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Schistosomiasis Model Incorporating Snail Predator as Biological Control Agent

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Abstract: Schistosomiasis is a parasitic disease caused by the schistosoma worm. A snail can act as the intermediate host for the parasite. Snail-population control is considered to be an effective way to control schistosomiasis spread. In this paper, we discuss the schistosomiasis model incorporating a snail predator as a biological control agent. We prove that the solutions of the model are non-negative and bounded. The existence condition of equilibrium points is investigated. We determine the basic reproduction number when the predator goes to extinction and when the predator survives. The local stability condition of disease-free equilibrium point is proved using linearization, and the Lienard–Chipart and Routh–Hurwitz criteria. We use center-manifold theory to prove the local stability condition of the endemic equilibrium points. Furthermore, we constructed a Lyapunov function to investigate the global stability condition of the disease-free equilibrium points. To support the analytical results, we presented some numerical simulation results. Our findings suggest that a snail predator as a biological control agent can reduce schistosomiasis prevalence. Moreover, the snail-predator birth rate plays an essential role in controlling schistosomiasis spread.

Keywords: schistosomiasis model; snail predator; biological control agent; stability analysis; basic reproduction number

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1. Introduction

Schistosomiasis is a parasitic disease caused by schistosoma-worm infection [1]. The parasite has a complicated life cycle. Some types of snail can act as intermediate hosts, while humans and mammals such as cows, pigs, and mice can serve as reservoir hosts [2]. Consequently, the parasite can infect mammals as a human substitute to complete its life cycle. This is one of the several factors that make schistosomiasis very difficult to eradicate. Controlling the snail population is the most effective way to control the spread of schistosomiasis. Some researchers recommended the use of molluscicides to manage the snail host population [1,3,4]. However, molluscicides have some negative environmental effects [5]. Another method that can be used to manage the snail population is an intervention with snail predators or competitors [6–8]. Sokolow et al. [7] stated that releasing river prawn, which is a snail predator, into a water contact site can reduce the snail population and schistosomiasis transmission.

Mathematical modeling is used to study the dynamics of the spread of the disease. The first mathematical model that is related to schistosomiasis is discussed in [9]. Schistosomiasis models considering parasite density in the environment are discussed in [10–14]. The authors divided the parasite into two compartments, i.e., miracidiae and cercariae, which can infect snails and humans or mammals, respectively. In 2009, Chiyaka et al. [10] investigated the host–parasite dynamics of schistosomiasis. They proposed a schistosomiasis model that consists of six first-order differential equations. They found that control interventions that target transmission to humans are more effective than control strategies

that aim the transmission to snails are. Gao et al. [11] developed a schistosomiasis model to investigate some control strategies, i.e., health education, cercaria control, snail control, and drug treatment. They found that killing snails is the most effective method to manage schistosomiasis spread. Nur et al. [13] proposed a schistosomiasis model incorporating health education and molluscicide intervention. They found that the most effective way to control schistosomiasis prevalence is molluscicide intervention. Moreover, schistosomiasis cannot be eradicated if the only intervention is health education [13]. Diaby et al. [15] proposed a schistosomiasis model with biological control, i.e., competitor-resistant snails. They found that competitor-resistant snails can be used to manage the spread of schistosomiasis. Okamoto et al. [16,17] proposed mathematical models of vectorborne diseases. They found that biological control agents that can be used to control vectorborne diseases are natural predators, natural competitors, or parasites of the vector. Schistosomiasis is a waterborne disease. Hence, we study the dynamics of the spread of schistosomiasis when a snail predator is used as a biological control agent of snails. Different from the model discussed in [10–13,15–17], we propose a schistosomiasis model with a biological control agent of snails, namely, a snail predator.

2. Mathematical Preliminaries

In this section, we present a theorem that can be used to investigate the existence of backward and forward bifurcation. The theorem is very useful, especially in epidemic models. Theorem 1 can be used to investigate the local stability condition of endemic equilibrium [10,11,18]. The proof of Theorem 1 can be found in [19].

Theorem 1. *(Castillo-Chavez and Song [19]) Consider the following general system of ordinary differential equations with a parameter ω.*

$$\frac{d\vec{x}}{dt} = f(\vec{x}, \omega), \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \qquad f \in C^2(\mathbb{R}^n \times \mathbb{R}), \tag{1}$$

where 0 is an equilibrium for System (1), such that $f(0, \omega) \equiv 0$ for all ω. Assume

A1: $JM = D_x f(0,0) = \left(\frac{\partial f_i}{\partial x_j}(0,0)\right)$ *is the linearization matrix of (1) around equilibrium 0, and ω is evaluated at 0. Zero is a simple eigenvalue of $JM0$, and the other eigenvalues of JM have a negative real part.*

A2: Matrix JM has a right eigenvector $\vec{v}$ and a left eigenvector $\vec{w}$ corresponding to the zero eigenvalue. Let f_k be the kth component of f and

$$A = \sum_{k,i,j=1}^{n} w_k v_i v_j \frac{\partial^2 f_k(0,0)}{\partial x_i \partial x_j},$$
$$B = \sum_{k,i=1}^{n} w_k v_i \frac{\partial^2 f_k(0,0)}{\partial x_i \partial \omega}.$$

The dynamics of System (1) around 0 is totally determined by the signs of A and B.

(i) $A > 0, B > 0$. *When $\omega < 0$ with $|\omega| \ll 1$, 0 is asymptotically stable, and there is a positive unstable equilibrium. When $\omega > 0$ with $|\omega| \ll 1$, 0 is unstable, and there is a negative asymptotically stable equilibrium;*

(ii) $A < 0, B < 0$. *When $\omega < 0$ with $|\omega| \ll 1$, 0 is unstable. When $\omega > 0$ with $|\omega| \ll 1$, 0 is asymptotically stable, and there is a positive unstable equilibrium;*

(iii) $A > 0, B < 0$. *When $\omega < 0$ with $|\omega| \ll 1$, 0 is unstable, and there is a negative asymptotically stable equilibrium. When $\omega > 0$ with $|\omega| \ll 1$, 0 is stable and a positive unstable equilibrium appears;*

(iv) $A < 0, B > 0$. *When ω changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium becomes positive and asymptotically stable.*

On the basis of Theorem 1, forward bifurcation occurs at $\omega = 0$ if $A < 0$ and $B > 0$. Moreover, backward bifurcation occurs at $\omega = 0$ if $A > 0$ and $B > 0$.

3. Model Formulation and Basic Properties

3.1. Model Formulation

On the basis of the life cycle of schistosoma worms [2,20,21], it is clear that there is a latent period. Hence, the human population is divided into three compartments, i.e., susceptible humans (S_h), exposed humans (E_h), and infectious humans (I_h). In this model, we assumed that there was no latent period in snail population. Thus, the snail population was only divided into two compartments, i.e., susceptible snails (S_v) and infectious snails (I_v). We only consider two stages of schistosoma-worm development, i.e., the cercaria and miracidia stages. Therefore, the parasite is divided into two compartments, i.e., miracidiae (M) and cercariae (C). On the basis of the epidemiology of schistosomiasis [20–22], miracidiae can infect snails and cercariae can infect humans. It was assumed that there was only one type of snail predator in the environment, and the intermediate host (snail) was the only food for the predator. A reduction in parasites in the environment due to direct interaction with humans and snails was neglected because infectious humans can excrete large amounts of eggs that can hatch and release miracidiae, while infectious snails can excrete large amounts of cercariae [20]. We assumed that there was no recovery for infectious snails and no disease-related death in the snail and human populations. The transition and interaction between compartments are shown in Figure 1. The description of all parameters is given in Table 1.

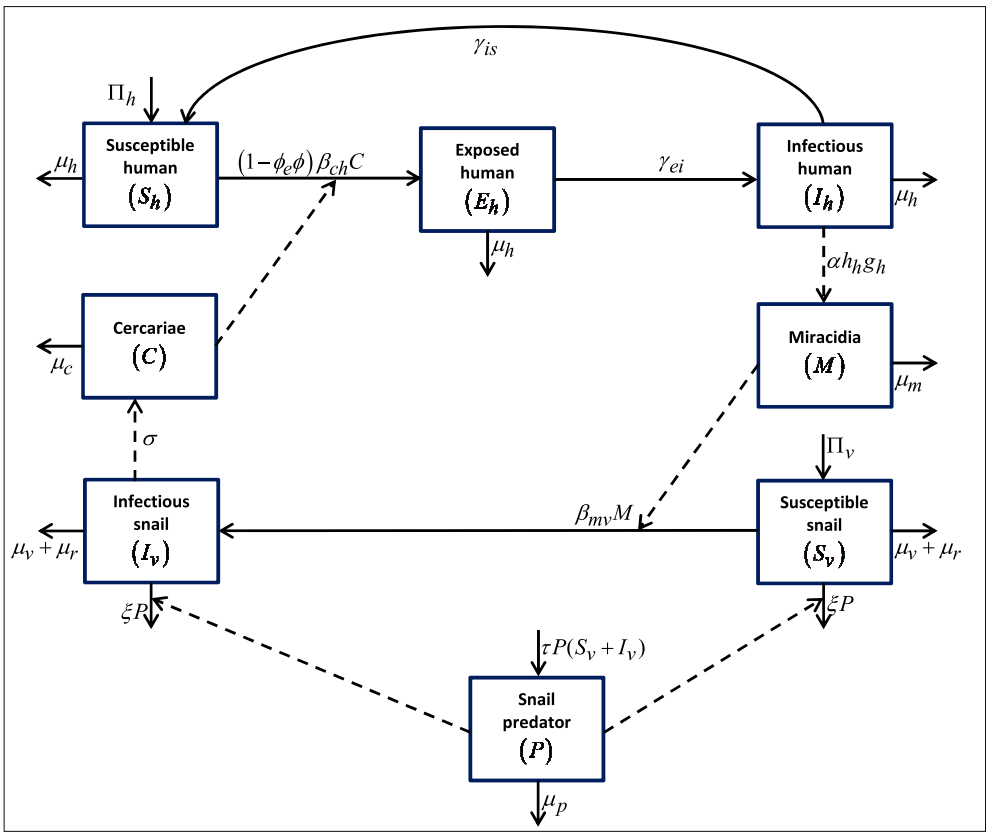

Figure 1. Compartment diagram.

Table 1. Description of model parameters.

Parameter	Description
Π_h	Recruitment rate of humans
ϕ_e	Effectiveness of education in reducing contact between humans and cercariae
ϕ	Education coverage
β_{ch}	Cercaria infection rate on susceptible humans
γ_{is}	Recovery rate of infectious humans
μ_h	Natural death rate of humans
$1/\gamma_{ei}$	Latent period
Π_v	Recruitment rate of snails
β_{mv}	Miracidia infection rate on susceptible snails
μ_v	Natural death rate of snails
μ_r	Molluscicide-related death rate of snails
ξ	Predation rate
τ	Conversion rate
σ	Cercaria production rate
α	Parasite-egg hatch rate
h_h	Number of eggs per gram of stool
g_h	Average weight of human stool per day
μ_p	Natural death rate of snail predators
μ_c	Natural death rate of cercariae
μ_m	Natural death rate of miracidiae

On the basis of Figure 1, we constructed a schistosomiasis model as follows:

$$\begin{aligned}
\frac{dS_h}{dt} &= \Pi_h - (1-\phi_e\phi)\beta_{ch}CS_h + \gamma_{is}I_h - \mu_h S_h,\\
\frac{dE_h}{dt} &= (1-\phi_e\phi)\beta_{ch}CS_h - (\gamma_{ei}+\mu_h)E_h,\\
\frac{dI_h}{dt} &= \gamma_{ei}E_h - (\gamma_{is}+\mu_h)I_h,\\
\frac{dS_v}{dt} &= \Pi_v - \beta_{mv}MS_v - (\mu_v+\mu_r)S_v - \xi PS_v,\\
\frac{dI_v}{dt} &= \beta_{mv}MS_v - (\mu_v+\mu_r)I_v - \xi PI_v,\\
\frac{dC}{dt} &= \sigma I_v - \mu_c C,\\
\frac{dM}{dt} &= \alpha h_h g_h I_h - \mu_m M,\\
\frac{dP}{dt} &= \tau P(S_v+I_v) - \mu_p P,
\end{aligned} \tag{2}$$

where $\phi \in [0,1)$, $\phi_e \in [0,1)$ and the other parameters are positive.

3.2. Invariant Region

In this subsection, we prove that the solutions of System (2) with non-negative initial value are non-negative and bounded.

Theorem 2. *Solutions of System (2) are non-negative for all non-negative initial conditions.*

Proof. We prove this theorem by using a similar method as that used in [18,23]. From System (2), we have

$$\begin{aligned}
\frac{dS_h}{dt}\Big|_{S_h=0,E_h\geq 0,I_h\geq 0,S_v\geq 0,I_v\geq 0,C\geq 0,M\geq 0,P\geq 0} &= \Pi_h + \gamma_{is} I_h > 0,\\
\frac{dE_h}{dt}\Big|_{E_h=0,S_h\geq 0,I_h\geq 0,S_v\geq 0,I_v\geq 0,C\geq 0,M\geq 0,P\geq 0} &= (1-\phi_e\phi)\beta_{ch} C S_h \geq 0,\\
\frac{dI_h}{dt}\Big|_{I_h=0,S_h\geq 0,E_h\geq 0,S_v\geq 0,I_v\geq 0,C\geq 0,M\geq 0,P\geq 0} &= \gamma_{ei} E_h \geq 0,\\
\frac{dS_v}{dt}\Big|_{S_v=0,S_h\geq 0,E_h\geq 0,I_h\geq 0,I_v\geq 0,C\geq 0,M\geq 0,P\geq 0} &= \Pi_v > 0,\\
\frac{dI_v}{dt}\Big|_{I_v=0,S_h\geq 0,E_h\geq 0,I_h\geq 0,S_v\geq 0,C\geq 0,M\geq 0,P\geq 0} &= \beta_{mv} M S_v \geq 0,\\
\frac{dC}{dt}\Big|_{C=0,S_h\geq 0,E_h\geq 0,I_h\geq 0,S_v\geq 0,I_v\geq 0,M\geq 0,P\geq 0} &= \sigma I_v \geq 0,\\
\frac{dM}{dt}\Big|_{M=0,S_h\geq 0,E_h\geq 0,I_h\geq 0,S_v\geq 0,I_v\geq 0,C\geq 0,P\geq 0} &= \alpha h_h g_h I_h \geq 0,\\
\frac{dP}{dt}\Big|_{P=0,S_h\geq 0,E_h\geq 0,I_h\geq 0,S_v\geq 0,I_v\geq 0,C\geq 0,M\geq 0} &= 0.
\end{aligned}$$

On the basis of Lemma 2 in [24], $\mathbb{R}^8_{+0}$ is an invariant region of System (2). Hence, the solutions of System (2) with initial values in $\mathbb{R}^8_{+0}$ remain in $\mathbb{R}^8_{+0}$. This completes the proof. □

Theorem 3. *Solutions of System (2) with non-negative initial value are bounded.*

Proof. On the basis of the assumptions that are used when formulating the model, we have $N_h = S_h + E_h + I_h$ and $N_v = S_v + I_v$. Here, N_h and N_v are the total number of humans and total number of snails, respectively. On the basis of System (2), we have

$$\begin{aligned}
\frac{dN_h}{dt} &= \Pi_h - \mu_h N_h,\\
\frac{dN_v}{dt} &= \Pi_v - (\mu_v + \mu_r) N_v - \xi P N_v,\\
\frac{dP}{dt} &= \tau N_v P - \mu_p P.
\end{aligned} \tag{3}$$

First, we prove that N_h is bounded. It is clear that the solution of $\frac{dN_h}{dt} = \Pi_h - \mu_h N_h$ is

$$N_h(t) = \frac{\Pi_h}{\mu_h} + \left(N_h(0) - \frac{\Pi_h}{\mu_h}\right) e^{-\mu_h t}.$$

It easy to see that $0 \leq N_h(t) \leq \frac{\Pi_h}{\mu_h}$ for $t \geq 0$ if $0 \leq N_h(0) \leq \frac{\Pi_h}{\mu_h}$. Hence, N_h is bounded. $\lim_{t\to\infty} N_h(t) = \frac{\Pi_h}{\mu_h}$. Now, let $W = N_v + P$. We prove that W is bounded. The last two equations of (3) give

$$\begin{aligned}
\frac{dW}{dt} &= \frac{dN_v}{dt} + \frac{dP}{dt}\\
&= (\Pi_v - (\mu_v + \mu_r) N_v - \xi P N_v) + (\tau N_v P - \mu_p P)\\
&\leq \Pi_v - (\xi - \tau) P N_v - \mu (N_v + P)\\
&\leq \Pi_v - \mu W,
\end{aligned} \tag{4}$$

where $\xi \geq \tau$ and $\mu = \min\{(\mu_v + \mu_r), \mu_p\}$. On the basis of Gronwall's lemma, we obtain $W = N_v + P \leq \frac{\Pi_v}{\mu}$. Thus, N_v and P are bounded. Now, we can show that C and M are bounded. We proved that $N_h \leq \frac{\Pi_h}{\mu_h}$, which implies that $I_h \leq \frac{\Pi_h}{\mu_h}$. Moreover, we have $N_v + P \leq \frac{\Pi_v}{\mu}$, which means that $I_v \leq \frac{\Pi_v}{\mu}$. Hence, from System (2), we have

$$\begin{aligned}
\frac{dC}{dt} &= \sigma I_v - \mu_c C\\
&\leq \sigma \frac{\Pi_v}{\mu} - \mu_c C,\\
\frac{dM}{dt} &= \alpha h_h g_h I_h - \mu_m M\\
&\leq \alpha h_h g_h \frac{\Pi_h}{\mu_h} - \mu_m M.
\end{aligned} \tag{5}$$

According to Gronwall's lemma, we obtain $C \leq \frac{\sigma\Pi_v}{\mu\mu_c}$ and $M \leq \frac{\alpha h_h g_h \Pi_h}{\mu_h \mu_m}$. Thus, the solutions of System (2) are bounded. □

Therefore, System (2) is well-posed with invariant region Θ_+

$$\Theta_+ = \{(S_h,E_h,I_h,S_v,I_v,C,M,P)\in\mathbb{R}^8_{+0}: N_h \leq \tfrac{\Pi_h}{\mu_h}; N_v+P \leq \tfrac{\Pi_v}{\mu}; C \leq \tfrac{\sigma\Pi_v}{\mu\mu_c}; M \leq \tfrac{\alpha h_h g_h \Pi_h}{\mu_h \mu_m}\}$$

3.3. Equilibrium Points and Basic Reproduction Number

System (2) has four equilibrium points.

- Disease-free equilibrium point E_0^a.

$$E_0^a = (S_h^{a*}, E_h^{a*}, I_h^{a*}, S_v^{a*}, I_v^{a*}, C^{a*}, M^{a*}, P^{a*}) = \left(\frac{\Pi_h}{\mu_h}, 0, 0, \frac{\Pi_v}{\mu_v+\mu_r}, 0, 0, 0, 0\right).$$

E_0^a always exists in $\mathbb{R}^8_{+0}$.

- Endemic equilibrium point E_1^a.

$$E_1^a = (S_h^{a**}, E_h^{a**}, I_h^{a**}, S_v^{a**}, I_v^{a**}, C^{a**}, M^{a**}, 0),$$

where

$$\begin{aligned}
S_h^{a**} &= \frac{\Pi_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c}{((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{a**}+\mu_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c},\\
E_h^{a**} &= \frac{\Pi_h(\mu_h+\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{a**}}{((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{a**}+\mu_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c},\\
I_h^{a**} &= \frac{\Pi_h\gamma_{ei}(1-\phi_e\phi)\beta_{ch}\sigma I_v^{a**}}{((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{a**}+\mu_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c},\\
S_v^{a**} &= \frac{\Pi_v}{\mu_v+\mu_r} - I_v^{a**},\\
I_v^{a**} &= \frac{(R_e^a-1)\mu_h(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m(\mu_v+\mu_r)}{\beta_{mv}\Pi_h\gamma_{ei}(1-\phi_e\phi)\alpha h_h g_h\beta_{ch}\sigma+((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})\mu_m(1-\phi_e\phi)\beta_{ch}\sigma(\mu_v+\mu_r)},\\
C^{a**} &= \frac{\sigma I_v^{a**}}{\mu_c},\\
M^{a**} &= \frac{\alpha h_h g_h \Pi_h\gamma_{ei}(1-\phi_e\phi)\beta_{ch}\sigma I_v^{a**}}{\mu_m(((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{a**}+\mu_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c)},\\
R_e^a &= \frac{\Pi_h\Pi_v\gamma_{ei}(1-\phi_e\phi)\beta_{ch}\sigma\alpha h_h g_h\beta_{mv}}{\mu_h(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m(\mu_v+\mu_r)^2}.
\end{aligned}$$

It is clear that E_1^a exists in $\mathbb{R}^8_{+0}$ if $R_e^a > 1$.

- Disease-free equilibrium point E_0^b.

$$E_0^b = (S_h^{b*}, E_h^{b*}, I_h^{b*}, S_v^{b*}, I_v^{b*}, C^{b*}, M^{b*}, P^{b*}) = \left(\frac{\Pi_h}{\mu_h}, 0, 0, \frac{\mu_p}{\tau}, 0, 0, 0, \frac{\Pi_v\tau-(\mu_v+\mu_r)\mu_p}{\xi\mu_p}\right),$$

E_0^b exists in $\mathbb{R}^8_{+0}$ if $\frac{\Pi_v\tau}{(\mu_v+\mu_r)\mu_p} > 1$.

- Endemic equilibrium point E_1^b.

$$E_1^b = (S_h^{b**}, E_h^{b**}, I_h^{b**}, S_v^{b**}, I_v^{b**}, C^{b**}, M^{b**}, P^{b**}),$$

where

$$
\begin{aligned}
S_h^{b**} &= \frac{\Pi_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c}{((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{b**}+\mu_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c},\\
E_h^{b**} &= \frac{\Pi_h(\mu_h+\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{b**}}{((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{b**}+\mu_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c},\\
I_h^{b**} &= \frac{\Pi_h\gamma_{ei}(1-\phi_e\phi)\beta_{ch}\sigma I_v^{b**}}{((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{b**}+\mu_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c},\\
S_v^{b**} &= \frac{\mu_p}{\tau}-I_v^{b**},\\
I_v^{b**} &= \frac{(R_e^b-1)\mu_h(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m((\mu_v+\mu_r)+\xi P^{b**})}{\beta_{mv}\Pi_h\gamma_{ei}(1-\phi_e\phi)\alpha h_h g_h\beta_{ch}\sigma+((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})\mu_m(1-\phi_e\phi)\beta_{ch}\sigma((\mu_v+\mu_r)+\xi P^{b**})},\\
C^{b**} &= \frac{\sigma I_v^{b**}}{\mu_c},\\
M^{b**} &= \frac{\alpha h_h g_h\Pi_h\gamma_{ei}(1-\phi_e\phi)\beta_{ch}\sigma I_v^{b**}}{\mu_m(((\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})-\gamma_{ei}\gamma_{is})(1-\phi_e\phi)\beta_{ch}\sigma I_v^{b**}+\mu_h(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_c)},\\
P^{b**} &= \frac{\Pi_v\tau-(\mu_v+\mu_r)\mu_p}{\xi\mu_p}\\
R_e^b &= \frac{\Pi_h\mu_p\gamma_{ei}(1-\phi_e\phi)\beta_{ch}\sigma\alpha h_h g_h\beta_{mv}}{\mu_h(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m\tau(\mu_v+\mu_r+\xi P^{b**})}.
\end{aligned}
$$

It is clear that E_1^b exists in $\mathbb{R}^8_{+0}$ if $R_e^b > 1$ and $\frac{\Pi_v\tau}{(\mu_v+\mu_r)\mu_p} > 1$.

The basic reproduction numbers are determined by using a next-generation matrix [25]. Here, we regard E_h, I_h, I_v, C, and M as the infected compartments. Thus, we have

$$
\mathcal{F} = \begin{pmatrix} (1-\phi_e\phi)\beta_{ch}CS_h \\ 0 \\ \beta_{mv}MS_v \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} (\gamma_{ei}+\mu_h)E_h \\ (\gamma_{is}+\mu_h)I_h-\gamma_{ei}E_h \\ (\mu_v+\mu_r+\xi P)I_v \\ \mu_c C-\sigma I_v \\ \mu_m M-\alpha h_h g_h I_h \end{pmatrix}.
$$

$\mathcal{F}$ and $\mathcal{V}$ represent the new infection terms and transition terms, respectively. The Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ at arbitrary equilibrium point $(S_h^*, 0, 0, S_v^*, 0, 0, 0, P^*)$ are, respectively, given by

$$
F = \begin{pmatrix} 0 & 0 & 0 & \beta_{ch}(1-\phi_e\phi)S_h^* & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_{mv}S_v^* \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

and

$$
V = \begin{pmatrix} \mu_h+\gamma_{ei} & 0 & 0 & 0 & 0 \\ -\gamma_{ei} & \mu_h+\gamma_{is} & 0 & 0 & 0 \\ 0 & 0 & \mu_v+\mu_r+\xi P^* & 0 & 0 \\ 0 & 0 & -\sigma & \mu_c & 0 \\ 0 & -\alpha h_h g_h & 0 & 0 & \mu_m \end{pmatrix}.
$$

The basic reproduction number is the spectral radius of FV^{-1}.

$$FV^{-1} = \begin{pmatrix} 0 & 0 & \frac{\beta_{ch}(1-\phi_e\phi)S_h^*\sigma}{\mu_c((\mu_v+\mu_r)+\xi P^*)} & \frac{\beta_{ch}(1-\phi_e\phi)S_h^*}{\mu_c} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_{mv}S_v^*\alpha h_h g_h\gamma_{ei}}{(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_m} & \frac{\beta_{mv}S_v^*\alpha h_h g_h}{(\mu_h+\gamma_{is})\mu_m} & 0 & 0 & \frac{\beta_{mv}S_v^*}{\mu_m} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

The characteristic polynomial of (6) is

$$\mathbf{P}(\lambda) = \lambda^3\left(\lambda^2 - \frac{(1-\phi_e\phi)\beta_{ch}S_h^*\sigma\beta_{mv}S_v^*\alpha h_h g_h\gamma_{ei}}{((\mu_v+\mu_r)+\xi P^*)\mu_c\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})}\right).$$

Thus, the spectral radius of FV^{-1} is

$$\rho(FV^{-1}) = \sqrt{\frac{(1-\phi_e\phi)\beta_{ch}S_h^*\sigma\beta_{mv}S_v^*\alpha h_h g_h\gamma_{ei}}{((\mu_v+\mu_r)+\xi P^*)\mu_c\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})}}. \quad (7)$$

By substituting E_0^a into (7), we obtain the basic reproduction number when the predator becomes extinct.

$$R_0^a = \sqrt{\frac{(1-\phi_e\phi)\beta_{ch}\Pi_h\sigma\beta_{mv}\Pi_v\alpha h_h g_h\gamma_{ei}}{\mu_h(\mu_v+\mu_r)^2\mu_c\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})}}.$$

$R_0^a > 0$ and $R_e^a = (R_0^a)^2$. In agreement with the existence condition of E_1^a, if $R_0^a > 1$ then E_1^a exists in $\mathbb{R}^8_{+0}$. It is easy to see that R_0^a is completely independent from parameters that are related to the snail predator. This makes sense because E_0^a and E_1^a describe the condition when the predator becomes extinct.

After substituting E_0^b into (7), we obtain the basic reproduction number when the predator survives, namely,

$$\begin{aligned} R_0^b &= \sqrt{\frac{(1-\phi_e\phi)\beta_{ch}\Pi_h\sigma\beta_{mv}\mu_p\alpha h_h g_h\gamma_{ei}}{\mu_h((\mu_v+\mu_r)+\xi P^{b*})\mu_c\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\tau}} \\ &= \sqrt{\frac{(1-\phi_e\phi)\beta_{ch}\Pi_h\sigma\beta_{mv}\mu_p\alpha h_h g_h\gamma_{ei}}{\mu_h\left((\mu_v+\mu_r)+\xi\left(\frac{\Pi_v\tau-(\mu_v+\mu_r)\mu_p}{\xi\mu_p}\right)\right)\mu_c\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\tau}} \\ &= \sqrt{\frac{(1-\phi_e\phi)\beta_{ch}\Pi_h\sigma\beta_{mv}\mu_p^2\alpha h_h g_h\gamma_{ei}}{\mu_h\Pi_v\mu_c\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\tau^2}}. \end{aligned}$$

$R_0^b > 0$ and $R_e^b = (R_0^b)^2$. In line with the existence condition of E_1^b, if $R_0^b > 1$ and $\frac{\Pi_v\tau}{(\mu_v+\mu_r)\mu_p} > 1$, then E_1^b exists in $\mathbb{R}^8_{+0}$. R_0^b is independent to ξ. Furthermore, it is clear that R_0^b decreases as τ increases. The higher the natural death rate of the snail predator is, the higher R_0^b is.

4. Stability Analysis

In this section, we investigate the stability condition of all equilibrium points. The Jacobian matrix of System (2) at E_0^a is

$$J(E_0^a)=\begin{pmatrix} -\mu_h & 0 & \gamma_{is} & 0 & 0 & -(1-\phi_e\phi)\beta_{ch}S_h^{a*} & 0 & 0 \\ 0 & -(\gamma_{ei}+\mu_h) & 0 & 0 & 0 & (1-\phi_e\phi)\beta_{ch}S_h^{a*} & 0 & 0 \\ 0 & \gamma_{ei} & -(\gamma_{is}+\mu_h) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\mu_v+\mu_r) & 0 & 0 & -\beta_{mv}S_v^{a*} & -\xi S_v^{a*} \\ 0 & 0 & 0 & 0 & -(\mu_v+\mu_r) & 0 & \beta_{mv}S_v^{a*} & 0 \\ 0 & 0 & 0 & 0 & \sigma & -\mu_c & 0 & 0 \\ 0 & 0 & \alpha h_h g_h & 0 & 0 & 0 & -\mu_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau S_v^{a*}-\mu_p \end{pmatrix} \tag{8}$$

The eigenvalues of (8) are zeroes of $L(\lambda)$

$$\begin{aligned} L(\lambda) &= [(\lambda+D_1)(\lambda+D_2)(\lambda+D_3)(\lambda+D_4)(\lambda+D_5)-\gamma_{ei}\alpha h_h g_h(1-\phi_e\phi)\beta_{ch}S_h^{a*}\sigma\beta_{mv}S_v^{a*}] \\ &\quad \times(\lambda+\mu_h)(\lambda+(\mu_v+\mu_r))(\lambda-(\tau S_v^{a*}-\mu_p)) \\ &= [L_1(\lambda)](\lambda+\mu_h)(\lambda+(\mu_v+\mu_r))(\lambda-(\tau S_v^{a*}-\mu_p)), \end{aligned} \tag{9}$$

where

$$\begin{aligned} L_1(\lambda) &= \lambda^5+l_1\lambda^4+l_2\lambda^3+l_3\lambda^2+l_4\lambda+l_5, \\ l_1 &= \sum_{g=1}^{5} D_g, \\ l_2 &= \sum_{1\le g<h}^{5} D_g D_h, \\ l_3 &= \sum_{1\le g<h<i}^{5} D_g D_h D_i, \\ l_4 &= \sum_{1\le g<\ldots<j}^{5} D_g D_h D_i D_j, \\ l_5 &= (\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})(\mu_v+\mu_r)\mu_c\mu_m-\gamma_{ei}\alpha h_h g_h(1-\phi_e\phi)\beta_{ch}S_h^{a*}\sigma\beta_{mv}S_v^{a*} \\ &= (1-(R_0^a)^2)(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})(\mu_v+\mu_r)\mu_c\mu_m, \end{aligned} \tag{10}$$

$D_1=\mu_h+\gamma_{ei}>0, D_2=\mu_h+\gamma_{is}>0, D_3=\mu_v+\mu_r>0, D_4=\mu_c>0$, and $D_5=\mu_m>0$. Clearly, l_i for $i=1,2,3,4,5$ are elementary symmetric functions [26]. It is obvious that (8) has two negative eigenvalues, i.e., $\lambda_1=-\mu_h$ and $\lambda_2=-(\mu_v+\mu_r)$. Moreover, $\lambda_3=\tau S_v^{a*}-\mu_p<0$ if $\frac{\tau S_v^{a*}}{\mu_p}=\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)}<1$. The other eigenvalues are zeroes of $L_1(\lambda)$. It is easy to see that $l_i>0$ for $i=1,2,\ldots,5$ if $R_0^a<1$. If $R_0^a=1$ then $l_5=0$. Hence, if $\frac{\tau S_v^{a*}}{\mu_p}=\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)}<1$ and $R_0^a=1$ then one eigenvalue of $J(E_0^a)$ is zero. Following [27,28], we use a Routh–Hurwitz array shown in Table 2 to determine the Hurwitz determinant.

Table 2. Routh–Hurwitz array associated with characteristic polynomial $L_1(\lambda)$.

	Column 1	**Column 2**	**Column 3**	**Column 4**
λ^5	1	l_2	l_4	0
λ^4	$\Delta_1=l_1$	l_3	l_5	0
λ^3	$r_{(2,1)}=\frac{l_1l_2-l_3}{l_1}$	$r_{(2,2)}=\frac{l_1l_4-l_5}{l_1}$	0	0
λ^2	$r_{(3,1)}=\frac{r_{(2,1)}l_3-l_1r_{(2,2)}}{r_{(2,1)}}$	l_5	0	0
λ^1	$r_{(4,1)}=\frac{r_{(2,2)}r_{(3,1)}-l_5r_{(2,1)}}{r_{(3,1)}}$	0	0	0
λ^0	$r_{(5,1)}=l_5$	0	0	0

The Hurwitz determinant of order-i (Δ_i) is given by $\Delta_i=r_{(i,1)}\Delta_{i-1}$ for $i=2,3,4,5$. Hence we obtain

$$\begin{aligned}
\Delta_1 &= l_1,\\
\Delta_2 &= \left(\tfrac{l_1 l_2 - l_3}{l_1}\right)\Delta_1\\
&= l_1 l_2 - l_3,\\
\Delta_3 &= \left(\tfrac{l_3(l_1 l_2 - l_3) - l_1(l_1 l_4 - l_5)}{l_1 l_2 - l_3}\right)\Delta_2\\
&= l_3(l_1 l_2 - l_3) - l_1(l_1 l_4 - l_5),
\end{aligned}$$

$$\begin{aligned}
\Delta_4 &= \left(\tfrac{l_4[(l_1 l_2 - l_3)l_3 - l_1(l_1 l_4 - l_5)] - l_5[l_2(l_2 l_1 - l_3) - (l_1 l_4 - l_5)]}{l_3(l_1 l_2 - l_3) - l_1(l_1 l_4 - l_5)}\right)\Delta_3\\
&= l_4[(l_1 l_2 - l_3)l_3 - l_1(l_1 l_4 - l_5)] - l_5[l_2(l_2 l_1 - l_3) - (l_1 l_4 - l_5)],\\
\Delta_5 &= l_5\Delta_4\\
&= l_5(l_4[(l_1 l_2 - l_3)l_3 - l_1(l_1 l_4 - l_5)] - l_5[l_2(l_2 l_1 - l_3) - (l_1 l_4 - l_5)]).
\end{aligned}$$

We prove this theorem by using the Liénard–Chipart criterion [29]. According to the criterion, all roots of $L_1(\lambda)$ have a negative real part if $l_1, \Delta_2, l_3, \Delta_4, l_5 > 0$. It is clear that $l_i > 0$ always holds for $i = 1, 3$. Furthermore, $l_5 > 0$ if $R_0^a < 1$. Now, we only need to check Δ_2 and Δ_4. First, we set

$$R_0^a = \sqrt{\frac{R_{01}^a}{R_{02}^a}},$$

where

$$\begin{aligned}
R_{01}^a &= \gamma_{ei}\alpha h_h g_h(1 - \phi_e\phi)\beta_{ch}S_h^{a*}\sigma\beta_{mv}S_v^{a*}\\
R_{02}^a &= D_1 D_2 D_3 D_4 D_5.
\end{aligned}$$

It is clear that $R_{01}^a > 0$ and $R_{02}^a > 0$. Therefore, $R_0^a < 1$ implies $R_{01}^a < R_{02}^a$. We now investigate Δ_2 and Δ_4.

$$\begin{aligned}
\Delta_2 &= l_1 l_2 - l_3\\
&= D_1(D_2 + D_3 + D_4 + D_5)(D_1 + D_2 + D_3 + D_4 + D_5)\\
&\quad + D_2(D_2 + D_3 + D_4 + D_5)(D_3 + D_4 + D_5)\\
&\quad + (D_3 + D_4 + D_5)(D_4 + D_5) + D_4(D_4 D_5 + D_5 D_5),\\
\Delta_4 &= l_4[(l_1 l_2 - l_3)l_3 - l_1(l_1 l_4 - l_5)] - l_5[l_2(l_2 l_1 - l_3) - (l_1 l_4 - l_5)]\\
&= \Xi + 12R_{01}^a R_{02}^a - R_{01}^a R_{01}^a\\
&= \Xi + R_{01}^a 12R_{02}^a - R_{01}^a R_{01}^a\\
&= \Xi + R_{01}^a(12R_{02}^a - R_{01}^a).
\end{aligned}$$

Since $D_i > 0$ for $i = 1, 2, 3, 4, 5$, it is clear that $\Delta_2 > 0$. We also proved that $\Xi > 0$ (see Appendix A). Thus, $\Delta_4 > 0$ if $R_0^a < 1$. Therefore, on the basis of the Liénard–Chipart criterion [29], all zeroes of $L_1(\lambda)$ have a negative real part if $R_0^a < 1$. Hence, all eigenvalues of $J(E_0^a)$ have a negative real part if $R_0^a < 1$ and $\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} < 1$. Therefore, E_0^a is asymptotically stable if $R_0^a < 1$ and $\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} < 1$. If $R_0^a > 1$, then there is one sign change in the sequence of $L_1(\lambda)$ coefficients. On the basis of Descartes' sign rule [26], there is exactly one positive real root if $R_0^a > 1$. Hence, E_0^a is unstable if $R_0^a > 1$.

Theorem 4. *Disease-free equilibrium point E_0^a is asymptotically stable if $R_0^a < 1$ and $\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} < 1$. If $R_0^a > 1$ or $\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} > 1$ then E_0^a is unstable.*

Now, we present the local stability condition of E_1^a.

Theorem 5. *Endemic equilibrium point E_1^a is asymptotically stable if $R_0^a > 1$ (near 1) and $\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} < 1$.*

Proof. We use Theorem 1 to prove this theorem. Consider β_{ch} as bifurcation parameter. We determine the bifurcation point when $R_0^a = 1$. The bifurcation point that is obtained is $\beta_{ch}^* = \frac{\mu_h(\mu_v+\mu_r)^2\mu_c\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})}{(1-\phi_e\phi)\Pi_h\sigma\beta_{mv}\Pi_v\alpha h_h g_h\gamma_{ei}}$. When $\beta_{ch} = \beta_{ch}^*$, the characteristic polynomial (9) becomes

$$L(\lambda) = [L_1(\lambda)](\lambda+\mu_h)(\lambda+(\mu_v+\mu_r))(\lambda-(\tau S_v^{a*}-\mu_p))\lambda, \quad (11)$$

where

$$L_1(\lambda) = \lambda^4 + l_1\lambda^3 + l_2\lambda^2 + l_3\lambda + l_4. \quad (12)$$

It is clear that characteristic polynomial (11) has simple zero root ($\lambda_1 = 0$) and two negative roots, i.e., $\lambda_2 = -\mu_h$ and $\lambda_3 = -(\mu_v+\mu_r)$. Moreover, $\lambda_4 = \tau S_v^{a*} - \mu_p < 0$ if $\frac{\tau S_v^{a*}}{\mu_p} = \frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} < 1$. Using the Routh–Hurwitz array in Tabel 3, we determine the conditions that guarantee that the other roots of (11) have a negative real part.

Table 3. Routh–Hurwitz array associated with characteristic polynomial (12).

	Column 1	**Column 2**	**Column 3**	**Column 4**
λ^4	1	l_2	l_4	0
λ^3	l_1	l_3	0	0
λ^2	$r_1 = \frac{l_1 l_2 - l_3}{l_1}$	l_4	0	0
λ^1	$r_2 = \frac{r_1 l_3 - l_1 l_4}{r_1}$	0	0	0
λ^0	l_4	0	0	0

$l_1, r_1, r_2, l_4 > 0$ (see Appendix B). Hence, on the basis of the Routh–Hurwitz criterion [27], all roots of (12) have a negative real part. These results imply that, if $\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} < 1$, then $J(E_0^a, \beta_{ch}^*)$ has one zero eigenvalue, and the other eigenvalues have a negative real part. Thus, Assumption A1 in Theorem 1 is satisfied if $\frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} < 1$. The right eigenvector of $J(E_0^a, \beta_{ch}^*)$ corresponding to a zero eigenvalue is

$$\vec{v}^a = \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \\ v_4^a \\ v_5^a \\ v_6^a \\ v_7^a \\ v_8^a \end{pmatrix} = \begin{pmatrix} \frac{(\gamma_{is}\gamma_{ei}-(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is}))(1-\phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{\mu_h(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})} \\ \frac{(1-\phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{\mu_h+\gamma_{ei}} \\ \frac{\gamma_{ei}(1-\phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})} \\ \frac{-\beta_{mv} S_v^{a*}\alpha h_h g_h \gamma_{ei}(1-\phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})(\mu_v+\mu_r)\mu_m} \\ \frac{\beta_{mv} S_v^{a*}\alpha h_h g_h \gamma_{ei}(1-\phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})(\mu_v+\mu_r)\mu_m} \\ v_6^a \\ \frac{\alpha h_h g_h \gamma_{ei}(1-\phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_m} \\ 0 \end{pmatrix},$$

where v_6^a is arbitrarily positive. It is clear that $v_1^a < 0$ and $v_4^a < 0$. The left eigenvector of $J(E_0^a, \beta_{ch}^*)$, corresponding to a zero eigenvalue is

$$\vec{w^a}^T = \begin{pmatrix} w_1^a \\ w_2^a \\ w_3^a \\ w_4 \\ w_5^a \\ w_6^a \\ w_7^a \\ w_8^a \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\gamma_{ei}\alpha h_h g_h \sigma \beta_{mv} S_v^{a*} w_6^a}{(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})(\mu_v+\mu_r)\mu_m} \\ \frac{\alpha h_h g_h \sigma \beta_{mv} S_v^{a*} w_6^a}{(\mu_h+\gamma_{is})(\mu_v+\mu_r)\mu_m} \\ 0 \\ \frac{\sigma w_6^a}{\mu_v+\mu_r} \\ w_6^a \\ \frac{\beta_{mv} S_v^{a*}\sigma w_6^a}{\mu_m(\mu_v+\mu_r)} \\ 0 \end{pmatrix},$$

where w_6^a is determined, such that $\vec{w}^a.\vec{v}^a = 1$. It is straightforward to show that $w_6^a > 0$. Let

$$\begin{array}{cccc} x_1 = S_h & , & x_2 = E_h & , & x_3 = I_h & , & x_4 = S_v, \\ x_5 = I_v & , & x_6 = C & , & x_7 = M & , & x_8 = P, \\ f_1 = \frac{dS_h}{dt} & , & f_2 = \frac{dE_h}{dt} & , & f_3 = \frac{dI_h}{dt} & , & f_4 = \frac{dS_v}{dt}, \\ f_5 = \frac{dI_v}{dt} & , & f_6 = \frac{dC}{dt} & , & f_7 = \frac{dM}{dt} & , & f_8 = \frac{dP}{dt}. \end{array}$$

Now, we compute A and B that are defined in Theorem 1. It is clear that the only nonzero terms of A and B are

$$\begin{aligned} w_2^a v_1^a v_6^a \frac{\partial^2 f_2\left(E_0^a, \beta_{ch}^*\right)}{\partial x_1 \partial x_6} =& \left(\frac{\gamma_{ei}\alpha h_h g_h \sigma \beta_{mv} S_v^{a*} w_6^a}{(\mu_h + \gamma_{is})(\mu_h + \gamma_{ei})(\mu_v + \mu_r)\mu_m} \right) \\ & \times \left(\frac{(\gamma_{is}\gamma_{ei} - (\mu_h + \gamma_{ei})(\mu_h + \gamma_{is}))(1 - \phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{\mu_h(\mu_h + \gamma_{ei})(\mu_h + \gamma_{is})} \right) v_6^a \beta_{ch}^* < 0, \end{aligned}$$

$$\begin{aligned} w_2^a v_6^a v_1^a \frac{\partial^2 f_2\left(E_0^a, \beta_{ch}^*\right)}{\partial x_6 \partial x_1} =& \left(\frac{\gamma_{ei}\alpha h_h g_h \sigma \beta_{mv} S_v^{a*} w_6^a}{(\mu_h + \gamma_{is})(\mu_h + \gamma_{ei})(\mu_v + \mu_r)\mu_m} \right) \\ & \times v_6^a \left(\frac{(\gamma_{is}\gamma_{ei} - (\mu_h + \gamma_{ei})(\mu_h + \gamma_{is}))(1 - \phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{\mu_h} \right) \beta_{ch}^* < 0, \end{aligned}$$

$$\begin{aligned} w_5^a v_4^a v_7^a \frac{\partial^2 f_5\left(E_0^a, \beta_{ch}^*\right)}{\partial x_4 \partial x_7} =& \left(\frac{\sigma w_6^a}{\mu_v + \mu_r} \right) \left(\frac{-\beta_{mv} S_v^{a*} \alpha h_h g_h \gamma_{ei}(1 - \phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{(\mu_h + \gamma_{is})(\mu_h + \gamma_{ei})(\mu_v + \mu_r)\mu_m} \right) \\ & \times \left(\frac{\alpha h_h g_h \gamma_{ei}(1 - \phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{(\mu_h + \gamma_{is})(\mu_h + \gamma_{ei})\mu_m} \right) \beta_{mv} < 0, \end{aligned}$$

$$\begin{aligned} w_5^a v_7^a v_4^a \frac{\partial^2 f_5\left(E_0^a, \beta_{ch}^*\right)}{\partial x_7 \partial x_4} =& \left(\frac{\sigma w_6^a}{\mu_v + \mu_r} \right) \left(\frac{-\beta_{mv} S_v^{a*} \alpha h_h g_h \gamma_{ei}(1 - \phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{(\mu_h + \gamma_{is})(\mu_h + \gamma_{ei})(\mu_v + \mu_r)\mu_m} \right) \\ & \times \left(\frac{\alpha h_h g_h \gamma_{ei}(1 - \phi_e\phi)\beta_{ch}^* S_h^{a*} v_6^a}{(\mu_h + \gamma_{is})(\mu_h + \gamma_{ei})\mu_m} \right) \beta_{mv} < 0, \end{aligned}$$

$$w_2^a v_6^a \frac{\partial^2 f_2\left(E_0^a, \beta_{ch}^*\right)}{\partial x_6 \partial \beta_{ch}} = \left(\frac{\gamma_{ei}\alpha h_h g_h \sigma \beta_{mv} S_v^{a*} w_6^a}{(\mu_h + \gamma_{is})(\mu_h + \gamma_{ei})(\mu_v + \mu_r)\mu_m} \right) v_6^a \frac{\Pi_h}{\mu_h} > 0.$$

Hence, we obtain

$$\begin{aligned} A &= w_2^a v_1^a v_6^a \frac{\partial^2 f_2\left(E_0^a, \beta_{ch}^*\right)}{\partial x_1 \partial x_6} + w_2^a v_6^a v_1^a \frac{\partial^2 f_2\left(E_0^a, \beta_{ch}^*\right)}{\partial x_6 \partial x_1} + w_5^a v_4^a v_7^a \frac{\partial^2 f_5\left(E_0^a, \beta_{ch}^*\right)}{\partial x_4 \partial x_7} + w_5^a v_7^a v_4^a \frac{\partial^2 f_5\left(E_0^a, \beta_{ch}^*\right)}{\partial x_7 \partial x_4} < 0, \\ B &= w_2^a v_6^a \frac{\partial^2 f_2\left(E_0^a, \beta_{ch}^*\right)}{\partial x_6 \partial \beta_{ch}} > 0. \end{aligned}$$

According to Theorem 1, forward bifurcation occurs at $R_0^a = 1$. Consequently, endemic equilibrium point E_1^a, which exists when $R_0^a > 1$, is asymptotically stable if $R_0^a > 1$ (near 1) and $\frac{\tau \Pi_v}{\mu_p(\mu_v + \mu_r)} < 1$. □

Theorem 6. *Disease-free equilibrium point E_0^a is globally asymptotically stable if $R_0^a \leq 1$ and $\frac{\tau \Pi_v}{\mu_p(\mu_v + \mu_r)} < 1$.*

Proof. Consider the following Lyapunov function:

$$Z = z_1\left(S_h - S_h^{a*} - S_h^{a*}\ln\frac{S_h}{S_h^{a*}}\right) + z_2E_h + z_3I_h + z_4\left(S_v - S_v^{a*} - S_v^{a*}\ln\frac{S_v}{S_v^{a*}}\right) + z_5I_v + z_6P + z_7C + z_8M,$$

where

$$z_1 = z_2 = \frac{\beta_{mv}S_v^{a*}\alpha h_h g_h\gamma_{ei}}{\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})}, \quad z_3 = \frac{\beta_{mv}S_v^{a*}\alpha h_h g_h}{\mu_m(\mu_h+\gamma_{is})}, \quad z_4 = z_5 = 1,$$
$$z_6 = \frac{\xi}{\tau}, \qquad z_7 = \frac{\mu_v+\mu_r}{\sigma}, \qquad z_8 = \frac{\beta_{mv}S_v^{a*}}{\mu_m}.$$

The derivative of Z with respect to t is

$$\begin{aligned}\frac{dZ}{dt} &= z_1\left(1-\frac{S_h^{a*}}{S_h}\right)\frac{dS_h}{dt} + z_2\frac{dE_h}{dt} + z_3\frac{dI_h}{dt} + z_4\left(1-\frac{S_v^{a*}}{S_v}\right)\frac{dS_v}{dt} + z_5\frac{dI_v}{dt} + z_6\frac{dP}{dt} + z_7\frac{dC}{dt} + z_8\frac{dM}{dt}\\ &= \frac{-z_1\mu_h}{S_h}\left(S_h - S_h^{a*}\right)^2 + \frac{z_1\gamma_{is}}{S_h}I_h\left(S_h - S_h^{a*}\right) - \frac{z_4(\mu_h+\mu_r)}{S_v}\left(S_v - S_v^{a*}\right)^2\\ &\quad + \left(\frac{\tau S_v^{a*}}{\mu_p} - 1\right)\frac{\xi\mu_p}{\tau}P + \left((R_0^a)^2 - 1\right)\frac{(\mu_v+\mu_r)\mu_c}{\sigma}C.\end{aligned}$$

$S_h \le S_h^{a*}$ always holds. It is obvious that $\frac{dZ}{dt} \le 0$ if $R_0^a \le 1$ and $\frac{\tau S_v^{a*}}{\mu_p} = \frac{\tau\Pi_v}{\mu_p(\mu_v+\mu_r)} < 1$. $\frac{dZ}{dt} = 0$ if and only if $S_v = S_v^{a*}$, $P = 0, C = 0$, $S_h = S_h^{a*}$, $I_h = 0$, and $I_v = 0$. Hence, the largest invariant set in $\left\{(S_h, E_h, I_h, S_v, I_v, C, M, P) | \frac{dZ}{dt} = 0\right\}$ is a singleton set $\left\{E_0^a\right\}$. Thus [30], E_0^a is globally asymptotically stable if $R_0^a \le 1$ and $\frac{\tau S_v^{a*}}{\mu_p} < 1$. □

Now, we determine the local stability condition of E_0^b. The Jacobian matrix of System (2) at E_0^b is

$$J(E_0^b) = \begin{pmatrix} -\mu_h & 0 & \gamma_{is} & 0 & 0 & -(1-\phi_e\phi)\beta_{ch}S_h^{b*} & 0 & 0\\ 0 & -(\gamma_{ei}+\mu_h) & 0 & 0 & 0 & (1-\phi_e\phi)\beta_{ch}S_h^{b*} & 0 & 0\\ 0 & \gamma_{ei} & -(\gamma_{is}+\mu_h) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -(\mu_v+\mu_r+\xi P^{b*}) & 0 & 0 & -\beta_{mv}S_v^{b*} & -\xi S_v^{b*}\\ 0 & 0 & 0 & 0 & -(\mu_v+\mu_r+\xi P^{b*}) & 0 & \beta_{mv}S_v^{b*} & 0\\ 0 & 0 & 0 & 0 & \sigma & -\mu_c & 0 & 0\\ 0 & 0 & \alpha h_h g_h & 0 & 0 & 0 & -\mu_m & 0\\ 0 & 0 & 0 & \tau P^{b*} & \tau P^{b*} & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

The eigenvalues of (13) are the solutions of $H(\lambda) = 0$.

$$H(\lambda) = (\lambda+\mu_h)H_1(\lambda), \tag{14}$$

where

$$\begin{aligned} H_1(\lambda) &= \lambda^7 + h_1\lambda^6 + h_2\lambda^5 + h_3\lambda^4 + h_4\lambda^3 + h_5\lambda^2 + h_6\lambda + h_7,\\ h_1 &= (\mu_v+\mu_r+\xi P^{b*}) + q_1,\\ h_2 &= \tau P^{b*}\xi S_v^{b*} + (\mu_v+\mu_r+\xi P^{b*})q_1 + q_2,\\ h_3 &= \tau P^{b*}\xi S_v^{b*}q_1 + (\mu_v+\mu_r+\xi P^{b*})q_2 + q_3,\\ h_4 &= \tau P^{b*}\xi S_v^{b*}q_2 + (\mu_v+\mu_r+\xi P^{b*})q_3 + q_4,\\ h_5 &= \tau P^{b*}\xi S_v^{b*}q_3 + (\mu_v+\mu_r+\xi P^{b*})q_4 + (\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m(\mu_v+\mu_r+\xi P^{b*})\\ &\quad -\gamma_{ei}\alpha h_h g_h(1-\phi_e\phi)\beta_{ch}S_h^{b*}\sigma\beta_{mv}S_v^{b*}\\ &= \tau P^{b*}\xi S_v^{b*}q_3 + (\mu_v+\mu_r+\xi P^{b*})q_4\\ &\quad +\left(1-(R_0^b)^2\right)(\mu_v+\mu_r+\xi P^{b*})(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m, \end{aligned}$$

$$
\begin{aligned}
h_6 &= \tau P^{b*}\xi S_v^{b*} q_4 + (\mu_v+\mu_r+\xi P^{b*})(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m(\mu_v+\mu_r+\xi P^{b*}) \\
&\quad -(\mu_v+\mu_r+\xi P^{b*})\gamma_{ei}\alpha h_h g_h(1-\phi_e\phi)\beta_{ch}S_h^{b*}\sigma\beta_{mv}S_v^{b*} \\
&= \tau P^{b*}\xi S_v^{b*} q_4 + \left(1-(R_0^b)^2\right)(\mu_v+\mu_r+\xi P^{b*})^2(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m, \\
h_7 &= \tau P^{b*}\xi S_v^{b*}(\mu_v+\mu_r+\xi P^{b*})(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m \\
&\quad -\gamma_{ei}\alpha h_h g_h(1-\phi_e\phi)\beta_{ch}S_h^{b*}\tau P^{b*}\xi S_v^{b*}\sigma\beta_{mv}S_v^{b*} \\
&= \left(1-(R_0^b)^2\right)(\mu_v+\mu_r+\xi P^{b*})(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c\mu_m\tau P^{b*}\xi S_v^{b*}, \\
q_1 &= \sum_{g=1}^{5} D_g, \\
q_2 &= \sum_{1\le g<h}^{5} D_g D_h, \\
q_3 &= \sum_{1\le g<h<i}^{5} D_g D_h D_i, \\
q_4 &= \sum_{1\le g<\ldots<j}^{5} D_g D_h D_i D_j.
\end{aligned}
$$

$D_1=\mu_h+\gamma_{ei}$, $D_2=\mu_h+\gamma_{is}$, $D_3=\mu_v+\mu_r+\xi P^{b*}$, $D_4=\mu_c$, and $D_5=\mu_m$. It is clear that (13) has one negative eigenvalue, i.e., $\lambda_1=-\mu_h$. The other eigenvalues are zeros of $H_1(\lambda)$. If $R_0^b=1$, then $h_7=0$. Hence, if $R_0^b=1$, one eigenvalue of $J(E_0^b)$ is zero. In the following, we apply a Routh–Hurwitz array shown in Table 4 to investigate the local stability condition of E_0^b.

Table 4. Routh–Hurwitz array associated to characteristic polynomial $H_1(\lambda)$.

	Column 1	**Column 2**	**Column 3**	**Column 4**	**Column 5**
λ^7	1	h_2	h_4	h_6	0
λ^6	h_1	h_3	h_5	h_7	0
λ^5	$k_1=\frac{h_1h_2-h_3}{h_1}$	$k_2=\frac{h_1h_4-h_5}{h_1}$	$k_3=\frac{h_6h_1-h_7}{h_1}$	0	0
λ^4	$k_4=\frac{k_1h_3-h_1k_2}{k_1}$	$k_5=\frac{k_1h_5-h_1k_3}{k_1}$	h_7	0	0
λ^3	$k_6=\frac{k_2k_4-k_1k_5}{k_4}$	$k_7=\frac{k_3k_4-k_1h_7}{k_4}$	0	0	0
λ^2	$k_8=\frac{k_5k_6-k_4k_7}{k_6}$	h_7	0	0	0
λ^1	$k_9=\frac{k_7k_8-k_6h_7}{k_8}$	0	0	0	0
λ^0	h_7	0	0	0	0

It is clear that $h_1>0$. On the basis of the Routh–Hurwitz criterion [27,28], all roots of $H_1(\lambda)$ have a negative real part if the other entries in Column 1 are also positive. k_1 is always positive. It is obvious that $h_7>0$ if $R_0^b<1$. Hence, all roots of $H(\lambda)$ have a negative real part, which implies that E_0^b is asymptotically stable if $k_4>0$, $k_6>0$, $k_8>0$, $k_9>0$, and $R_0^b<1$. Notice that $h_7<0$ if $R_0^b>1$. Hence, E_0^b is unstable if $R_0^b>1$.

Theorem 7. *Disease-free equilibrium point E_0^b is asymptotically stable if $k_4>0$, $k_6>0$, $k_8>0$, $k_9>0$, and $R_0^b<1$. If $R_0^b>1$, then E_0^b is unstable.*

Now, we investigate the local stability condition of E_1^b. We use the method that is presented in [19] to study the stability condition of E_1^b. Consider β_{mv} as a bifurcation parameter. We determine the bifurcation point that is equivalent to $R_0^b=1$. We obtain $\beta_{mv}^*=\frac{\mu_h(\mu_v+\mu_r+\xi P^{b*})\mu_c\mu_m(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\tau}{(1-\phi_e\phi)\beta_{ch}\Pi_h\sigma\mu_p\alpha h_h g_h\gamma_{ei}}$. If $\beta_{mv}=\beta_{mv}^*$, characteristic polynomial (14) becomes

$$H(\lambda) = \lambda(\lambda + \mu_h)H_1(\lambda), \tag{15}$$

where

$$H_1(\lambda) = \lambda^6 + h_1'\lambda^5 + h_2'\lambda^4 + h_3'\lambda^3 + h_4'\lambda^2 + h_5'\lambda + h_6', \tag{16}$$

and

$$\begin{aligned}
h_1' &= (\mu_v + \mu_r + \xi P^{b*}) + q_1,\\
h_2' &= \tau P^{b*}\xi S_v^{b*} + (\mu_v + \mu_r + \xi P^{b*})q_1 + q_2,\\
h_3' &= \tau P^{b*}\xi S_v^{b*} q_1 + (\mu_v + \mu_r + \xi P^{b*})q_2 + q_3,\\
h_4' &= \tau P^{b*}\xi S_v^{b*} q_2 + (\mu_v + \mu_r + \xi P^{b*})q_3 + q_4,\\
h_5' &= \tau P^{b*}\xi S_v^{b*} q_3 + (\mu_v + \mu_r + \xi P^{b*})q_4,\\
h_6' &= \tau P^{b*}\xi S_v^{b*} q_4.
\end{aligned}$$

It is clear that characteristic polynomial (15) has one zero root and one negative root, i.e., $\lambda_1 = 0$ and $\lambda_2 = -\mu_h$. Using the Routh–Hurwitz array in Table 5, we determine the conditions that guarantee that the other roots of (15) have a negative real part.

Table 5. Routh–Hurwitz array associated to characteristic polynomial (16).

	Column 1	**Column 2**	**Column 3**	**Column 4**	**Column 5**
λ^6	1	h_2'	h_4'	h_6'	0
λ^5	h_1'	h_3'	h_5'	0	0
λ^4	$kz_1 = \frac{h_1'h_2' - h_3'}{h_1'}$	$kz_2 = \frac{h_1'h_4' - h_5'}{h_1'}$	h_6'	0	0
λ^3	$kz_3 = \frac{kz_1h_3' - h_1'kz_2}{kz_1}$	$kz_4 = \frac{kz_1h_5' - h_1h_6'}{kz_1}$	0	0	0
λ^2	$kz_5 = \frac{kz_2kz_3 - kz_1kz_4}{kz_3}$	h_6'	0	0	0
λ^1	$kz_6 = \frac{kz_4kz_5 - kz_3h_6'}{kz_5}$	0	0	0	0
λ^0	h_6'	0	0	0	0

On the basis of the Routh–Hurwitz criterion [27], all roots of (16) have a negative real part if $h_1', kz_1, kz_3, kz_5, kz_6, h_6' > 0$. We recognize that $h_1', kz_1, h_6' > 0$. These results imply that if $kz_3, kz_5, kz_6 > 0$, then $J\left(E_0^b, \beta_{mv}^*\right)$ has one zero eigenvalue, and the other eigenvalues have a negative real part. Hence, Assumption A1 in Theorem 1 is satisfied if $kz_3, kz_5, kz_6 > 0$. The right eigenvector of $J\left(E_0^b, \beta_{mv}^*\right)$ corresponding to zero eigenvalue is

$$\vec{v^b} = \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \\ v_4^b \\ v_5^b \\ v_6^b \\ v_7^b \\ v_8^b \end{pmatrix} = \begin{pmatrix} \frac{(\gamma_{is}\gamma_{ei} - (\mu_h + \gamma_{ei})(\mu_h + \gamma_{is}))(1-\phi_e\phi)\beta_{ch}S_h^{b*}\sigma\beta_{mv}^*S_v^{b*}v_7^b}{\mu_h(\mu_h + \gamma_{ei})(\mu_h + \gamma_{is})\mu_c(\mu_v + \mu_r + \xi P^{b*})} \\ \frac{(1-\phi_e\phi)\beta_{ch}S_h^{b*}\sigma\beta_{mv}^*S_v^{b*}v_7^b}{(\mu_h + \gamma_{ei})\mu_c(\mu_v + \mu_r + \xi P^{*})} \\ \frac{\gamma_{ei}(1-\phi_e\phi)\beta_{ch}S_h^{b*}\sigma\beta_{mv}^*S_v^{b*}v_7^b}{(\mu_h + \gamma_{is})(\mu_h + \gamma_{ei})\mu_c(\mu_v + \mu_r + \xi P^{b*})} \\ \frac{-\beta_{mv}^*S_v^{b*}v_7^b}{(\mu_v + \mu_r + \xi P^{b*})} \\ \frac{\beta_{mv}^*S_v^{b*}v_7^b}{(\mu_v + \mu_r + \xi P^{b*})} \\ \frac{\sigma\beta_{mv}^*S_v^{b*}v_7^b}{\mu_c(\mu_v + \mu_r + \xi P^{b*})} \\ v_7^b \\ 0 \end{pmatrix},$$

where v_7^b is arbitrarily positive. It is clear that $v_1^b < 0$ and $v_4^b < 0$. The left eigenvector of $J\left(E_0^b, \beta_{mv}^*\right)$ corresponding to a zero eigenvalue is

$$\vec{w^b}^T = \begin{pmatrix} w_1^b \\ w_2^b \\ w_3^b \\ w_4^b \\ w_5^b \\ w_6^b \\ w_7^b \\ w_8^b \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\gamma_{ei}\alpha h_h g_h \beta_{mv}^* S_v^{b*} w_5^b}{(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_m} \\ \frac{\alpha h_h g_h \beta_{mv}^* S_v^{b*} w_5^b}{(\mu_h+\gamma_{is})\mu_m} \\ 0 \\ w_5^b \\ \frac{(1-\phi_e\phi)\beta_{ch} S_h^{b*} \gamma_{ei}\alpha h_h g_h \beta_{mv}^* S_v^{b*} w_5^b}{\mu_c(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_m} \\ \frac{\beta_{mv}^* S_v^{b*} w_5^b}{\mu_m} \\ 0 \end{pmatrix},$$

where w_5^b is calculated, such that $\vec{w^b}.\vec{v^b} = 1$. It is straightforward to show that $w_5^b > 0$. Now, we set

$$\begin{array}{cccc} x_1 = S_h & , & x_2 = E_h & , & x_3 = I_h & , & x_4 = S_v, \\ x_5 = I_v & , & x_6 = C & , & x_7 = M & , & x_8 = P, \\ f_1 = \frac{dS_h}{dt} & , & f_2 = \frac{dE_h}{dt} & , & f_3 = \frac{dI_h}{dt} & , & f_4 = \frac{dS_v}{dt}, \\ f_5 = \frac{dI_v}{dt} & , & f_6 = \frac{dC}{dt} & , & f_7 = \frac{dM}{dt} & , & f_8 = \frac{dP}{dt}. \end{array}$$

Thus, the only nonzero terms of A and B that are described in Theorem 1 are

$$\begin{aligned} w_2^b v_1^b v_6^b \frac{\partial^2 f_2\left(E_0^b, \beta_{mv}^*\right)}{\partial x_1 \partial x_6} =& \left(\frac{\gamma_{ei}\alpha h_h g_h \beta_{mv}^* S_v^{b*} w_5^b}{(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_m}\right) \\ &\times \left(\frac{(\gamma_{is}\gamma_{ei} - (\mu_h+\gamma_{ei})(\mu_h+\gamma_{is}))(1-\phi_e\phi)\beta_{ch} S_h^{b*} \sigma \beta_{mv}^* S_v^{b*} v_7^b}{\mu_h(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c(\mu_v+\mu_r+\xi P^{b*})}\right) \\ &\times \left(\frac{\sigma \beta_{mv}^* S_v^{b*} v_7^b}{\mu_c(\mu_v+\mu_r+\xi P^{b*})}\right)\beta_{ch} < 0, \end{aligned}$$

$$\begin{aligned} w_2^b v_6^b v_1^b \frac{\partial^2 f_2\left(E_0^b, \beta_{mv}^*\right)}{\partial x_6 \partial x_1} =& \left(\frac{\gamma_{ei}\alpha h_h g_h \beta_{mv}^* S_v^{b*} w_5^b}{(\mu_h+\gamma_{is})(\mu_h+\gamma_{ei})\mu_m}\right)\left(\frac{\sigma \beta_{mv}^* S_v^{b*} v_7^b}{\mu_c(\mu_v+\mu_r+\xi P^{b*})}\right) \\ &\times \left(\frac{(\gamma_{is}\gamma_{ei} - (\mu_h+\gamma_{ei})(\mu_h+\gamma_{is}))(1-\phi_e\phi)\beta_{ch} S_h^{b*} \sigma \beta_{mv}^* S_v^{b*} v_7^b}{\mu_h(\mu_h+\gamma_{ei})(\mu_h+\gamma_{is})\mu_c(\mu_v+\mu_r+\xi P^{b*})}\right)\beta_{ch} < 0, \end{aligned}$$

$$\begin{aligned} w_5^b v_4^b v_7^b \frac{\partial^2 f_5\left(E_0^b, \beta_{mv}^*\right)}{\partial x_4 \partial x_7} &= w_5^b \left(\frac{-\beta_{mv}^* S_v^{b*} v_7^b}{(\mu_v+\mu_r+\xi P^{b*})}\right) v_7^b \beta_{mv}^* < 0, \\ w_5^b v_7^b v_4^b \frac{\partial^2 f_5\left(E_0^b, \beta_{mv}^*\right)}{\partial x_7 \partial x_4} &= w_5^b v_7^b \left(\frac{-\beta_{mv}^* S_v^{b*} v_7^b}{(\mu_v+\mu_r+\xi P^{b*})}\right) \beta_{mv}^* < 0, \\ w_5^b v_7^b \frac{\partial^2 f_5\left(E_0^b, \beta_{mv}^*\right)}{\partial x_7 \partial \beta_{mv}} &= w_5^b v_7^b \frac{d_p}{\tau} > 0. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} A &= w_2^b v_1^b v_6^b \frac{\partial^2 f_2\left(E_0^b, \beta_{mv}^*\right)}{\partial x_1 \partial x_6} + w_2^b v_6^b v_1^b \frac{\partial^2 f_2\left(E_0^b, \beta_{mv}^*\right)}{\partial x_6 \partial x_1} + w_5^b v_4^b v_7^b \frac{\partial^2 f_5\left(E_0^b, \beta_{mv}^*\right)}{\partial x_4 \partial x_7} + w_5^b v_7^b v_4^b \frac{\partial^2 f_5\left(E_0^b, \beta_{mv}^*\right)}{\partial x_7 \partial x_4} < 0, \\ B &= w_5^b v_7^b \frac{\partial^2 f_5\left(E_0^b, \beta_{mv}^*\right)}{\partial x_7 \partial \beta_{mv}} > 0. \end{aligned}$$

According to Theorem 1, forward bifurcation occurs at $R_0^b = 1$. Consequently, endemic equilibrium point E_1^b that exists when $R_0^b > 1$ is asymptotically stable if $kz_3, kz_5, kz_6 > 0$, and $R_0^b > 1$ (close to 1).

Theorem 8. *Endemic equilibrium point E_1^b is asymptotically stable if $kz_3 > 0$, $kz_5 > 0$, $kz_6 > 0$, and $R_0^b > 1$ (close to 1).*

Now, we present the global stability condition of E_0^b.

Theorem 9. *Disease-free equilibrium point E_0^b is globally asymptotically stable if $R_0^b \leq 1$.*

Proof. Consider a Lyapunov function as follows:

$$\begin{aligned} Q &= q_1\left(S_h - S_h^{b*} - S_h^{b*} \ln \frac{S_h}{S_h^{b*}}\right) + q_2 E_h + q_3 I_h + q_4\left(S_v - S_v^{b*} - S_v^{b*} \ln \frac{S_v}{S_v^{b*}}\right) + q_5 I_v \\ &\quad + q_6\left(P - P^{b*} - P^{b*} \ln \frac{P}{P^{b*}}\right) + q_7 C + q_8 M, \end{aligned}$$

where

$$\begin{aligned} &q_1 = q_2 = \frac{\beta_{mv} S_v^{b*} \alpha h_h g_h \gamma_{ei}}{\mu_m(\mu_h + \gamma_{ei})(\mu_h + \gamma_{is})}, \quad q_3 = \frac{\beta_{mv} S_v^{b*} \alpha h_h g_h}{\mu_m(\mu_h + \gamma_{is})}, \quad q_4 = q_5 = 1, \\ &q_6 = \frac{\xi}{\tau}, \quad q_7 = \frac{(1 - \phi_e \phi)\beta_{ch} S_h^{b*} \beta_{mv} S_v^{b*} \alpha h_h g_h \gamma_{ei}}{\mu_c \mu_m(\mu_h + \gamma_{ei})(\mu_h + \gamma_{is})}, \quad q_8 = \frac{\beta_{mv} S_v^{b*}}{\mu_m}. \end{aligned}$$

The time derivative of Q is

$$\begin{aligned} \frac{dQ}{dt} &= q_1\left(1 - \frac{S_h^{b*}}{S_h}\right)\frac{dS_h}{dt} + q_2 \frac{dE_h}{dt} + q_3 \frac{dI_h}{dt} + q_4\left(1 - \frac{S_v^{b*}}{S_v}\right)\frac{dS_v}{dt} + q_5 \frac{dI_v}{dt} + q_6\left(1 - \frac{P^{b*}}{P}\right)\frac{dP}{dt} + q_7 \frac{dC}{dt} + q_8 \frac{dM}{dt} \\ &= \frac{-q_1 \mu_h}{S_h}\left(S_h - S_h^{b*}\right)^2 + \frac{q_1 \gamma_{is}}{S_h} I_h\left(S_h - S_h^{b*}\right) - \frac{q_4(\mu_h + \mu_r)}{S_v}\left(S_v - S_v^{b*}\right)^2 \\ &\quad + \left((R_0^b)^2 - 1\right)(\mu_v + \mu_r + \xi P^{b*}) I_v + q_4 \xi P^{b*} S_v^{b*}\left(2 - \left(\frac{S_v^{b*}}{S_v} + \frac{S_v}{S_v^{b*}}\right)\right). \end{aligned}$$

Arithmetic mean is always greater than the geometric mean. Hence, $\left(2 - \left(\frac{S_v^{b*}}{S_v} + \frac{S_v}{S_v^{b*}}\right)\right) < 0$ always holds. It is clear that $\left(2 - \left(\frac{S_v^{b*}}{S_v} + \frac{S_v}{S_v^{b*}}\right)\right) = 0$ if $S_v = S_v^{b*}$. Furthermore, $S_h \leq S_h^{b*}$ always holds. Thus, it is obvious that $\frac{dQ}{dt} \leq 0$ if $R_0^b \leq 1$. $\frac{dQ}{dt} = 0$ if and only if $S_v = S_v^{b*}$, $S_h = S_h^{b*}$, $I_h = 0$, and $I_v = 0$. Hence, the largest invariant set in $\left\{(S_h, E_h, I_h, S_v, I_v, C, M, P) | \frac{dQ}{dt} = 0\right\}$ is a singleton set $\left\{E_0^b\right\}$. Thus [30], E_0^b is globally asymptotically stable if $R_0^b \leq 1$. □

We proved the global stability condition of the disease-free equilibrium points, E_0^a and E_0^b, by formulating suitable Lyapunov functions. Since we have difficulty in finding a suitable Lyapunov function for the endemic equilibrium point, we only investigate the local stability condition of the endemic equilibrium points (E_1^a and E_1^b). When the local stability condition of the endemic equilibrium point is met, the endemic equilibrium point is asymptotically stable, but may only attract a very small part of the state space. Therefore, studying the size of the basin of attraction of the endemic equilibrium point is relevant [31]. However, we do not discuss the basin of attraction of the endemic equilibrium point in this article. A method for numerical estimates of the size and shape of the basin of attraction, as well as the systems' return time to the attractor can be seen in [31,32].

5. Numerical Simulations

To verify and support the previous qualitative-analysis results, we performed some numerical simulations using the parameter values presented in Table 6. Those parameter values were taken from the literature if available. Otherwise, they are given as assumptions.

Table 6. Parameter values.

Symbol	Parameter value	Units	Source
Π_h	$\frac{1000\times 1\%}{365}$	human$\times$ day^{-1}	Assumed
ϕ_e	0.9		Assumed
ϕ	0.9		Assumed
γ_{is}	$\frac{1}{10\times 7}$	day^{-1}	[13]
μ_h	$\frac{1}{365\times 65}$	day^{-1}	[13]
γ_{ei}	$\frac{1}{6\times 7}$	day^{-1}	[13]
Π_v	1	snail$\times$ day^{-1}	[12]
μ_v	0.033	day^{-1}	[12]
μ_r	0.0001	day^{-1}	Estimated
ξ	0.01	predator$^{-1}\times$ day^{-1}	Assumed
σ	600	cercariae$\times$ snail$^{-1}\times$ day^{-1}	[20]
α	0.01	miracidia$\times$ egg^{-1}	[12]
h_h	513	egg$\times$ gram^{-1}	[12]
g_h	160	gram$\times$ human$^{-1}\times$ day^{-1}	[12]
μ_p	$\frac{1}{365\times 2}$	day^{-1}	Estimated
μ_c	1	day^{-1}	[12]
μ_m	2	day^{-1}	[12]

Numerical simulations were performed by varying β_{ch}, β_{mv}, and τ. Here, we present the numerical-simulation results with initial values of $S_h(0) = 900$, $E_h(0) = 0$, $I_h(0) = 100$, $S_v(0) = 100$, $I_v(0) = 10$, $P(0) = 50$, $C(0) = 100$, $M(0) = 100$.

5.1. Predator Becomes Extinct

In this subsection, we show the numerical results using $\tau = 0.00001$ and $\beta_{mv} = \frac{0.0004}{365}$. Here, we have $\frac{\tau\Pi_v}{(\mu_v+\mu_r)\mu_p} = 0.22054 < 1$; thus, the existence condition of the equilibrium points says that the predator goes extinct. We can also show basic reproduction number $R_0^a = 1$ corresponds to the critical value of cercaria infection rate on susceptible humans, i.e., $\beta_{ch}^* = 4.71859 \times 10^{-7}$. If we set $\beta_{ch} = 1.71859 \times 10^{-7} < \beta_{ch}^*$, then we obtain $R_0^a = 0.60350 < 1$. On the basis of Theorems 4 and 6, disease-free equilibrium point (E_0^a) is asymptotically stable. This situation was confirmed by our simulation shown in Figure 2a where I_h, I_v, and P were convergent to zero. For the second simulation, we set $\beta_{ch} = 4.71859 \times 10^{-6} > \beta_{ch}^*$ which led to $R_0^a = 3.16227 > 1$. Theorem 5 states that endemic equilibrium point (E_1^a) is asymptotically stable. The stability of (E_1^a) is clearly shown in Figure 2b, namely, I_h and I_v converge to a positive equilibrium, whereas P goes to zero. To more clearly see the effect of the cercaria infection rate on humans, we performed numerical simulations by varying β_{ch} from 2.35929×10^{-7} to 7.07789×10^{-7}, which corresponds to R_0^a between 0.70710 to 1.22474. Figure 2c indicates the occurrence of forward bifurcation driven by R_0^a or β_{ch}. If $R_0^a < 1$, then E_0^a is asymptotically stable, while E_1^a does not exist. Conversely, if $R_0^a > 1$, then E_0^a is unstable, and there appears E_1^a, which is asymptotically stable. Bifurcation point $R_0^a = 1$ is achieved when $\beta_{ch} = \beta_{ch}^* = 4.71859 \times 10^{-7}$.

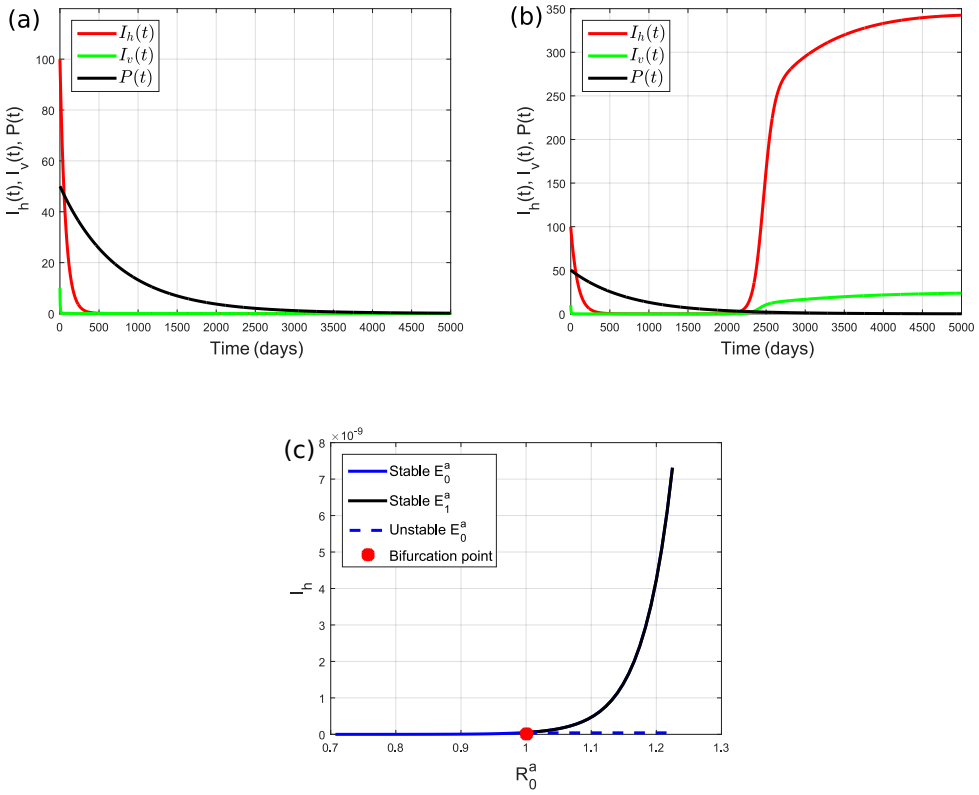

Figure 2. Dynamics of I_h, I_v, P for (**a**) $\beta_{ch} = 1.71859 \times 10^{-7}$ that corresponds to $(R_0^a < 1)$, and (**b**) $\beta_{ch} = 4.71859 \times 10^{-6}$ that corresponds to $(R_0^a > 1)$. (**a**,**b**) Reducing cercaria infection rate on humans by limiting the number of contacts between susceptible humans and contaminated areas or by using self-protective gear when entering contaminated areas can decrease schistosomiasis prevalence. (**c**) Forward bifurcation diagram of System (2) in the case of the snail predator becoming extinct, showing that schistosomiasis is eradicated if $R_0^a < 1$, but it becomes endemic if $R_0^a > 1$.

5.2. Predator Survives

We next performed numerical simulations using $\tau = 0.0001$ and $\beta_{ch} = 1.914 \times 10^{-5}$, which implies that $\frac{\tau \Pi_v}{(\mu_v + \mu_r)\mu_p} = 2.20543 > 1$. On the basis of the existence condition of the equilibrium points, the predator survives. In this case, the basic reproduction number is unity $(R_0^b = 1)$ if the miracidia transmission rate on snails reaches its critical value, i.e., $\beta_{mv}^* = 1.31409 \times 10^{-7}$. Hence, if we set $\beta_{mv} = 0.31409 \times 10^{-7} < \beta_{mv}^*$, we obtain $k_1 = 2.43742 > 0$, $k_4 = 0.37218 > 0$, $k_6 = 0.01988 > 0$, $k_8 = 0.00045 > 0$, $k_9 = 3.1 \times 10^{-6} > 0$, and $R_0^b = 0.48889 < 1$. On the basis of Theorems 7 and 9, disease-free equilibrium point (E_0^b) is asymptotically stable. This behavior was confirmed by our simulation, where the numerical solution was convergent to (E_0^b); see Figure 3a. On the other hand, if we take $\beta_{mv} = 4.31409 \times 10^{-7} > \beta_{mv}^*$, such that $R_0^b = 1.81188 > 1$, then Theorem 8 shows that endemic equilibrium (E_1^b) is asymptotically stable. The stability of (E_1^b) is confirmed by Figure 3b, which shows that the numerical solution converges to (E_1^b). When the predator goes extinct, schistosomiasis prevalence is higher than that when the predator survives. This means that the existence of a snail predator in the environment can reduce schistosomiasis prevalence.

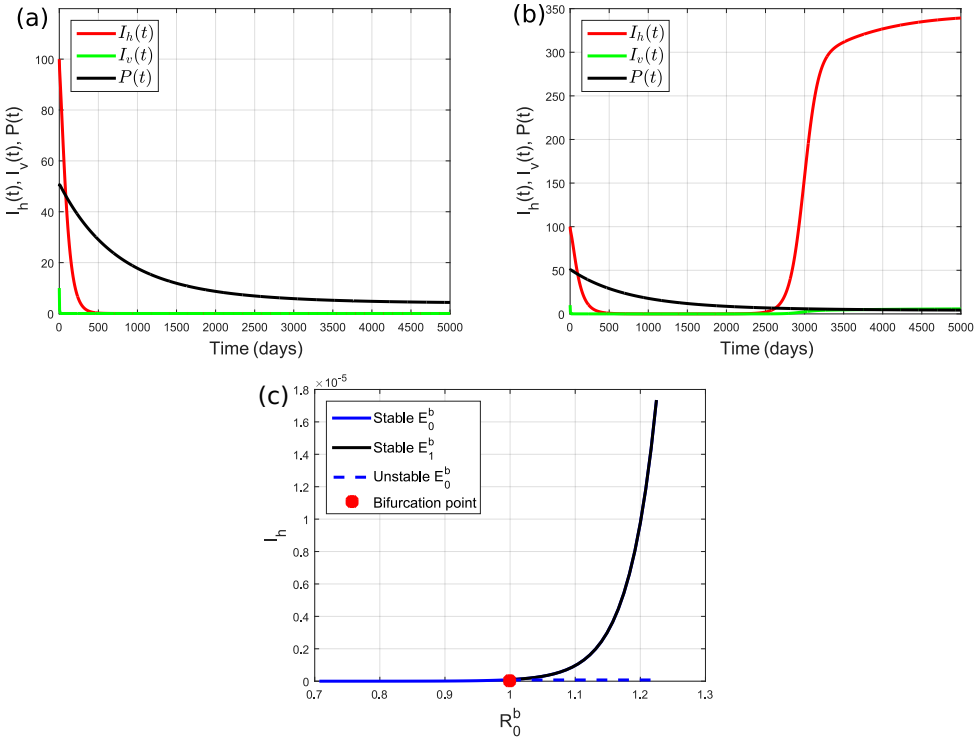

Figure 3. Dynamics of I_h, I_v, P for (**a**) $\beta_{mv} = 0.31409 \times 10^{-7}$, which corresponds to $R_0^b < 1$; and (**b**) $\beta_{mv} = 4.31409 \times 10^{-7}$, which corresponds to $R_0^b > 1$. (**a**,**b**) Reducing miracidia transmission rate on snails by improving irrigation facilities (e.g., lining canals with cement) can reduce schistosomiasis prevalence. (**c**) Forward bifurcation diagram of System (2) in the case of snail predator surviving, showing that schistosomiasis is eradicated if $R_0^b < 1$, but it becomes endemic if $R_0^b > 1$.

Figure 3a,b indicate that there is an exchange of stability between the disease-free equilibrium point and the disease endemic equilibrium point when the miracidia transmission rate on snails is changed from $\beta_{mv} = 0.31409 \times 10^{-7} < \beta_{mv}^*$ to $\beta_{mv} = 4.31409 \times 10^{-7} > \beta_{mv}^*$. To more clearly see the effect of β_{mv}, we plotted in Figure 3c the steady state of I_h against R_0^b. R_0^b between 0.70710 to 1.22474 corresponding to β_{mv} from 0.65704×10^{-7} to 1.97114×10^{-7}; $R_0^b = 1$ is obtained when $\beta_{mv} = \beta_{mv}^* = 1.31409 \times 10^{-7}$. Figure 3c shows that System (2) experiences forward bifurcation. Thus, E_0^b is asymptotically stable, and E_1^b does not exist when $R_0^b < 1$. If $R_0^b > 1$, E_0^b becomes unstable, and E_1^b exists and it is asymptotically stable.

5.3. Impact of Snail Predator as Biological Control Agent

We now investigate the impact of a snail predator as biological control agent. Here, we study the effect of predation rate (ξ) by using parameter values as in Table 6: $\tau = 0.0001$, $\beta_{ch} = 0.1$ and $\beta_{mv} = 4.31409 \times 10^{-7}$. According to [33], the predation rate is the proportion of prey killed per predator per time. It suggests that the predation rate is related to the effectiveness of the predator in hunting and killing snails. In this simulation, the basic reproduction number does not depend on the predation rate and is given by $R_0^b = 18.11886$. Figure 4 shows that all numerical solutions using $\xi = 0.1, \xi = 0.15$, and $\xi = 0.3$ are convergent to the same endemic equilibrium point. However, detailed observation shows that the schistosomiasis prevalence in the beginning of the intervention decreased as predation rate (ξ) increased. Several studies that are related to satiation-based predation models show that the predation rate is related to the gut capacity of the predator [34]. Hence, we should use snail predators that can effectively hunt and kill snails, and have high gut capacity.

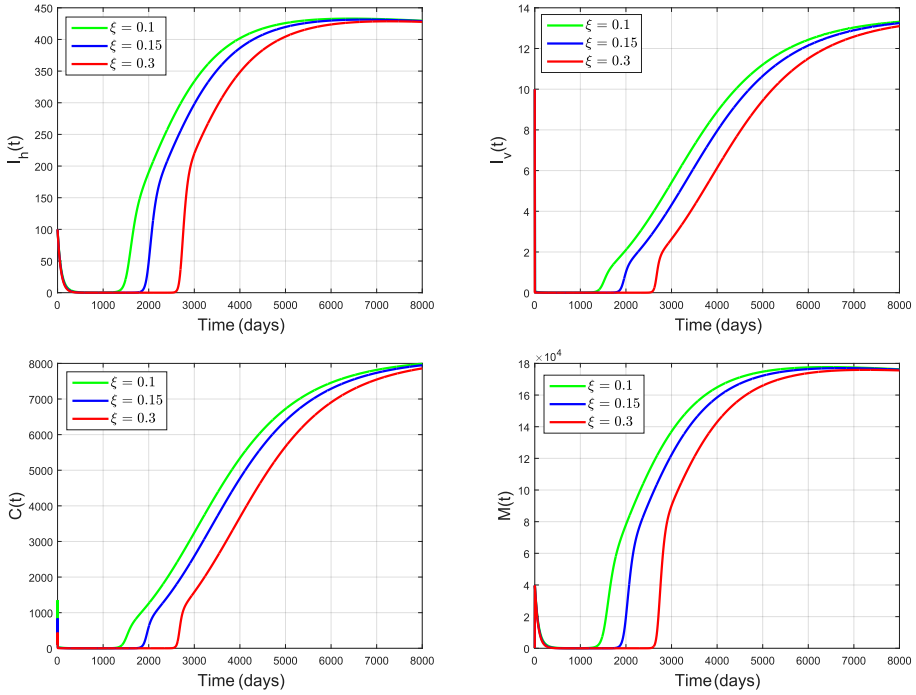

Figure 4. Dynamics of I_h, I_v, C, M for $\xi = 0.1, 0.15, 0.3$, showing that the effectiveness of predators in hunting and killing snails plays an important role in reducing schistosomiasis prevalence at the beginning of intervention.

We next investigate the influence of conversion rate τ by performing simulations using the same parameter values as those in Figure 4, but with varying τ and fixed $\xi = 0.01$. According to [35], the conversion rate is related to the efficiency in turning predation into new predators. It implies that the conversion rate is related to the birth rate of snail predator. By choosing $\tau = 0.0001$, $\tau = 0.00015$, and $\tau = 0.0003$, we get the basic reproduction number $(R_0^b) = 18.118869$, $(R_0^b) = 12.07924$, $(R_0^b) = 6.03962$, respectively. Hence, the basic reproduction number decreases as τ increases. The dynamics of I_h, I_v, C, and M for $\tau = 0.0001, 0.00015$, and 0.0003 is shown in Figure 5. Values of the steady state of I_h, I_v, C, and M were smaller for larger values of τ. Thus, schistosomiasis prevalence decreases as τ increases. So, the snail-predator birth rate plays an essential role in controlling schistosomiasis spread. Figures 4 and 5 show that using a natural predator of snails as a biological control agent can decrease schistosomiasis prevalence. These results are in line with the results in [7].

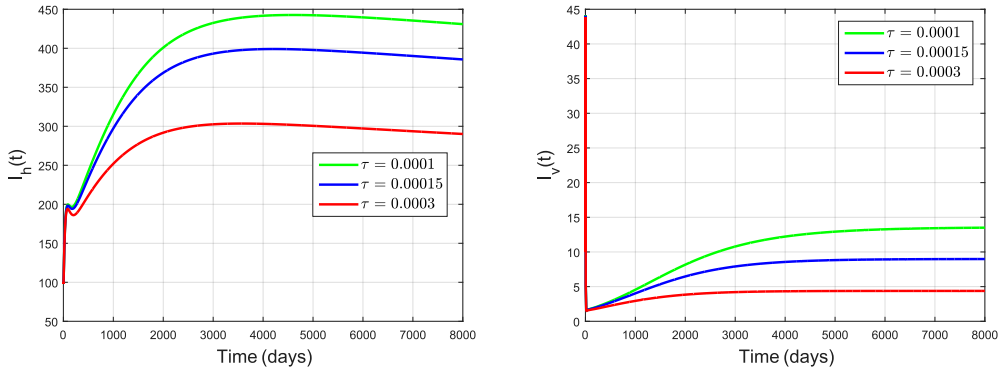

Figure 5. *Cont.*

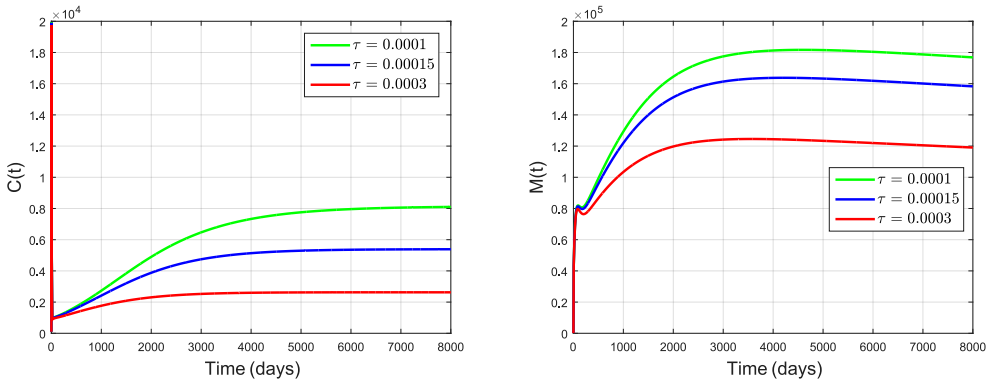

Figure 5. Dynamics of I_h, I_v, C, M for $\tau = 0.0001, 0.00015, 0.0003$, showing that the conversion rate of the snail predator that is related to its birth rate plays an essential role in reducing schistosomiasis prevalence.

6. Conclusions

In this work, a deterministic schistosomiasis model incorporating a snail predator as a biological control agent was discussed. The existence and stability conditions of all equilibrium points were investigated. Our findings suggest that a snail predator as a biological control agent can reduce the prevalence of schistosomiasis. Moreover, our results showed that the snail-predator birth rate plays an important role in controlling schistosomiasis spread. Despite the contributions of our work, it has some limitations. Due to the lack of data related to the parameters of our model, we only chose parameter values by considering the results of qualitative analysis, i.e., the existence and stability conditions of the equilibrium points. For further research, sensitivity and cost-effectiveness analysis of the interventions discussed in our proposed model will be considered.

Author Contributions: Conceptualization, T. and A.S.; methodology, W.N., A.S., and W.M.K.; software, W.N.; validation, T., A.S. and W.M.K.; formal analysis, W.N. and A.S.; investigation, W.N.; resources, T.; data curation, W.N.; writing—original draft preparation, W.N.; writing—review and editing, T., A.S. and W.M.K.; visualization, W.N.; supervision, T., A.S. and W.M.K.; project administration, T.; funding acquisition, T. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest: The authors declare no conflict of interest in this paper.

Appendix A

Term of Ξ.

$$
\begin{aligned}
\Xi \;=\; & \eta_1\eta_2\eta_3 D_1^4\big[D_2^3+(\eta_3+D_5)D_2^2+(D_3D_4+\eta_3D_5)D_2+D_3D_4D_5\big]\\
&+D_1^3\big[\big(D_2^4+2D_2^3(\eta_3+D_5)\big)\eta_1\eta_2\eta_3\big]\\
&+D_1^3D_2^2\big[\eta_1^2D_3^4+4\eta_1^2D_3^3+2(\eta_1+D_5)(D_4+\eta_1)\eta_1D_3^2+(\eta_4+3D_4D_5)\eta_1^2D_3+\eta_5\big]\\
&+D_1^3D_2\big[\eta_1^2D_3^4+2\eta_1\eta_4D_3^3+(\eta_4+3D_4D_5)\eta_1^2D_3^2+2\eta_5D_3+\eta_1\eta_6\big]\\
&+D_1^3\big[\eta_1D_4D_5D_3^4+2\eta_1D_4D_5D_3^3+\eta_5D_3^2+\eta_1\eta_6D_3+\eta_1^2R_{01}^a\big]\\
&+D_1^2\big[\eta_1\eta_2\eta_3(\eta_3+D_5)D_2^4\big]\\
&+D_1^2D_2^3\big[\eta_1D_3^4+4\eta_1^2D_3^3+2(\eta_1+D_4)(\eta_1+D_5)\eta_1D_3^2+(\eta_4+3D_4D_5)\eta_1^2D_3+\eta_5\big]\\
&+D_1^2D_2^2\big[2\eta_1^2D_3^4+2(\eta_1+D_4)(\eta_1+D_5)\eta_1D_3^3+2(\eta_4+2D_4D_5)\eta_1^2D_3^2\big]\\
&+D_1^2D_2^2\big[\eta_8D_3+2\eta_1\eta_6\big]\\
&+D_1^2D_2\big[\eta_1\eta_4D_3^4+(\eta_4+3D_4D_5)\eta_1^2D_3^3\big]\\
&+D_1^2D_2\big[(3\eta_5+D_4^4D_5+2D_4^3D_5^2+2D_4^2D_5^3+D_4D_5^4+R_{01}^a)D_3^2\big]\\
&+D_1^2D_2\big[\eta_1(3\eta_6+D_4^3D_5^2+D_4^2D_5^3+R_{01}^a)D_3+\eta_7\big]\\
&+D_1^2\big[D_4D_5\eta_1^2D_3^4+\eta_5D_3^3+2\eta_1\eta_6D_3^2+\eta_7D_3+R_{01}^a\eta_1\eta_4\big]\\
&+D_1D_2^4\big[\eta_1^2D_3^3+\eta_1\eta_4D_3^2+2D_4D_5\eta_1^2D_3+D_4^2D_5^2\eta_1\big]\\
&+D_1D_2^3\big[\eta_1^2D_3^4+2\eta_1\eta_4D_3^3+(\eta_4+3D_4D_5)\eta_1^2D_3^2+2\eta_5D_3+\eta_1\eta_6\big]\\
&+D_1D_2^2\big[\eta_1\eta_4D_3^4+(\eta_4+3D_4D_5)\eta_1^2D_3^3+\eta_8D_3^2\big]\\
&+D_1D_2^2\big[\eta_1(3\eta_6+D_4^3D_5^2+D_4^2D_5^3+R_{01}^a)D_3+\eta_7\big]\\
&+D_1D_2\big[2D_4D_5\eta_1^2D_3^4+2\eta_5D_3^3+\eta_1(3\eta_6+D_4^3D_5^2+D_4^2D_5^3+R_{01}^a)D_3^2\big]\\
&+D_1D_2\big[(2D_4^4D_5^3+2D_4^3D_5^4+7R_{01}^a(D_4^2+D_5^2))D_3+R_{01}^a(\eta_1+D_5)(\eta_1+D_4)\eta_1\big]\\
&+D_1\big[D_4^2D_5^2\eta_1D_3^4+\eta_7D_3^2+R_{01}^a(\eta_1+D_5)(\eta_1+D_4)\eta_1D_3+2R_{01}^aD_4D_5\eta_1^2\big]\\
&+D_2^4D_4D_5\eta_1\eta_2\eta_3+D_2^3\big[D_4D_5\eta_1(D_3^4+\eta_1D_3^3+\eta_5D_3^2)+\eta_1\eta_6D_3+\eta_1^2R_{01}^a\big]\\
&+D_2^2\big[D_4D_5\eta_1^2D_3^4+\eta_5D_3^3+2\eta_1\eta_6D_3^2+\eta_7D_3+R_{01}^a\eta_1\eta_4\big]\\
&+D_2\big[D_4^2D_5^2\eta_1D_3^4+\eta_1\eta_6D_3^3+\eta_7D_3^2++R_{01}^a(\eta_1+D_5)(\eta_1+D_4)\eta_1D_3+2R_{01}^aD_4D_5\eta_1^2\big]\\
&+\eta_1^2R_{01}^aD_3^3+R_{01}\eta_1\eta_4D_3^2+3R_{01}^aD_4D_5\eta_1^2D_3+R_{01}^aD_4^2D_5^2\eta_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_1 &= D_4+D_5,\\
\eta_2 &= D_3+D_5,\\
\eta_3 &= D_3+D_4,\\
\eta_4 &= D_4^2+3D_4D_5+D_5^2,\\
\eta_5 &= D_4^4D_5+4D_4^3D_5^2+4D_4^2D_5^3+D_4D_5^4+R_{01}^a,\\
\eta_6 &= D_4^3D_5^2+D_4^2D_5^3+2R_{01}^a,\\
\eta_7 &= D_4^4D_5^3+D_4^3D_5^4+4R_{01}^a(D_4^2+D_5^2)+7R_{01}^aD_4D_5,\\
\eta_8 &= (3\eta_5+D_4^4D_5+2D_4^3D_5^2+2D_4^2D_5^3+D_4D_5^4+R_{01}^a).
\end{aligned}
$$

It is clear that $\Xi>0$ because $D_i>0$ for $i=1,2,3,4,5$.

Appendix B

On the basis of (10), $l_1>0$ and $l_4>0$. When we derived Theorem 4, we proved that $\Delta_2=l_1l_2-l_3>0$. Hence, $r_1=\frac{l_1l_2-l_3}{l_1}>0$. We also can show that $r_2>0$.

$$
\begin{aligned}
r_2 \;=\; & \big(\zeta_1D_3^2+\zeta_1^2D_3+\zeta_2\zeta_3\zeta_4\big)D_2^3+\big(\zeta_1D_3^3+2\zeta_1^2D_3^2+\zeta_5D_3+\zeta_2\zeta_3\zeta_4\zeta_1\big)D_2^2\\
&+\big(\zeta_1^2D_3^3+\zeta_5D_3^2+\zeta_1\zeta_6D_3+\zeta_2\zeta_3\zeta_4\zeta_7\big)D_2+\zeta_2\zeta_3\zeta_4\big(D_3^3+\zeta_1D_3^2+\zeta_7D_3+D_1D_2D_3\big),
\end{aligned}
$$

where

$$
\begin{aligned}
&\zeta_1=D_1+D_4+D_5,\\
&\zeta_2=D_4+D_5,\\
&\zeta_3=D_1+D_5,\\
&\zeta_4=D_1+D_4,\\
&\zeta_5=D_1^2(D_1+4D_4+4D_5)+D_4^2(4D_1+D_4+4D_5)+D_5^2(4D_1+4D_4+D_5)+7D_1D_4D_5,\\
&\zeta_6=2D_1^2D_4+2D_4^2D_5+2D_1D_4^2+3D_1D_4D_5+2D_1D_5^2+2D_4^2D_5+2D_4D_5^2,\\
&\zeta_7=D_1\zeta_2+D_4D_5.
\end{aligned}
$$

It is clear that $r_2>0$ since $D_i>0$ for $i=1,2,3,4,5$.

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