has a Gaussian asymptotic distribution based on the central limit theorem. Nevertheless, the causes do not seem to jointly act by a simple summation. It is more natural to suppose that each cause gives an impulse, and then the effect depends on both the impulse strength and value of the metric at the moment when the impulse is working.

Let $M_1, \dots, M_k$ be independent random variables associated with the magnitude of k impulses that work sequentially according to their subscripts. Furthermore, let O be a generic metric that is obtained from these impulses. Suppose that O_{j+1} corresponds to value of the metric O at the instant immediately before, which increases by the $(j + 1)$ th impulse M_{j+1} that is proportional to some function g of the mentioned value of the metric O stated as

$$O_{j+1} = O_j + M_{j+1} g(O_j), \quad j = 0, 1, \dots \quad (2)$$

Thus, O_{j+1} is the accumulated value of the metric after applying M_{j+1} . The formulation expressed in (2) is usually known as the proportionate-effect law. From this law, observe that the BS distribution may be reached when $g(O_j) = 1$ in (2). Note that the BS model was built to describe the fatigue life of materials subject to cyclic stress. It causes damage that is accumulated over time by summing several small damages. If the cumulative damage is greater than the rupture value (threshold) of the material, it fails [5]. Table 1 shows a conceptual comparison for environmental, fatigue, medical, neural frameworks that follow the proportionate-effect law.

Table 1. Conceptual comparison for environmental, fatigue, medical, neural frameworks.

Framework	Item	Cause	Threshold	Effect	Random Variable
Environmental	Chemical element	Contamination	Target	Emergence	Content
Fatigue	Material	Damage	Rupture	Failure	Fatigue life
Health metric	Patient	Impulse	To die	Death	Death time
Neural spiking	Neuron	Voltage	Firing	Spike	Inter-spike time

Next, we put the Frost–Dugdale differential equation in a general framework by considering

$$\frac{do}{dm} = \frac{oS^3}{c} = c_1 f(o^{1/2} b \Delta S) \approx c_0 + c_2 g(o), \quad (3)$$

with o being the generic metric and do its differential, m being the number of impulses and dm its differential, S being the strength exercised in each impulse, f being an empirical function and c, c_1 being constants. Observe that $o^{1/2} b \Delta S$ stated in (3) is a strength factor related to o , with b being an associated parameter and ΔS being the strength exercised in each impulse. Furthermore, in the approximation established in (3), c_0 is a baseline, c_2 contains other constants, and g is a function of the metric o . Consider the elements o, c_0 and c_2 as random variables to perform a model closer to reality.

Observe how the differential-equation stated in (3) and the proportionate-effect law given in (2) are similar. Assuming (2), apply the central limit theorem, and consider that the increment $\Delta O_j = O_{j+1} - O_j$ in the $(j + 1)$ th impulse provides a small contribution to the growth of the metric. Hence, if the summation is switched by integration, $\sum_{j=1}^k M_j$ is closely a Gaussian distributed random variable, with

$$\sum_{j=1}^k M_j = \sum_{j=1}^k \frac{\Delta O_j}{g(O_j)} \approx \int_{O_s}^{O_k} \frac{dx}{g(x)} = \log(g(O_k)) - \log(g(O_s)), \quad (4)$$

and O_s is an initial value of the metric and O_k its final value after applying k impulses.

To generate the BS model in a general framework, consider that the mean of O_j is η and its variance is ϱ^2 . Hence, a continuous extension of (4) using the time t is established by

$$I(O(t)) = \int_{O(t_0)}^{O(t)} \frac{dx}{g(x)} \sim N(t\eta, t\varrho^2), \quad (5)$$

with $O(t)$ being the value of the metric at t and $O(t_0)$ its starting value. Now, suppose that $O_c > O(t_0)$ is a critical value of the metric at which the event of interest happens and so $T = \inf\{t: O(t) > O_c\}$. Thus, based on (5) and employing $\{T \leq t\} \equiv \{O(t) > O_c\}$, we get the cumulative distribution function (CDF) of T by

$$F_T(t) = \Phi\left(\frac{t\eta - I(O_c)}{\varrho\sqrt{t}}\right), \quad t > 0. \quad (6)$$

Using (6), observe that the function g in the model given in (2) establishes the dependence of the metric on its previous value. In this case, a power function for g can be suitable obtaining $g(x) = x^\varsigma$, where ς is an indicator associated with the metric used for each framework (see Table 1), and $\varsigma \geq 0$, but $\varsigma \neq 1$. Hence, based on (5) and considering $g(x) = x^\varsigma$, we have that

$$I(O(t)) = \int_{O(t_0)}^{O(t)} \frac{dx}{x^\varsigma} = \frac{1}{\varsigma - 1} \left(\frac{1}{O(t_0)^{\varsigma-1}} - \frac{1}{O(t)^{\varsigma-1}} \right) = \frac{O(t_0)^{1-\varsigma} - O(t)^{1-\varsigma}}{\varsigma - 1} \sim N(t\eta, t\varrho^2). \quad (7)$$

Then, from (7)–(6) and by symmetry, we get $O(t)^{1-\varsigma} \sim N(O_s^{1-\varsigma} + (1-\varsigma)t\eta, (1-\varsigma)^2 t\varrho^2)$. Therefore, the CDF of T at t is expressed as

$$F_T(t) = \begin{cases} \Phi\left(\frac{O_c^{1-\varsigma} - O_s^{1-\varsigma} + (\varsigma-1)t\eta}{(\varsigma-1)\sqrt{t}\varrho}\right), & \text{if } \varsigma > 1; \\ \Phi\left(\frac{O_s^{1-\varsigma} - O_c^{1-\varsigma} + (1-\varsigma)t\eta}{(1-\varsigma)\sqrt{t}\varrho}\right), & \text{if } \varsigma < 1. \end{cases} \quad (8)$$

From (8), observe that the BS distribution is obtained for $\varsigma = 0$. Nonetheless, if $\varsigma = 1$, it is the case of the lognormal distribution ($g(x) = x$) in the proportionate-effect law stated in (4), which cannot be obtained from (8). Hence, the CDF of T formulated in (8) corresponds the BS distribution and not to the lognormal distribution.

To be specific and as an example, the setting discussed above might be placed in a framework of pollution, doing an analogy between the proportionate-effect law and a contaminant concentration model. Such an example is valid as a justification to model data on indoor and outdoor environmental quality, water contamination in lakes, groundwater, and soils, and trace metals in human tissue, blood, and feces.

To contextualize the model stated in (8) for environmental contamination, suppose the following. We assume a non-random framework. Consider that an environment has a pollutant concentration X_0 and an air mass V_0 at an initial time $t = 0$. Hence, $q_0 = X_0 V_0$ corresponds to the total pollutants in the environment. If no chemical reactions are present, conservation of matter requires the total pollutant amount after dilution to be equal to

the total pollutant amount before dilution. In this way, when V_1 is the volume of the mixture after the first dilution and X_1 the pollutant concentration in that diluted mixture, we have that

$$q_0 = X_0 V_0 = X_1 V_1. \quad (9)$$

Solving the equation stated in (9) for X_1 , the content of X_1 is inversely proportional to the ratio of the final volume to the initial volume given by $X_1 = (V_0/V_1) X_0$. Therefore, if V_0/V_1 is established by a dimensionless dilution D_1 that is treated as a random variable, then $X_1 = D_1 X_0$ is also a random variable, where X_0 is non-random. Such a random framework is closer to reality due to the fact that contaminants vary over time because of climatology, geography, topography, and human activity.

Let D_j denote the j th dilution. Then, the concentration X_2 in the second dilution is formulated as

$$X_2 = D_2 X_1 = D_2 D_1 X_0.$$

In addition, let N be the number of successive dilutions in a random dilution process. Then, the final concentration after N dilutions is defined as

$$X_N = X_0 \prod_{j=1}^N D_j. \quad (10)$$

By applying the logarithm function in the expression given in (10), we get that

$$\sum_{j=1}^N \log(D_j) = \log(X_N) - \log(X_0) = \int_{X_0}^{X_N} \frac{dx}{x}. \quad (11)$$

Alternatively, the formulation defined in (11) may be expressed by means of a differential equation of first-order as usual in the mass balance model, after permitting the change rate of the concentration to be small enough. Once again, observe how the equations stated in (4) and (11) if $g(x) = x$ are similar. As mentioned, this formulation is associated with a lognormal distribution. Nevertheless, a more general expression for (11) might be assumed by considering

$$\sum_{j=1}^N Y_j \approx \int_{X_0}^{X_N} \frac{dx}{g(x)}. \quad (12)$$

Note that the expression stated in (12) should conduct a BS model upon certain conditions, with this expression being similar to (4). Such a setting permits us to consider a statistical distribution to model contaminant concentration data, which should not be symmetrical but skewed to the right.

3. The BS Distribution under Bimodality

In this section, we obtain the bimodal BS distribution of a random variable Y based on the product of the $BS(\alpha, \beta)$ PDF and a fourth-degree polynomial function as a competitor of two other versions of bimodal BS distributions proposed in [26,27], which are named bimodal BS distributions of types 1 and 2, denoted by BBS1 and BBS2, respectively. The new bimodal BS distribution, which we name type 3, is denoted by $Y \sim BBS3(\alpha, \beta, \delta)$, with $\delta \geq 0$ being the bimodality parameter.

3.1. Probability Density Function

A bimodal BS distribution with parameters α_1 , β_1 , and δ_1 of a random variable Y , denoted by $Y \sim BBS1(\alpha_1, \beta_1, \delta_1)$, was postulated in [26]. This bimodal BS distribution of type 1 has a PDF stated as

$$f_Y(y; \alpha_1, \beta_1, \delta_1) = \frac{y^{-\frac{3}{2}}(y + \beta_1)}{4\alpha_1\beta_1^{\frac{1}{2}}\Phi(-\delta_1)} \phi(|a(y)| + \delta_1), \quad y > 0, \alpha_1 > 0, \beta_1 > 0, -\infty < \delta_1 < \infty, \quad (13)$$

with $a(y)$ being defined as in (1) and Φ being the standard normal CDF.

A second bimodal BS distribution with parameters α_2 , β_2 , and δ_2 of a random variable Y , denoted by $Y \sim \text{BBS2}(\alpha_2, \beta_2, \delta_2)$, was postulated in [27]. This bimodal BS distribution of type 2 has a PDF as a competitor of (13) given by

$$f_Y(y; \alpha_2, \beta_2, \delta_2) = \frac{(1 - \delta_2 a(y))^2 + 1}{2 + \delta_2^2} \phi(a(y)) \frac{y^{-\frac{3}{2}}(y + \beta_2)}{2\alpha_2\beta_2^{\frac{1}{2}}}, \quad y > 0, \alpha_2 > 0, \beta_2 > 0, -\infty < \delta_2 < \infty, \quad (14)$$

with $a(y)$ being defined as in (1). Next, we propose the bimodal BS distribution of type 3 as a competitor of the classical unimodal BS distribution, as well as of the bimodal BS distributions of types 1 and 2, with PDFs established in (1), (13), and (14), respectively.

Proposition 1. Let $Y \sim \text{BBS3}(\alpha, \beta, \delta)$. Then, the PDF Y is stated as

$$f_Y(y; \alpha, \beta, \delta) = \frac{1 + \delta \left(\frac{y - \mu}{\sigma} \right)^4}{1 + \kappa \delta} f_X(y; \alpha, \beta), \quad y > 0, \alpha > 0, \beta > 0, \delta \geq 0, \quad (15)$$

where f_X is the PDF of $X \sim \text{BS}(\alpha, \beta)$ given in (1), and

$$\begin{aligned} \mu &= E(X) = \beta \left(1 + \frac{\alpha^2}{2} \right), \\ \sigma &= \text{SD}(X) = \sqrt{\text{Var}(X)} = \alpha \beta \left(1 + \frac{5\alpha^2}{4} \right)^{1/2}, \\ \kappa &= \text{CK}(X) = \frac{48 + 360\alpha^2 + 633\alpha^4}{(5\alpha^2 + 4)^2}, \end{aligned}$$

with SD denoting the standard deviation, α, β being shape parameters, and δ being the bimodality parameter.

Proof. Note that

$$\begin{aligned} \int_0^\infty f_Y(y; \alpha, \beta, \delta) dy &= \int_0^\infty \kappa_0 \left(1 + \delta \left(\frac{y - \mu}{\sigma} \right)^4 \right) f_X(y; \alpha, \beta) dy \\ &= \kappa_0 \left(\int_0^\infty f_X(y; \alpha, \beta) dy + \int_0^\infty \delta \left(\frac{y - \mu}{\sigma} \right)^4 f_X(y; \alpha, \beta) dy \right) \\ &= \kappa_0 \left(1 + \delta E \left(\left(\frac{Y - \mu}{\sigma} \right)^4 \right) \right) \\ &= \kappa_0 (1 + \kappa \delta) = 1, \end{aligned}$$

where $\kappa_0 = 1/(1 + \kappa \delta)$ and f_X is the PDF of $X \sim \text{BS}(\alpha, \beta)$. $\square$

Observe that the BBS3 distribution is obtained following the idea given in [29]. Thus, to obtain an interpretation of the modality parameter δ , we present below a proposition that generalizes a classical continuous distribution allowing us to generate unimodality or bimodality in the distribution.

Proposition 2. Let $f(y; \mu, \sigma)$ be the PDF of a continuous distribution with finite mean μ , variance σ^2 , and CK κ . Then, it is possible to build a new distribution with PDF stated as

$$h(y; \mu, \sigma, \delta) = \omega(y; \mu, \sigma, \delta) f(y; \mu, \sigma), \quad (16)$$

with $0 \leq \delta < \infty$ being the parameter that controls the unimodality or the bimodality of the distribution and where

$$\omega(y; \mu, \sigma, \delta) = \frac{1}{1 + \kappa \delta} \left(1 + \delta \left(\frac{y - \mu}{\sigma} \right)^4 \right).$$

Note that the function stated in (16) is a genuine PDF.

Proof. The result is directly obtained by taking into account that $f(y; \mu, \sigma) \geq 0$ and integrating over the support R_Y of the random variable Y to have $\int_{R_Y} f_Y(y) dy = 1$. $\square$

Observe that the parameter δ defined in the PDF given in (16) controls the unimodality or bimodality of the corresponding distribution. Additionally, by taking $\delta = 0$, the parent PDF $f(y; \mu, \sigma)$ is obtained as a special case. For a shape analysis of the BBS3 PDF compared to the BS distribution, see Figure 1.

Figure 1. PDF (a) and CDF (b) plots of the BBS3 and BS distributions with $\alpha = 0.8$, $\beta = 0.3$ and $\delta = 0.1$ (solid line), $\alpha = 0.8$, $\beta = 0.3$ and $\delta = 0$ (dashed line), and $\alpha = 0.8$, $\beta = 0.3$ and $\delta = 0.8$ (dotted line).

3.2. Cumulative Distribution Function and BBS3 Properties

Next, we give some elementary properties of the BBS3 distribution. Note that this distribution is closed under reciprocation and has the standard BS distribution as a particular case.

Proposition 3. Let $Y \sim \text{BBS3}(\alpha, \beta, \delta)$. Then, we have that

- (i) $Y^{-1} \sim \text{BBS3}(\alpha, \beta^{-1}, \delta)$.
- (ii) If $\delta = 0$, $Y \sim \text{BS}(\alpha, \beta)$.

The following proposition states the CDF of the BBS3 distribution, which depend on the standard BS CDF.

Proposition 4. Let $Y \sim \text{BBS3}(\alpha, \beta, \delta)$. Then, the CDF of Y is given by

$$F_Y(t; \alpha, \beta, \delta) = \frac{1}{1 + \kappa \delta} \left(F_X(t) + \frac{\delta}{2\alpha\beta^{\frac{1}{2}}\sigma^4\sqrt{2\pi}} \times \int_{-\infty}^t y^{-\frac{3}{2}}(y + \beta)(y - \mu)^4 e^{-\frac{1}{2\alpha^2} \left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}} \right)^2} dx \right),$$

where F_X is the CDF of $X \sim \text{BS}(\alpha, \beta)$, and μ, σ, κ are defined in Proposition 1.

Proof. Let $Y \sim \text{BBS3}(\alpha, \beta, \delta)$ and F_X be the CDF of $X \sim \text{BS}(\alpha, \beta)$. Then,

$$\begin{aligned} F_Y(t; \alpha, \beta, \delta) &= \int_0^\infty f_Y(y; \alpha, \beta, \delta) dy \\ &= \frac{1}{1 + \kappa \delta} \int_{-\infty}^t \left(1 + \delta \left(\frac{y - \mu}{\sigma} \right)^4 \right) f_X(y; \alpha, \beta) dy \\ &= \frac{1}{1 + \kappa \delta} \left(F_X(t) + \frac{\delta}{2\alpha\beta^{\frac{1}{2}}\sigma^4\sqrt{2\pi}} \times \int_{-\infty}^t y^{-\frac{3}{2}}(y + \beta)(y - \mu)^4 e^{-\frac{1}{2\alpha^2} \left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}} \right)^2} dx \right). \end{aligned}$$

□

Figure 1a shows the PDFs of the BBS3 and BS distributions for different values of α, β and δ . Here, we can see the bimodality of the BBS3 distribution and the unimodality of the BS distribution. Figure 1b shows the graphical plots of the corresponding CDFs for different values of their parameters.

3.3. Cumulant Generating Function

The following proposition states the cumulant generating function of the BBS3 distribution, which depends on the moment generating function of the standard BS distribution. From the cumulant generating function, the cumulants of a distribution may be obtained, which correspond to quantities that provide an alternative to the moments of the distribution. Note that the r th cumulant is computed as

$$c_r = C^{(r)}(0), \quad (17)$$

where $C(t)$ is the cumulant generating function of the BBS3 distribution given in the following proposition, and $C^{(r)}(0)$ defined in (17) is obtained by taking the r th derivative of $C(t)$ with respect to t and evaluating it at $t = 0$.

Proposition 5. Let $Y \sim \text{BBS3}(\alpha, \beta, \delta)$. Then, the cumulant generating function of Y is obtained as

$$C(t) = \log(E(e^{tY})), \quad -\infty < t < \infty,$$

where

$$\begin{aligned} E(e^{tY}) &= \frac{1}{1 + \kappa \delta} \left(E(e^{tX}) \right. \\ &\quad \left. + \frac{\delta}{2\alpha\beta^{\frac{1}{2}}\sigma^4\sqrt{2\pi}} \int_0^\infty y^{-\frac{3}{2}}(y + \beta)(y - \mu)^4 e^{-\frac{1}{2\alpha^2} \left(\sqrt{\frac{y}{\beta}} - \sqrt{\frac{\beta}{y}} \right)^2 - ty} dy \right), \end{aligned}$$

with $E(e^{tX})$ being the moment generating function of $X \sim \text{BS}(\alpha, \beta)$.

Proof.

$$\begin{aligned}
 E(e^{tY}) &= \int_0^\infty e^{tY} f_Y(y; \alpha, \beta, \delta) dy \\
 &= \frac{1}{1 + \kappa \delta} \int_{-\infty}^t \left(1 + \delta \left(\frac{y - \mu}{\sigma} \right)^4 \right) e^{ty} f_X(y; \alpha, \beta) dy \\
 &= \frac{1}{1 + \kappa \delta} \left(E(e^{tX}) + \frac{\delta}{2\alpha\beta^{\frac{1}{2}}\sigma^4\sqrt{2\pi}} \right. \\
 &\quad \left. \times \int_0^\infty y^{-\frac{3}{2}}(y + \beta)(y - \mu)^4 e^{-\frac{1}{2\alpha^2} \left(\sqrt{\frac{y}{\beta}} - \sqrt{\frac{\beta}{y}} \right)^2 - ty} dy \right).
 \end{aligned}$$

□

The r th moment μ'_r is an r th degree polynomial function in the first r cumulants, where

$$\begin{aligned}
 \mu'_1 &= c_1, \\
 \mu'_2 &= c_2 + c_1^2, \\
 \mu'_3 &= c_3 + 3c_2c_1 + c_1^3, \\
 \mu'_4 &= c_4 + 4c_3c_1 + 3c_2^2 + 6c_2c_1^2 + c_1^4, \\
 \mu'_5 &= c_5 + 5c_4c_1 + 10c_3c_2 + 10c_3c_1^2 + 15c_2^2c_1 + 10c_2c_1^3 + c_1^5, \\
 \mu'_6 &= c_6 + 6c_5c_1 + 15c_4c_2 + 15c_4c_1^2 + 10c_3^2 + 60c_3c_2c_1 + 20c_3c_1^3 \\
 &\quad + 15c_2^3 + 45c_2^2c_1^2 + 15c_2c_1^4 + c_1^6.
 \end{aligned}$$

Observe that the "prime" states a difference between the moments $\mu'_r = E(Y^r)$ and the central moments $\mu_r = E((Y - \mu)^r)$. To obtain the central moments as functions of the cumulants, we must eliminate from these polynomials all the terms in which c_1 is a factor. Thus, from the above expressions, we generate the first six moments as $\mu_1 = 0$, $\mu_2 = c_2$, $\mu_3 = c_3$, $\mu_4 = c_4 + 3c_2^2$, $\mu_5 = c_5 + 10c_3c_2$, and $\mu_6 = c_6 + 15c_4c_2 + 10c_3^2 + 15c_2^3$. Similarly, the r th cumulant c_r is an r th degree polynomial function in the first r raw (non-central) moments so that the first six cumulants are stated as

$$\begin{aligned}
 c_1 &= \mu'_1, \\
 c_2 &= \mu'_2 - \mu_1'^2, \\
 c_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3, \\
 c_4 &= \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4, \\
 c_5 &= \mu'_5 - 5\mu'_4\mu'_1 - 10\mu'_3\mu'_2 + 20\mu'_3\mu_1'^2 + 30\mu_2'^2\mu'_1 - 60\mu'_2\mu_1'^3 + 24\mu_1'^5, \\
 c_6 &= \mu'_6 - 6\mu'_5\mu'_1 - 15\mu'_4\mu'_2 + 30\mu'_4\mu_1'^2 - 10\mu_3'^2 + 120\mu'_3\mu_2'\mu'_1 \\
 &\quad - 120\mu'_3\mu_1'^3 + 30\mu_2'^3 - 270\mu_2'^2\mu_1'^2 + 360\mu'_2\mu_1'^4 - 120\mu_1'^6.
 \end{aligned}$$

To obtain the cumulants c_r for $r > 1$ as functions of the central moments, we must remove from these polynomials all the terms in which μ_1' is a factor. Hence, from the above expressions, we generate the first six cumulants as $c_2 = \mu_2$, $c_3 = \mu_3$, $c_4 = \mu_4 - 3\mu_2^2$, $c_5 = \mu_5 - 10\mu_3\mu_2$, and $c_6 = \mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3$. This indicates that the first cumulant is the mean, the second cumulant is the variance, and the third cumulant is the third central moment. However, from the fourth cumulant, they are not equal to the central moments.

3.4. Moments

The following proposition states the moments of the BBS3 distribution. Essentially, these moments depend on the moments of the standard BS distribution.

Proposition 6. Let $Y \sim \text{BBS3}(\alpha, \beta, \delta)$. Then, for $r = 1, 2, \dots$, we have that

$$\begin{aligned}\mu_r &= E(Y^r) = \frac{1}{1 + \kappa \delta} \left(\left(1 + \frac{\delta \mu^4}{\sigma^4} \right) E(X^r) \right. \\ &\quad \left. + \frac{\delta}{\sigma^4} \left(E(X^{r+4}) - 4\mu E(X^{r+3}) + 6\mu^2 E(X^{r+2}) - 4\mu^3 E(X^{r+1}) \right) \right),\end{aligned}$$

where $E(X^r)$ is the r th moment of $X \sim \text{BS}(\alpha, \beta)$, and μ, σ, κ are given in Proposition 1.

Proof. Let $Y \sim \text{BBS3}(\alpha, \beta, \delta)$ and F_X be the CDF of $X \sim \text{BS}(\alpha, \beta)$. Then,

$$\begin{aligned}f_Y(y; \alpha, \beta, \delta) &= \frac{1 + \delta \left(\frac{y - \mu}{\sigma} \right)^4}{1 + \kappa \delta} f_X(y; \alpha, \beta) \\ &= \frac{1}{1 + \kappa \delta} \left(1 + \frac{\delta}{\sigma^4} (y - \mu)^4 \right) f_X(y; \alpha, \beta) \\ &= \frac{1}{1 + \kappa \delta} \left(1 + \frac{\delta}{\sigma^4} (y^4 - 4\mu y^3 + 6\mu^2 y^2 - 4\mu^3 y + \mu^4) \right) f_X(y; \alpha, \beta).\end{aligned}$$

Thus, we have that

$$\begin{aligned}\mu_r &= E(Y^r) \\ &= \frac{1}{1 + \kappa \delta} \int_0^\infty y^r \left(1 + \frac{\delta}{\sigma^4} (y^4 - 4\mu y^3 + 6\mu^2 y^2 - 4\mu^3 y + \mu^4) \right) f_X(y; \alpha, \beta) dy \\ &= \frac{1}{1 + \kappa \delta} \left(\left(1 + \frac{\delta \mu^4}{\sigma^4} \right) E(X^r) \right. \\ &\quad \left. + \frac{\delta}{\sigma^4} \left(E(X^{r+4}) - 4\mu E(X^{r+3}) + 6\mu^2 E(X^{r+2}) - 4\mu^3 E(X^{r+1}) \right) \right).\end{aligned}$$

□

Observe that the six first moments of $X \sim \text{BS}(\alpha, \beta)$ are given by

$$\begin{aligned}E(X) &= \frac{1}{2} \beta (\alpha^2 + 2) \\ E(X^2) &= \frac{1}{2} \beta^2 (3\alpha^4 + 4\alpha^2 + 2) \\ E(X^3) &= \frac{1}{2} \beta^3 (15\alpha^6 + 18\alpha^4 + 9\alpha^2 + 2) \\ E(X^4) &= \frac{1}{2} \beta^4 (105\alpha^8 + 120\alpha^6 + 60\alpha^4 + 16\alpha^2 + 2) \\ E(X^5) &= \frac{\beta^5}{2} (945\alpha^{10} + 1050\alpha^8 + 525\alpha^6 + 150\alpha^4 + 25\alpha^2 + 2) \\ E(X^6) &= \frac{\beta^6}{2} (10,395\alpha^{12} + 11,340\alpha^{10} + 5670\alpha^8 + 1680\alpha^6 + 315\alpha^4 + 36\alpha^2 + 2)\end{aligned}$$

which allow us to obtain the corresponding moments of $Y \sim \text{BBS3}(\alpha, \beta, \delta)$ from Proposition 6.

4. Inference

In this section, we discuss statistical inference based on the moment and maximum likelihood estimators of the BBS3 distribution parameters. For more details about moment estimation for the standard BS distribution, see [12,30].

4.1. Moment Estimators

Let $\mathbf{Y} = (Y_1, \dots, Y_n)^\top$ be a random sample of size n from $Y \sim \text{BBS3}(\theta)$, with $\theta = (\alpha, \beta, \delta)$, and $\mathbf{y} = (y_1, \dots, y_n)^\top$ the observed value of $\mathbf{Y}$. Then, the moment estimator of θ can be obtained from the equations stated as

$$\begin{aligned}\bar{y} &= \frac{1}{1 + \kappa \delta} \left(\left(1 + \frac{\delta \mu^4}{\sigma^4} \right) E(X) + \frac{\delta}{\sigma^4} \left\{ E(X^5) - 4\mu E(X^4) + 6\mu^2 E(X^3) - 4\mu^3 E(X^2) \right\} \right), \\ \bar{y}^2 &= \frac{1}{1 + \kappa \delta} \left(\left(1 + \frac{\delta \mu^4}{\sigma^4} \right) E(X^2) + \frac{\delta}{\sigma^4} \left(E(X^6) - 4\mu E(X^5) + 6\mu^2 E(X^4) - 4\mu^3 E(X^3) \right) \right), \\ \bar{y}^3 &= \frac{1}{1 + \kappa \delta} \left(\left(1 + \frac{\delta \mu^4}{\sigma^4} \right) E(X^3) + \frac{\delta}{\sigma^4} \left(E(X^7) - 4\mu E(X^6) + 6\mu^2 E(X^5) - 4\mu^3 E(X^4) \right) \right),\end{aligned}$$

where $\bar{y}^r = \sum_{i=1}^n y_i^r / n$, for $r = 1, 2, 3$.

4.2. Maximum Likelihood Estimators

Let $\mathbf{Y} = (Y_1, \dots, Y_n)^\top$ be a random sample of size n from $Y \sim \text{BBS3}(\theta)$, with $\theta = (\alpha, \beta, \delta)$, and $\mathbf{y} = (y_1, \dots, y_n)^\top$ the observed value of $\mathbf{Y}$. Then, the maximum likelihood estimator of θ can be obtained from the corresponding log-likelihood function defined as

$$\begin{aligned}l(\theta; \mathbf{y}) &\propto \sum_{i=1}^n \log(1 + \delta z_i^4) - n \log(1 + \kappa \delta) - \sum_{i=1}^n \frac{a^2(y_i)}{2} - \frac{3}{2} \sum_{i=1}^n \log(y_i) + \sum_{i=1}^n \log(y_i + \beta) \\ &\quad - n \log(\alpha) - \frac{n}{2} \log(\beta),\end{aligned}\tag{18}$$

where $z_i = (y_i - \mu)/\sigma$, for $i = 1, \dots, n$. Thus, the score vector associated with the log-likelihood function stated in (18) has elements given by

$$\begin{aligned}S_\alpha &= \sum_{i=1}^n \frac{\alpha \sigma^2 - \delta z_i^3 (5z_i \alpha^4 \beta^2 + 4\alpha^2 \sigma \beta + 4z_i \sigma^2)}{\alpha \sigma^2 (1 + \delta z_i^4)} - \frac{n \delta \alpha^3 \beta^2 (2532 \alpha^4 \beta^2 + 732 \alpha^2 \beta^2 - 5 \kappa \sigma^2)}{\sigma^4 (1 + \kappa \delta)} \\ &\quad + \frac{1}{\alpha} \sum_{i=1}^n a(y_i)^2 - \frac{n}{\alpha}, \\ S_\beta &= \sum_{i=1}^n \frac{2\sigma \beta - \delta (\beta \alpha^2 + 2\beta + 2z_i \sigma)}{2\sigma \beta (1 + \delta z_i^4)} + \frac{1}{2\alpha^2 \beta} \sum_{i=1}^n \left(\frac{y_i}{\beta} - \frac{\beta}{y_i} \right) + \sum_{i=1}^n \frac{1}{y_i + \beta} - \frac{n}{2\beta}, \\ S_\delta &= \sum_{i=1}^n \frac{z_i^4}{(1 + \delta z_i^4)} - \frac{n \kappa}{(1 + \kappa \delta)}.\end{aligned}$$

By solving the estimation equations obtained from the score vector numerically, we obtain maximum likelihood estimators of the parameters α , β and δ .

5. Numerical Applications

In this section, we use simulated and real-world data to show our numerical results. An algorithm is employed to simulate bimodal BS distributed data and then evaluate the performance of the estimators of the new bimodal BS distribution parameters. Hence, a computational implementation in the R software produced by the authors of this study is utilized to illustrate the obtained results with real-world data sets from environmental and medical sciences to show potential applications. The data analyzed in this paper and the computational codes are available from the authors under request.

5.1. Simulation Algorithm and Computer Characteristics

Even though some software provide built-in random number generators, there are probability distributions that are not covered by these software. This is the case of the

BBS3 distribution. We use the acceptance–rejection method to generate random numbers from the $BBS3(\alpha, \beta, \delta)$ distribution with PDF defined in (15) according to Algorithm 1. The results in a sequence of n random numbers are stored within an array that we name n_vector . Since the BBS3 distribution has a non-finite support, we use a constant $l_1 > 0$ to limit the BBS3 values generated. Furthermore, we employ a constant $l_2 > 0$ corresponding to the maximum value of the BBS3 PDF f , which must be evaluated at the true parameters. In addition to the parameters α, β, γ of the BBS3 distribution, to build the algorithm, we need to define:

- n : The length of the n_vector .
- Y : A random variable with $BBS3(\alpha, \beta, \delta)$ distribution.
- $f_Y(y)$: The BBS3 PDF with $y > 0$.
- l_1 : A lower limit for the BBS3 numbers to be generated with $l_1 > 0$.
- l_2 : The maximum value of f_Y with $l_2 > 0$.
- U_1 : A random variable with a uniform distribution in $(0, l_1)$, $U(0, l_1)$ in short.
- U_2 : A random variable with a $U(0, l_2)$ distribution.

Algorithm 1: Acceptance–rejection algorithm to generate numbers from the $BBS3(\alpha, \beta, \delta)$ distribution

```

1 begin
  Input :  $n, \alpha, \beta, \gamma$ 
  Output:  $n\_vector$ ,
2  Set  $l_2 = \max_{y>0} \{f_Y(y)\}$ ;
3  Generate a value  $u_1$  from  $U_1 \sim U(0, l_1)$ ;
4  Obtain a value  $u_2$  from  $U_2 \sim U(0, l_2)$ ;
5  Set  $y = u_1$  from  $Y \sim BBS3(\alpha, \beta, \delta)$  if  $u_2 \leq f(u_1)$  and append  $y$  to  $n\_vector$ ;
  otherwise, go back to step 3;
6  Repeat steps 3–5 until the length of  $n\_vector$  is equal to  $n$ ;
7 end

```

The computational experiments were carried out in the R programming language, using a computer with the following characteristics: (i) OS: Windows 10 Pro for 64 bits; (ii) RAM: 8 Gigabytes; and (iii) Processor: Intel(R) Core(TM) i7-8550U CPU @ 1.99 GigaHertz. Algorithm 1 was executed 100 times with $n = 100,000$, and an average processing time equal to 2.9 seconds was obtained. When executing this algorithm 5000 times, with $n = 50,200$, the average processing times were 0.035 and 0.039 seconds, respectively.

5.2. Results of the Simulation Study

We use Algorithm 1 to conduct a Monte Carlo simulation study. The results of this study are reported in Table 2 to assess the behavior of the maximum likelihood estimators based on 5000 generated samples of sizes $n = 50, 100, 150, 200$ from a $BBS3(\alpha, \beta, \delta)$ distributed population. For each sample generated, maximum likelihood estimates were computed numerically using a Newton–Raphson procedure. Then, empirical means, SDs, average length (ALI) and covering probabilities (C) of 95% confidence intervals are reported. Note that the empirical bias becomes smaller as the sample size n increases, as expected. Table 3 shows the results of a simulation with 5000 samples but with smaller sizes ($n = 10, 25, 40$) and using moment estimation.

5.3. Illustrative Example I with Real-World Data

The first data set corresponds to the duration of the eruption for the Old Faithful geyser in Yellowstone National Park, Wyoming, USA; the data were reported in [31]. Descriptive statistics for the data set are the following: 272 (sample size), 43 (minimum), 96 (maximum), 76 (median), 70.897 (mean), 13.595 (SD), 19.176 (CV), -0.414 (CS) and 1.844 (CK). Table 4 reports the maximum likelihood estimates, standard errors (SEs), and log-likelihood (log-lik) values for the BBS1, BBS2, and BBS3 models. Furthermore, we report the values of

the Akaike (AIC) and Bayesian information (BIC) criteria as well as the p -values of the Kolmogorov–Smirnov (KS) test. Figure 2a shows the histogram with estimated PDFs, which indicates that the BBS3 model fits the data better than the BBS1 and BBS2 models. This result is supported by Figure 2b with the estimated CDF and by Figure 3 based on the theoretical versus empirical quantiles (QQ) plots.

Table 2. Empirical means, SDs, ALIs, and covering probabilities of 95% confidence intervals based on simulated data from the BBS3(α, β, δ) distribution for its parameter estimators using the maximum likelihood method.

n	α	β	δ	Mean($\hat{\alpha}$)	SD($\hat{\alpha}$)	ALI($\hat{\alpha}$)	C($\hat{\alpha}$)	Mean($\hat{\beta}$)	SD($\hat{\beta}$)	ALI($\hat{\beta}$)	C($\hat{\beta}$)	Mean($\hat{\delta}$)	SD($\hat{\delta}$)	ALI($\hat{\delta}$)	C($\hat{\delta}$)
50	0.1	0.2	0.8	0.0992	0.0067	0.0261	94.90	0.2003	0.0036	0.0143	94.80	1.0105	0.5855	2.2953	93.44
100	0.1	0.2	0.8	0.0995	0.0047	0.0183	95.22	0.2001	0.0024	0.0093	95.04	0.8951	0.3361	1.3174	94.30
150	0.1	0.2	0.8	0.0998	0.0038	0.0148	95.00	0.2001	0.0020	0.0078	95.34	0.8505	0.2367	0.9278	94.36
200	0.1	0.2	0.8	0.0998	0.0033	0.0128	95.06	0.2001	0.0017	0.0066	94.98	0.8489	0.2089	0.8188	94.64
50	0.2	0.3	0.5	0.1974	0.0165	0.0646	94.36	0.3019	0.0138	0.0541	94.70	0.6083	0.3473	1.3615	94.18
100	0.2	0.3	0.5	0.1983	0.0114	0.0446	94.58	0.3014	0.0094	0.0369	94.48	0.5549	0.1918	0.7518	93.60
150	0.2	0.3	0.5	0.1991	0.0095	0.0371	94.46	0.3006	0.0076	0.0299	94.94	0.5307	0.1422	0.5576	94.36
200	0.2	0.3	0.5	0.1994	0.0083	0.0325	94.72	0.3005	0.0067	0.0262	94.76	0.5212	0.1203	0.4714	95.16
50	0.5	0.5	0.5	0.4890	0.0601	0.2357	95.08	0.5220	0.0829	0.3248	93.90	0.6472	0.4457	1.7472	94.56
100	0.5	0.5	0.5	0.4947	0.0395	0.1550	95.06	0.5102	0.0537	0.2105	94.20	0.5576	0.2155	0.8447	94.28
150	0.5	0.5	0.5	0.4965	0.0323	0.1267	94.92	0.5067	0.0431	0.1690	94.64	0.5322	0.1494	0.5857	94.02
200	0.5	0.5	0.5	0.4971	0.0273	0.1071	94.56	0.5050	0.0365	0.1429	94.56	0.5233	0.1263	0.4951	94.32
50	0.5	0.8	0.2	0.4939	0.0439	0.1722	94.96	0.8191	0.0957	0.3751	94.26	0.2211	0.0969	0.3797	94.96
100	0.5	0.8	0.2	0.4978	0.0301	0.1179	94.90	0.8079	0.0643	0.2520	94.68	0.2075	0.0547	0.2143	94.86
150	0.5	0.8	0.2	0.4980	0.0245	0.0960	94.62	0.8061	0.0516	0.2024	94.56	0.2051	0.0440	0.1723	94.98
200	0.5	0.8	0.2	0.4986	0.0216	0.0846	95.12	0.8053	0.0451	0.1767	94.96	0.2035	0.0380	0.1492	95.04
50	0.5	0.8	0.4	0.4898	0.0560	0.2195	94.94	0.8309	0.1239	0.4858	93.92	0.4934	0.3101	1.2156	95.00
100	0.5	0.8	0.4	0.4955	0.0374	0.1466	94.30	0.8139	0.0823	0.3226	94.70	0.4366	0.1479	0.5797	94.48
150	0.5	0.8	0.4	0.4970	0.0305	0.1196	94.62	0.8100	0.0652	0.2558	94.76	0.4199	0.1089	0.4269	94.42
200	0.5	0.8	0.4	0.4975	0.0258	0.1011	94.74	0.8073	0.0554	0.2170	94.68	0.4151	0.0908	0.3559	94.52
50	0.8	0.8	0.1	0.7923	0.0670	0.2626	94.82	0.8212	0.1195	0.4685	94.62	0.1091	0.0397	0.1557	93.66
100	0.8	0.8	0.1	0.7961	0.0478	0.1872	94.68	0.8116	0.0830	0.3254	94.88	0.1037	0.0251	0.0983	94.82
150	0.8	0.8	0.1	0.7971	0.0386	0.1513	95.06	0.8079	0.0663	0.2600	94.62	0.1026	0.0200	0.0783	95.00
200	0.8	0.8	0.1	0.7971	0.0333	0.1303	94.80	0.8065	0.0564	0.2210	94.82	0.1023	0.0170	0.0667	94.68

Table 3. Empirical means, SDs, ALIs, and covering probabilities of 95% confidence intervals based on simulated data from the BBS3(α, β, δ) distribution for its parameter estimators using the moment method.

n	α	β	δ	Mean($\hat{\alpha}$)	SD($\hat{\alpha}$)	ALI($\hat{\alpha}$)	C($\hat{\alpha}$)	Mean($\hat{\beta}$)	SD($\hat{\beta}$)	ALI($\hat{\beta}$)	C($\hat{\beta}$)	Mean($\hat{\delta}$)	SD($\hat{\delta}$)	ALI($\hat{\delta}$)	C($\hat{\delta}$)
10	0.1	0.2	0.8	0.0928	0.0233	0.0912	97.44	0.2014	0.0143	0.0559	94.94	0.8100	0.0722	0.2829	95.06
25	0.1	0.2	0.8	0.0973	0.0123	0.0482	96.16	0.2007	0.0091	0.0357	95.02	0.8073	0.0624	0.2447	95.70
40	0.1	0.2	0.8	0.0984	0.0091	0.0357	94.88	0.2005	0.0074	0.0289	95.30	0.8044	0.0528	0.2070	96.56
10	0.2	0.3	0.5	0.1873	0.0491	0.1927	94.76	0.3094	0.0495	0.1939	94.88	0.5334	0.1666	0.6532	95.62
25	0.2	0.3	0.5	0.1953	0.0313	0.1228	95.10	0.3035	0.0309	0.1210	94.82	0.5232	0.1346	0.5276	95.08
40	0.2	0.3	0.5	0.1970	0.0249	0.0978	95.24	0.3025	0.0244	0.0956	94.88	0.5193	0.1175	0.4604	95.30
10	0.5	0.8	0.4	0.4925	0.0509	0.1996	98.22	0.7449	0.1350	0.5291	96.18	0.1278	1.3174	5.1644	98.66
25	0.5	0.8	0.4	0.4963	0.0482	0.1890	99.04	0.7608	0.1079	0.4231	93.90	0.1715	1.1857	4.6480	98.82
40	0.5	0.8	0.4	0.4973	0.0461	0.1806	98.98	0.7675	0.0948	0.3717	90.64	0.2090	0.8960	3.5124	97.96
10	0.5	0.5	0.5	0.5017	0.0509	0.1995	98.06	0.4818	0.0842	0.3302	90.06	0.4354	0.4059	1.5909	88.36
25	0.5	0.5	0.5	0.4994	0.0327	0.1282	98.38	0.4961	0.0466	0.1825	96.06	0.4963	0.2398	0.9402	95.68
40	0.5	0.5	0.5	0.4996	0.0171	0.0670	95.84	0.4986	0.0336	0.1316	96.50	0.5030	0.1781	0.6980	96.34

Table 4. Maximum likelihood estimates (and SEs in parenthesis) of the indicated distribution parameters with eruption data and model selection indicators.

Estimate/Indicator	BBS1 (SE)	BBS2 (SE)	BBS3 (SE)
$\hat{\alpha}$	0.0898 (0.0045)	0.1257 (0.0039)	0.1042 (0.0027)
$\hat{\beta}$	65.0852 (0.3740)	66.9851 (0.4785)	63.8997 (0.4378)
$\hat{\delta}$	−2.1242 (0.1392)	−3.8569 (0.0225)	1.7480 (0.4438)
Log-likelihood	−1054.180	−1050.573	−1036.824
AIC	2114.360	2107.146	2079.648
BIC	2129.325	2122.111	2090.465
KS statistic	0.1764	0.0808	0.0735
KS <i>p</i> -value	0.0004	0.3359	0.454

Figure 2. Histogram with estimated BBS1—dotted line, BBS2—dashed line, and BBS3—solid line, PDFs (a); and plots of empirical CDF versus the BBS1—dotted line, BBS2—dashed line, and BBS3—solid line, theoretical CDFs (b) with eruption data. **Figure 3.** QQ plots of the BBS1 (a), BBS2 (b) and BBS3 (c) distributions with eruption data.

Once the best model to describe these data has been selected, that is, the BBS3 distribution, and its parameters estimated, diverse statistical analyses can be conducted by practitioners and users of statistics. For example, by estimating the mean duration of the eruption of geysers, certain measures may be established to reduce the risk against people visiting the Yellowstone National Park in Wyoming, USA. In our case, the mean duration of the eruption of geysers in Yellowstone Park is 49.205 minutes. This information is

obtained from the estimates reported in Table 4 and the expressions taken from Section 3.4 specified as

$$\mu = E(Y) = \frac{1}{1 + \kappa \delta} \left(\left(1 + \frac{\delta \mu^4}{\sigma^4} \right) E(X^r) + \frac{\delta}{\sigma^4} \left(E(X^5) - 4\mu E(X^4) + 6\mu^2 E(X^3) - 4\mu^3 E(X^2) \right) \right),$$

where

$$\begin{aligned} E(X) &= \frac{1}{2}\beta(\alpha^2 + 2) \\ E(X^2) &= \frac{1}{2}\beta^2(3\alpha^4 + 4\alpha^2 + 2) \\ E(X^3) &= \frac{1}{2}\beta^3(15\alpha^6 + 18\alpha^4 + 9\alpha^2 + 2) \\ E(X^4) &= \frac{1}{2}\beta^4(105\alpha^8 + 120\alpha^6 + 60\alpha^4 + 16\alpha^2 + 2) \\ E(X^5) &= \frac{\beta^5}{2}(945\alpha^{10} + 1050\alpha^8 + 525\alpha^6 + 150\alpha^4 + 25\alpha^2 + 2) \\ E(X^6) &= \frac{\beta^6}{2}(10,395\alpha^{12} + 11,340\alpha^{10} + 5670\alpha^8 + 1680\alpha^6 + 315\alpha^4 + 36\alpha^2 + 2) \end{aligned}$$

5.4. Illustrative Example II with Real-World Data

Fleroxacin is a fluoroquinolone derivative for a broad antibacterial spectrum and powerful in vitro activity against various gram-negative and many other gram-positive varieties. This antibacterial element was studied in [32] with the aim of estimating the representative values of clearance on systemic availability (D/F) and volume of distribution on systemic availability (I/F), after administering therapeutic doses of fleroxacin. In addition, the factors that influence the disposition of fleroxacin were detected and to what degree. The individuals studied were 172 volunteers, men and women, healthy, without infections and within a wide range of ages. Among the data that were analyzed in [32] are the following measurements (in ml/min) of clearance on D/F and creatinine clearance (Dcr). In this second illustration with real data, we consider a data set corresponding to the Dcr. Table 5 provides descriptive statistics of the Dcr data. Table 6 reports values of the maximum likelihood estimates, SEs, log-lik, AIC, and BIC as well as the KS p -values for the BBS1, BBS2, and BBS3 models. Figure 4 shows the good agreement of the BBS3 model in relation to the BBS2 and BBS2 models, such as supported by the QQ plots of Figure 5.

Table 5. Summary statistics with Dcr data.

n	Mean	SD	CS	CK
172	69.623	38.0311	−0.2112	2.1637

Table 6. Maximum likelihood estimates (and SEs in parenthesis) of the indicated distribution parameters with Dcr data and model selection indicators.

Estimate/Indicator	BBS1(SE)	BBS1 (SE)	BBS3 (SE)
$\hat{\alpha}$	0.4308 (0.0252)	0.6976 (0.0283)	0.6784 (0.0442)
$\hat{\beta}$	27.9125 (0.9800)	31.7897 (1.4594)	15.6625 (1.5963)
$\hat{\delta}$	−2.5689 (0.1831)	−2.6272 (0.3830)	0.4213 (0.1031)
Log-likelihood	−870.4175	−851.217	−838.790
AIC	1746.835	1708.434	1683.580
BIC	1756.277	1717.876	1692.602
KS statistic	0.0698	0.0640	0.0407
KS p -value	0.4329	0.4949	0.7521

Figure 4. Histogram with estimated BBS1—dotted line, BBS2—dashed line, and BBS3—solid line, PDFs (a); and plots of empirical CDF versus the BBS1—dotted line, BBS2—dashed line, and BBS3—solid line, theoretical CDFs (b) with Dcr data.

Figure 5. QQ plots of the BBS1 (a), BBS2 (b) and BBS3 (c) distributions with Dcr data.

Similarly to Example I, once the best model to describe these data has been selected, that is, the BBS3 distribution, and its parameters estimated, we obtain the mean creatinine clearance, which is 70.602 mL/min. Once again, this information is obtained from the estimates reported in Table 6 and the expressions taken from Section 3.4 specified in Example I.

6. Conclusions, Limitations and Future Investigation

In this paper, we have proposed a new bimodal Birnbaum–Saunders distribution, which is widely flexible. We have estimated the model parameters with the moment and maximum likelihood methods and used information criteria for model selection to assess the adequacy of the new bimodal Birnbaum–Saunders distribution in comparison with other competitor distributions.

We have conducted a Monte Carlo simulation study to empirically evaluate the statistical performance of the moment and maximum likelihood estimators of the new bimodal Birnbaum–Saunders distribution. Furthermore, we have studied coverage probabilities and the average length of confidence intervals of the corresponding parameters based on the asymptotic normality of these estimators. This simulation study reported a good statistical performance of such estimators.

Two data analyses were conducted related to environmental and medical sciences with two competing models. Both of these analyses reported a suitable performance of the new model that was superior to all of the competing models, providing evidence that the bimodal Birnbaum–Saunders distribution is a good alternative for modeling bimodal data. Such results reported that the bimodal Birnbaum–Saunders model may be a new setting for analyzing this type of data. The new approach is an addition to the tools of applied statisticians and diverse practitioners interested in the modeling of data. From these applications, we have obtained helpful information that can be employed by practitioners and users of statistics.

Some limitations of our methodology are the instability of the maximum likelihood estimators for small samples. The bootstrap and Jackknife methods and the application of bias correction, mainly for small sample sizes, can be proposed to improve the statistical performance of these estimators. In addition, the use of explanatory variables in the modeling is of interest. Thus, their effect on the different parameters of the new bimodal Birnbaum–Saunders distribution is a topic to be studied in a future work, as detailed next. Other topics of future research based on this new distribution are related to studying how multivariate, regression, quantile, spatial, temporal, partial least squares, principal components, and sampling structures can be inserted in the modeling of bimodal data [33–44]. In addition, Tobit and Cobb–Douglas type frameworks may be analyzed in the present thematic [45,46]. Moreover, censored observations might be studied in the context of our investigation [7,47].

The authors are working on these and other issues associated with the present investigation, and the corresponding findings are expected to be reported in future works.

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