



Article Inertial Extragradient Methods for Solving Split Equilibrium Problems

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Abstract: This paper presents two inertial extragradient algorithms for finding a solution of split pseudomonotone equilibrium problems in the setting of real Hilbert spaces. The weak and strong convergence theorems of the introduced algorithms are presented under some constraint qualifications of the scalar sequences. The discussions on the numerical experiments are also provided to demonstrate the effectiveness of the proposed algorithms.

Keywords: split equilibrium problems; pseudomonotone bifunction; extragradient method; inertial method

MSC: 47J25; 65K15; 90C33

1. Introduction

The equilibrium problem is a problem of finding a point $x^* \in C$ such that

$$f(x^*, y) \ge 0, \forall y \in C, \tag{1}$$

where *C* is a nonempty closed convex subset of a real Hilbert space *H*, and $f : H \times H \to \mathbb{R}$ is a bifunction. The solution set of the equilibrium problem (1) will be represented by EP(f, C). It is well known that the equilibrium problem (1) can be applied to many mathematical problems, such as optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems, saddle point problems, and fixed point problems (see [1–4], and the references therein). It is pointed out that one of the most popular methods for solving the equilibrium problem (1), when *f* is a monotone bifunction, is the proximal point method (see [5]). However, the proximal point method cannot be guaranteed for a weaker assumption, such as *f* is a pseudomonotone bifunction. To overcome this drawback, Tran et al. [6] proposed the following so-called extragradient method for solving the equilibrium problem, when the bifunction *f* is pseudomonotone and satisfies Lipschitz-type continuous conditions with positive constants c_1 and c_2 :

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\left\{\lambda f(x_k, y) + \frac{1}{2} \|y - x_k\|^2 : y \in C\right\}, \\ x_{k+1} = \arg\min\left\{\lambda f(y_k, y) + \frac{1}{2} \|y - x_k\|^2 : y \in C\right\}, \end{cases}$$
(2)



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Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. They proved that the sequence $\{x_k\}$ generated by (2) converges weakly to a solution of the equilibrium problem (1).

Meanwhile, the inertial-type methods have received a lot of attention from many researchers. This method originates from an implicit discretization method (heavy ball method) of the second-order dynamical in time [7,8] and can be regarded as a method of speeding up the convergence properties. In general, the main feature of the inertial-type methods is that the next iterate is constructed by the two previous iterates. The inertial techniques have been proposed for solving the equilibrium problems, for instance, [9,10] and the references therein. In 2019, by using the ideas of inertial and extragradient methods, Vinh and Muu [11] proposed the following method for solving the equilibrium problem, when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous conditions with positive constants c_1 and c_2 :

$$\begin{cases} x_0, x_1 \in C, \\ w_k = x_k + \theta_k (x_k - x_{k-1}), \\ y_k = \arg\min\left\{\lambda f(w_k, y) + \frac{1}{2} \|y - w_k\|^2 : y \in C\right\}, \\ x_{k+1} = \arg\min\left\{\lambda f(y_k, y) + \frac{1}{2} \|y - w_k\|^2 : y \in C\right\}, \end{cases}$$
(3)

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and θ_k is a suitable parameter. They proved that the sequence $\{x_k\}$ generated by (3) converges weakly to a solution of the equilibrium problem (1). Observe that, in the case of $\theta_k = 0$, for all $k \in \mathbb{N}$, the algorithm (3) is nothing but the algorithm (2). Moreover, in [11], the authors proposed the following method, when the bifunction *f* is pseudomonotone and satisfies Lipschitz-type continuous conditions with positive constants c_1 and c_2 :

$$\begin{cases} x_0, x_1 \in C, \\ w_k = x_k + \theta_k (x_k - x_{k-1}), \\ y_k = \arg\min\left\{\lambda f(w_k, y) + \frac{1}{2} \|y - w_k\|^2 : y \in C\right\}, \\ z_k = \arg\min\left\{\lambda f(y_k, y) + \frac{1}{2} \|y - w_k\|^2 : y \in C\right\}, \\ x_{k+1} = (1 - \beta_k - \gamma_k)w_k + \beta_k z_k, \end{cases}$$
(4)

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\{\beta_k\}, \{\gamma_k\} \subset (0, 1)$ such that $\sum_{k=0}^{\infty} \gamma_k = \infty$, $\lim_{k \to \infty} \gamma_k = 0$, $\inf_{k \to \infty} \beta_k (1 - \beta_k - \gamma_k) > 0$, and θ_k is a suitable parameter. They proved that the sequence $\{x_k\}$ generated by (4) converges strongly to the minimum-norm element of the solution of the equilibrium problem (1).

On the other hand, Censer and Elfving [12] proposed the following split feasibility problems:

Find
$$x^* \in C$$
 such that $Ax^* \in Q$, (5)

where *C* and *Q* are two nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2 , respectively, and $A : H_1 \rightarrow H_2$ is a bounded linear operator. Many important problems arising from real-world problems can be formulated as the split feasibility problems which had been used for studying signal processing, medical image reconstruction, intensity-modulated radiation therapy, sensor networks, and data compression (see [12–15] and the references therein).

In 2012, He [16] (see also Moudafi [17]) introduced the split equilibrium problems, as the generalization of the split feasibility problems (5), as follows:

$$\begin{cases} \text{Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \ge 0, \forall y \in C, \\ \text{and} \quad u^* := Ax^* \in Q \quad \text{solves} \quad g(u^*, v) \ge 0, \forall v \in Q, \end{cases}$$
(6)

where *C*, *Q* are two nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2 , respectively, $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ are bifunctions, and $A : H_1 \to H_2$ is a bounded linear operator. To solve the split equilibrium problems (6), He [16] proposed the following proximal point method, when the bifunctions *f* and *g* are monotone:

$$\begin{cases} x_{0} \in C, \\ y_{k} \in C \text{ such that } f(y_{k}, y) + \frac{1}{r_{k}} \langle y - y_{k}, y_{k} - x_{k} \rangle \geq 0, \forall y \in C, \\ u_{k} \in Q \text{ such that } g(u_{k}, v) + \frac{1}{r_{k}} \langle v - u_{k}, u_{k} - Ay_{k} \rangle \geq 0, \forall v \in Q, \\ x_{k+1} = P_{C}(y_{k} + \eta A^{*}(u_{k} - Ay_{k})), \end{cases}$$
(7)

where $\eta \in (0, 1/||A||^2)$, $\{r_k\} \subset (0, +\infty)$ with $\liminf_{k\to\infty} r_k > 0$, and A^* is the adjoint operator of A. He proved that the sequence $\{x_k\}$ generated by (7) converges weakly to a solution of the split equilibrium problems (6). Here, the algorithm (7) will be called the PPA Algorithm. After that, under the setting of $f: H_1 \times H_1 \to \mathbb{R}$ and $g: H_2 \times H_2 \to \mathbb{R}$, Kim and Dinh [18] proposed the following the extragradient method for finding a solution of the split equilibrium problems, when the bifunctions f and g are pseudomonotone and satisfy Lipschitz-type continuous conditions with positive constants c_1 and c_2 :

$$\begin{cases} x_{0} \in C, \\ y_{k} = \arg\min\left\{\lambda_{k}f(x_{k}, y) + \frac{1}{2}||y - x_{k}||^{2} : y \in C\right\}, \\ z_{k} = \arg\min\left\{\lambda_{k}f(y_{k}, y) + \frac{1}{2}||y - x_{k}||^{2} : y \in C\right\}, \\ u_{k} = \arg\min\left\{\mu_{k}g(Az_{k}, u) + \frac{1}{2}||u - Az_{k}||^{2} : u \in Q\right\}, \\ v_{k} = \arg\min\left\{\mu_{k}g(u_{k}, u) + \frac{1}{2}||u - Az_{k}||^{2} : u \in Q\right\}, \\ x_{k+1} = P_{C}(z_{k} + \eta A^{*}(v_{k} - Az_{k})), \end{cases}$$

$$(8)$$

where $\eta \in (0, 1/||A||^2)$, and $\{\lambda_k\}$, $\{\mu_k\} \subset [\underline{\rho}, \overline{\rho}]$ with $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$. They proved that the sequence $\{x_k\}$ generated by (8) converges weakly to a solution of the split equilibrium problems. Here, the algorithm (8) will be called the PEA Algorithm. We point out that the algorithm (8) cannot be applied for solving the problem (6) under the setting of $g : Q \times Q \to \mathbb{R}$, since we can not guarantee if Az_k belongs to the considered closed convex set Q.

In this paper, we will continue developing methods for solving the split equilibrium problems (6). That is, some new iterative algorithms will be introduced for finding the solutions of the split equilibrium problems, when the considered bifunctions are pseudomonotone. Some numerical examples and comparison of the introduced methods with the aforesaid algorithms will be discussed.

This paper is organized as follows: In Section 2, some definitions and properties will be reviewed for use in subsequent sections. Section 3 will present two inertial extragradient algorithms and prove their convergence theorems. In Section 4, we will discuss the performance of the two introduced algorithms by comparing to the well-known algorithms.

2. Preliminaries

This section will present the definitions and some important basic properties that will be used in this paper. Let *H* be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and its corresponding $\|\cdot\|$. The symbols \rightarrow and \rightarrow will be denoted for the strong convergence and the weak convergence in *H*, respectively.

First, we will recall definitions and facts for concerning the equilibrium problems.

Definition 1 ([1,3,19]). *Let C be a nonempty closed convex subset of H*. *A bifunction* $f : H \times H \rightarrow \mathbb{R}$ *is said to be:*

(i) monotone on C if

$$f(x,y) + f(y,x) \le 0, \forall x,y \in C;$$

(ii) pseudomonotone on C if

$$f(x,y) \ge 0 \Rightarrow f(y,x) \le 0, \forall x,y \in C;$$

(iii) Lipshitz-type continuous on H with constants $L_1 > 0$ and $L_2 > 0$ if

$$f(x,y) + f(y,z) \ge f(x,z) - L_1 ||x - y||^2 - L_2 ||y - z||^2, \forall x, y, z \in H.$$

Remark 1. A monotone bifunction is a pseudomonotone bifunction, but the converse is not true in general, for instance, see [20].

For a nonempty closed convex subset *C* of *H* and a bifunction $f : H \times H \rightarrow \mathbb{R}$, we are concerned with the following assumptions in this paper:

Assumption 1. *f* is weakly continuous on $C \times C$ in the sense that, if $x \in C$, $y \in C$, and $\{x_k\} \subset C$, $\{y_k\} \subset C$ are two sequences that converge weakly to *x* and *y* respectively, then $f(x_k, y_k)$ converges to f(x, y).

Assumption 2. $f(x, \cdot)$ is convex and subdifferentiable on *C*, for each fixed $x \in C$.

Assumption 3. *f* is psuedomonotone on *C* and f(x, x) = 0, for each $x \in C$.

Assumption 4. *f* is Lipshitz-type continuous on H with constants $L_1 > 0$ and $L_2 > 0$.

Remark 2. We note that the solution set EP(f, C) is closed and convex, when the bifunction f satisfies the Assumptions 1–3 (see [6,21,22] for more detail).

The following lemma is important in order to obtain the main results of this paper.

Lemma 1 ([23]). Let $f : H \times H \to \mathbb{R}$ be satisfied Assumptions 2–4. Assume that EP(f, C) is a nonempty set and $0 < \lambda_0 < \min\left\{\frac{1}{2L_1}, \frac{1}{2L_2}\right\}$. Let $x_0 \in H$. If y_0 and z_0 are constructed by

$$\begin{cases} y_0 = \arg\min\left\{\lambda_0 f(x_0, w) + \frac{1}{2} \|w - x_0\|^2 : w \in C\right\},\\ z_0 = \arg\min\left\{\lambda_0 f(y_0, w) + \frac{1}{2} \|w - x_0\|^2 : w \in C\right\}, \end{cases}$$

then,

 $\begin{array}{ll} (i) & \lambda_0 \left[f(x_0, w) - f(x_0, y_0) \right] \ge \langle y_0 - x_0, y_0 - w \rangle, \, \forall w \in C; \\ (ii) & \| z_0 - q \|^2 \le \| x_0 - q \|^2 - (1 - 2\lambda_0 L_1) \| x_0 - y_0 \|^2 - (1 - 2\lambda_0 L_2) \| y_0 - z_0 \|^2, \, \forall q \in EP(f, C). \end{array}$

Next, we recall some basic facts in the functional analysis which will be used in the sequel. For a Hilbert space *H*, we know that

$$|x+y||^2 \le ||x||^2 + 2\langle y, x+y \rangle,$$
 (9)

and

$$\|\alpha x + \beta y + \gamma z\|^{2} = \alpha \|x\|^{2} + \beta \|y\|^{2} + \gamma \|z\|^{2} - \alpha \beta \|x - y\|^{2} - \beta \gamma \|y - z\|^{2} - \alpha \gamma \|x - z\|^{2},$$
(10)

for each $x, y, z \in H$, and for each $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$ (see [11]).

For each $x \in H$, we denote the metric projection of x onto a nonempty closed convex subset C of H by $P_C(x)$, that is,

$$||x - P_C(x)|| \le ||y - x||, \forall y \in C.$$

Lemma 2 (see [24,25]). let C be a nonempty closed convex subset of H. Then,

- (*i*) $P_{C}(x)$ is singleton and well-defined for each $x \in H$;
- (*ii*) $z = P_C(x)$ *if and only if* $\langle x z, y z \rangle \le 0$, $\forall y \in C$;
- (iii) P_C is a nonexpansive operator, that is,

$$||P_C(x) - P_C(y)|| \le ||x - y||, \forall x, y \in H.$$

For a function $g : H \to \mathbb{R}$, the subdifferential of g at $z \in H$ is defined by

$$\partial g(z) = \{ w \in H : g(y) - g(z) \ge \langle w, y - z \rangle, \forall y \in H \}.$$

The function *g* is said to be subdifferentiable at *z* if $\partial g(z) \neq \emptyset$.

Lemma 3 (see [24]). For any $z \in H$, the subdifferential $\partial g(z)$ of a continuous convex function g is a nonempty, weakly closed, and bounded convex set.

We end this section by recalling some auxiliary results for obtaining the convergence theorems.

Lemma 4 ([26]). Let *H* be a Hilbert space and $\{x_k\}$ a sequence in *H* such that there exists a nonempty set $S \subset H$ satisfying

- (i) For each $z \in S$, $\lim_{k \to \infty} ||x_k z||$ exists;
- (ii) $\omega_w(x_k) \subset S$, where $\omega_w(x_k) = \{x \in H : \text{there is a subsequence } \{x_{k_n}\} \text{ of } \{x_k\} \text{ such that } x_{k_n} \rightharpoonup x\}.$

Then, there exists $x^* \in S$ such that the sequence $\{x_k\}$ converges weakly to x^* .

Lemma 5 ([27]). Let $\{a_k\}$ and $\{b_k\}$ be sequences of non-negative real numbers such that $a_{k+1} \le a_k + b_k$, $\forall k \in \mathbb{N}$. If $\sum_{k=1}^{\infty} b_k < \infty$, then $\lim_{k \to \infty} a_k$ exists.

Lemma 6 ([28,29]). Let $\{a_k\}$ and $\{c_k\}$ be sequences of non-negative real numbers such that

$$a_{k+1} \leq (1-\delta_k)a_k + b_k + c_k, \forall k \in \mathbb{N} \cup \{0\},$$

where $\{\delta_k\}$ is a sequence in (0,1) and $\{b_k\}$ is a sequence in \mathbb{R} . Assume $\sum_{k=0}^{\infty} c_k < \infty$. Then the following results hold:

(i) If there is M > 0 such that $b_k \leq \delta_k M$, for all $k \in \mathbb{N} \cup \{0\}$, then $\{a_k\}$ is a bounded sequence;

(ii) If
$$\sum_{k=0}^{\infty} \delta_k = \infty$$
 and $\limsup_{k \to \infty} (b_k / \delta_k) \le 0$, then $\lim_{k \to \infty} a_k = 0$

Lemma 7 ([30]). Let $\{a_k\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ such that $a_{k_i} < a_{k_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_n\}$ of positive integers such that $\lim_{n\to\infty} m_n = \infty$ and the following properties hold:

$$a_{m_n} \leq a_{m_n+1} \text{ and } a_n \leq a_{m_n+1}$$

for all (sufficiently large) numbers $n \in \mathbb{N}$. Indeed, m_n is the largest number k in the set $\{1, 2, ..., n\}$ such that

$$a_k < a_{k+1}$$

3. Main Results

Let H_1 and H_2 be two real Hilbert spaces and C and Q be nonempty closed convex subsets of H_1 and H_2 , respectively. Suppose that $f: H_1 \times H_1 \to \mathbb{R}$ and $g: H_2 \times H_2 \to \mathbb{R}$

are bifunctions which satisfy Assumptions 1–4 with some positive constants $\{c_1, c_2\}$ and $\{d_1, d_2\}$, respectively. Let us recall the split equilibrium problems:

Find
$$x^* \in C$$
 such that $f(x^*, y) \ge 0, \forall y \in C$,
and $u^* := Ax^* \in Q$ solves $g(u^*, v) \ge 0, \forall v \in Q$, (11)

where $f: H_1 \times H_1 \to \mathbb{R}$ and $g: H_2 \times H_2 \to \mathbb{R}$ are bifunctions, and $A: H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^* . From now on, the solution set of problem (11) will be denoted by Ω . That is,

$$\Omega := EP(f, C) \cap A^{-1}(EP(g, Q)).$$

3.1. Inertial Extragradient Method

Now, we introduce Algorithm 1 for solving the split equilibrium problems (11).

Algorithm 1: Inertial Extragradient Method (IEM)

Initialization. Choose parameters $\alpha \in [0,1)$, $\eta \in \left(0, \frac{1}{\|A\|^2}\right)$, $\{\lambda_k\}$ with $0 < \inf \lambda_k \le \sup \lambda_k < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\{\mu_k\}$ with $0 < \inf \mu_k \le \sup \mu_k < \min\left\{\frac{1}{2d_1}, \frac{1}{2d_2}\right\}$, and $\{\epsilon_k\} \subset [0,\infty)$ such that $\sum_{k=0}^{\infty} \epsilon_k < \infty$. Pick $x_0, x_1 \in C$ and set k = 1.

Step 1. Choose θ_k such that $0 \le \theta_k \le \overline{\theta}_k$, where

$$\overline{\theta}_{k} = \begin{cases} \min\left\{\alpha, \frac{\epsilon_{k}}{\|x_{k} - x_{k-1}\|}\right\}, & \text{if } x_{k} \neq x_{k-1}, \\ \alpha, & \text{otherwise,} \end{cases}$$

and compute

$$w_k = x_k + \theta_k (x_k - x_{k-1}).$$

Step 2. Solve the strongly convex program

$$y_k = \arg\min\bigg\{\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\bigg\}.$$

Step 3. Solve the strongly convex program

$$z_k = \arg\min\bigg\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\bigg\}.$$

Step 4. Solve the strongly convex program

$$u_k = \arg\min\left\{\mu_k g(Az_k, u) + \frac{1}{2} \|u - Az_k\|^2 : u \in Q\right\}.$$

Step 5. Solve the strongly convex program

$$v_k = \arg\min\left\{\mu_k g(u_k, u) + \frac{1}{2}\|u - Az_k\|^2 : u \in Q\right\}.$$

Step 6. The next approximation x_{k+1} is defined by

$$x_{k+1} = P_C(z_k + \eta A^*(v_k - Az_k)).$$

Step 7. Put k := k + 1 and go to **Step 1**.

Remark 3. We pointed out that the term $\theta_k(x_k - x_{k-1})$, which is included in the IEM Algorithm, is intended to speed up convergence properties and is called the inertial effect. We emphasize that the choice of parameter θ_k may lead to the superior numerical behavior of the IEM Algorithm. Moreover, we observe that if $\theta_k = 0$, for each $k \in \mathbb{N}$, then the IEM Algorithm reduces to the PEA Algorithm (8), which was presented in [18].

Theorem 1. Suppose that the solution set Ω is nonempty. Then, the sequence $\{x_k\}$ which is generated by the IEM Algorithm converges weakly to an element of Ω .

Proof. Let $p \in \Omega$. That is, $p \in EP(f, C)$, and $Ap \in EP(g, Q)$. Then, by Lemma 1 (ii), we have

$$||z_k - p||^2 \le ||w_k - p||^2 - (1 - 2\lambda_k c_1) ||w_k - y_k||^2 - (1 - 2\lambda_k c_2) ||y_k - z_k||^2.$$
(12)

This implies that

$$||z_k - p|| \le ||w_k - p||.$$
(13)

By the definition of w_k , we have

$$\begin{aligned} \|w_{k} - p\|^{2} &= \|(1 + \theta_{k})(x_{k} - p) - \theta_{k}(x_{k-1} - p)\|^{2} \\ &= (1 + \theta_{k})\|x_{k} - p\|^{2} - \theta_{k}\|x_{k-1} - p\|^{2} + \theta_{k}(1 + \theta_{k})\|x_{k} - x_{k-1}\|^{2} \\ &\leq (1 + \theta_{k})\|x_{k} - p\|^{2} - \theta_{k}\|x_{k-1} - p\|^{2} + 2\theta_{k}\|x_{k} - x_{k-1}\|^{2}. \end{aligned}$$
(14)

Thus, in view of (12) and (14), we obtain

$$||z_{k} - p||^{2} - ||x_{k} - p||^{2} \leq \theta_{k}(||x_{k} - p||^{2} - ||x_{k-1} - p||^{2}) + 2\theta_{k}||x_{k} - x_{k-1}||^{2} - (1 - 2\lambda_{k}c_{1})||w_{k} - y_{k}||^{2} - (1 - 2\lambda_{k}c_{2})||y_{k} - z_{k}||^{2}.$$
 (15)

On the other hand, by Lemma 1 (ii), we obtain

$$\|v_k - Ap\|^2 \le \|Az_k - Ap\|^2 - (1 - 2\mu_k d_1)\|Az_k - u_k\|^2 - (1 - 2\mu_k d_2)\|u_k - v_k\|^2.$$
(16)

This implies that

$$|v_k - Ap|| \le ||Az_k - Ap||.$$
(17)

By the definition of x_{k+1} and the nonexpansivity of P_C , we have

$$\|x_{k+1} - p\|^2 \leq \|(z_k - p) + \eta A^*(v_k - Az_k)\|^2$$

= $\|z_k - p\|^2 + \eta^2 \|A\|^2 \|v_k - Az_k\|^2 + 2\eta \langle A(z_k - p), v_k - Az_k \rangle.$ (18)

Consider,

$$2\langle A(z_k - p), v_k - Az_k \rangle = 2\langle v_k - Ap, v_k - Az_k \rangle - 2 \|v_k - Az_k\|^2$$

= $\|v_k - Ap\|^2 - \|v_k - Az_k\|^2 - \|Az_k - Ap\|^2.$ (19)

Using this one together with (18), we obtain

$$\|x_{k+1} - p\|^{2} \leq \|z_{k} - p\|^{2} - \eta(1 - \eta \|A\|^{2}) \|v_{k} - Az_{k}\|^{2} + \eta(\|v_{k} - Ap\|^{2} - \|Az_{k} - Ap\|^{2}).$$

$$(20)$$

Combining with (17) implies that

$$\|x_{k+1} - p\|^2 \leq \|z_k - p\|^2 - \eta(1 - \eta \|A\|^2) \|v_k - Az_k\|^2.$$
(21)

Thus, by the choice of η , we have

$$||x_{k+1} - p|| \leq ||z_k - p||.$$
(22)

Now, the relations (13) and (22) imply that

$$||x_{k+1} - p|| \leq ||w_k - p||.$$
(23)

So, it follows from the definition of w_k that

$$||x_{k+1} - p|| \leq ||x_k - p|| + \theta_k ||x_k - x_{k-1}||.$$
(24)

Due to the properties of the sequences $\{\theta_k\}$ and $\{\epsilon_k\}$, we observe that

$$\sum_{k=1}^{\infty} \theta_k \|x_k - x_{k-1}\| \le \sum_{k=1}^{\infty} \overline{\theta}_k \|x_k - x_{k-1}\| \le \sum_{k=1}^{\infty} \epsilon_k < \infty.$$

$$(25)$$

Then, by (24), (25), and Lemma 5, we show $\lim_{k\to\infty} ||x_k - p||$ exists. Consequently, the sequence $\{x_k\}$ is bounded. In addition, in view of (15) and (22), we see that

$$(1 - 2\lambda_k c_1) \|w_k - y_k\|^2 + (1 - 2\lambda_k c_2) \|y_k - z_k\|^2 \leq \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + \theta_k (\|x_k - p\|^2) - \|x_{k-1} - p\|^2) + 2\theta_k \|x_k - x_{k-1}\|^2.$$
(26)

Thus, by the choices of the control sequences $\{\lambda_k\}$ together with the existence of $\lim_{k\to\infty} ||x_k - p||$ and $\lim_{k\to\infty} \theta_k ||x_k - x_{k-1}|| = 0$, we have

$$\lim_{k \to \infty} \|w_k - y_k\| = 0, \tag{27}$$

and

$$\lim_{k \to \infty} \|y_k - z_k\| = 0.$$
⁽²⁸⁾

These imply that

$$\lim_{k \to \infty} \|w_k - z_k\| = 0.$$
⁽²⁹⁾

Moreover, by using $\lim_{k \to \infty} \theta_k ||x_k - x_{k-1}|| = 0$, we have

$$\lim_{k \to \infty} \|w_k - x_k\| = 0.$$
(30)

Using this one together with (27), we obtain

$$\lim_{k \to \infty} \|x_k - y_k\| = 0. \tag{31}$$

Thus, it follows from (28) that

$$\lim_{k \to \infty} \|x_k - z_k\| = 0.$$
(32)

On the other hand, by using (21), we see that

$$\eta(1 - \eta \|A\|^{2}) \|v_{k} - Az_{k}\|^{2} \leq \|z_{k} - p\|^{2} - \|x_{k+1} - p\|^{2} \\ \leq (\|z_{k} - x_{k}\| + \|x_{k} - p\| - \|x_{k+1} - p\|)(\|z_{k} - p\| + \|x_{k+1} - p\|).$$
(33)

Thus, by the existence of $\lim_{k\to\infty} ||x_k - p||$ and (32), we have

$$\lim_{k \to \infty} \|v_k - Az_k\| = 0. \tag{34}$$

Furthermore, in view of (16), we obtain

$$(1 - 2\mu_k d_1) \|Az_k - u_k\|^2 + (1 - 2\mu_k d_2) \|u_k - v_k\|^2 \leq \|Az_k - Ap\|^2 - \|v_k - Ap\|^2 \\ = \|Az_k - v_k\|(\|Az_k - Ap\|) \\ + \|v_k - Ap\|).$$

Thus, applying (34) to the above inequality, we have

$$\lim_{k \to \infty} \|Az_k - u_k\| = 0, \tag{35}$$

and

$$\lim_{k \to \infty} \|u_k - v_k\| = 0.$$
(36)

Now, we will complete the proof of this theorem by using Lemma 4. Notice that, it remains to show $\omega_w(x_k) \subset \Omega$. Let $x^* \in \omega_w(x_k)$ and $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_n} \rightharpoonup x^*$, as $n \rightarrow \infty$.

We know that, by using (30)–(32), we also have $w_{k_n} \rightarrow x^*$, $y_{k_n} \rightarrow x^*$, and $z_{k_n} \rightarrow x^*$, as $n \rightarrow \infty$. The latter fact also implies that $Az_{k_n} \rightarrow Ax^*$, as $n \rightarrow \infty$. Using this one together with (35), we obtain $u_{k_n} \rightarrow Ax^*$, as $n \rightarrow \infty$. Since *C* and *Q* are closed and convex sets, so *C* and *Q* are weakly closed, therefore, $x^* \in C$ and $Ax^* \in Q$. By Lemma 1 (i), we have

$$\lambda_{k_n}[f(w_{k_n}, y) - f(w_{k_n}, y_{k_n})] \ge \langle y_{k_n} - w_{k_n}, y_{k_n} - y \rangle, \forall y \in C,$$

and

$$\mu_{k_n}[g(Az_{k_n}, u) - g(Az_{k_n}, u_{k_n})] \ge \langle u_{k_n} - Az_{k_n}, u_{k_n} - u \rangle, \forall u \in Q.$$

These imply that

$$f(w_{k_n}, y) - f(w_{k_n}, y_{k_n}) \ge -\frac{1}{\lambda_{k_n}} ||y_{k_n} - w_{k_n}|| ||y_{k_n} - y||, \forall y \in C,$$

and

$$g(Az_{k_n}, u) - g(Az_{k_n}, u_{k_n}) \ge -\frac{1}{\mu_{k_n}} ||u_{k_n} - Az_{k_n}|| ||u_{k_n} - u||, \forall u \in Q.$$

Thus, by using (27), (35), and the weak continuity of f and g, we have

$$f(x^*, y) \ge 0, \forall y \in C$$

and

$$g(Ax^*, u) \ge 0, \forall u \in Q.$$

Then, we show that $x^* \in \Omega$. This shows that $\omega_w(x_k) \subset \Omega$. Hence, by Lemma 4, we can conclude that the sequence $\{x_k\}$ converges weakly to an element of Ω . This completes the proof. \Box

3.2. Mann-Type Inertial Extragradient Method

In order to obtain a strong convergence result, we propose Algorithm 2 by using the Mann-type techniques (see [11,31]).

Algorithm 2: Mann-type Inertial Extragradient Method (MIEM)

Initialization. Choose parameters $\alpha \in [0,1)$, $\eta \in \left(0, \frac{1}{\|A\|^2}\right)$, $\{\lambda_k\}$ with $0 < \inf \lambda_k \le \sup \lambda_k < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\{\mu_k\}$ with $0 < \inf \mu_k \le \sup \mu_k < \min\left\{\frac{1}{2d_1}, \frac{1}{2d_2}\right\}$, and $\{\epsilon_k\} \subset [0,\infty)$, $\{\beta_k\} \subset (0,1)$, $\{\gamma_k\} \subset (0,1)$ such that $\inf_{k\to\infty} \beta_k(1-\beta_k-\gamma_k) > 0$, $\sum_{k=0}^{\infty} \gamma_k = \infty$, $\lim_{k\to\infty} \gamma_k = 0$, $\sum_{k=0}^{\infty} \epsilon_k < \infty$, and $\epsilon_k = o(\gamma_k)$, where $\epsilon_k = o(\gamma_k)$ means that the sequence $\{\epsilon_k\}$ is an infinitesimal of higher order than $\{\alpha_k\}$.

an infinitesimal of higher order than $\{\gamma_n\}$. Pick $x_0, x_1 \in C$ and set k = 1. **Step 1.** Choose θ_k such that $0 \le \theta_k \le \overline{\theta}_k$, where

$$\overline{\theta}_{k} = \begin{cases} \min\left\{\alpha, \frac{\epsilon_{k}}{\|x_{k} - x_{k-1}\|}\right\}, & \text{if } x_{k} \neq x_{k-1}, \\ \alpha, & \text{otherwise,} \end{cases}$$

and compute

$$w_k = x_k + \theta_k (x_k - x_{k-1}).$$

Step 2. Solve the strongly convex program

$$y_k = \arg\min\left\{\lambda_k f(w_k, y) + \frac{1}{2} \|y - w_k\|^2 : y \in C\right\}.$$

Step 3. Solve the strongly convex program

$$z_k = \arg\min\bigg\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\bigg\}.$$

Step 4. Solve the strongly convex program

$$u_k = \arg\min\left\{\mu_k g(Az_k, u) + \frac{1}{2}||u - Az_k||^2 : u \in Q\right\}.$$

Step 5. Solve the strongly convex program

$$v_k = \arg\min\left\{\mu_k g(u_k, u) + \frac{1}{2}||u - Az_k||^2 : u \in Q\right\}.$$

Step 6. Compute

$$t_k = P_C(z_k + \eta A^*(v_k - Az_k)).$$

Step 7. The next approximation x_{k+1} is defined by

$$x_{k+1} = (1 - \beta_k - \gamma_k)w_k + \beta_k t_k.$$

Step 8. Put k := k + 1 and go to **Step 1**.

Theorem 2. Suppose that the solution set Ω is nonempty. Then, the sequence $\{x_k\}$ which is generated by the MIEM Algorithm converges strongly to the minimum-norm element of Ω .

Proof. Let $p \in \Omega$. That is, $p \in EP(f, C)$, and $Ap \in EP(g, Q)$. Following the proof of Theorem 1, we can check that

$$||z_k - p|| \le ||w_k - p||,\tag{37}$$

$$||t_k - p|| \le ||z_k - p||, \tag{38}$$

$$||v_k - Ap|| \le ||Az_k - Ap||, \tag{39}$$

$$||w_{k} - p||^{2} \leq ||x_{k} - p||^{2} + \theta_{k}(||x_{k} - p||^{2} - ||x_{k-1} - p||^{2}) + 2\theta_{k}||x_{k} - x_{k-1}||^{2},$$
(40)

and

$$||t_k - p||^2 \leq ||z_k - p||^2 - \eta (1 - \eta ||A||^2) ||v_k - Az_k||^2.$$
(41)

By the definition of x_{k+1} , we obtain

$$\begin{aligned} \|x_{k+1} - p\| &= \|(1 - \beta_k - \gamma_k)(w_k - p) + \beta_k(t_k - p) - \gamma_k p\| \\ &\leq (1 - \beta_k - \gamma_k)\|w_k - p\| + \beta_k\|t_k - p\| + \gamma_k\|p\|. \end{aligned}$$

It follows from (37) and (38) that

$$||x_{k+1} - p|| \le (1 - \gamma_k) ||w_k - p|| + \gamma_k ||p||$$

Thus, by the definition of w_k , we have

$$\begin{aligned} \|x_{k+1} - p\| &\leq (1 - \gamma_k) \|x_k - p\| + (1 - \gamma_k) \theta_k \|x_k - x_{k-1}\| + \gamma_k \|p\| \\ &= (1 - \gamma_k) \|x_k - p\| + \gamma_k (\sigma_k + \|p\|), \end{aligned}$$
(42)

where $\sigma_k = (1 - \gamma_k) \frac{\theta_k}{\gamma_k} ||x_k - x_{k-1}||$. Due to the choices of the sequence $\{\theta_k\}$, we obtain that

$$\sigma_k = (1 - \gamma_k) \frac{\theta_k}{\gamma_k} \| x_k - x_{k-1} \| \le (1 - \gamma_k) \frac{\epsilon_k}{\gamma_k}$$

Thus, by the properties of $\epsilon_k = o(\gamma_k)$ and $\lim_{k \to \infty} \gamma_k = 0$, we have

$$\lim_{\to\infty} \sigma_k = 0. \tag{43}$$

This implies that the sequence $\{\sigma_k\}$ is a null sequence. Put $M = \max\{\|p\|, \sup_{k \in \mathbb{N}} \sigma_k\}$.

Then, by (42) and Lemma 6 (i), the sequence $\{||x_k - p||\}$ is bounded. Consequently, $\{x_k\}$ is a bounded sequence.

In addition, by the definition of x_{k+1} and (10), we have

$$||x_{k+1} - p||^{2} = ||(1 - \beta_{k} - \gamma_{k})(w_{k} - p) + \beta_{k}(t_{k} - p) + \gamma_{k}(-p)||^{2}$$

$$\leq (1 - \beta_{k} - \gamma_{k})||w_{k} - p||^{2} + \beta_{k}||t_{k} - p||^{2} + \gamma_{k}||p||^{2}$$

$$-\beta_{k}(1 - \beta_{k} - \gamma_{k})||w_{k} - t_{k}||^{2}.$$
(44)

Thus, by using (38), (40), and Lemma 1 (ii), we obtain that

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq (1 - \beta_k - \gamma_k) \|w_k - p\|^2 + \beta_k \|z_k - p\|^2 + \gamma_k \|p\|^2 \\ &- \beta_k (1 - \beta_k - \gamma_k) \|w_k - t_k\|^2 \\ &\leq (1 - \gamma_k) \|w_k - p\|^2 - \beta_k (1 - \beta_k - \gamma_k) \|w_k - t_k\|^2 + \gamma_k \|p\|^2 \\ &- \beta_k (1 - 2\lambda_k c_1) \|w_k - y_k\|^2 - \beta_k (1 - 2\lambda_k c_2) \|y_k - z_k\|^2 \\ &\leq (1 - \gamma_k) \|x_k - p\|^2 + \theta_k (1 - \gamma_k) (\|x_k - p\|^2 - \|x_{k-1} - p\|^2) \\ &+ 2\theta_k (1 - \gamma_k) \|x_k - x_{k-1}\|^2 - \beta_k (1 - \beta_k - \gamma_k) \|w_k - t_k\|^2 + \gamma_k \|p\|^2 \\ &- \beta_k (1 - 2\lambda_k c_1) \|w_k - y_k\|^2 - \beta_k (1 - 2\lambda_k c_2) \|y_k - z_k\|^2. \end{aligned}$$

This implies that

$$\beta_{k}(1-\beta_{k}-\gamma_{k})\|w_{k}-t_{k}\|^{2}+\beta_{k}(1-2\lambda_{k}c_{1})\|w_{k}-y_{k}\|^{2}+\beta_{k}(1-2\lambda_{k}c_{2})\|y_{k}-z_{k}\|^{2} \leq \|x_{k}-p\|^{2}-\|x_{k+1}-p\|^{2}+\theta_{k}(1-\gamma_{k})(\|x_{k}-p\|^{2}-\|x_{k-1}-p\|^{2}) +2\theta_{k}(1-\gamma_{k})\|x_{k}-x_{k-1}\|^{2}+\gamma_{k}\|p\|^{2}.$$
(45)

Next, we will show that $\{x_k\}$ converges strongly to $\tilde{p} := P_{\Omega}(0)$. We consider the following two possible cases.

Case 1. Suppose that there exists $k_0 \in \mathbb{N}$ such that $||x_{k+1} - \tilde{p}|| \le ||x_k - \tilde{p}||$, for all $k \ge k_0$. This means that $\{||x_k - \tilde{p}||\}_{k \ge k_0}$ is a nonincreasing sequence. Consequently, by using this one together with the boundness property of $\{||x_k - \tilde{p}||\}$, we have that the limit of $||x_k - \tilde{p}||$ exists. Since $\lim_{k \to \infty} \theta_k ||x_k - x_{k-1}|| = 0$ and the properties of the control sequences $\{\beta_k\}, \{\gamma_k\}, \{\lambda_k\}, \{\theta_k\}$, it follows from (45) that

$$\lim_{k \to \infty} \|w_k - t_k\| = 0, \tag{46}$$

$$\lim_{k \to \infty} \|w_k - y_k\| = 0, \tag{47}$$

and

$$\lim_{k \to \infty} \|y_k - z_k\| = 0. \tag{48}$$

These imply that

$$\lim_{k \to \infty} \|z_k - t_k\| = 0.$$
(49)

Moreover, since $\lim_{k\to\infty} \theta_k ||x_k - x_{k-1}|| = 0$, we obtain

$$\lim_{k \to \infty} \|w_k - x_k\| = 0.$$
⁽⁵⁰⁾

Using this one together with (47), we obtain

$$\lim_{k \to \infty} \|x_k - y_k\| = 0.$$
⁽⁵¹⁾

It follows from (48) that

$$\lim_{k \to \infty} \|x_k - z_k\| = 0.$$
⁽⁵²⁾

On the other hand, in view of (41), we see that

$$\begin{aligned} \eta(1-\eta \|A\|^2) \|v_k - Az_k\|^2 &\leq \|z_k - \tilde{p}\|^2 - \|t_k - \tilde{p}\|^2 \\ &\leq \|z_k - t_k\|(\|z_k - \tilde{p}\| + \|t_k - \tilde{p}\|). \end{aligned}$$

Thus, by using (49), we have

$$\lim_{k \to \infty} \|v_k - Az_k\| = 0.$$
⁽⁵³⁾

Furthermore, by Lemma 1 (ii), we obtain that

$$(1 - 2\mu_k d_1) \|Az_k - u_k\|^2 + (1 - 2\mu_k d_2) \|u_k - v_k\|^2 \leq \|Az_k - A\tilde{p}\|^2 - \|v_k - A\tilde{p}\|^2$$

= $\|Az_k - v_k\|(\|Az_k - A\tilde{p}\|)$
+ $\|v_k - A\tilde{p}\|).$

Then, applying (53) to the above inequality, we have

$$\lim_{k \to \infty} \|Az_k - u_k\| = 0, \tag{54}$$

 $\lim_{k \to \infty} \|u_k - v_k\| = 0.$ (55)

Now, let $x^* \in \omega_w(x_k)$ and $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_n} \rightharpoonup x^*$, as $n \rightarrow \infty$. Following the line proof of Theorem 1, we can show that $x^* \in \Omega$. This means that $\omega_w(x_k) \subset \Omega$. Put $s_k = (1 - \beta_k)w_k + \beta_k t_k$. The relations (37) and (38) imply that

$$\begin{aligned} \|s_k - \tilde{p}\| &\leq (1 - \beta_k) \|w_k - \tilde{p}\| + \beta_k \|t_k - \tilde{p}\| \\ &\leq \|w_k - \tilde{p}\|. \end{aligned}$$
(56)

By the definition of x_{k+1} , we see that

 $\begin{aligned} x_{k+1} &= s_k - \gamma_k w_k \\ &= (1 - \gamma_k) s_k - \gamma_k (w_k - s_k) \\ &= (1 - \gamma_k) s_k - \gamma_k \beta_k (w_k - t_k). \end{aligned}$

It follows from (9) that

$$\|x_{k+1} - \tilde{p}\|^{2} = \|(1 - \gamma_{k})(s_{k} - \tilde{p}) - \gamma_{k}\beta_{k}(w_{k} - t_{k}) - \gamma_{k}\tilde{p}\|^{2} \leq (1 - \gamma_{k})\|s_{k} - \tilde{p}\|^{2} - 2\gamma_{k}\beta_{k}\langle w_{k} - t_{k}, x_{k+1} - \tilde{p}\rangle -2\gamma_{k}\langle \tilde{p}, x_{k+1} - \tilde{p}\rangle.$$
(57)

Thus, by using (56), we have

$$\|x_{k+1} - \tilde{p}\|^{2} \leq (1 - \gamma_{k}) \|w_{k} - \tilde{p}\|^{2} + \gamma_{k} [-2\beta_{k} \langle w_{k} - t_{k}, x_{k+1} - \tilde{p} \rangle + 2 \langle x_{k+1} - \tilde{p}, -\tilde{p} \rangle].$$
(58)

Consider,

$$\|w_{k} - \tilde{p}\|^{2} \leq (\|x_{k} - \tilde{p}\| + \theta_{k}\|x_{k} - x_{k-1}\|)^{2} \leq \|x_{k} - \tilde{p}\|^{2} + 2\theta_{k}\|x_{k} - \tilde{p}\|\|x_{k} - x_{k-1}\| + \theta_{k}\|x_{k} - x_{k-1}\|^{2} \leq \|x_{k} - \tilde{p}\|^{2} + 3M\theta_{k}\|x_{k} - x_{k-1}\|,$$
(59)

where $M = \sup_{k \in \mathbb{N}} \{ \|x_k - \tilde{p}\|, \|x_k - x_{k-1}\| \}$. This, together with (58), implies that

$$\begin{aligned} \|x_{k+1} - \tilde{p}\|^{2} &\leq (1 - \gamma_{k}) \|x_{k} - \tilde{p}\|^{2} + 3M\theta_{k}(1 - \gamma_{k}) \|x_{k} - x_{k-1}\| \\ &+ \gamma_{k} [-2\beta_{k} \langle w_{k} - t_{k}, x_{k+1} - \tilde{p} \rangle + 2 \langle x_{k+1} - \tilde{p}, -\tilde{p} \rangle] \\ &= (1 - \gamma_{k}) \|x_{k} - \tilde{p}\|^{2} + \gamma_{k} [3M(1 - \gamma_{k}) \frac{\theta_{k}}{\gamma_{k}} \|x_{k} - x_{k-1}\| \\ &- 2\beta_{k} \langle w_{k} - t_{k}, x_{k+1} - \tilde{p} \rangle + 2 \langle x_{k+1} - \tilde{p}, -\tilde{p} \rangle]. \end{aligned}$$
(60)

Thus, by the properties of $\tilde{p} := P_{\Omega}(0)$ and $x^* \in \omega_w(x_k) \subset \Omega$, we obtain

$$\limsup_{k \to \infty} \langle x_{k+1} - \tilde{p}, -\tilde{p} \rangle = \lim_{n \to \infty} \langle x_{k_n+1} - \tilde{p}, -\tilde{p} \rangle = \langle x^* - \tilde{p}, -\tilde{p} \rangle \le 0.$$
(61)

Hence, by (43), (46), (60), (61), and Lemma 6 (ii), we have

$$\lim_{k \to \infty} \|x_k - \tilde{p}\| = 0.$$
(62)

This completes the proof for the first case.

and

Case 2. Suppose that there exists a subsequence $\{||x_{k_i} - \tilde{p}||\}$ of $\{||x_k - \tilde{p}||\}$ such that

$$\|x_{k_i} - \tilde{p}\| < \|x_{k_i+1} - \tilde{p}\|, \forall i \in \mathbb{N}.$$

According to Lemma 7, there exists a nondecreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $\lim_{n\to\infty}m_n=\infty, \text{ and }$

$$\|x_{m_n} - \tilde{p}\| \le \|x_{m_n+1} - \tilde{p}\|$$
 and $\|x_n - \tilde{p}\| \le \|x_{m_n+1} - \tilde{p}\|, \forall n \in \mathbb{N}.$ (63)

It follows from (45) that

$$\beta_{m_n}(1-\beta_{m_n}-\gamma_{m_n})\|w_{m_n}-t_{m_n}\|^2 + \beta_{m_n}(1-2\lambda_{m_n}c_1)\|w_{m_n}-y_{m_n}\|^2 +\beta_{m_n}(1-2\lambda_{m_n}c_2)\|y_{m_n}-z_{m_n}\|^2 \leq \|x_{m_n}-\tilde{p}\|^2 - \|x_{m_n+1}-\tilde{p}\|^2 + \theta_{m_n}(1-\gamma_{m_n})(\|x_{m_n}-\tilde{p}\|^2 - \|x_{m_n-1}-\tilde{p}\|^2) +2\theta_{m_n}(1-\gamma_{m_n})\|x_{m_n}-x_{m_n-1}\|^2 + \gamma_{m_n}\|\tilde{p}\|^2 \leq \theta_{m_n}(1-\gamma_{m_n})\|x_{m_n}-x_{m_n-1}\|(\|x_{m_n}-\tilde{p}\|+\|x_{m_n-1}-\tilde{p}\|) +2\theta_{m_n}(1-\gamma_{m_n})\|x_{m_n}-x_{m_n-1}\|^2 + \gamma_{m_n}\|\tilde{p}\|^2.$$
(64)

Following the line proof of Case 1, we can show that

$$\lim_{n \to \infty} \|w_{m_n} - t_{m_n}\| = 0, \lim_{n \to \infty} \|x_{m_n} - y_{m_n}\| = 0, \lim_{n \to \infty} \|x_{m_n} - z_{m_n}\| = 0,$$
(65)

$$\lim_{n \to \infty} \|v_{m_n} - Az_{m_n}\| = 0, \lim_{n \to \infty} \|Az_{m_n} - u_{m_n}\| = 0, \lim_{n \to \infty} \|u_{m_n} - v_{m_n}\| = 0,$$
(66)

$$\limsup_{n \to \infty} \langle x_{m_n+1} - \tilde{p}, -\tilde{p} \rangle \le 0, \tag{67}$$

and

$$\|x_{m_{n}+1} - \tilde{p}\|^{2} \leq (1 - \gamma_{m_{n}}) \|x_{m_{n}} - \tilde{p}\|^{2} + \gamma_{m_{n}} [3M(1 - \gamma_{m_{n}}) \frac{\theta_{m_{n}}}{\gamma_{m_{n}}} \|x_{m_{n}} - x_{m_{n}-1}\| - 2\beta_{m_{n}} \langle w_{m_{n}} - t_{m_{n}}, x_{m_{n}+1} - \tilde{p} \rangle + 2 \langle x_{m_{n}+1} - \tilde{p}, -\tilde{p} \rangle],$$
(68)

where $M = \sup_{n \in \mathbb{N}} \{ \|x_{m_n} - \tilde{p}\|, \|x_{m_n} - x_{m_n-1}\| \}$. This, together with (63), implies that

$$\|x_{m_{n}+1} - \tilde{p}\|^{2} \leq (1 - \gamma_{m_{n}}) \|x_{m_{n}+1} - \tilde{p}\|^{2} + \gamma_{m_{n}} [3M(1 - \gamma_{m_{n}}) \frac{\theta_{m_{n}}}{\gamma_{m_{n}}} \|x_{m_{n}} - x_{m_{n}-1}\| - 2\beta_{m_{n}} \langle w_{m_{n}} - t_{m_{n}}, x_{m_{n}+1} - \tilde{p} \rangle + 2 \langle x_{m_{n}+1} - \tilde{p}, -\tilde{p} \rangle].$$

$$(69)$$

Using this one together with (63) again, we obtain

$$\|x_{n} - \tilde{p}\|^{2} \leq 3M(1 - \gamma_{m_{n}}) \frac{\theta_{m_{n}}}{\gamma_{m_{n}}} \|x_{m_{n}} - x_{m_{n}-1}\| - 2\beta_{m_{n}} \langle w_{m_{n}} - t_{m_{n}}, x_{m_{n}+1} - \tilde{p} \rangle$$

$$+ 2 \langle x_{m_{n}+1} - \tilde{p}, -\tilde{p} \rangle.$$
(70)

Thus, by using (43), (65), and (67), we have

$$\limsup_{n\to\infty}\|x_n-\tilde{p}\|^2\leq 0.$$

Hence, we can conclude that the sequence $\{x_n\}$ converges strongly to $\tilde{p} := P_{\Omega}(0)$. This completes the proof. \Box

4. Numerical Experiments

This section will show some examples and numerical results to support Theorems 1 and 2. We will compare the introduced algorithms, IEM and MIEM, with the PPA Algorithm (7) in Example 1 and the PEA Algorithm (8) in Example 2. The numerical

experiments are written in Matlab R2015b and performed on a laptop with AMD Dual Core R3-2200U CPU @ 2.50 GHz and RAM 4.00 GB. In both Examples 1 and 2, for each considered matrix, the $\|\cdot\|$ means the spectral norm.

Example 1. Let $H_1 = \mathbb{R}^n$ and $H_2 = \mathbb{R}^m$ be two real Hilbert spaces with the Euclidean norm. We consider the bifunctions \tilde{f} and \tilde{g} which are generated from the Nash–Cournot oligopolistic equilibrium models of electricity markets (see [22,32]),

$$\tilde{f}(x,y) = \langle P_1 x + Q_1 y, y - x \rangle, \quad \forall x, y \in \mathbb{R}^n,$$
(71)

$$\tilde{g}(u,v) = \langle P_2 u + Q_2 v, v - u \rangle, \quad \forall u, v \in \mathbb{R}^m,$$
(72)

where $P_1, Q_1 \in \mathbb{R}^{n \times n}$ and $P_2, Q_2 \in \mathbb{R}^{m \times m}$ are matrices such that Q_1, Q_2 are symmetric positive semidefinite and $Q_1 - P_1, Q_2 - P_2$ are negative semidefinite. Observe that $\tilde{f}(x,y) + \tilde{f}(y,x) = (x-y)^T (Q_1 - P_1)(x-y), \forall x, y \in \mathbb{R}^n$. Moreover, from the property of $Q_1 - P_1$, we have that \tilde{f} is a monotone operator. Similarly, we also have that \tilde{g} is monotone.

Next, we consider the two bifunctions f and g, which are given by

$$f(x,y) = \begin{cases} \tilde{f}(x,y), & \text{if } (x,y) \in C \times C, \\ 0, & \text{otherwise,} \end{cases}$$
(73)

and

$$g(u,v) = \begin{cases} \tilde{g}(u,v), & \text{if } (u,v) \in Q \times Q, \\ 0, & \text{otherwise,} \end{cases}$$
(74)

where $C = \prod_{i=1}^{n} [-5,5]$ and $Q = \prod_{j=1}^{m} [-20,20]$ are the constrained boxes. We note that f and g are Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2} ||P_1 - Q_1||$ and $d_1 = d_2 = \frac{1}{2} ||P_2 - Q_2||$, respectively (see [6]). Choose $b_1 = \max\{c_1, d_1\}$, and $b_2 = \max\{c_2, d_2\}$. Then, both bifunctions f and g are Lipschitz-type continuous with constants b_1 and b_2 .

For this numerical experiment, the matrices P_1 , Q_1 , P_2 , and Q_2 are randomly generated in the interval [-5,5] such that they satisfy the required properties above and the linear operator $A: \mathbb{R}^n \to \mathbb{R}^m$ is a $m \times n$ matrix, in which each of its entries is randomly generated in the interval [-2,2]. Note that the solution set Ω is nonempty because of $0 \in \Omega$. We will work with the following control parameters: $\alpha = 0.5$, $\eta = \frac{1}{2||A||^2}$, $\lambda_k = \mu_k = \frac{1}{4\max\{b_1, b_2\}}$, $\epsilon_k = \frac{1}{(k+1)^2}$, $\gamma_k = \frac{1}{k+1}$, and $\beta_k = 0.5(1 - \gamma_k)$. The following five cases of the parameter θ_k are considered:

Case 1. $\theta_k = 0$.

Case 2. $\theta_k = 0.25 \ \overline{\theta}_k$.

- Case 3. $\theta_k = 0.5 \ \overline{\theta}_{\underline{k}}$.
- Case 4. $\theta_k = 0.75 \,\overline{\theta}_k$.
- *Case 5.* $\theta_k = \overline{\theta}_k$.

The function quadprog in Matlab Optimization Toolbox was used to solve vectors y_k , z_k , u_k , and v_k . We randomly generated starting points $x_0 = x_1 \in \mathbb{R}^n$ in the interval [-5,5]. The IEM and MIEM algorithms were tested along with the PPA Algorithm (7) by using the stopping criteria $||x_{k+1} - x_k|| < 10^{-8}$. We randomly generated 10 starting points and the presented results are the average, where n = 10 and m = 20.

From Table 1, we may suggest that the parameter $\theta_k = \overline{\theta}_k$ provides better CPU times and iteration numbers than other cases. Moreover, iteration numbers of the IEM and MIEM algorithms with the parameter, $\theta_k \neq 0$, are mostly better than those of the PPA Algorithm. Meanwhile, CPU times of the PPA Algorithm are better than those of the IEM and MIEM Algorithms. However, we would like to remind that, by Remark 1, the class of pseudomonotone bifunction is strictly larger than the class of monotone bifunction, and both IEM and MIEM Algorithms can solve the split equilibrium problems for pseudomonotone bifunctions while the PPA Algorithm may not be applied.

	Average CPU Times (s)			Average Iterations		
Cases	IEM	MIEM	PPA	IEM	MIEM	PPA
1	3.1609	4.1125		179.6	237.5	
2	2.7578	3.4859		164.6	207.9	
3	2.5641	2.9828	1.7781	148.7	177.9	177.6
4	2.3875	2.4531		131.8	146.7	
5	2.0031	1.8766		110.9	112.6	

Table 1. The numerical results of Example 1.

Example 2. Let $H_1 = \mathbb{R}^n$ and $H_2 = \mathbb{R}^m$ be two real Hilbert spaces with the Euclidean norm. We consider a classical form of the bifunction which given by the Cournot–Nash models (see [33]),

1 -

$$f(x,y) = \langle A_1 x + a_1^n(y+x), y-x \rangle, \quad \forall x, y \in \mathbb{R}^n,$$

$$\tilde{a}(y,y) = \langle A_2 y + a_1^m(y+y), y-y \rangle, \quad \forall y, y \in \mathbb{R}^m,$$
(75)
$$\tilde{a}(y,y) = \langle A_2 y + a_1^m(y+y), y-y \rangle, \quad \forall y, y \in \mathbb{R}^m,$$
(75)

$$\tilde{g}(u,v) = \langle A_2 u + a_2^m(v+u), v-u \rangle, \quad \forall u,v \in \mathbb{R}^m,$$
(76)

where

$$A_{1} = \begin{pmatrix} 0 & a_{1} & a_{1} & \cdots & a_{1} \\ a_{1} & 0 & a_{1} & \cdots & a_{1} \\ a_{1} & a_{1} & 0 & \cdots & a_{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1} & a_{1} & \vdots & \cdots & 0 \end{pmatrix}_{n \times n}, A_{2} = \begin{pmatrix} 0 & a_{2} & a_{2} & \cdots & a_{2} \\ a_{2} & 0 & a_{2} & \cdots & a_{2} \\ a_{2} & a_{2} & 0 & \cdots & a_{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2} & a_{2} & \vdots & \cdots & 0 \end{pmatrix}_{m \times m}$$

where a_1 and a_2 are positive real numbers. We know that the bifunctions \hat{f} and \tilde{g} are pseudomonotone and they are not monotone on C and Q, respectively (see [34]).

Here, the numerical experiment is considered under the following setting: the boxes C and Q, the bifunctions f and g, the linear operator A, and the control parameters are given as in Example 1. Notice that the solution set Ω is nonempty because of $0 \in \Omega$. We observe that f and g are Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2}||A_1||$ and $d_1 = d_2 = \frac{1}{2}||A_2||$, respectively. Choose $b_1 = \max\{c_1, d_1\}$, and $b_2 = \max\{c_2, d_2\}$. Then, both bifunctions f and g are Lipschitz-type continuous with constants b_1 and b_2 . In addition, the positive real numbers a_1 and a_2 are randomly generated in the interval (1, 2) and (3, 4), respectively. The following four cases of the parameter θ_k are considered:

Case 1. $\theta_k = 0.25 \,\overline{\theta}_k$. Case 2. $\theta_k = 0.5 \,\overline{\theta}_k$. Case 3. $\theta_k = 0.75 \,\overline{\theta}_k$. Case 4. $\theta_k = \overline{\theta}_k$.

The function quadprog in Matlab Optimization Toolbox was used to solve vectors y_k , z_k , u_k , and v_k . The starting points $x_0 = x_1 \in \mathbb{R}^n$ are randomly generated in the interval [-5,5]. We compare the IEM and MIEM Algorithms with the PEA Algorithm (8) by using the stopping criteria $||x_{k+1} - x_k|| < 10^{-8}$. We randomly 10 starting points and the presented results are the average, where n = 10 and m = 20.

Table 2 shows that the parameter θ_k from case 4, as $\theta_k = \theta_k$, yields better CPU times and iteration numbers than other cases. In addition, we see that CPU times and iteration numbers of the IEM and MIEM Algorithms are mostly better than those of the PEA Algorithm.

	Avera	Average Iterations				
Cases	IEM	MIEM	PEA	IEM	MIEM	PEA
1	1.9609	2.5922		79.7	111.4	
2	1.7250	2.2891	7 1001	71.5	94.6	00 7
3	1.5281	1.8844	2.1091	64.4	78.4	00.2
4	1.4094	1.5313		59.2	63.6	

Table 2. The numerical results of Example 2.

5. Conclusions

We present two algorithms for solving the split equilibrium problems, when the bifunctions are pseudomonotone and Lipschitz-type continuous in the framework of real Hilbert spaces. We consider both inertial and extragradient methods for introducing a sequence which is convergent to a solution of the considered problem. Some numerical experiments in which the bifunctions are generated from the Nash-Cournot oligopolistic equilibrium models of electricity markets and the Cournot-Nash models, respectively, are performed to illustrate the convergence of introduced algorithms and compare them with some algorithms. In the numerical experiment Example 1, one may observe that the time per iteration ratio of IEM and MIEM is around 0.018, while it is around 0.011 for PPA. This means both the IEM and MIEM Algorithms take approximately 1.63 times the CPU time of the PPA Algorithm in each iteration for solving the considered monotone type problem. When we consider the pseudomonotone type problem in Example 2, the time per iteration of IEM, MIEM, and PEA are about the same at 0.024. This information may lead to a conclusion that a key advantage of the inertial technique is trying to reduce the number of iterations rather than reduce the CPU time per iteration. On the other hand, we emphasize that the exact Lipschitz-type constants of the bifunctions are needed in order to control input parameters of the two introduced algorithms. However, the Lipschitz-type constants of the bifunctions are often unknown, and even in nonlinear problems they are difficult to approximate. For a future research direction, it would be very interesting to develop the algorithm but without the prior knowledge of the Lipschitz-type constants of the bifunctions.

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