

Article

# Time-Inhomogeneous Feller-Type Diffusion Process in Population Dynamics

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**Abstract:** The time-inhomogeneous Feller-type diffusion process, having infinitesimal drift  $\alpha(t)x + \beta(t)$  and infinitesimal variance  $2r(t)x$ , with a zero-flux condition in the zero-state, is considered. This process is obtained as a continuous approximation of a birth-death process with immigration. The transition probability density function and the related conditional moments, with their asymptotic behaviors, are determined. Special attention is paid to the cases in which the intensity functions  $\alpha(t)$ ,  $\beta(t)$ ,  $r(t)$  exhibit some kind of periodicity due to seasonal immigration, regular environmental cycles or random fluctuations. Various numerical computations are performed to illustrate the role played by the periodic functions.

**Keywords:** diffusion approximation; transient and asymptotic densities; conditional moments; periodic intensity functions

**MSC:** 60J60; 60J70; 92D25



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## 1. Introduction and Background

One-dimensional diffusion processes are used to model the time evolution of dynamical systems in biology, genetics, physics, engineering, neuroscience, economics, finance, queueing and other fields (cf. for instance, Bharucha-Reid [1], Cox and Miller [2], Ricciardi [3,4], Tuckwell [5], Gardiner [6]). For many applications, it is often useful to consider the class of time-inhomogeneous linear diffusion processes, that includes the Feller-type diffusion process and the Ornstein-Uhlenbeck diffusion process. In this paper, we focus on the time-inhomogeneous Feller-type diffusion process with a zero-flux condition in the zero-state.

The Feller-type diffusion process  $\{X(t), t \geq t_0\}$ ,  $t_0 \geq 0$ , is an one-dimensional time-inhomogeneous diffusion process with linear infinitesimal drift and linear infinitesimal variance

$$A_1(x, t) = \alpha(t)x + \beta(t), \quad A_2(x, t) = 2r(t)x, \quad (1)$$

defined in the state-space  $[0, +\infty)$ , with  $\alpha(t) \in \mathbb{R}$ ,  $\beta(t) \geq 0$ ,  $r(t) > 0$  continuous functions for all  $t \geq t_0$ . Hence,  $X(t)$  satisfies the following stochastic differential equation:

$$dX(t) = [\alpha(t)X(t) + \beta(t)] dt + \sqrt{2r(t)X(t)} dW(t), \quad X(t_0) = x_0,$$

where  $W(t)$  is a standard Wiener process.

Feller diffusion process is widely used in population dynamics to model the growth of a population (cf. Feller [7], Ricciardi et al. [8], Pugliese and Milner [9], Masoliver and Perelló [10], Masoliver [11]). Indeed, in population dynamics the Feller-type diffusion process arises as a continuous approximation of a birth-death process with immigration. In these cases  $\alpha(t)$ , related to the growth intensity function, can be positive, negative or zero

at different time instants. In particular,  $\alpha(t)$  is positive (negative) when the birth intensity function is greater (less) than the death intensity function, whereas  $\alpha(t) = 0$  if the birth intensity function is equal to the death intensity function. Instead, the function  $\beta(t)$  is related to the immigration intensity function. In particular,  $\beta(t) > 0$  indicates the presence of immigrations, whereas  $\beta(t) = 0$  denotes the absence of the immigration phenomena. The function  $r(t)$  takes into account the environmental fluctuations and describes the noise intensity.

The Feller diffusion process is also used in queueing systems to describe the number of customers in a queue (cf. Di Crescenzo and Nobile [12]), in neurobiology to analyze the input-output behavior of single neurons (see, for instance, Sacerdote [13], Ditlevsen and Lánský [14], Buonocore et al. [15], Giorno et al. [16], Nobile and Pirozzi [17], D’Onofrio et al. [18]), in mathematical finance to model asset prices, market indices, interest rates and stochastic volatility (see, Tian and Zhang [19], Cox et al. [20], Maghsoodi [21], Peng and Schellhorn [22], Linetsky [23], Di Nardo and D’Onofrio [24]).

In many real applications, the transition probability density function (pdf) plays a relevant role for the description of the evolution of the dynamic system. In the sequel, we assume that a zero-flux condition is placed in the zero-state of  $X(t)$  to ensure that the total probability mass is conserved in  $[0, +\infty)$ .

The transition pdf  $f(x, t|x_0, t_0)$  of  $X(t)$  is solution of Fokker-Planck equation

$$\frac{\partial f(x, t|x_0, t_0)}{\partial t} = -\frac{\partial}{\partial x} \{ [\alpha(t)x + \beta(t)] f(x, t|x_0, t_0) \} + r(t) \frac{\partial^2}{\partial x^2} [x f(x, t|x_0, t_0)], \quad (2)$$

to solve with the initial delta condition

$$\lim_{t \downarrow t_0} f(x, t|x_0, t_0) = \delta(x - x_0) \quad (3)$$

and a zero-flux condition in the zero-state:

$$\lim_{x \downarrow 0} \{ [\alpha(t)x + \beta(t)] f(x, t|x_0, t_0) - r(t) \frac{\partial}{\partial x} [x f(x, t|x_0, t_0)] \} = 0. \quad (4)$$

Indeed, denoting by

$$\begin{aligned} j(x, t|x_0, t_0) &= A_1(x, t) f(x, t|x_0, t_0) - \frac{1}{2} \frac{\partial}{\partial x} [A_2(x, t) f(x, t|x_0, t_0)] \\ &= [\alpha(t)x + \beta(t)] f(x, t|x_0, t_0) - r(t) \frac{\partial}{\partial x} [x f(x, t|x_0, t_0)], \quad x \geq 0 \end{aligned} \quad (5)$$

the probability flux (or current) of  $X(t)$ , the Fokker-Planck Equation (2) can be re-written as  $\partial f / \partial t = -\partial j / \partial x$  and the zero-flux condition (4) corresponds to requiring that  $\int_0^{+\infty} f(x, t|x_0, t_0) dx = 1$  for all  $t \geq t_0$ .

We remark that the time-inhomogeneous Feller-type diffusion process with an absorbing boundary at the zero-state is considered in Lavigne and Roques [25] and in Giorno and Nobile [26], in which the first-passage time problem through the zero-state is also analyzed.

### Plan of the Paper

The paper is organized in five sections and six appendices in which the proofs of the main results are reported. In Section 2, starting from the forward equations for the transition probabilities of the time-inhomogeneous birth-death process with immigration, we describe the continuous approximation that leads to the Fokker-Planck Equation (2), with the initial condition (3) and the zero-flux condition in the zero-state (4). In Section 3, for the time-inhomogeneous Feller-type diffusion process  $X(t)$ , we give some preliminary results concerning the moment generating function of the transition pdf  $f(x, t|x_0, t_0)$ . Some special situations are analyzed: (i) the absence of immigration with  $\beta(t) = 0$ , (ii) the proportional case in which  $\beta(t) = \zeta r(t)$ , with  $\zeta > 0$ , and (iii) the time-homogeneous case.

Sections 4 and 5 contain the main results of the paper concerning the analysis of transient and asymptotic behavior of the Feller-type diffusion process in the general case. Specifically, in Section 4, the transition pdf  $f(x, t|x_0, t_0)$  is obtained for the time-inhomogeneous Feller-type process in the general case for  $x_0 = 0$  (Section 4.1) and for  $x_0 > 0$  (Section 4.2). Finally, in Section 5, particular attention is paid to the periodic cases by assuming that the growth intensity function  $\alpha(t)$ , the immigration intensity function  $\beta(t)$  and the noise intensity function  $r(t)$  have some kind of periodicity. The asymptotic behaviors of the transition pdf and of the moments are also discussed in the following cases: periodic immigration intensity function, periodic growth intensity function, periodic immigration and growth intensity functions and periodic immigration, growth and noise intensity functions. Various numerical computations are performed making use of MATHEMATICA to analyze the role played by the involved periodic functions. Specifically, for some choices of the periodic functions  $\alpha(t)$ ,  $\beta(t)$  and  $r(t)$ , of the interest in population dynamics, the transition densities, the conditional means and variances and their asymptotic behaviors are discussed and compared.

### 2. Diffusion Approximation of Birth-Death Process with Immigration

In this section, we show that the Feller-type diffusion process  $X(t)$  can be obtained starting from a linear time-inhomogeneous birth-death process  $N(t)$  with immigration by using a standard limit procedure (cf. for instance, Bhattacharya and Waymire [27]). Specifically, we prove that, under suitable assumptions, the discrete scaled process converges weakly to  $X(t)$ .

Let  $\{N(t), t \geq t_0\}$  be a time-inhomogeneous linear birth-death process with immigration having state-space  $\mathbb{N}_0$ , conditioned to start from  $j \in \mathbb{N}_0$  at time  $t_0$ . The transition probabilities of  $N(t)$  satisfy the Kolmogorov forward equations and the related initial condition:

$$\begin{aligned} \frac{dp_{j,0}(t|t_0)}{dt} &= -\nu(t) p_{j,0}(t|t_0) + \mu(t) p_{j,1}(t|t_0), \\ \frac{dp_{j,n}(t|t_0)}{dt} &= [\lambda(t)(n - 1) + \nu(t)] p_{j,n-1}(t|t_0) + \mu(t) (n + 1) p_{j,n+1}(t|t_0) \\ &\quad - \{[\lambda(t) + \mu(t)] n + \nu(t)\} p_{j,n}(t|t_0), \quad n \in \mathbb{N}, \\ \lim_{t \downarrow t_0} p_{j,n}(t|t_0) &= \delta_{j,n}, \end{aligned} \tag{6}$$

where  $\lambda(t) > 0$ ,  $\mu(t) > 0$  and  $\nu(t) \geq 0$  are bounded and continuous functions for  $t \geq t_0$  representing birth, death and immigration intensity functions, respectively, and  $\delta_{j,n}$  is the Kronecker delta function. For  $t \geq t_0$  and  $j \in \mathbb{N}_0$ , the probability generating function of the process  $N(t)$  is (cf. Giorno and Nobile [28]):

$$\begin{aligned} G(z, t|j, t_0) &= \sum_{n=0}^{+\infty} z^n p_{j,n}(t|t_0) = \left[ \frac{1 + (z - 1) e^{\Lambda(t|t_0) - M(t|t_0)} [1 - H(t|t_0)]}{1 - (z - 1) e^{\Lambda(t|t_0) - M(t|t_0)} H(t|t_0)} \right]^j \\ &\quad \times \exp \left\{ (z - 1) \int_{t_0}^t \frac{\nu(u) e^{\Lambda(t|u) - M(t|u)}}{1 - (z - 1) e^{\Lambda(t|u) - M(t|u)} [H(t|u) - H(u|t_0)]} du \right\}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \Lambda(t|t_0) &= \int_{t_0}^t \lambda(\tau) d\tau, \quad M(t|t_0) = \int_{t_0}^t \mu(\tau) d\tau, \\ H(t|t_0) &= \int_{t_0}^t \lambda(\tau) e^{M(\tau|t_0) - \Lambda(\tau|t_0)} d\tau. \end{aligned} \tag{8}$$

By virtue of (7), the transition probabilities  $p_{j,n}(t|t_0)$  are obtained in Giorno and Nobile [28]. Furthermore, for  $t \geq t_0$  and  $j \in \mathbb{N}_0$ , the conditional mean and the conditional variance of  $N(t)$  are

$$\begin{aligned} E[N(t)|N(t_0) = j] &= j e^{\Lambda(t|t_0) - M(t|t_0)} + \int_{t_0}^t \nu(u) e^{\Lambda(t|u) - M(t|u)} du, \\ \text{Var}[N(t)|N(t_0) = j] &= j e^{\Lambda(t|t_0) - M(t|t_0)} \left\{ 1 - e^{\Lambda(t|t_0) - M(t|t_0)} \right. \\ &\quad \left. + 2 e^{\Lambda(t|t_0) - M(t|t_0)} H(t|t_0) \right\} + \int_{t_0}^t \nu(u) e^{\Lambda(t|u) - M(t|u)} du \\ &\quad + 2 e^{\Lambda(t|t_0) - M(t|t_0)} \int_{t_0}^t \nu(u) e^{\Lambda(t|u) - M(t|u)} [H(t|t_0) - H(u|t_0)] du, \end{aligned} \tag{9}$$

with  $\Lambda(t|t_0)$ ,  $M(t|t_0)$  and  $H(t|t_0)$  defined in (8).

We now consider a diffusion approximation of  $N(t)$ . To this purpose, we use a suitable procedure, similar to that adopted in Di Crescenzo et al. [29,30] and Dharmaraja et al. [31]. We rename the intensity functions related to  $N(t)$ , by setting

$$\lambda(t) = \alpha_1(t) + \frac{r(t)}{\epsilon}, \quad \mu(t) = \alpha_2(t) + \frac{r(t)}{\epsilon}, \quad \nu(t) = \frac{\beta(t)}{\epsilon}, \tag{10}$$

where  $\epsilon$  is a positive scaling parameter. In (10),  $\alpha_1(t) > 0$ ,  $\alpha_2(t) > 0$ ,  $r(t) > 0$  and  $\beta(t) \geq 0$  are bounded and continuous functions for  $t \geq t_0$ . Let us consider the Markov process  $\{N_\epsilon(t), t \geq t_0\}$ , with  $N_\epsilon(t) = \epsilon N(t)$ , having state-space  $\{0, \epsilon, 2\epsilon, \dots\}$ . For  $\epsilon \downarrow 0$ , the scaled process  $N_\epsilon(t)$  converges weakly to a diffusion process  $\{X(t), t \geq t_0\}$  having state-space  $[0, +\infty)$ . Indeed, with reference to the system (6), substituting  $p_{j,n}(t|t_0)$  with  $f(n\epsilon, t|j\epsilon, t_0)\epsilon$  and setting  $x = n\epsilon$  and  $x_0 = j\epsilon$ , we have:

$$\begin{aligned} \frac{\partial f(x, t|x_0, t_0)}{\partial t} &= \frac{\lambda(t)(x - \epsilon) + \nu(t)\epsilon}{\epsilon} f(x - \epsilon, t|x_0, t_0) \\ &\quad - \frac{[\lambda(t) + \mu(t)]x + \nu(t)\epsilon}{\epsilon} f(x, t|x_0, t_0) + \frac{\mu(t)(x + \epsilon)}{\epsilon} f(x + \epsilon, t|x_0, t_0), \\ \lim_{\epsilon \rightarrow 0} \left\{ \epsilon \frac{\partial f(x, t|x_0, t_0)}{\partial t} + [\lambda(t)x + \nu(t)\epsilon]f(x, t|x_0, t_0) \right. \\ &\quad \left. - \mu(t)(x + \epsilon)f(x + \epsilon, t|x_0, t_0) \right\} = 0, \end{aligned}$$

with the intensity functions  $\lambda(t)$ ,  $\mu(t)$  and  $\nu(t)$  defined in (10). Expanding  $f(x - \epsilon, t|x_0, t_0)$  and  $f(x + \epsilon, t|x_0, t_0)$  as Taylor series, taking the limit as  $\epsilon \downarrow 0$  and setting  $\alpha(t) = \alpha_1(t) - \alpha_2(t)$ , we obtain (2) and (4) with the delta initial condition (3). Hence, making use of the considered approximating procedure, the Fokker-Planck Equation (2) can be obtained from the second of (6), whereas the zero-flux condition (4) can be derived from the first equation of (6). Moreover, from (10), it follows that  $\alpha(t) = \lambda(t) - \mu(t)$ , so that  $\alpha(t)$  can be positive, negative or null for each fixed  $t$ .

### 3. Moment Generating Function and Transition PDF in Special Cases

Let  $\{X(t), t \geq t_0\}$ ,  $t_0 \geq 0$ , be the time-inhomogeneous Feller-type diffusion process with infinitesimal drift and infinitesimal variance given in (1), defined in the state-space  $[0, +\infty)$ , with a zero-flux condition in the zero-state. In this section, we determine the moment generating function, the conditional mean and the conditional variance. Furthermore, the explicit expression of the transition pdf  $f(x, t|x_0, t_0)$  is obtained in the following special situations: (i) in the absence of immigration, (ii) for  $\beta(t) = \zeta r(t)$ , with  $\zeta > 0$ , and (iii) for the time-homogeneous process.

### 3.1. Moment Generating Function, Conditional Mean and Conditional Variance

For  $t \geq t_0$  and  $x_0 \geq 0$ , we consider the moment generating function:

$$M(s, t|x_0, t_0) = \int_0^{+\infty} e^{-sx} f(x, t|x_0, t_0) dx, \quad \text{Re } s > 0. \tag{11}$$

Multiplying both sides of (2) by  $e^{-sx}$ , integrating with respect to  $x$  over the interval  $[0, +\infty)$  and making use of the boundary condition (4), we obtain the following partial differential equation

$$\frac{\partial M(s, t|x_0, t_0)}{\partial t} - s [\alpha(t) - s r(t)] \frac{\partial M(s, t|x_0, t_0)}{\partial s} + s \beta(t) M(s, t|x_0, t_0) = 0, \tag{12}$$

to solve with the initial condition

$$\lim_{t \downarrow t_0} M(s, t|x_0, t_0) = e^{-s x_0}, \tag{13}$$

derived from (11) by using the initial condition (3).

**Proposition 1.** For  $t \geq t_0$ , by assuming that  $\alpha(t) \in \mathbb{R}$ ,  $\beta(t) \geq 0$  and  $r(t) > 0$ , the moment generating function of the Feller-type diffusion process  $X(t)$  with a zero-flux condition in the zero-state is:

$$M(s, t|x_0, t_0) = \exp \left\{ -s \int_{t_0}^t \frac{\beta(u) e^{A(t|u)}}{1 + s e^{A(t|t_0)} [R(t|t_0) - R(u|t_0)]} du \right\} \times \exp \left\{ -\frac{s x_0 e^{A(t|t_0)}}{1 + s e^{A(t|t_0)} R(t|t_0)} \right\}, \quad x_0 \geq 0, \tag{14}$$

where

$$A(t|t_0) = \int_{t_0}^t \alpha(z) dz, \quad R(t|t_0) = \int_{t_0}^t r(\tau) e^{-A(\tau|t_0)} d\tau. \tag{15}$$

**Proof.** The proof is given in Appendix A.  $\square$

The expression of the moment generating function, given in (14), allows to determine the conditional mean and the conditional variance of the time-inhomogeneous Feller-type diffusion process  $X(t)$ . Indeed, for  $t \geq t_0$  and  $x_0 \geq 0$  one has:

$$\begin{aligned} E[X(t)|X(t_0) = x_0] &= x_0 e^{A(t|t_0)} + \int_{t_0}^t \beta(u) e^{A(t|u)} du, \\ \text{Var}[X(t)|X(t_0) = x_0] &= 2 x_0 e^{2A(t|t_0)} R(t|t_0) \\ &\quad + 2 e^{A(t|t_0)} \int_{t_0}^t \beta(u) e^{A(t|u)} [R(t|t_0) - R(u|t_0)] du. \end{aligned} \tag{16}$$

We note that the conditional mean in (16) coincides with the solution of the linear first-order differential equation:

$$\frac{dx(t)}{dt} = \alpha(t) x(t) + \beta(t), \quad x(t_0) = x_0.$$

Moreover, Equations (14) and (16) can be also derived from (7) and (9), respectively, making use of the diffusion approximation described in Section 2. Indeed, by virtue of (10) with  $\alpha(t) = \alpha_1(t) - \alpha_2(t)$ , from (7) and (9) one has:

$$\begin{aligned}
 M(s, t|x_0, t_0) &= \lim_{\varepsilon \downarrow 0} G\left(e^{-s\varepsilon}, t \mid \frac{x_0}{\varepsilon}, t_0\right), \\
 E[X(t)|X(t_0) = x_0] &= \lim_{\varepsilon \downarrow 0} \left\{ \varepsilon E\left[N(t)|N(t_0) = \frac{x_0}{\varepsilon}\right] \right\}, \\
 \text{Var}[X(t)|X(t_0) = x_0] &= \lim_{\varepsilon \downarrow 0} \left\{ \varepsilon^2 \text{Var}\left[N(t)|N(t_0) = \frac{x_0}{\varepsilon}\right] \right\}.
 \end{aligned}$$

### 3.2. Absence of Immigration

We assume that the immigration intensity function  $\beta(t) = 0$ ,  $\alpha(t) \in \mathbb{R}$  and  $r(t) > 0$  for  $t \geq 0$ .

**Proposition 2.** *If  $\beta(t) = 0$ , for  $t \geq t_0$  the moment generating function (14) becomes:*

$$M^*(s, t|x_0, t_0) = \exp\left\{-\frac{s x_0 e^{A(t|t_0)}}{1 + s e^{A(t|t_0)} R(t|t_0)}\right\}, \quad x_0 \geq 0. \tag{17}$$

Furthermore, the transition pdf of  $X(t)$  with a zero-flux condition in the zero-state is:

$$f^*(x, t|x_0, t_0) = \begin{cases} \delta(x), & x_0 = 0, \\ \exp\left\{-\frac{x_0}{R(t|t_0)}\right\} \delta(x) + \frac{1}{R(t|t_0)} \sqrt{\frac{x_0 e^{-A(t|t_0)}}{x}} \\ \quad \times \exp\left\{-\frac{x_0 + x e^{-A(t|t_0)}}{R(t|t_0)}\right\} I_1\left[\frac{2\sqrt{x x_0 e^{-A(t|t_0)}}}{R(t|t_0)}\right], & x_0 > 0, \end{cases} \tag{18}$$

with  $A(t|t_0)$  and  $R(t|t_0)$  defined in (15) and where

$$I_\nu(z) = \sum_{k=0}^{+\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k+\nu}, \quad \nu \in \mathbb{R} \tag{19}$$

denotes the modified Bessel function of the first kind and  $\Gamma(\zeta)$  is the Euler gamma function.

**Proof.** The proof is given in Appendix B.  $\square$

Equation (18) is in agreement with the expression given in Masoliver [11] and in Gan and Waxman [32]. We now consider the random variable  $T(x_0, t_0)$  describing the first-passage time through the zero-state starting from  $x_0 > 0$  at time  $t_0$ . We note that (18) can be rewritten as

$$f^*(x, t|x_0, t_0) = P\{T(x_0, t_0) < t\} \delta(x) + f_a(x, t|x_0, t_0), \quad x_0 > 0,$$

where

$$P\{T(x_0, t_0) < t\} = \exp\left\{-\frac{x_0}{R(t|t_0)}\right\}$$

is the first-passage time probability through the zero-state starting from  $x_0 > 0$  and  $f_a(x, t|x_0, t_0)$  denotes the transition pdf of the considered Feller process in the presence of an absorbing boundary in the zero-state (cf. for instance, Giorno and Nobile [26]):

$$f_a(x, t|x_0, t_0) = \frac{1}{R(t|t_0)} \sqrt{\frac{x_0 e^{-A(t|t_0)}}{x}} \exp\left\{-\frac{x_0 + x e^{-A(t|t_0)}}{R(t|t_0)}\right\} I_1\left[\frac{2\sqrt{x x_0 e^{-A(t|t_0)}}}{R(t|t_0)}\right].$$

From (18) it follows:

$$\int_0^{+\infty} f^*(x, t|x_0, t_0) dx = P\{T(x_0, t_0) < t\} + \int_0^{+\infty} f_a(x, t|x_0, t_0) dx = 1.$$

Setting  $\beta(t) = 0$  in (16), we obtain the conditional mean and the conditional variance of  $X(t)$  in the absence of immigration. Moreover, making use of (18), the  $k$ -th conditional moment can be evaluated:

$$\begin{aligned} E[X^k(t)|X(t_0) = x_0] &= \int_0^{+\infty} x^k f_X^*(x, t|x_0, t_0) dx \\ &= (k - 1)! [R(t|t_0) e^{A(t|t_0)}]^k \sum_{j=1}^k \binom{k}{j} \frac{1}{(j - 1)!} \left[ \frac{x_0}{R(t|t_0)} \right]^j, \quad k = 1, 2, \dots \end{aligned} \tag{20}$$

We note that if  $x_0 = 0$ , from (20) one has  $E[X^k(t)|X(t_0) = 0] = 0$ , according to the first expression of (18).

### 3.3. Proportional Case

For  $t \geq 0$ , we assume that the functions  $\beta(t)$  and  $r(t)$  are proportional:

$$\beta(t) = \xi r(t), \quad \xi > 0. \tag{21}$$

**Proposition 3.** Under the assumption (21), if  $\alpha(t) \in \mathbb{R}$  and  $\beta(t) > 0$  for  $t \geq t_0$  one has:

$$M(s, t|x_0, t_0) = \left[ 1 + s e^{A(t|t_0)} R(t|t_0) \right]^{-\xi} \exp \left\{ - \frac{s x_0 e^{A(t|t_0)}}{1 + s e^{A(t|t_0)} R(t|t_0)} \right\}, \quad x_0 \geq 0. \tag{22}$$

Furthermore, the transition pdf of  $X(t)$  with a zero-flux condition in the zero-state is:

$$f(x, t|x_0, t_0) = \begin{cases} \frac{1}{x \Gamma(\xi)} \left[ \frac{x e^{-A(t|t_0)}}{R(t|t_0)} \right]^\xi \exp \left\{ - \frac{x e^{-A(t|t_0)}}{R(t|t_0)} \right\}, & x_0 = 0, \\ \frac{e^{-A(t|t_0)}}{R(t|t_0)} \left[ \frac{x e^{-A(t|t_0)}}{x_0} \right]^{(\xi-1)/2} \exp \left\{ - \frac{x_0 + x e^{-A(t|t_0)}}{R(t|t_0)} \right\} \\ \quad \times I_{\xi-1} \left[ \frac{2\sqrt{x x_0 e^{-A(t|t_0)}}}{R(t|t_0)} \right], & x_0 > 0, \end{cases} \tag{23}$$

with  $A(t|t_0)$  and  $R(t|t_0)$  defined in (15) and  $I_\nu(z)$  given in (19).

**Proof.** The proof is given in Appendix C.  $\square$

Since for fixed  $\nu$ , when  $z \rightarrow 0$  (cf. Abramowitz and Stegun [33], p. 375, no 9.6.7)

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^\nu \quad \nu \neq -1, -2, \dots, \tag{24}$$

the first formula of (23) follows from the second expression as  $x_0 \downarrow 0$ . Furthermore, by virtue of (24), from (23) for  $x_0 \geq 0$  one has:

$$\lim_{x \downarrow 0} f(x, t|x_0, t_0) = \begin{cases} +\infty, & 0 < \xi < 1, \\ \frac{e^{-A(t|t_0)}}{R(t|t_0)} \exp \left\{ - \frac{x_0}{R(t|t_0)} \right\}, & \xi = 1, \\ 0, & \xi > 1. \end{cases} \tag{25}$$

Relation (25) shows that if  $\xi > 1$  the zero-state behaves as an entrance boundary that cannot be reached from the interior of the state-space, while it is possible to starts right there.

By setting  $\beta(t) = \zeta r(t)$ , with  $\zeta > 0$ , in (16), we obtain the conditional mean and the conditional variance of  $X(t)$  in the proportional case. Moreover, making use of (23), the  $k$ -th conditional moment can be evaluated:

$$E[X^k(t)|X(t_0) = x_0] = k! \left[ R(t|t_0) e^{A(t|t_0)} \right]^k \left\{ \frac{(\zeta)_k}{k!} + \sum_{i=1}^k \frac{(\zeta)_{k-i}}{i(k-i)!} \sum_{j=1}^i \binom{i}{j} \frac{1}{(j-1)!} \left[ \frac{x_0}{R(t|t_0)} \right]^j \right\}, \quad k = 1, 2, \dots, \tag{26}$$

where  $(\zeta)_n$  denotes the Pochhammer symbol, defined as  $(\zeta)_0 = 1$  and  $(\zeta)_n = \zeta(\zeta + 1) \cdots (\zeta + n - 1)$  for  $n = 1, 2, \dots$ . In particular, by setting  $x_0 = 0$  in (26) one has:

$$E[X^k(t)|X(t_0) = 0] = (\zeta)_k \left[ R(t|t_0) e^{A(t|t_0)} \right]^k, \quad k = 1, 2, \dots$$

### 3.4. Time-Homogeneous Feller Process

We consider the time-homogeneous Feller process, obtained from (1) by setting  $\alpha(t) = \alpha, \beta(t) = \beta, r(t) = r$ , with  $\alpha \in \mathbb{R}, \beta \geq 0$  and  $r > 0$ . This process is analyzed by Feller [34,35]. The explicit expression of transition pdf in the presence of a zero-flux condition in the zero-state for  $\beta > 0$  is given in Karlin and Taylor [36] and in Giorno et al. [37]. From (15) we have:

$$A(t|t_0) = \alpha(t - t_0), \quad R(t|t_0) = \begin{cases} r(t - t_0), & \alpha = 0, \\ \frac{r}{\alpha} (1 - e^{-\alpha(t-t_0)}), & \alpha \neq 0. \end{cases} \tag{27}$$

In the absence of immigration, i.e., when  $\beta = 0$ , the transition pdf can be obtained from (18) making use of (27). When  $\alpha = 0, \beta > 0$  and  $r > 0$ , by virtue of (27), one has:

$$f(x, t|x_0, t_0) = \begin{cases} \frac{1}{x \Gamma(\beta/r)} \left[ \frac{x}{r(t-t_0)} \right]^{\beta/r} \exp\left\{ -\frac{x}{r(t-t_0)} \right\}, & x_0 = 0, \\ \frac{1}{r(t-t_0)} \left( \frac{x}{x_0} \right)^{(\beta-r)/(2r)} \exp\left\{ -\frac{x_0+x}{r(t-t_0)} \right\} \\ \quad \times I_{\beta/r-1} \left[ \frac{2\sqrt{xx_0}}{r(t-t_0)} \right], & x_0 > 0, \end{cases} \tag{28}$$

whereas if  $\alpha \neq 0, \beta > 0$  and  $r > 0$  one obtains:

$$f(x, t|x_0, t_0) = \begin{cases} \frac{1}{x \Gamma(\beta/r)} \left[ \frac{\alpha x}{r(e^{\alpha(t-t_0)} - 1)} \right]^{\beta/r} \exp\left\{ -\frac{\alpha x}{r(e^{\alpha(t-t_0)} - 1)} \right\}, & x_0 = 0, \\ \frac{\alpha}{r(e^{\alpha(t-t_0)} - 1)} \left[ \frac{x e^{-\alpha(t-t_0)}}{x_0} \right]^{(\beta-r)/(2r)} \exp\left\{ -\frac{\alpha [x+x_0 e^{\alpha(t-t_0)}]}{r(e^{\alpha(t-t_0)} - 1)} \right\} \\ \quad \times I_{\beta/r-1} \left[ \frac{2\alpha \sqrt{xx_0 e^{\alpha(t-t_0)}}}{r(e^{\alpha(t-t_0)} - 1)} \right], & x_0 > 0. \end{cases} \tag{29}$$

Note that (28) and (29) can be derived from (23) by setting  $\zeta = \beta/r, A(t|t_0)$  and  $R(t|t_0)$  as in (27). Moreover, by carrying out the same choices in (26), the conditioned moments are also obtained.

When  $\alpha < 0, \beta > 0$  and  $r > 0$ , the time-homogeneous Feller process admits a steady-state behavior:

$$W(x) = \lim_{t \rightarrow +\infty} f(x, t|x_0, t_0) = \frac{1}{x \Gamma(\beta/r)} \left( \frac{|\alpha| x}{r} \right)^{\beta/r} \exp\left\{ -\frac{|\alpha| x}{r} \right\}, \quad x > 0,$$

that is a gamma density of parameters  $\beta/r$  and  $r/|\alpha|$ . We note that

$$\lim_{x \downarrow 0} W(x) = \begin{cases} +\infty, & \beta < r, \\ |\alpha|/r, & \beta = r, \\ 0, & \beta > r. \end{cases}$$

The steady-state density  $W(x)$  is a decreasing function of  $x$  when  $\beta \leq r$ , whereas  $W(x)$  has a single maximum in  $x = (\beta - r)/|\alpha|$  for  $\beta > r$ . Furthermore, the asymptotic moments are:

$$E(X^k) = \lim_{t \rightarrow +\infty} E[X^k(t)|X(t_0) = x_0] = \left(\frac{r}{|\alpha|}\right)^k \frac{\Gamma(k + \beta/r)}{\Gamma(\beta/r)}, \quad k = 1, 2, \dots$$

#### 4. Transition PDF and Conditional Moments in the General Case

In this section, we obtain the transition pdf and its moments for the Feller-type diffusion process (1) with a zero-flux condition in the zero-state in the general case. Furthermore, the special cases considered in the Section 3 are now derived from the general case.

From (14), for  $t \geq t_0$  we note that

$$M(s, t|x_0, t_0) = \begin{cases} M(s, t|0, t_0), & x_0 = 0, \\ M(s, t|0, t_0) M^*(s, t|x_0, t_0), & x_0 > 0, \end{cases}$$

where  $M(s, t|0, t_0)$  is the moment generating function of  $X(t)$  for  $x_0 = 0$ , whereas  $M^*(s, t|x_0, t_0)$  is the moment generating function given in (17).

Therefore, to determine the transition pdf  $f(x, t|x_0, t_0)$  of the time-inhomogeneous Feller process, we proceed as follows:

- (1) we determine the transition pdf  $f(x, t|0, t_0)$  for  $x \geq 0$  and  $t \geq t_0$ ;
- (2) we calculate the transition density  $f(x, t|x_0, t_0)$  for  $x_0 > 0$ ,  $x \geq 0$  and  $t \geq t_0$  as a convolution between  $f(x, t|0, t_0)$  and the transition pdf  $f^*(x, t|x_0, t_0)$ , given in (18), of the Feller-type process in the absence of immigration.

##### 4.1. General Case: $x_0 = 0$

To determine the transition pdf of time-inhomogeneous Feller-type process with a zero-flux condition in the zero-state, we set  $x_0 = 0$  in (14), so that for  $t \geq t_0$  we obtain:

$$M(s, t|0, t_0) = \exp\left\{-s \int_{t_0}^t \frac{\beta(u) e^{A(t|u)}}{1 + s e^{A(t|t_0)} [R(t|t_0) - R(u|t_0)]} du\right\}, \quad (30)$$

with  $A(t|t_0)$  and  $R(t|t_0)$  defined in (15).

In the sequel, we denote by  $B_n(d_1, d_2, \dots, d_n)$  the complete Bell polynomials, recursively defined as follows:

$$B_0 = 1, \quad B_{n+1}(d_1, d_2, \dots, d_{n+1}) = \sum_{i=0}^n \binom{n}{i} B_{n-i}(d_1, d_2, \dots, d_{n-i}) d_{i+1}, \quad n \in \mathbb{N}_0, \quad (31)$$

with

$$d_k = -\frac{k!}{[R(t|t_0)]^k} \int_{t_0}^t \beta(u) e^{-A(u|t_0)} [R(u|t_0)]^{k-1} du, \quad k = 1, 2, \dots \quad (32)$$

In particular, from (31) and (32) one has

$$\begin{aligned} B_1(d_1) &= d_1 = -\frac{1}{R(t|t_0)} \int_{t_0}^t \beta(u) e^{-A(u|t_0)} du, \\ B_2(d_1, d_2) &= d_1^2 + d_2 = \frac{1}{R^2(t|t_0)} \left\{ \left[ \int_{t_0}^t \beta(u) e^{-A(u|t_0)} du \right]^2 \right. \\ &\quad \left. - 2 \int_{t_0}^t \beta(u) e^{-A(u|t_0)} R(u|t_0) du \right\}. \end{aligned} \quad (33)$$

Furthermore, we consider the Laguerre polynomials

$$L_n(y) = \frac{e^y}{n!} \frac{d^n}{dy^n} (e^{-y} y^n) = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k y^k}{k!}, \quad n = 0, 1, \dots \tag{34}$$

whose derivative (cf. Gradshteyn and Ryzhik [38], p. 1001, n. 8.971.3) is:

$$\frac{dL_n(y)}{dy} = \frac{n L_n(y) - n L_{n-1}(y)}{y}. \tag{35}$$

**Proposition 4.** Under the assumptions of Proposition 1, for  $t \geq t_0$  and  $x_0 = 0$  the transition pdf of the time-inhomogeneous Feller-type diffusion process  $X(t)$  with a condition of zero-flux in the zero-state is

$$f(x, t|0, t_0) = \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \delta(x) + \exp\left\{-\frac{x e^{-A(t|t_0)}}{R(t|t_0)}\right\} \Phi(x, t|t_0), \tag{36}$$

where

$$\Phi(z, t|t_0) = \sum_{n=1}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \frac{d}{dz} L_n\left[\frac{z e^{-A(t|t_0)}}{R(t|t_0)}\right], \quad z > 0, \tag{37}$$

with  $A(t|t_0)$  and  $R(t|t_0)$  defined in (15),  $B_n(d_1, d_2, \dots, d_n)$  given in (31) and (32),  $L_n(y)$  and  $dL_n(y)/dy$  defined in (34) and (35), respectively.

**Proof.** The proof is given in Appendix D.  $\square$

**Proposition 5.** Under the assumptions of Proposition 1, for  $t \geq t_0$  the  $k$ -th conditional moment of  $X(t)$  with  $X(t_0) = 0$  is:

$$E[X^k(t)|X(t_0) = 0] = k! \left[ R(t|t_0) e^{A(t|t_0)} \right]^k \sum_{n=1}^k (-1)^n \binom{k-1}{n-1} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \tag{38}$$

for  $k = 1, 2, \dots$

**Proof.** The proof is given in Appendix E.  $\square$

We note that, by virtue of (33), from (38) follows (16) for  $x_0 = 0$ .

#### 4.2. General Case: $x_0 > 0$

We determine the transition pdf of  $X(t)$  when  $x_0 > 0$ .

**Proposition 6.** For  $t \geq t_0$  and  $x_0 > 0$ , the transition pdf of the time-inhomogeneous Feller-type diffusion process  $X(t)$  with a condition of zero-flux in the zero-state is:

$$\begin{aligned} f(x, t|x_0, t_0) &= f^*(x, t|x_0, t_0) \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \\ &+ \exp\left\{-\frac{x_0 + x e^{-A(t|t_0)}}{R(t|t_0)}\right\} \left\{ \Phi(x, t|t_0) + \frac{1}{R(t|t_0)} \right. \\ &\times \left. \int_0^x \sqrt{\frac{x_0 e^{-A(t|t_0)}}{x-z}} I_1 \left[ \frac{2\sqrt{(x-z) x_0 e^{-A(t|t_0)}}}{R(t|t_0)} \right] \Phi(z, t|t_0) dz \right\}, \end{aligned} \tag{39}$$

where  $f^*(x, t|x_0, t_0)$  is the transition pdf in the absence of immigration, defined in (18), and  $\Phi(x, t|t_0)$  is given (37).

**Proof.** For  $x_0 > 0$ , the moment generating function (14) for the Feller process  $X(t)$  can be written as  $M(s, t|x_0, t_0) = M(s, t|0, t_0) M^*(s, t|x_0, t_0)$ , where  $M(s, t|0, t_0)$  is the moment generating function of  $X(t)$  for  $x_0 = 0$ , whereas  $M^*(s, t|x_0, t_0)$  is the moment generating function given (17). Therefore, for  $t \geq t_0$  the transition pdf  $f(x, t|x_0, t_0)$  is given by the following convolution:

$$f(x, t|x_0, t_0) = \int_0^x f(x - z, t|0, t_0) f^*(z, t|x_0, t_0) dz, \quad x_0 > 0 \tag{40}$$

from which, recalling (18) and (36), Equation (39) follows.  $\square$

**Proposition 7.** Under the assumptions of Proposition 6, for  $t \geq t_0$  the  $k$ -th conditional moment of  $X(t)$  with  $X(t_0) = x_0 > 0$  is:

$$\begin{aligned} E[X^k(t)|X(t_0) = x_0] &= k! \left[ R(t|t_0) e^{A(t|t_0)} \right]^k \left\{ \sum_{n=1}^k (-1)^n \binom{k-1}{n-1} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \right. \\ &+ \sum_{i=1}^{k-1} \frac{1}{i} \left[ \sum_{j=1}^i \binom{i}{j} \frac{1}{(j-1)!} \left[ \frac{x_0}{R(t|t_0)} \right]^j \right] \left[ \sum_{n=1}^{k-i} (-1)^n \binom{k-i-1}{n-1} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \right] \\ &\left. + \frac{1}{k} \sum_{j=1}^k \binom{k}{j} \frac{1}{(j-1)!} \left[ \frac{x_0}{R(t|t_0)} \right]^j \right\} \tag{41} \end{aligned}$$

for  $k = 2, 3, \dots$ , whereas for  $k = 1$  the first formula of (16) holds.

**Proof.** The proof is given in Appendix F.  $\square$

Note that (36) and (38) follow taking the limit as  $x_0 \downarrow 0$  in (39) and (41), respectively.

**Remark 1** (Absence of immigration). We assume that  $\beta(t) = 0$ . We prove that (18) can be obtained from (36) and (39).

Indeed, from (31) and (32) one has

$$d_n = 0, \quad B_0 = 1, \quad B_n(d_1, d_2, \dots, d_n) = 0, \quad n = 1, 2, \dots,$$

so that, recalling (37), one has:

$$\sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} = 1, \quad \Phi(z, t|t_0) = 0.$$

Hence, from (36) we have  $f(x, t|0, t_0) = \delta(x)$ , that coincides with the first expression of (18), whereas from (39) we obtain the second formula of (18). Furthermore, by setting  $\beta(t) = 0$  in (41), we have (20).

**Remark 2** (Proportional case). We assume that (21) holds, i.e.,  $\beta(t) = \xi r(t)$ , with  $\xi > 0$ . We prove that (23) can be obtained from (36) and (39).

Indeed, by virtue of (31) and (32) one has

$$d_n = -\xi (n-1)!, \quad B_0 = 1, \quad B_n(d_1, d_2, \dots, d_n) = (-\xi)_n, \quad n = 1, 2, \dots$$

Recalling the binomial series

$$\sum_{n=0}^{+\infty} \binom{n+b-1}{n} x^n = \sum_{n=0}^{+\infty} \binom{b}{n} \frac{x^n}{n!} = (1-x)^{-b},$$

it follows:

$$\sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} = \sum_{n=0}^{+\infty} \frac{(-\xi)_n}{n!} = 0, \quad \xi > 0. \tag{42}$$

Furthermore, since (cf. Erdelyi et al. [39], p. 213, no. 16)

$$\sum_{n=0}^{+\infty} \frac{(-\xi)_n}{n!} L_n(y) = \frac{y^\xi}{\Gamma(\xi + 1)}, \quad y > 0, \xi \geq 0,$$

from (37) one has:

$$\begin{aligned} \Phi(z, t|t_0) &= \sum_{n=1}^{+\infty} \frac{(-\xi)_n}{n!} \frac{d}{dz} L_n \left[ \frac{z e^{-A(t|t_0)}}{R(t|t_0)} \right] = \frac{d}{dz} \sum_{n=0}^{+\infty} \frac{(-\xi)_n}{n!} L_n \left[ \frac{z e^{-A(t|t_0)}}{R(t|t_0)} \right] \\ &= \frac{z^{\xi-1}}{\Gamma(\xi)} \left[ \frac{e^{-A(t|t_0)}}{R(t|t_0)} \right]^\xi. \end{aligned} \tag{43}$$

Hence, making use of (42) and (43) in (36), the first expression of (23) follows. Moreover, from (39), by virtue of (42) and (43), it follows:

$$\begin{aligned} f(x, t|x_0, t_0) &= \frac{1}{\Gamma(\xi)} \exp \left\{ -\frac{x_0 + x e^{-A(t|t_0)}}{R(t|t_0)} \right\} \left[ \frac{e^{-A(t|t_0)}}{R(t|t_0)} \right]^\xi \\ &\times \left\{ x^{\xi-1} + \frac{1}{R(t|t_0)} \int_0^x z^{\xi-1} \sqrt{\frac{x_0 e^{-A(t|t_0)}}{x-z}} I_1 \left[ \frac{2\sqrt{(x-z) x_0 e^{-A(t|t_0)}}}{R(t|t_0)} \right] dz \right\} \end{aligned}$$

that leads to the second expression of (23), being

$$\int_0^x (x-z)^{\xi-1} z^{-1/2} I_1(2a\sqrt{z}) dz = -\frac{x^{\xi-1}}{a} + \Gamma(\xi) a^{-\xi} x^{(\xi-1)/2} I_{\xi-1}(2a\sqrt{x}), \quad \text{Re } a \geq 0.$$

Finally, recalling that

$$\sum_{n=1}^k (-1)^n \binom{k-1}{n-1} \frac{(-\xi)_n}{n!} = \frac{(\xi)_k}{k!}, \quad k = 1, 2, \dots,$$

from (41) one obtains (26).

### 5. Periodic Intensity Functions

Periodic immigration and periodic growth intensity functions play an important role in the description of the evolution of dynamic systems influenced by seasonal immigration or other regular environmental cycles. Furthermore, the population dynamics can be affected by noise of periodic intensity. Therefore, in this section we assume that the growth intensity function  $\alpha(t)$ , or the immigration intensity function  $\beta(t)$  or the noise intensity  $r(t)$  have some kind of periodicity (cf. for instance, Coleman et al. [40], Keeling and Rohani [41]).

#### 5.1. Periodic Immigration Intensity Function

We consider the time-inhomogeneous Feller process  $X(t)$  such that

$$A_1(x, t) = \alpha x + \xi r(t), \quad A_2(x, t) = 2r(t)x, \tag{44}$$

with  $\alpha \in \mathbb{R}$ ,  $\xi > 0$  and a zero-flux condition in the zero-state. We assume that  $r(t)$  is a periodic function of period  $Q_1$ . From (15) for  $n = 0, 1, \dots$  one has  $A(t + n Q_1|t_0) = \alpha(t + n Q_1 - t_0)$  and

$$R(t + n Q_1|t_0) = \int_{t_0}^{t+nQ_1} r(\tau) e^{-A(\tau|t_0)} d\tau = e^{\alpha t_0} \left\{ \frac{1 - e^{-n\alpha Q_1}}{1 - e^{-\alpha Q_1}} \int_{t_0}^{t_0+Q_1} r(\tau) e^{-\alpha \tau} d\tau + e^{-n\alpha Q_1} \int_{t_0}^t r(\tau) e^{-\alpha \tau} d\tau \right\}.$$

If  $\alpha < 0$ , the process  $X(t)$  admits an asymptotic behavior. In this case, from (23), one has:

$$W(x, t) = \lim_{n \rightarrow +\infty} f(x, t + n Q_1|x_0, t_0) = \frac{1}{x \Gamma(\xi)} [\psi_1(t) x]^\xi e^{-\psi_1(t)x}, \quad x \geq 0, \quad (45)$$

where

$$\psi_1(t) = \lim_{n \rightarrow +\infty} \frac{e^{-A(t+nQ_1|t_0)}}{R(t+nQ_1|t_0)} = \frac{e^{|\alpha|Q_1} - 1}{R(t+Q_1|t)}, \quad \alpha < 0. \quad (46)$$

We note that (45) is a gamma density of parameters  $\xi$  and  $[\psi_1(t)]^{-1}$  for all  $t \geq 0$ , so that for  $\alpha < 0$  it follows:

$$M_k(t) = \lim_{n \rightarrow +\infty} E\{[X(t + n Q_1)]^k | X(t_0) = x_0\} = [\psi_1(t)]^{-k} (\xi)_k, \quad k = 1, 2, \dots, \quad (47)$$

with  $\psi_1(t)$  given in (46).

**Example 1.** The dynamic of a population influenced by seasonal immigration and regular environmental cycles, can be described by the time-inhomogeneous Feller process (44), with

$$r(t) = v \left[ 1 + c \sin\left(\frac{2\pi t}{Q_1}\right) \right], \quad t \geq 0, \quad (48)$$

where  $v > 0$  is the average of the periodic function  $r(t)$  of period  $Q_1$ ,  $c$  is the amplitude of the oscillations, with  $0 \leq c < 1$ . These choices of parameters ensure that the both the immigration intensity function and the environment noise are positive functions. From (15), for  $t \geq t_0$  one has  $A(t|t_0) = \alpha(t - t_0)$  and

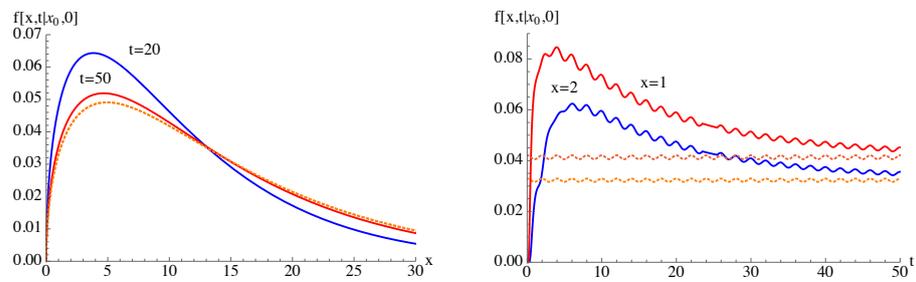
$$R(t|t_0) = \begin{cases} v(t - t_0) + \frac{c v Q_1}{2\pi} \left[ \cos\left(\frac{2\pi t_0}{Q_1}\right) - \cos\left(\frac{2\pi t}{Q_1}\right) \right], & \alpha = 0, \\ \frac{v}{\alpha} \left( 1 - e^{-\alpha(t-t_0)} \right) + \frac{c v Q_1}{4\pi^2 + Q_1^2 \alpha^2} \left\{ 2\pi \cos\left(\frac{2\pi t_0}{Q_1}\right) + \alpha Q_1 \sin\left(\frac{2\pi t_0}{Q_1}\right) - e^{-\alpha(t-t_0)} \left[ 2\pi \cos\left(\frac{2\pi t}{Q_1}\right) + \alpha Q_1 \sin\left(\frac{2\pi t}{Q_1}\right) \right] \right\}, & \alpha \neq 0. \end{cases}$$

For  $\alpha < 0$ , the asymptotic density and the asymptotic moments are given in (45) and (47), respectively, with

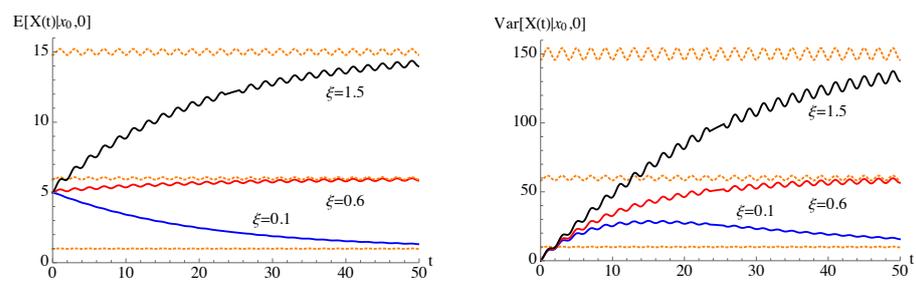
$$\psi_1(t) = \frac{1}{v} \left\{ \frac{1}{|\alpha|} - \frac{c Q_1}{4\pi^2 + Q_1^2 \alpha^2} \left[ 2\pi \cos\left(\frac{2\pi t}{Q_1}\right) + \alpha Q_1 \sin\left(\frac{2\pi t}{Q_1}\right) \right] \right\}^{-1}.$$

In Figures 1–5, we consider the process (44); we assume that  $r(t) = 0.5[1 + 0.9 \sin(\pi t)]$  and that at the initial time  $t_0 = 0$  the size of population is  $X(t_0) = x_0 = 5$ . We recall that  $\alpha < 0$  ( $\alpha > 0$ ) means that the birth intensity of the population is less (greater) than the death intensity. In Figures 1 and 2 we assume that  $\alpha = -0.05$ , so that the process admits an asymptotic behavior. Specifically, in Figure 1, for  $\xi = 1.5$  the transition pdf of  $X(t)$  is plotted as function of  $x$  on the left and as function of  $t$  on the right; the dotted functions indicate the corresponding asymptotic densities given in (45). Furthermore, in Figure 2, the conditional mean and variance of the population size and the related

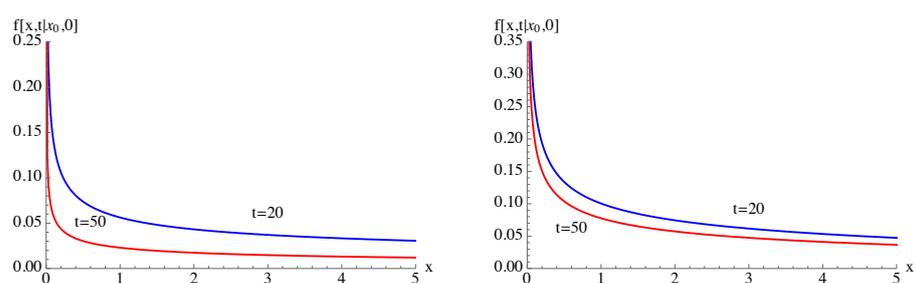
asymptotic behaviors are shown as function of  $t$  for various choices of  $\zeta$ . In Figures 3–5 we assume that  $\alpha = 0.05$  (on the left) and  $\alpha = 0$  (on the right), so that  $X(t)$  does not admit an asymptotic behavior being  $\alpha \geq 0$ . In particular, in Figures 3, for  $\zeta = 0.6$  the transition densities of the process (44) are plotted as function of  $x$ . Finally, in Figures 4 and 5, the conditional mean and variance are plotted as function of  $t$  for various choices of  $\zeta$ ; we note that the mean and the variance of the population size don't have an upper bound as  $t$  increases.



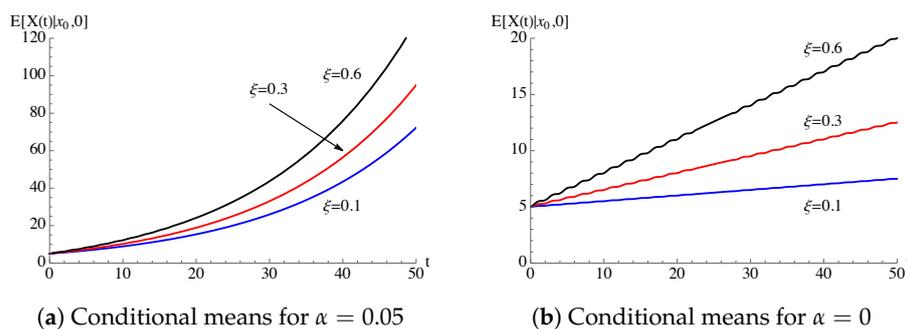
(a) Transition densities as function of  $x$  (b) Transition densities as function of  $t$   
**Figure 1.** For the process (44), with  $x_0 = 5$ ,  $\alpha = -0.05$ ,  $\zeta = 1.5$  and  $r(t) = 0.5[1 + 0.9 \sin(\pi t)]$ , the transition densities are plotted. The dotted functions indicate the corresponding asymptotic densities.



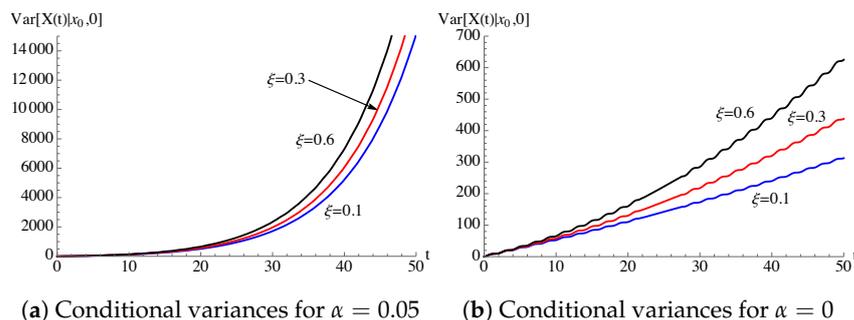
(a) Conditional means (b) Conditional variances  
**Figure 2.** For the process of Figure 1, the conditional means and variances are plotted as function of  $t$  for various choices of  $\zeta$ . The dotted functions indicate the corresponding asymptotic means and variances.



(a) Transition densities for  $\alpha = 0.05$  (b) Transition densities for  $\alpha = 0$   
**Figure 3.** For the process (44), with  $x_0 = 5$ ,  $\zeta = 0.6$  and  $r(t) = 0.5[1 + 0.9 \sin(\pi t)]$ , the transition densities are plotted as function of  $x$ .



**Figure 4.** For the process of Figure 3, the conditional means are plotted as function of  $t$  for various choices of  $\zeta$ .



**Figure 5.** For the process of Figure 3, the conditional variances are plotted as function of  $t$  for various choices of  $\zeta$ .

5.2. Periodic Growth Intensity Function

We consider the time-inhomogeneous Feller process  $X(t)$  such that

$$A_1(x, t) = \alpha(t) x + \zeta r, \quad A_2(x) = 2 r x, \tag{49}$$

with  $r > 0, \zeta > 0$  and a zero-flux condition in the zero-state. We assume that  $\alpha(t)$  is a periodic function of period  $Q_2$  and let

$$\bar{\alpha} = \frac{A(t + Q_2|t)}{Q_2} = \frac{1}{Q_2} \int_t^{t+Q_2} \alpha(\tau) d\tau \tag{50}$$

be the mean of  $\alpha(t)$  in the period  $Q_2$ . From (15), for  $n = 0, 1, \dots$  one has:

$$\begin{aligned} A(t + n Q_2|t_0) &= \int_{t_0}^{t+n Q_2} \alpha(\tau) d\tau = n A(t_0 + Q_2|t_0) + A(t|t_0) = n \bar{\alpha} Q_2 + A(t|t_0), \\ R(t + n Q_2|t_0) &= r \int_{t_0}^{t+n Q_2} e^{-A(\tau|t_0)} d\tau \\ &= r \left\{ \frac{1 - e^{-n \bar{\alpha} Q_2}}{1 - e^{-\bar{\alpha} Q_2}} \int_{t_0}^{t_0+Q_2} e^{-A(\tau|t_0)} d\tau + e^{-n \bar{\alpha} Q_2} \int_{t_0}^t e^{-A(\tau|t_0)} d\tau \right\}. \end{aligned}$$

If  $\bar{\alpha} < 0$ , the process  $X(t)$  admits an asymptotic behavior and, from (23), one has:

$$W(x, t) = \lim_{n \rightarrow +\infty} f(x, t + n Q_2|x_0, t_0) = \frac{1}{x \Gamma(\bar{\zeta})} [\psi_2(t) x]^\zeta e^{-\psi_2(t) x}, \quad x \geq 0, \tag{51}$$

where

$$\psi_2(t) = \lim_{n \rightarrow +\infty} \frac{e^{-A(t+n Q_2|t_0)}}{R(t + n Q_2|t_0)} = \frac{e^{|\bar{\alpha}| Q_2} - 1}{R(t + Q_2|t)}, \quad \bar{\alpha} < 0. \tag{52}$$

We note again that (51) is a gamma density of parameters  $\zeta$  and  $[\psi_2(t)]^{-1}$  for all  $t \geq 0$ ; hence, for  $\bar{\alpha} < 0$  we have:

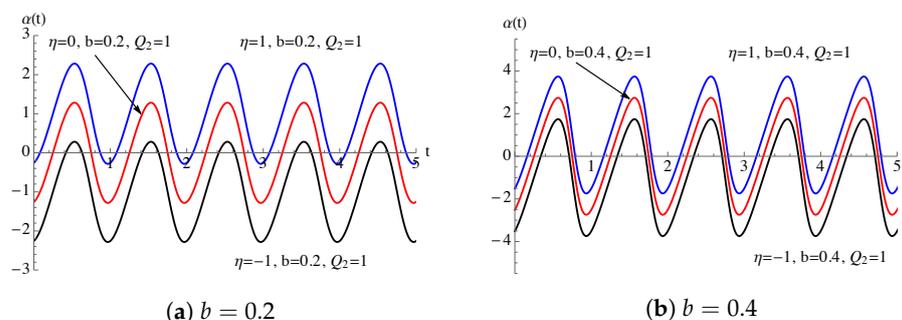
$$M_k(t) = \lim_{n \rightarrow +\infty} E\{[X(t + n Q_2)]^k | X(t_0) = x_0\} = [\psi_2(t)]^{-k} (\zeta)_k, \quad k = 1, 2, \dots, \quad (53)$$

with  $\psi_2(t)$  given in (52).

**Example 2.** For the time-inhomogeneous Feller process (49), we consider the flexible growth intensity function

$$\alpha(t) = \eta - \frac{2\pi b}{Q_2} \frac{\cos\left(\frac{2\pi t}{Q_2}\right)}{1 + b \sin\left(\frac{2\pi t}{Q_2}\right)}, \quad t \geq 0, \quad (54)$$

where  $\eta \in \mathbb{R}$ ,  $Q_2$  is the period of  $\alpha(t)$  and  $b$  determines the amplitude of the oscillations, with  $0 \leq b < 1$ . As shown in Figure 6, the growth intensity function (54) can be positive, negative or zero at different time instants; furthermore, different choices of the parameter  $b$  make the function (54) less or more asymmetric in a period. Hence, the variety of shapes exhibits by  $\alpha(t)$  allows to model several population growth trends.



**Figure 6.** The growth intensity function  $\alpha(t)$ , given in (54), is plotted as function of  $t$  for some choices of the parameters.

From (15), making use of (54), for  $t \geq t_0$  one obtains the following cumulative growth intensity function

$$A(t|t_0) = \eta (t - t_0) - \ln \left[ 1 + b \sin\left(\frac{2\pi t}{Q_2}\right) \right] + \ln \left[ 1 + b \sin\left(\frac{2\pi t_0}{Q_2}\right) \right] \quad (55)$$

and hence

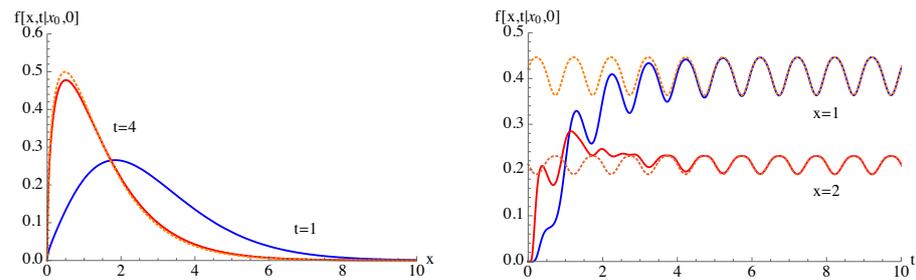
$$R(t|t_0) = \begin{cases} \frac{r}{1 + b \sin\left(\frac{2\pi t_0}{Q_2}\right)} \left\{ t - t_0 - \frac{b Q_2}{2\pi} \left[ \cos\left(\frac{2\pi t}{Q_2}\right) - \cos\left(\frac{2\pi t_0}{Q_2}\right) \right] \right\}, & \eta = 0, \\ \frac{r}{1 + b \sin\left(\frac{2\pi t_0}{Q_2}\right)} \left\{ \frac{1 - e^{-\eta(t-t_0)}}{\eta} - \frac{2\pi b Q_2}{4\pi^2 + Q_2^2 \eta^2} \left[ e^{-\eta(t-t_0)} \cos\left(\frac{2\pi t}{Q_2}\right) \right. \right. \\ \left. \left. + \frac{Q_2 \eta}{2\pi} e^{-\eta(t-t_0)} \sin\left(\frac{2\pi t}{Q_2}\right) - \cos\left(\frac{2\pi t_0}{Q_2}\right) - \frac{Q_2 \eta}{2\pi} \sin\left(\frac{2\pi t_0}{Q_2}\right) \right] \right\}, & \eta \neq 0. \end{cases} \quad (56)$$

From (50) and (54) we have  $\bar{\alpha} = \eta$ , so that  $X(t)$  admits an asymptotic behavior for  $\eta < 0$ ; the asymptotic density and the asymptotic moments are given in (51) and (53), respectively, with

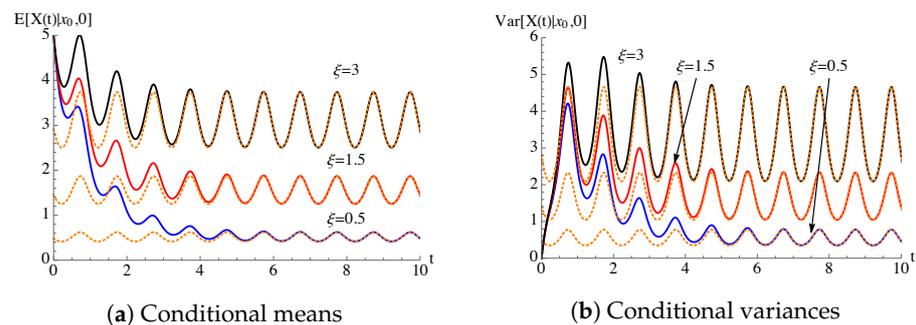
$$\psi_2(t) = \frac{1}{r} \left[ 1 + b \sin\left(\frac{2\pi t}{Q_2}\right) \right] \left\{ \frac{1}{|\eta|} - \frac{2\pi b Q_2}{4\pi^2 + Q_2^2 \eta^2} \left[ \cos\left(\frac{2\pi t}{Q_2}\right) + \frac{Q_2 \eta}{2\pi} \sin\left(\frac{2\pi t}{Q_2}\right) \right] \right\}^{-1}.$$

In Figures 7–11, we consider the process (49) with  $r = 1$  and  $\alpha(t) = \eta - 0.4 \pi \cos(2\pi t) [1 + 0.2 \sin(2\pi t)]^{-1}$ ; furthermore, we assume that at the initial time  $t_0 = 0$  the size of population is  $X(t_0) = x_0 = 5$ . We note that  $\eta < 0$  ( $\eta > 0$ ) indicates that the average of birth intensity

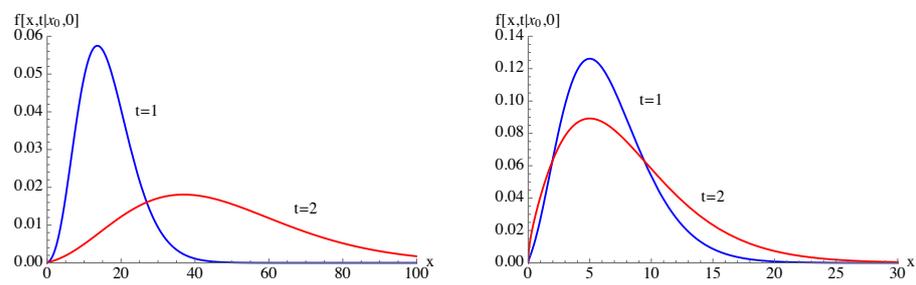
of the population is less (greater) than the average of the death intensity. In Figures 7 and 8 we assume that  $\eta = -1$ , so that the process admits an asymptotic behavior. In particular, in Figure 7, for  $\zeta = 1.5$  the transition pdf of  $X(t)$  is plotted as function of  $x$  on the left and as function of  $t$  on the right; the dotted functions indicate the corresponding asymptotic densities, given in (51). Furthermore, in Figure 8 the conditional mean and variance and the related asymptotic behaviors are plotted as function of  $t$  for various choices of  $\zeta$ . Instead, in Figures 9–11 we assume that  $\eta = 1$  on the left and  $\eta = 0$  on the right. Specifically, in Figure 9, the transition densities of the process (49), for  $\zeta = 1.5$  are shown as function of  $x$ . Finally, in Figures 10 and 11 the conditional mean and variance of the population size are plotted as function of  $t$  for various choices of  $\zeta$ .



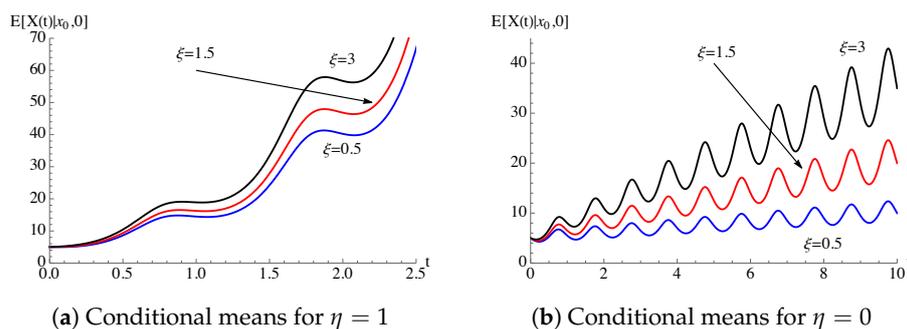
(a) Transition densities as function of  $x$  (b) Transition densities as function of  $t$   
**Figure 7.** For the process (49), with  $x_0 = 5, r = 1, \zeta = 1.5, \eta = -1$  and  $\alpha(t) = \eta - 0.4 \pi \cos(2 \pi t) [1 + 0.2 \sin(2 \pi t)]^{-1}$ , the transition densities are plotted. The dotted functions indicate the corresponding asymptotic densities.



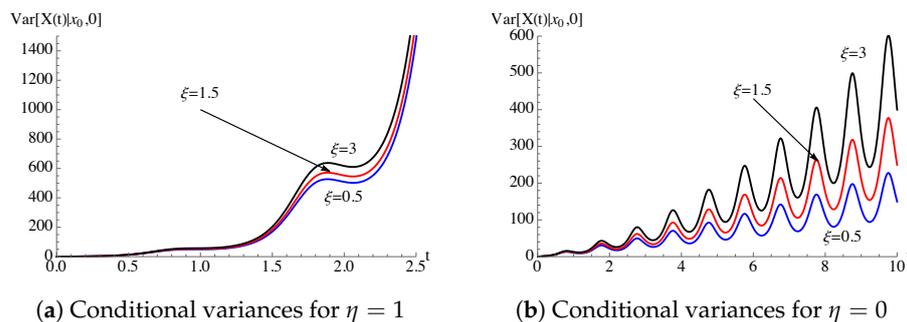
(a) Conditional means (b) Conditional variances  
**Figure 8.** For the process of Figure 7, the conditional means and variances are plotted as function of  $t$  for various choices of  $\zeta$ . The dotted functions indicate the related asymptotic means and variances.



(a) Transition densities for  $\eta = 1$  (b) Transition densities for  $\eta = 0$   
**Figure 9.** For the process of Figure 7, the transition densities are plotted as function of  $x$ .



(a) Conditional means for  $\eta = 1$  (b) Conditional means for  $\eta = 0$   
**Figure 10.** For the process of Figure 7, the conditional means are plotted as function of  $t$  for various choices of  $\xi$ .



(a) Conditional variances for  $\eta = 1$  (b) Conditional variances for  $\eta = 0$   
**Figure 11.** For the process of Figure 7, the conditional variances are plotted as function of  $t$  for various choices of  $\xi$ .

### 5.3. Periodic Immigration and Growth Intensity Functions

We consider the process  $X(t)$  such that

$$A_1(x, t) = \alpha(t) x + \xi r(t), \quad A_2(x, t) = 2 r(t) x, \tag{57}$$

with  $\xi > 0$  and a zero-flux condition in the zero-state. We assume that  $r(t)$  and  $\alpha(t)$  are periodic functions of periods  $Q_1$  and  $Q_2$ , respectively. We denote by  $Q = \text{LCM}(Q_1, Q_2)$  the least common multiple between  $Q_1$  and  $Q_2$  and let

$$\hat{\alpha} = \frac{A(t + Q|t)}{Q} = \frac{1}{Q} \int_t^{t+Q} \alpha(\tau) d\tau = \frac{1}{Q_2} \int_t^{t+Q_2} \alpha(\tau) d\tau = \bar{\alpha}$$

be the mean of  $\alpha(t)$  in  $Q$ . From (15) for  $n = 0, 1, \dots$  one has:

$$\begin{aligned} A(t + n Q|t_0) &= \int_{t_0}^{t+nQ} \alpha(\tau) d\tau = n A(t_0 + Q|t_0) + A(t|t_0) = n \hat{\alpha} Q + A(t|t_0), \\ R(t + n Q|t_0) &= \int_{t_0}^{t+nQ} r(\tau) e^{-A(\tau|t_0)} d\tau \\ &= \frac{1 - e^{-n \hat{\alpha} Q}}{1 - e^{-\hat{\alpha} Q}} \int_{t_0}^{t_0+Q} r(\tau) e^{-A(\tau|t_0)} d\tau + e^{-n \hat{\alpha} Q} \int_{t_0}^t r(\tau) e^{-A(\tau|t_0)} d\tau. \end{aligned}$$

If  $\hat{\alpha} < 0$ , the process  $X(t)$  admits an asymptotic behavior and, from (23), one obtains the gamma density:

$$W(x, t) = \lim_{n \rightarrow +\infty} f(x, t + n Q|x_0, t_0) = \frac{1}{x \Gamma(\xi)} [\psi_3(t) x]^\xi e^{-\psi_3(t) x}, \quad x \geq 0, \tag{58}$$

where

$$\psi_3(t) = \lim_{n \rightarrow +\infty} \frac{e^{-A(t+nQ|t_0)}}{R(t+nQ|t_0)} = \frac{e^{\hat{\alpha} Q} - 1}{R(t + Q|t)}, \quad \hat{\alpha} < 0. \tag{59}$$

Hence, for  $\hat{\alpha} < 0$  we have:

$$M_k(t) = \lim_{n \rightarrow +\infty} E\{[X(t + nQ)]^k | X(t_0) = x_0\} = [\psi_3(t)]^{-k}(\xi)_k, \quad k = 1, 2, \dots, \quad (60)$$

with  $\psi_3(t)$  given in (59).

**Example 3.** We consider a population described by the time-inhomogeneous Feller process (57) and we assume that the noise intensity  $r(t)$  is defined in (48) and the growth intensity function  $\alpha(t)$  is chosen as in (54). In this case, the cumulative growth intensity function  $A(t|t_0)$  is given in (55) and

$$R(t|t_0) = \frac{\nu}{1 + b \sin\left(\frac{2\pi t_0}{Q_2}\right)} \int_{t_0}^t e^{-\eta(\tau-t_0)} \left[1 + c \sin\left(\frac{2\pi\tau}{Q_1}\right)\right] \left[1 + b \sin\left(\frac{2\pi\tau}{Q_2}\right)\right] d\tau. \quad (61)$$

From (50) and (54) we have  $\hat{\alpha} = \eta$ . For  $\eta < 0$ , the asymptotic density and the asymptotic moments are given in (58) and (60), respectively, with

$$\psi_3(t) = \frac{1}{\nu} \left[1 + b \sin\left(\frac{2\pi t}{Q_2}\right)\right] \left\{ \frac{1}{|\eta|} - c Q_1 B_1(t) - b Q_2 B_2(t) - \frac{bc Q_1 Q_2}{2} [C(t) - D(t)] \right\}^{-1},$$

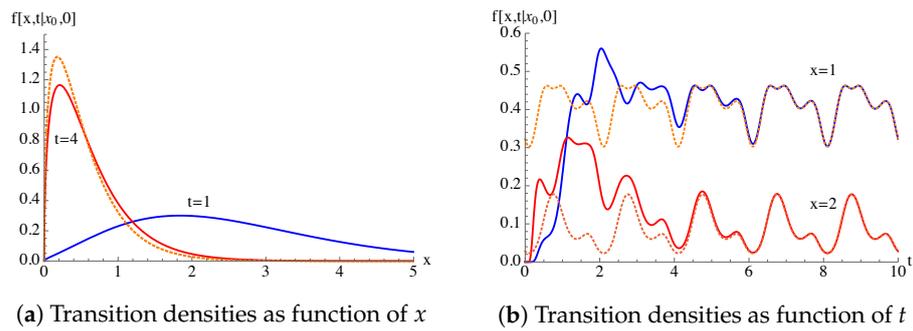
where

$$B_i(t) = \frac{2\pi \cos\left(\frac{2\pi t}{Q_i}\right) + Q_i \eta \sin\left(\frac{2\pi t}{Q_i}\right)}{4\pi^2 + Q_i^2 \eta^2}, \quad i = 1, 2,$$

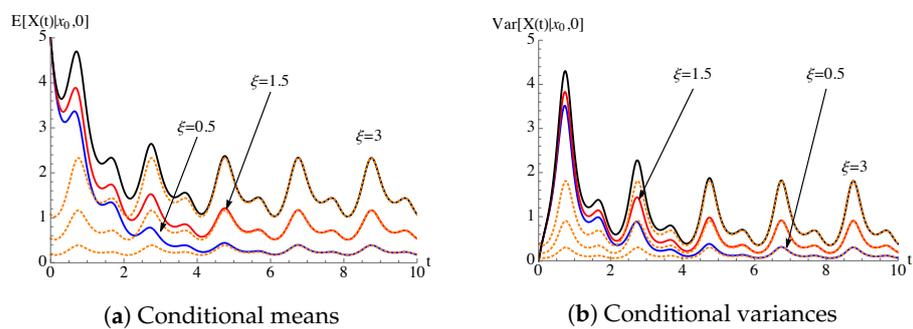
$$C(t) = \frac{Q_1 Q_2 \eta \cos\left(2\pi \frac{Q_1 - Q_2}{Q_1 Q_2} t\right) - 2\pi (Q_1 - Q_2) \sin\left(2\pi \frac{Q_1 - Q_2}{Q_1 Q_2} t\right)}{4\pi^2 (Q_1 - Q_2)^2 + Q_1^2 Q_2^2 \eta^2},$$

$$D(t) = \frac{Q_1 Q_2 \eta \cos\left(2\pi \frac{Q_1 + Q_2}{Q_1 Q_2} t\right) - 2\pi (Q_1 + Q_2) \sin\left(2\pi \frac{Q_1 + Q_2}{Q_1 Q_2} t\right)}{4\pi^2 (Q_1 + Q_2)^2 + Q_1^2 Q_2^2 \eta^2}.$$

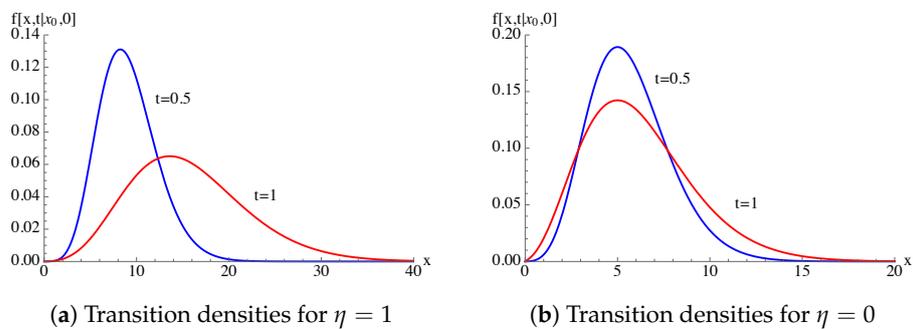
In Figures 12–16, we assume that the noise intensity function is  $r(t) = 0.5[1 + 0.9 \sin(\pi t)]$  and the growth intensity function is  $\alpha(t) = \eta - 0.4 \pi \cos(2\pi t) [1 + 0.2 \sin(2\pi t)]^{-1}$ ; since  $r(t)$  has period  $Q_1 = 2$  and  $\alpha(t)$  has period  $Q_2 = 1$ , one has  $Q = \text{LCM}(Q_1, Q_2) = 2$ . Furthermore, the size of population at time  $t_0 = 0$  is  $X(0) = 5$ . In Figures 12 and 13 we assume that  $\eta = -1$ , so that the process admits an asymptotic behavior. Specifically, in Figure 12, for  $\xi = 1.5$  the transition pdf of the process (57) is plotted as function of  $x$  on the left and as function of  $t$  on the right; the dotted functions indicate the corresponding asymptotic densities, given in (51). Furthermore, in Figure 13 the conditional mean and variance and the related asymptotic behaviors are plotted as function of  $t$  for various choices of  $\xi$ . Comparing Figures 8 and 13 we can highlight the effect of the periodic noise intensity on the conditional mean and variance of the population size for a fixed periodic growth intensity function. In Figures 14–16, we consider  $\eta = 1$  on the left and  $\eta = 0$  on the right; in these cases the process does not exhibit an asymptotic behavior, being  $\eta \geq 0$ . In Figure 14, for  $\xi = 1.5$  the transition densities of the process are plotted as function of  $x$ . Finally, in Figures 15 and 16 the conditional mean and variance of the population size are shown as function of  $t$  for various choices of  $\xi$ .



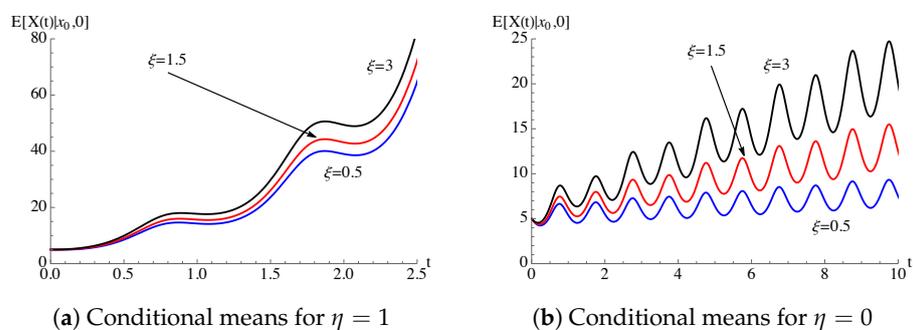
**Figure 12.** For the process (57), with  $x_0 = 5$ ,  $\zeta = 1.5$ ,  $\eta = -1$ ,  $\alpha(t) = \eta - 0.4\pi \cos(2\pi t)[1 + 0.2 \sin(2\pi t)]^{-1}$  and  $r(t) = 0.5[1 + 0.9 \sin(\pi t)]$ , the transition densities are plotted. The dotted functions indicate the related asymptotic densities.



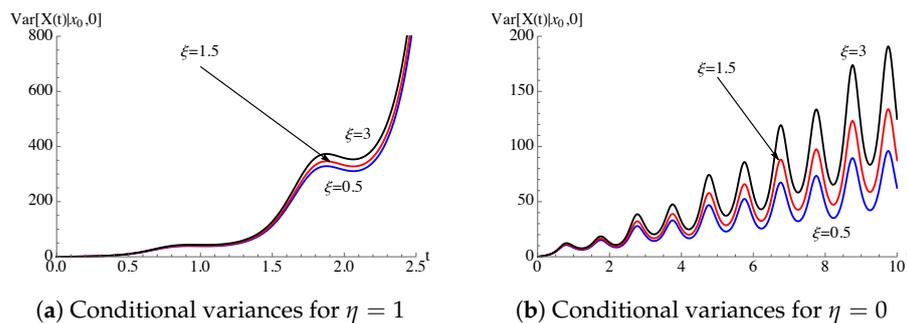
**Figure 13.** For the process of Figure 12, the conditional means and the conditional variances are plotted as function of  $t$  for various choices of  $\zeta$ . The dotted functions indicate the corresponding asymptotic means and variances.



**Figure 14.** For the process of Figure 12, the transition densities are plotted as function of  $x$ .



**Figure 15.** For the process of Figure 12, the conditional means are plotted as function of  $t$  for various choices of  $\zeta$ .



(a) Conditional variances for  $\eta = 1$  (b) Conditional variances for  $\eta = 0$   
**Figure 16.** For the process of Figure 12, the conditional variances are plotted as function of  $t$  for various choices of  $\xi$ .

5.4. Periodic Immigration, Growth and Noise Intensity Functions

We consider the time-inhomogeneous Feller process  $X(t)$  having infinitesimal moments (1), with a zero-flux condition in the zero-state. We assume that  $r(t)$ ,  $\alpha(t)$  and  $\beta(t)$  are periodic functions of periods  $Q_1$ ,  $Q_2$  and  $Q_3$ , respectively. Some, but not all, of these functions can be constant. We denote by  $Q$  the least common multiple of the periods related to the periodic functions and by  $\hat{\alpha}$  the mean of  $\alpha(t)$  in  $Q$ , so that relations in (58) hold.

If  $\hat{\alpha} < 0$ , the process  $X(t)$  admits an asymptotic behavior. In this case, from (16) one obtains the asymptotic mean and variance:

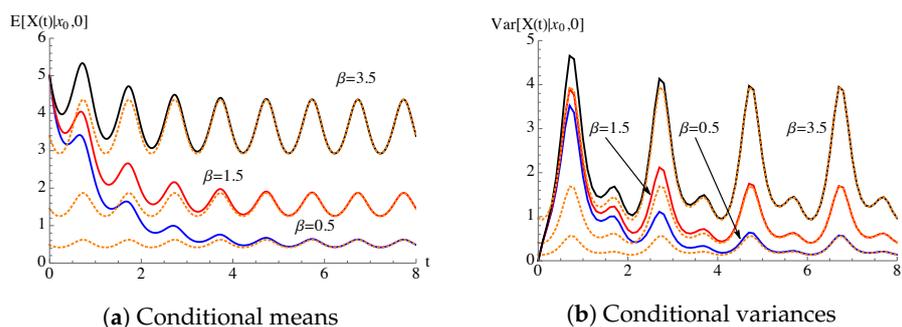
$$M_1(t) = \lim_{n \rightarrow +\infty} E\{X(t + nQ) | X(t_0) = x_0\} = \frac{1}{e^{|\hat{\alpha}|Q} - 1} \int_t^{t+Q} \beta(u) e^{-A(u|t)} du, \tag{62}$$

$$V(t) = \lim_{n \rightarrow +\infty} \text{Var}\{X(t + nQ) | X(t_0) = x_0\} = \frac{2}{(e^{|\hat{\alpha}|Q} - 1)^2 (1 + e^{|\hat{\alpha}|Q})} \times \int_t^{t+Q} \beta(u) e^{-A(u|t)} \left[ R(u|t) + e^{|\hat{\alpha}|Q} \int_u^{t+Q} r(\tau) e^{-A(\tau|t)} d\tau \right] du. \tag{63}$$

Note that if  $\beta(t) = \zeta r(t)$ , from (62) and (63) one has  $M_1(t) = \zeta [\psi_3(t)]^{-1}$  and  $V(t) = \zeta^2 [\psi_3(t)]^{-2}$ , with  $\psi_3(t)$  defined in (59).

**Example 4.** We consider the process (1) and we assume that  $\beta(t) = \beta > 0$ ,  $r(t)$  is given in (48) and  $\alpha(t)$  is defined in (54). In this case, the immigration rate is constant, whereas the growth and the noise intensity functions are periodic functions with different periods. Expressions (55) and (61) for  $A(t|t_0)$  and  $R(t|t_0)$  hold. Furthermore, from (50) and (54) we have  $\hat{\alpha} = \eta$ , so that for  $\eta < 0$  the process admits an asymptotic behavior.

We assume that the noise intensity function  $r(t)$  has period  $Q_1 = 2$  and the growth intensity function  $\alpha(t)$  has period  $Q_2 = 1$ , so that  $Q = \text{LCM}(Q_1, Q_2) = 2$ . In Figure 17, the conditional mean and variance and the related asymptotic behaviors are plotted as function of  $t$  for various choices of  $\beta$ . We note that the conditional mean is not affected to the periodicity of  $r(t)$ , whereas the conditional variance depends on the different periodicities of the growth intensity function  $\alpha(t)$  and of the noise intensity function  $r(t)$ .



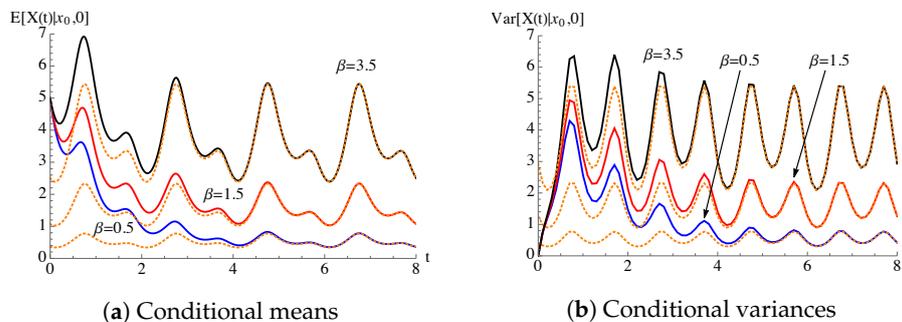
**Figure 17.** For the process (1), with  $x_0 = 5, \eta = -1, \alpha(t) = \eta - 0.4 \pi \cos(2 \pi t) [1 + 0.2 \sin(2 \pi t)]^{-1}$  and  $r(t) = 0.5 [1 + 0.9 \sin(\pi t)]$ , the conditional means and variances are plotted as function of  $t$  for various choices of  $\beta$ . The dotted functions indicate the corresponding asymptotic means and variances.

**Example 5.** We consider the process (1) and we assume that  $r(t) = r > 0, \alpha(t)$  is defined in (54) and

$$\beta(t) = \beta \left[ 1 + c \sin\left(\frac{2\pi t}{Q_1}\right) \right], \quad t \geq 0, \tag{64}$$

where  $\beta > 0$  is the average of the periodic function  $\beta(t)$  of period  $Q_1, c$  is the amplitude of the oscillations, with  $0 \leq c < 1$ . Differently from Example 4, the noise intensity is constant, whereas the growth and the immigration intensity functions are periodic functions with different periods. Expressions (55) and (56) for  $A(t|t_0)$  and  $R(t|t_0)$  hold. Furthermore, from (50) and (54) we have  $\hat{\alpha} = \eta$ , so that for  $\eta < 0$  the process admits an asymptotic behavior.

We assume that the immigration intensity function  $\beta(t)$ , given in (64), has period  $Q_1 = 2$  and the growth intensity function  $\alpha(t)$  has period  $Q_2 = 1$ , so that  $Q = \text{LCM}(Q_1, Q_2) = 2$ . In Figure 18, the conditional mean and variance and the related asymptotic behaviors are plotted as function of  $t$  for various choices of  $\beta$ . We note that both the mean and the variance depend on the periodicities of the intensity functions  $\alpha(t)$  and  $\beta(t)$ .



**Figure 18.** For the process (1), with  $x_0 = 5, \eta = -1, r = 1, \alpha(t) = \eta - 0.4 \pi \cos(2 \pi t) [1 + 0.2 \sin(2 \pi t)]^{-1}$  and  $\beta(t) = \beta [1 + 0.9 \sin(\pi t)]$ , the conditional means and variances are plotted as function of  $t$  for various choices of  $\beta$ . The dotted functions indicate the related asymptotic means and variances.

**6. Concluding Remarks**

In this paper, we considered a time-inhomogeneous Feller-type diffusion process  $\{X(t), t \geq t_0\}, t_0 \geq 0$ , with infinitesimal drift  $A_1(x, t) = \alpha(t) x + \beta(t)$  and infinitesimal variance  $A_2(x, t) = 2 r(t) x$ , defined in the state-space  $[0, +\infty)$ , with a zero-flux condition in the zero-state. We have assumed that  $\alpha(t) \in \mathbb{R}, \beta(t) \geq 0, r(t) > 0$  for all  $t \geq t_0$ . This diffusion process plays a relevant role in several biological applications and can be derived as the continuous approximation of the time-inhomogeneous birth-death process with immigration. In the general case, the transition density and the conditional moments are explicitly obtained. Some special situations are analyzed: (i) the absence of immigration with  $\beta(t) = 0$ , (ii) the proportional case in which  $\beta(t) = \zeta r(t)$ , with  $\zeta > 0$ , and (iii) the time-homogeneous case. Sometimes in the dynamics of populations it is necessary to consider

periodic intensity functions. Indeed, periodic immigration and periodic growth intensity functions play an important role in the description of the evolution of dynamic systems influenced by seasonal immigration or other regular environmental cycles. Furthermore, the population dynamics can be affected by noise of periodic intensity. Therefore, we have assumed that the growth intensity function  $\alpha(t)$ , or the immigration intensity function  $\beta(t)$  or the noise intensity  $r(t)$  have some kind of periodicity. In these cases, the asymptotic behaviors of the transition pdf and of the moments are discussed. Various numerical computations are performed to analyze how the population dynamics is affected by the periodic intensity functions.

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### Appendix A. Proof of Proposition 1

We solve (12) with initial condition (13) making use of the method of characteristics (cf. for instance, Williams [42]). We consider the differential equations:

$$\frac{dt}{d\zeta} = 1, \quad \frac{ds}{d\zeta} = -s [\alpha(t) - sr(t)], \quad \frac{dM}{d\zeta} = -s\beta(t)M, \tag{A1}$$

and the initial conditions:

$$t(w, \zeta = t_0) = t_0, \quad s(w, \zeta = t_0) = w, \quad M(w, \zeta = t_0) = e^{-wx_0}. \tag{A2}$$

The first expression of (A1), with the first initial condition in (A2), leads to  $t = \zeta$ . Solving the second equation in (A1) with  $t = \zeta$  and recalling of the second formula of (A2), it follows:

$$s = \frac{w e^{-A(\zeta|t_0)}}{1 - w R(\zeta|t_0)}. \tag{A3}$$

Then, solving the third equation in (A1), with the related initial condition in (A2) and with  $t = \zeta$  and  $s$  given in (A3), we obtain:

$$M(w, \zeta) = e^{-wx_0} \exp \left\{ -w \int_{t_0}^{\zeta} \frac{\beta(u) e^{-A(u|t_0)}}{1 - w R(u|t_0)} du \right\}. \tag{A4}$$

From (A3) with  $\zeta = t$ , we also have

$$w = \frac{s e^{A(t|t_0)}}{1 + s e^{A(t|t_0)} R(t|t_0)}. \tag{A5}$$

Finally, Equation (14) follows from (A4), recalling that  $\zeta = t$  and making use of (A5).

**Appendix B. Proof of Proposition 2**

Setting  $\beta(t) = 0$  in (14), we immediately obtain (17), from which one has:

$$M^*(s, t|x_0, t_0) = \int_0^{+\infty} e^{-sx} f^*(x, t|x_0, t_0) dx$$

$$= \exp\left\{-\frac{x_0}{R(t|t_0)}\right\} \exp\left\{\frac{x_0}{R(t|t_0)} \frac{1}{1 + s e^{A(t|t_0)} R(t|t_0)}\right\}, \quad s > 0. \tag{A6}$$

Setting

$$1 + s e^{A(t|t_0)} R(t|t_0) = z, \quad \frac{x e^{-A(t|t_0)}}{R(t|t_0)} = y, \tag{A7}$$

in (A6), we obtain

$$\int_0^{+\infty} e^{-zy} \left\{ e^y f^* [R(t|t_0) e^{A(t|t_0)} y, t|x_0, t_0] \right\} dy$$

$$= \frac{e^{-A(t|t_0)}}{R(t|t_0)} \exp\left\{-\frac{x_0}{R(t|t_0)}\right\} \exp\left\{\frac{x_0}{R(t|t_0)} \frac{1}{z}\right\}. \tag{A8}$$

Since

$$\int_0^{+\infty} e^{-py} \left\{ \sqrt{\frac{a}{y}} I_1(2\sqrt{a}y) + \delta(y) \right\} dy = e^{a/p},$$

taking the inverse Laplace transforms on both sides of (A8), it follows:

$$f^* [R(t|t_0) e^{A(t|t_0)} y, t|x_0, t_0] = e^{-y} \frac{e^{-A(t|t_0)}}{R(t|t_0)} \exp\left\{-\frac{x_0}{R(t|t_0)}\right\}$$

$$\times \left\{ \delta(y) + \sqrt{\frac{x_0}{y R(t|t_0)}} I_1\left(2\sqrt{\frac{x_0 y}{R(t|t_0)}}\right) \right\}. \tag{A9}$$

Equation (18) follows from (A9), applying again the transformation  $x = R(t|t_0) e^{A(t|t_0)} y$  and recalling that  $\delta(ax) = \delta(x)/|a|$ .

**Appendix C. Proof of Proposition 3**

We note that

$$\frac{d}{du} \ln\left\{1 + s e^{A(t|t_0)} [R(t|t_0) - R(u|t_0)]\right\} = -\frac{s r(u) e^{A(t|u)}}{1 + s e^{A(t|t_0)} [R(t|t_0) - R(u|t_0)]}, \quad t_0 \leq u \leq t,$$

so that

$$\exp\left\{-\int_{t_0}^t \frac{s \beta(u) e^{A(t|u)}}{1 + s e^{A(t|t_0)} [R(t|t_0) - R(u|t_0)]} du\right\} = \exp\left\{-\xi \int_{t_0}^t \frac{s r(u) e^{A(t|u)}}{1 + s e^{A(t|t_0)} [R(t|t_0) - R(u|t_0)]} du\right\}$$

$$= [1 + s e^{A(t|t_0)} R(t|t_0)]^{-\xi}. \tag{A10}$$

Hence, making use of (A10) in (14), one obtains (22).

To derive (23), we distinguish two cases: (i)  $x_0 = 0$  and (ii)  $x_0 > 0$ .

Case (i) If  $x_0 = 0$ , the expression (22) becomes:

$$M(s, t|0, t_0) = \left[\frac{e^{-A(t|t_0)}}{R(t|t_0)}\right]^\xi \left[s + \frac{e^{-A(t|t_0)}}{R(t|t_0)}\right]^{-\xi}, \quad \xi > 0, \tag{A11}$$

which is recognized to be the moment generating function of a gamma distribution of parameters  $\xi$  and  $e^{-A(t|t_0)}/R(t|t_0)$  for all  $t \geq t_0$ . Indeed, since (cf. Erdélyi et al. [43], p. 144, no. 3)

$$\int_0^{+\infty} e^{-sx} x^{\nu-1} e^{-ax} dx = \Gamma(\nu) (s+a)^{-\nu}, \quad \text{Re } \nu > 0,$$

from (11), for  $t \geq t_0$  the first formula of (23) immediately follows.

Case (ii) If  $x > 0$ , from (11) and (22) it follows:

$$\int_0^{+\infty} e^{-sx} f(x, t|x_0, t_0) dx = \left[1 + s e^{A(t|t_0)} R(t|t_0)\right]^{-\xi} \exp\left\{-\frac{s x_0 e^{A(t|t_0)}}{1 + s e^{A(t|t_0)} R(t|t_0)}\right\},$$

so that, by virtue of (A7), one has:

$$\int_0^{+\infty} e^{-zy} \left\{ e^y f[y R(t|t_0) e^{A(t|t_0)}, t|x_0, t_0] \right\} dy = \frac{e^{-A(t|t_0)}}{R(t|t_0)} \exp\left\{-\frac{x_0}{R(t|t_0)}\right\} \times z^{-\xi} \exp\left\{\frac{x_0}{z R(t|t_0)}\right\}, \quad \xi > 0. \tag{A12}$$

Since (cf. Erdélyi et al. [43], p. 197, no. 18)

$$\int_0^{+\infty} e^{-zy} a^{-\nu/2} y^{\nu/2} I_\nu(2\sqrt{ay}) dy = z^{-\nu-1} e^{a/z}, \quad \text{Re } \nu > -1,$$

taking the inverse Laplace transforms on both sides of (A12), for  $t \geq t_0$  we obtain:

$$f[y R(t|t_0) e^{A(t|t_0)}, t|x_0, t_0] = e^{-y} \frac{e^{-A(t|t_0)}}{R(t|t_0)} \exp\left\{-\frac{x_0}{R(t|t_0)}\right\} \left[\frac{x_0}{R(t|t_0)}\right]^{(1-\xi)/2} \times y^{(\xi-1)/2} I_{\xi-1}\left[2\sqrt{\frac{x_0 y}{R(t|t_0)}}\right], \quad \xi > 0. \tag{A13}$$

The second expression of (23) follows from (A13), applying the transformation  $x = R(t|t_0) e^{A(t|t_0)} y$ .

**Appendix D. Proof of Proposition 4**

We assume that  $x \geq 0$  and  $t \geq t_0$ . Recalling (11) and by using (A7) in (30), we have:

$$\int_0^{+\infty} e^{-zy} \left\{ e^y f[y R(t|t_0) e^{A(t|t_0)}, t|0, t_0] \right\} dy = \frac{e^{-A(t|t_0)}}{R(t|t_0)} \times \exp\left\{-(z-1) \int_{t_0}^t \frac{\beta(u) e^{-A(u|t_0)}}{z R(t|t_0) - (z-1) R(u|t_0)} du\right\},$$

where

$$\begin{aligned} & \exp\left\{-(z-1) \int_{t_0}^t \frac{\beta(u) e^{-A(u|t_0)}}{z R(t|t_0) - (z-1) R(u|t_0)} du\right\} \\ &= \exp\left\{-\frac{z-1}{z} \int_{t_0}^t \frac{\beta(u) e^{-A(u|t_0)}}{R(t|t_0) \left[1 - \frac{z-1}{z} \frac{R(u|t_0)}{R(t|t_0)}\right]} du\right\} \\ &= \exp\left\{-\sum_{k=1}^{+\infty} \left(1 - \frac{1}{z}\right)^k \frac{1}{[R(t|t_0)]^k} \int_{t_0}^t \beta(u) e^{-A(u|t_0)} [R(u|t_0)]^{k-1} du\right\}. \tag{A14} \end{aligned}$$

Note that the last equality in (A14) follows being

$$0 < \frac{z-1}{z} \frac{R(u|t_0)}{R(t|t_0)} < 1, \quad t_0 \leq u \leq t.$$

Let  $B_n(d_1, d_2, \dots, d_n)$  be the complete Bell polynomials defined in (31), with  $d_k$  given in (32). Since (cf. for instance, Comtet [44]):

$$\exp\left\{\sum_{r=1}^{+\infty} \frac{d_r}{r!} \vartheta^r\right\} = \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \vartheta^n,$$

from (A14) one obtains:

$$\exp\left\{-(z-1) \int_{t_0}^t \frac{\beta(u) e^{-A(u|t_0)}}{z R(t|t_0) - (z-1) R(u|t_0)} du\right\} = \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \left(1 - \frac{1}{z}\right)^n. \tag{A15}$$

Making use of (A15) in (A14), it follows:

$$\begin{aligned} \int_0^{+\infty} e^{-zy} \left\{ e^y f[y R(t|t_0) e^{A(t|t_0)}, t|0, t_0] \right\} dy &= \frac{e^{-A(t|t_0)}}{R(t|t_0)} \left\{ \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \right. \\ &+ \left. \sum_{n=1}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \left[ \left(1 - \frac{1}{z}\right)^n - 1 \right] \right\}. \end{aligned} \tag{A16}$$

Recalling (34), one obtains

$$\int_0^{+\infty} e^{-zy} \frac{d}{dy} L_n(y) dy = \left(1 - \frac{1}{z}\right)^n - 1, \quad \text{Re } z > 0,$$

so that Equation (A16) leads to:

$$\begin{aligned} f[y R(t|t_0) e^{A(t|t_0)}, t|0, t_0] &= e^{-y} \frac{e^{-A(t|t_0)}}{R(t|t_0)} \left\{ \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \delta(y) \right. \\ &+ \left. \sum_{n=1}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \frac{d}{dy} L_n(y) \right\}. \end{aligned}$$

Finally, Equation (36) follows, applying again the transformation  $x = y R(t|t_0) e^{A(t|t_0)}$ .

**Appendix E. Proof of Proposition 5**

By virtue of (36), for  $k = 1, 2, \dots$  one has:

$$\begin{aligned} E[X^k(t) | X(t_0) = 0] &= \int_0^{+\infty} x^k f(x, t|0, t_0) dx \\ &= \sum_{n=1}^{+\infty} \frac{B_n(d_1, d_2, \dots, d_n)}{n!} \int_0^{+\infty} x^k \exp\left\{-\frac{x e^{-A(t|t_0)}}{R(t|t_0)}\right\} \frac{d}{dx} L_n\left[\frac{x e^{-A(t|t_0)}}{R(t|t_0)}\right] dx. \end{aligned} \tag{A17}$$

By using the transformation  $y = x e^{-A(t|t_0)} / R(t|t_0)$  on the right-hand side of (A17), one has:

$$\begin{aligned} \int_0^{+\infty} x^k \exp\left\{-\frac{x e^{-A(t|t_0)}}{R(t|t_0)}\right\} \frac{d}{dx} L_n\left[\frac{x e^{-A(t|t_0)}}{R(t|t_0)}\right] dx \\ = \left[ R(t|t_0) e^{A(t|t_0)} \right]^k \int_0^{+\infty} y^k e^{-y} \frac{d}{dy} L_n(y) dy, \quad k = 1, 2, \dots \end{aligned} \tag{A18}$$

Since

$$\int_0^{+\infty} y^k e^{-y} \frac{d}{dy} L_n(y) dy = \sum_{r=1}^n \binom{n}{r} \frac{(-1)^r}{(r-1)!} (k+r-1)! \\ = \begin{cases} (-1)^n k! \binom{k-1}{n-1}, & n = 1, 2, \dots, k \\ 0, & n = k+1, k+2, \dots, \end{cases}$$

from (A18) one obtains:

$$\int_0^{+\infty} x^k \exp\left\{-\frac{x e^{-A(t|t_0)}}{R(t|t_0)}\right\} \frac{d}{dx} L_n\left[\frac{x e^{-A(t|t_0)}}{R(t|t_0)}\right] dx \\ = \begin{cases} (-1)^n k! \binom{k-1}{n-1} [R(t|t_0) e^{A(t|t_0)}]^k, & n = 0, 1, \dots, k \\ 0, & n = k+1, k+2, \dots \end{cases} \tag{A19}$$

Relation (38) follows making use of (A19) in (A17).

### Appendix F. Proof of Proposition 7

Making use of (40), for  $k = 1, 2, \dots$  it follows:

$$E[X^k(t)|X(t_0) = x_0] = \int_0^{+\infty} x^k f(x, t|x_0, t_0) dx \\ = \int_0^{+\infty} dx x^k \int_0^x f^*(y, t|x_0, t_0) f(x-y, t|0, t_0) dy \\ = \sum_{i=0}^k \binom{k}{i} \int_0^{+\infty} y^i f^*(y, t|x_0, t_0) dy \int_0^{+\infty} z^{k-i} f(z, t|0, t_0) dz. \tag{A20}$$

For  $k = 1$ , from (A20) one has

$$E[X(t)|X(t_0) = x_0] = \int_0^{+\infty} y f^*(y, t|x_0, t_0) dy + \int_0^{+\infty} z f(z, t|0, t_0) dz,$$

that leads to the first expression of (16), making use of (20) and (38). Instead, for  $k = 2, 3, \dots$  from (A20) one obtains:

$$E[X^k(t)|X(t_0) = x_0] = \int_0^{+\infty} z^k f(z, t|0, t_0) dz + \int_0^{+\infty} y^k f^*(y, t|x_0, t_0) dy \\ + \sum_{i=1}^{k-1} \binom{k}{i} \int_0^{+\infty} y^i f^*(y, t|x_0, t_0) dy \int_0^{+\infty} z^{k-i} f(z, t|0, t_0) dz,$$

from which Equation (41) follows, recalling (20) and (38).

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