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# Multiple Dedekind Type Sums and Their Related Zeta Functions

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**Abstract:** The main purpose of this paper is to use the multiple twisted Bernoulli polynomials and their interpolation functions to construct multiple twisted Dedekind type sums. We investigate some properties of these sums. By use of the properties of multiple twisted zeta functions and the Bernoulli functions involving the Bernoulli polynomials, we derive reciprocity laws of these sums. Further developments and observations on these new Dedekind type sums are given.

**Keywords:** twisted $(h,q)$-Bernoulli numbers and polynomials; Barnes' type multiple zeta function; higher-order Bernoulli numbers and polynomials; Dedekind sums; reciprocity law



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## Contents



## 1. Introduction

Throughout this paper, we use the following notations:

- $\mathbb{Z}$ denotes the ring of integers;
- $\mathbb{N} := \{1,2,3,...\}$;
- $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$;
- for $q \in \mathbb{C}$, we denote

$$[x] = [x:q] = \begin{cases} \frac{1-q^x}{1-q}, & q \neq 1 \\ x, & q = 1 \end{cases}$$

- $\xi$ is an $r$-th root of 1 with $r \in \mathbb{N}$.

The purpose of this paper is not only to study the different types of higher-order *twisted* $(h,q)$-Bernoulli numbers and polynomials, which generalize those of [1], but also to study the relations between these numbers, polynomials, and Dedekind type sums and related areas.

The main motivation is the study of a $q$-analogue of the generalized Barnes' multiple zeta function (i.e., $q$-Barnes' multiple zeta function in twisted version) and to introduce a new type of Dedekind sum. We prove a Dedekind's type reciprocity law for these new sums. Our resulting generalized reciprocity formulas recover the results of Apostol [2] and Ota [3].

Many mathematicians have studied the so-called twisted Bernoulli numbers and polynomials and their interpolation functions. For more details, one can see Cenkci et al. [4], Hu-Min Soo [5], Bayad [6], Kim [7], Koblitz [8], and Simsek [9].

Let $h \in \mathbb{N}$ and $q, \xi \in \mathbb{C}$, we assume that $|q| < 1$. By using $p$-adic $q$-integral theory, Simsek [9] defined the generating function of the twisted $(h,q)$-Bernoulli numbers $B_{n,\xi}^{(h)}(q)$ using the following generating function:

$$\begin{aligned} \mathcal{F}_{\xi,q}^{(h)}(t) &:= \frac{\log q^h + t}{\xi q^h e^t - 1} \\ &= \sum_{n=0}^{\infty} B_{n,\xi}^{(h)}(q) \frac{t^n}{n!}. \end{aligned} \tag{1}$$

From the above equation, we have

$$B_{0,\xi}^{(h)}(q) = \frac{\log q^h}{\xi q^h - 1}, \; \xi q^h (B_{\xi}^{(h)}(q) + 1)^n - B_{n,\xi}^{(h)}(q) = \delta_{1,n}, \; n \geq 1\,,$$

where $\delta_{1,n}$ denotes the Kronecker symbol and the usual convention of symbolically replacing $(B_{\xi}^{(h)}(q))^n$ by $B_{n,\xi}^{(h)}(q)$ (cf. [9–12]). The link between the twisted $(h,q)$-extension of Bernoulli numbers and Frobenius–Euler numbers is given in [11] by the relation

$$B_{n,\xi}^{(h)}(q) = \frac{\left(\log q^h\right) H_n(\xi^{-1} q^{-h}) + n H_{n-1}(\xi^{-1} q^{-h})}{\xi q^h - 1},$$

where $H_n(u)$ denotes the Frobenius–Euler numbers, which are defined as follows.
For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius–Euler numbers $H_n(u)$ are defined by using of the following generating function:

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}\;;$$

$H_n(u)$ are rational fractions of polynomials and were studied in great detail by Frobenius [13], who was particularly interested in their relationship to Bernoulli numbers [14] and relation (2.7) in [15].

One can observe that for $\xi = 1$, $B_{n,\xi}^{(h)}(q) = B_n^{(h)}(q)$ (cf. [16]) and $\lim\limits_{q \to 1} B_n^{(h)}(q) = B_n$, $B_n$ are the Bernoulli numbers (cf. [2–32]).

The twisted $(h,q)$-Bernoulli polynomials are defined in [9] by using the generating function

$$\begin{aligned} \mathcal{F}_{\xi,q}^{(h)}(t,z) &= \frac{(t + \log q^h) e^{tz}}{\xi q^h e^t - 1} \\ &= \sum_{n=0}^{\infty} B_{n,\xi}^{(h)}(z,q) \frac{t^n}{n!}. \end{aligned} \tag{2}$$

We easily see that $B_{n,\xi}^{(h)}(0,q) = B_{n,\xi}^{(h)}(q)$. From (2), we have

$$B_{n,\xi}^{(h)}(z,q) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} B_{k,\xi}^{(h)}(q).$$

The twisted $(h,q)$-Bernoulli numbers and polynomials of order $v$ are given in [1,10,12]) by their generating functions:

$$\begin{aligned}\mathcal{F}_{\xi,q}^{(v)}(t \mid \vec{a}) &= (\log q^h + t)^v \prod_{j=1}^{v} \frac{a_j}{(\xi q^{he^t})^{a_j}-1} \\ &= \sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}(q \mid \vec{a}) \frac{t^n}{n!},\end{aligned} \tag{3}$$

and

$$\mathcal{F}_{\xi,q}^{(v)}(t,x \mid \vec{a}) = \mathcal{F}_{\xi,q}^{(v)}(t \mid \vec{a}) e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}(x,q \mid \vec{a}) \frac{t^n}{n!} \tag{4}$$

where $\vec{a} = (a_1, ..., a_v)$.

By using (4), we obtain

$$\mathcal{B}_{n,\xi}^{(h,v)}(z,q \mid \vec{a}) = \sum_{j=0}^{n} \binom{n}{j} z^{n-j} \mathcal{B}_{j,\xi}^{(h,v)}(q \mid \vec{a}).$$

Substituting $\vec{a} = (a, ..., a)$ (4), we see that

$$\begin{aligned}\mathcal{F}_{\xi,q}^{(v)}(t,x \mid \vec{a}) &= a^v (\log q^h + t)^v \left( \frac{1}{(\xi q^h e^t)^a - 1} \right)^v e^{tx} \\ &= \sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}(x,q \mid \vec{a}) \frac{t^n}{n!}.\end{aligned}$$

From the above, we arrive at the equality

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}(x,q \mid \vec{a}) \frac{t^n}{n!} = (-a)^v (\log q^h + t)^v \sum_{j=0}^{\infty} \binom{s+j-1}{j} (\xi q^h)^{aj} e^{t(x+aj)}. \tag{5}$$

From [10,12], we quote the distribution formula.

**Theorem 1.** *[Distribution relation]. Let $v$, $n$, $j \in \mathbb{N}$. We have*

$$\mathcal{B}_{n,\xi}^{(h,v)}(z,q \mid \vec{a}) = j^{n-1} \sum_{b_1,\cdots,b_v=0}^{j-1} (\xi q^h)^{\vec{a}.\vec{b}} \mathcal{B}_{n,\xi}^{(h,v)}\left( \frac{\vec{a}.\vec{b}+z}{j}, q \mid \vec{a} \right)$$

*where $\vec{a}.\vec{b} = \sum_{i=1}^{v} a_i b_i$, $\vec{a} = (a_1, \cdots, a_v)$, $\vec{b} = (b_1, \cdots, b_v)$.*

It is well-known that the distribution relation is useful for the construction of distribution on the ring of $p$-adic integers $\mathbb{Z}_p$. For more details, see [8], Chap. II. On the other hand, the above Theorem 1 and the results of Chapter II of Koblitz's book [8] illustrate that the twisted $(h,q)$-Bernoulli polynomials are $p$-adic in essence and have profound connections with the special values of certain zeta functions.

In our paper, we will construct these zeta functions, and their study will be further detailed in Sections 2 and 3.

Let us now specify the definition of the twisted $(h,q)$-Bernoulli numbers of order $v$ by using the following generating function:

$$\begin{aligned}\mathcal{F}_{\xi,q}^{(h,v)}(t \mid \vec{a}) &= (\log q^h + t)^v \prod_{j=1}^{v} \frac{a_j}{(\xi q^h)^{a_j} e^{a_j t} - 1} \\ &= \sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}(q \mid \vec{a}) \frac{t^n}{n!},\end{aligned} \tag{6}$$

where $\vec{a} = (a_1, \cdots, a_v)$ and $\mid t + \log(\xi q^h) \mid < \min\left\{\mid \frac{2\pi}{a_1} \mid, \cdots, \mid \frac{2\pi}{a_v} \mid\right\}$ (cf. [10,12]).

By using (1) and (6), we are now ready to give the relation between the numbers $\mathcal{B}_{n,\xi}^{(h,v)}(q \mid \vec{a})$ and $B_{n,\xi}^{(h)}(q)$ as follows. From (6), we obtain

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}(q \mid \vec{a}) \frac{t^n}{n!} = \prod_{j=1}^{v} \frac{\log q^{a_j h} + a_j t}{(\xi q^h)^{a_j} e^{a_j t} - 1},$$

By substituting (1) into the above, we find that

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}(q \mid \vec{a}) \frac{t^n}{n!} = \prod_{j=1}^{v} \sum_{n=0}^{\infty} B_{n,\xi^{a_j}}^{(h)}(q^{a_j}) \frac{a_j^n t^n}{n!}. \tag{7}$$

By substituting $v = 1$ into (7), we have

$$\mathcal{B}_{n,\xi}^{(h,1)}(q \mid a_1) = a_1 B_{n,\xi^{a_1}}^{(h)}(q^{a_1}).$$

Setting $v = 2$ and $\vec{a} = (a_1, a_2)$ in (7), we have

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,2)}(q \mid (a_1, a_2)) \frac{t^n}{n!} = \sum_{j=0}^{\infty} B_{j,\xi^{a_1}}^{(h)}(q^{a_1}) \frac{a_1^j t^j}{j!} \sum_{l=0}^{\infty} B_{l,\xi^{a_2}}^{(h)}(q^{a_2}) \frac{a_2^l t^l}{l!}.$$

By using the Cauchy product in the above, we obtain

$$\mathcal{B}_{n,\xi}^{(h,2)}(q \mid (a_1, a_2)) = \left({}^1B_{j,\xi^{a_1}}^{(h)}(q^{a_1})a_1 +{}^2 B_{l,\xi^{a_2}}^{(h)}(q^{a_2})a_2\right)^n, \tag{8}$$

where $\left({}^iB_{\xi^a}^{(h)}(q^a)\right)^j = B_{j,\xi^a}^{(h)}(q^a)$ and $\left({}^iB_{\xi^a}^{(h)}(q^a)\right)^j \left({}^lB_{\xi^a}^{(h)}(q^a)\right)^d \neq B_{j+d,\xi^a}^{(h)}(q^a)$ if $i \neq l$.

By (6), using geometric series, we find that

$$\begin{aligned}\mathcal{F}_{\xi,q}^{(h,v)}(t \mid \vec{a}) &= (-1)^v (\log q^h + t)^v \prod_{j=1}^{v} a_j \sum_{y_1,\cdots,y_v=0}^{\infty} (\xi q^h e^t)^{a_1 y_1 + \cdots + a_v y_v} \\ &= \sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(q \mid \vec{a}\right) \frac{t^n}{n!}.\end{aligned} \tag{9}$$

For $\vec{a} = (a, \cdots, a)$, Relation (6) is reduced to

$$\mathcal{F}_{\xi,q}^{(h,v)}(t \mid \vec{a}) = (\log q^h + t)^v a^v \left(\frac{1}{(\xi q^h)^a e^{at} - 1}\right)^v$$

where $\mid t + \log(\xi q^h) \mid < \mid \frac{2\pi}{a} \mid$.

Observe that if $\vec{a} = (1, \cdots, 1)$, then (6) reduces to

$$F_{\xi,q}^{(h,v)}(t \mid (1, \cdots, 1)) = \left(\frac{\log q^h + t}{\xi q^h e^t - 1}\right)^v = \sum_{n=0}^{\infty} B_{n,\xi}^{(h,v)}(q) \frac{t^n}{n!} \tag{10}$$

which is studied in [1].

Without loss of generality and also for the simplicity of the calculations, in this section, we treat in detail only the case where $\vec{a} = (a, ..., a)$. Here, our method can be extended to the general case.

By replacing $x$ by $xy$ in (4) with $y \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(xy, q \mid \vec{a}\right)\frac{t^n}{n!} = \frac{\left(\log q^{ha} + at\right)^v e^{xyt}}{((\xi q^h)^a e^{at} - 1)^v}.$$

After some calculations, we obtain

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(xy, q \mid \vec{a}\right)\frac{t^n}{n!} = \frac{(\log q^{ha} + at)^v e^{xyt}}{((\xi q^h)^{ay} e^{ayt} - 1)^v}\left(\sum_{j=0}^{y-1} (\xi q^h e^t)^{ja}\right)^v.$$

From the above, we find that

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(xy, q \mid \vec{a}\right)\frac{t^n}{n!}$$

is equal to

$$\frac{(\log q^{ha} + at)^v}{((\xi q^h)^{ay} e^{ayt} - 1)^v}\left(1 + (\xi q^h e^t)^a + (\xi q^h e^t)^{2a} + \cdots + (\xi q^h e^t)^{a(y-1)}\right)^v.$$

Thus, we have

$$\frac{((\xi q^h e^t)^{ay} - 1)^v}{(\log q^{ha} + at)^v}\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(xy, q \mid \vec{a}\right)\frac{t^n}{n!} =$$

$$\sum_{\substack{b_1, \cdots, b_{y-1} \geq 0 \\ b_1 + \cdots + b_{y-1} = v}} \binom{v}{b_1, \cdots, b_{y-1}} (\xi^a q^{ha})^{b_1 + 2b_2 + \cdots + (y-1)b_{y-1}} e^{\left(x + \frac{b_1 + 2b_2 + \cdots + (y-1)b_{y-1}}{y}\right)yt}.$$

By using (4), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(xy, q \mid \vec{a}\right)\frac{t^n}{n!} \\
&= \sum_{\substack{b_1, \cdots, b_{y-1} \geq 0 \\ b_1 + \cdots + b_{y-1} = v}} \binom{v}{b_1, \cdots, b_{y-1}} \frac{a^v \left(\xi^a q^{ha}\right)^{\sum_{k=1}^{y-1} kb_k} (\log q^h + t)^v e^{\left(x + \frac{\sum_{k=1}^{y-1} kb_k}{y}\right)yt}}{((\xi q^h)^{ay} e^{ayt} - 1)^v} \\
&= \sum_{n=0}^{\infty} y^n \sum_{\substack{b_1, \cdots, b_{y-1} \geq 0 \\ b_1 + \cdots + b_{y-1} = v}} \binom{v}{b_1, \cdots, b_{y-1}} (\xi^a q^{ha})^{e \sum_{k=1}^{y-1} kb_k} \mathcal{B}_{n,\xi^y}^{(h,v)}\left(x + \frac{\sum_{k=1}^{y-1} kb_k}{y}, q^y \mid \vec{a}\, t\right)\frac{t^n}{n!}.
\end{aligned}$$

By identifying the coefficients of $\frac{t^n}{n!}$ in the above formula, we obtain the $q$-Raabe multiplication formula for the polynomials $B_{n,\xi}^{(h,v)}(x, q \mid \vec{a})$.

**Proposition 1** ([10,12]). *Let $\overrightarrow{a}=(a,\cdots,a)$, $v,y\in\mathbb{N}$, $q\in\mathbb{C}$ with $|\,q\,|<1$ and $\xi^r=1$ with $\xi\neq 1$. Then, we have*

$$\begin{aligned}\mathcal{B}_{n,\xi}^{(h,v)}\left(xy,q\mid\overrightarrow{a}\right) &= y^n\sum_{\substack{b_1,\ldots,b_{y-1}\geq 0\\ b_1+\ldots+b_{y-1}=v}}\binom{v}{b_1,\cdots,b_{y-1}}\left(\xi q^h\right)^{a\sum_{k=1}^{y-1}kb_k}\\ &\quad\times\mathcal{B}_{n,\xi^y}^{(h,v)}\left(x+\frac{b_1+2b_2+3b_3+\cdots+(y-1)b_{y-1}}{y},q^y\mid\overrightarrow{a}\right).\end{aligned}$$

We can rewrite $\mathcal{F}_{\xi,q}^{(h,v)}(t\mid\overrightarrow{a})$, defined in (9), as follows:

$$\begin{aligned}\mathcal{F}_{\xi,q}^{(h,v)}(t\mid\overrightarrow{a}) &= (-1)^v\prod_{j=1}^{v}a_j\sum_{k=0}^{v}\binom{v}{k}(\log q^h)^k t^{v-k}\prod_{j=1}^{v}\frac{1}{(\xi q^h)^{a_j}e^{a_jt}-1} \qquad (11)\\ &= \sum_{n=0}^{\infty}\mathcal{B}_{n,\xi}^{(h,v)}(q\mid\overrightarrow{a})\frac{t^n}{n!}.\end{aligned}$$

Now, we give some identities related to our twisted Bernoulli polynomials of higher order. Let

$$\mathcal{F}_{\xi,q}^{(h,v)}(t,x\mid\overrightarrow{a})=(\log q+t)^v\frac{a^ve^{xt}}{((\xi qe^t)^a-1)^v}=\sum_{n=0}^{\infty}\mathcal{B}_{n,\xi}^{(v,h)}\left(x,q\mid\overrightarrow{a}\right)\frac{t^n}{n!},$$

where $\overrightarrow{a}=(a,\cdots,a)$. By the well-known identity

$$\frac{1}{(\xi q^h)^ae^{at}-1}=\frac{1+(\xi q^h)^ae^{at}+\cdots+(\xi q^h)^{a(m-1)}e^{a(m-1)t}}{(\xi q^h)^{am}e^{amt}-1},$$

we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{B}_{n,\xi}^{(h,v)}(mx,q\mid\overrightarrow{a})\frac{t^n}{n!}=\left(\log q^{ah}+at\right)^v\frac{e^{mxt}}{((\xi q^h)^ae^{at}-1)^v}\\ =&\left(\log q^{ah}+at\right)^ve^{mxt}\left(\sum_{j=0}^{m-1}(\xi^aq^{ah})^je^{ajt}\right)^v\left(\frac{1}{(\xi q^h)^{am}e^{amt}-1}\right)^v\\ =&\left(\log q^{ah}+at\right)^ve^{mxt}\left(\frac{1}{\xi^{am}q^{ahm}e^{amt}-1}\right)^v\\ &\times\left(1+\xi^aq^{ah}e^{at}+\cdots+\xi^{a(m-1)}q^{ah(m-1)}e^{a(m-1)t}\right)\\ =&\left(\log q^{ah}+at\right)^ve^{mxt}\left(\frac{1}{\xi^{ma}q^{ahm}e^{mat}-1}\right)^v\\ &\times\sum_{\alpha_1,..,\alpha_{m-1}\geq 0}\binom{v}{\alpha_1,\cdots,\alpha_{m-1}}(\xi^aq^{ah})^{\alpha_1}e^{\alpha_1at}(\xi^aq^{ah})^{2\alpha_2}e^{2\alpha_2at}\cdots\\ &\times(\xi^aq^{ah})^{\alpha_{m-1}a(m-1)}e^{\alpha_{m-1}a(m-1)t}\\ =&\left(\frac{\log q^{ah}+at}{(\xi q^h)^{ma}e^{mat}-1}\right)^ve^{mxt}\sum_{\alpha_1,..,\alpha_{m-1}\geq 0}\binom{v}{\alpha_1,\cdots,\alpha_{m-1}}\\ &\times(\xi^aq^{ah})^{\alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1}}e^{at(\alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1})}.\end{aligned}$$

From the above, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(mx, q \mid \overrightarrow{a}\right)\frac{t^n}{n!} \\
=\;& \frac{m^{-v}}{a^v}\left(\frac{\log q^{ahm}+atm}{(\xi q)^{am}e^{amt}-1}\right)^{v} \sum_{\alpha_1,\cdots,\alpha_{m-1}\geq 0}\binom{v}{\alpha_1,\cdots,\alpha_{m-1}} \\
&\times(\xi^a q^{ah})^{\alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1}} e^{atm\left(\frac{x}{a}+\frac{\alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1}}{m}\right)} \\
=\;& (am)^{-v}\sum_{\alpha_1,..,\alpha_{m-1}\geq 0}\binom{v}{\alpha_1,\cdots,\alpha_{m-1}}(\xi^a q^{ah})^{\alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1}} \\
&\times\sum_{n=0}^{\infty}\mathcal{B}_{n,\xi^{ma}}^{(h,\overrightarrow{a})}\left(\frac{x}{a}+\frac{\alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1}}{m}, q^{am}\right)\frac{a^n t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients of $t^n$ on both sides of the above, we arrive at the following result.

**Proposition 2.** *Let $\overrightarrow{a}=(a,\cdots,a)$, $v$, $m\in\mathbb{N}$, $q\in\mathbb{C}$ with $\mid q\mid<1$ and $\xi^r=1$ with $\xi\neq 1$. Then, we have*

$$
\begin{aligned}
\mathcal{B}_{n,\xi}^{(h,v)}\left(mx,q\mid\overrightarrow{a}\right) &= (ma)^{n-v}\sum_{\substack{\alpha_1,..,\alpha_{m-1}\geq 0\\ \alpha_1+..+\alpha_{m-1}=v}}\binom{v}{\alpha_1,\cdots,\alpha_{m-1}} \\
&\quad\times(\xi q^h)^{ay}\mathcal{B}_{n,\xi^{ma}}^{(h,v)}\left(\frac{x}{a}+\frac{y}{m},q^{ma}\mid\overrightarrow{a}\right),
\end{aligned}
$$

*where $y=\alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1}$.*

**Remark 1.** *We give some particular cases with $\overrightarrow{a}=(a,\cdots,a)$.*

- *Taking $q\to 1$, by Proposition 2, we obtain*

$$
B_{n,\xi}^{(h,v)}(mx\mid\overrightarrow{a})=(am)^{n-v}\sum_{\substack{\alpha_1,..,\alpha_{m-1}\geq 0\\ \alpha_1+..+\alpha_{m-1}=v}}\binom{v}{\alpha_1,..,\alpha_{m-1}}\xi^{ay}B_{n,\xi^{ma}}^{(h,v)}\left(\frac{x}{a}+\frac{y}{m}\mid\overrightarrow{a}\right).
$$

- *In addition, if $\xi=1$ in the above, we deduce the well-known Carlitz's multiplication formula [17]:*

$$
B_n^{(v)}(mx)=m^{n-v}\sum_{\substack{\alpha_1,\cdots,\alpha_{m-1}\geq 0\\ \alpha_1+\cdots+\alpha_{m-1}=v}}\binom{v}{\alpha_1,\cdots,\alpha_{m-1}}B_n^{(v)}\left(\frac{x}{a}+\frac{y}{m}\right).
$$

- *If $v=1$ in Proposition 2, then the Raabe formula for the usual Bernoulli polynomials is given by*

$$
B_n(mx)=m^{n-1}\sum_{j=0}^{m-1}B_n\left(x+\frac{j}{m}\right). \tag{12}
$$

- *In [23], Kim investigated several properties of symmetry for the p-adic invariant integral on $\mathbb{Z}_p$. By using symmetry for the p-adic invariant integral on $\mathbb{Z}_p$, Kim proved (12). If $a=1$ and $\xi=1$, $q^a\to\lambda$, then we arrive at Luo's formula [26]:*

$$
B_n^{(v)}(mx,\lambda)=m^{n-v}\sum_{\substack{\alpha_1,\cdots,\alpha_{m-1}\geq 0\\ \alpha_1+\cdots+\alpha_{m-1}=v}}\binom{v}{\alpha_1,\cdots,\alpha_{m-1}}\lambda^y B_n^{(v)}\left(x+\frac{y}{m},\lambda^m\right).
$$

By using (12), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(mx, q \mid \overrightarrow{a}\right)\frac{t^n}{n!} \\
= \;& (\log q^{ahm} + atm)^v \frac{m^{-v}}{a^v} \frac{e^{mxt}}{((\xi q^h)^{am} e^{amt} - 1)^v} \sum_{\alpha_1,..,\alpha_{m-1} \geq 0} \binom{v}{\alpha_1, \cdots, \alpha_{m-1}} (\xi^a q^{ah})^y e^{aty}.
\end{aligned}$$

From the above, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{B}_{n,\xi}^{(h,v)}\left(mx, q \mid \overrightarrow{a}\right)\frac{t^n}{n!} \\
= \;& (am)^{-v} \sum_{\alpha_1,..,\alpha_{m-1} \geq 0} \binom{v}{\alpha_1, \cdots, \alpha_{m-1}} (\xi q^h)^{ay} \sum_{n=0}^{\infty} \mathcal{B}_{n,\xi^{ma}}^{(h,v)}\left(q^{ma} \mid \overrightarrow{a}\right)\frac{t^n (am)^n}{n!} \\
& \times \sum_{n=0}^{\infty} \left(\frac{x}{a} + \frac{y}{m}\right)\frac{(amt)^n}{n!} \\
= \;& (am)^{-v+n} \sum_{\alpha_1,..,\alpha_{m-1} \geq 0} \binom{v}{\alpha_1, \cdots, \alpha_{m-1}} (\xi q^h)^{ay} \\
& \times \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{\left(\frac{x}{a} + \frac{y}{m}\right)^{n-j}}{(n-j)!} \mathcal{B}_{n,\xi^{ma}}^{(h,v)}\left(q^{ma} \mid \overrightarrow{a}\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus, we arrive at the following result.

**Corollary 1.** *Let $v$, $m \in \mathbb{N}$, $q \in \mathbb{C}$ with $\mid q \mid < 1$ and $\xi^r = 1$ with $\xi \neq 1$. We have*

$$\begin{aligned}
\frac{\mathcal{B}_{n,\xi}^{(h,v)}\left(mx, q \mid \overrightarrow{a}\right)}{(am)^{n-v}} = & \sum_{\substack{\alpha_1, \cdots, \alpha_{m-1} \geq 0 \\ \alpha_1 + \cdots + \alpha_{m-1} = v}} \sum_{j=0}^{n} \binom{n}{j} \binom{v}{\alpha_1, \cdots, \alpha_{m-1}} \\
& \times (\xi q^h)^{ay} \left(\frac{x}{a} + \frac{y}{m}\right)^{n-j} \mathcal{B}_{n,\xi^{ma}}^{(h,v)}\left(q^{ma} \mid \overrightarrow{a}\right),
\end{aligned}$$

*where $y = \alpha_1 + 2\alpha_2 + \cdots + (m-1)\alpha_{m-1}$.*

For $m = 1$, $v = 1$, from Corollary 1, we obtain

$$\mathcal{B}_{n,\xi}^{(h,1)}(x, q) = B_{n,\xi}^{(h)}(x, q) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} B_{n,\xi}^{(h)}(q),$$

(cf. [9]).

**Corollary 2.** *Let $v \in \mathbb{N}$, $q \in \mathbb{C}$ with $\mid q \mid < 1$ and $\xi^r = 1$ with $\xi \neq 1$. Then, we have the reduction formula*

$$B_{n,\xi}^{(h,v)}(x, q) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k,\xi}^{(h,v-1)}(q) B_{k,\xi}^{(h)}(x, q),\ n \in \mathbb{N}_0$$

*where $B_{n,\xi}^{(h,v)}(x, q) := \mathcal{B}_{n,\xi}^{(h,v)}\left(x, q \mid \overrightarrow{a}\right)$ and $B_{n,\xi}^{(h,v)}(q) := \mathcal{B}_{n,\xi}^{(h,v)}(0, q \mid \overrightarrow{a})$; $\overrightarrow{a} = (1, \cdots, 1)$ holds true between the twisted $(h, q)$-Bernoulli polynomials $B_{n,\xi}^{(h)}(x, q)$ and the twisted $(h, q)$-Bernoulli numbers $B_{n,\xi}^{(h,v-1)}(q)$ of order $(h, v-1)$.*

**Proof.** By using (10), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\xi}^{(h,v)}(x,q)\frac{t^n}{n!} &= \left(\frac{\log q^h + t}{q^h\xi e^t - 1}\right)^{v-1}\left(\frac{\log q^h + t}{q^h\xi e^t - 1}\right)e^{xt} \\
&= \sum_{n=0}^{\infty} B_{n,\xi}^{(h,v-1)}(q)\frac{z^n}{n!}\sum_{n=0}^{\infty} B_{n,\xi}^{(h)}(z,q)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{k,\xi}^{(h)}(x,q)\frac{t^k}{k!}B_{n-k,\xi}^{(h,v-1)}(q)\frac{t^{n-k}}{(n-k)!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} B_{n-k,\xi}^{(h,v-1)}(q)B_{k,\xi}^{(h)}(x,q)\binom{n}{k}\right)\frac{t^n}{n!}
\end{aligned}$$

here, we use the Cauchy product. By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we then arrive at the desired result. □

## 2. Twisted Barnes' Type $(h,q)$-Zeta Functions

In this section, we construct interpolation functions of the twisted $(h,q)$-Bernoulli polynomials and numbers of higher order. We also give some interesting identities related to these functions. Throughout this section, we study the complex $s$-plane. Let $q \in \mathbb{C}$ with $\mid q \mid < 1$ and $\xi^r = 1$ ($\xi$ is an $r$th root of 1) $\xi \neq 1$.

By (1) and (2), we define the following functions:

$$\begin{aligned}
\mathfrak{F}_{1,\xi,q}^{(h)}(t) &= \frac{t}{\xi q^h e^t - 1} \\
&= \sum_{n=0}^{\infty} b_{n,\xi}^{(h)}(q)\frac{t^n}{n!}.
\end{aligned} \tag{13}$$

Note that the numbers $b_{n,\xi}^{(h)}(q)$ are related to the so-called Apostol-Bernoulli numbers [26,27] and twisted $(h,q)$ Bernoulli numbers.

We decompose (2) as follows:

$$\begin{aligned}
F_{\xi,q}^{(h)}(t) &= \frac{\log q^h}{t}\mathfrak{F}_{1,\xi,q}^{(h)}(t) + \mathfrak{F}_{1,\xi,q}^{(h)}(t) \\
&= \sum_{n=0}^{\infty} B_{n,\xi}^{(h)}(q)\frac{t^n}{n!}.
\end{aligned}$$

From the above and the relation (13), we find that

$$B_{n,\xi}^{(h)}(q) = \left(\log q^{-(n+1)h}\right)b_{n+1,\xi}^{(h)}(q) + b_{n,\xi}^{(h)}(q).$$

Let us define

$$\mathrm{F}_{\xi,q}^{(h)}(t,z) = \mathfrak{F}_{1,\xi,q}^{(h)}(t)e^{tz} = \sum_{n=0}^{\infty} b_{n,\xi}^{(h)}(z,q)\frac{t^n}{n!} \tag{14}$$

observe that $b_{n,\xi}^{(h)}(0,q) = b_{n,\xi}^{(h)}(q)$.

For $0 < z \leq 1$, by applying the Mellin transformation to the above equation, we obtain

$$\begin{aligned}
\mathcal{G}_{q,\xi}^{(h)}(s,z) &= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-2}\mathrm{F}_{\xi,q}^{(h)}(-t,z)dt \\
&= \sum_{n=0}^{\infty}\frac{\left(\xi q^h\right)^n}{(n+z)^s},
\end{aligned} \tag{15}$$

it is easy to see that the series $s \to \sum_{n=0}^{\infty} \frac{\left(\xi q^h\right)^n}{(n+z)^s}$ converge for whole complex plane, because $|q| < 1$ and $\xi^r = 1$.

We easily see that

$$\mathcal{G}_{q,\xi}^{(h)}(s,1) = \mathcal{G}_{q,\xi}^{(h)}(s) = \sum_{n=1}^{\infty} \frac{\left(\xi q^h\right)^n}{n^s}. \tag{16}$$

Observe that for $\Re(s) > 1$, we have

$$\lim_{q\to 1} \mathcal{G}_{q,1}^{(h)}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which denotes the Riemann zeta function.

The function $\mathcal{G}_{q,\xi}^{(h)}(s,z)$ interpolates the polynomials $b_{n,\xi}^{(h)}(z,q)$ at negative integers as follows:

$$\mathcal{G}_{q,\xi}^{(h)}(1-n,z) = -\frac{b_{n,\xi}^{(h)}(z,q)}{n}, \; n \in \mathbb{Z}^+. \tag{17}$$

We easily see from the above that

$$\mathcal{G}_{q,\xi}^{(h)}(1-n) = -\frac{b_{n,\xi}^{(h)}(q)}{n}, \; n \in \mathbb{Z}^+. \tag{18}$$

We now construct higher-order interpolation functions.

The Mellin transform of (11) is given by

$$\mathcal{Z}_{q,\xi}^{(h,v)}(s \mid \overrightarrow{a}) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-v-1} \mathcal{F}_{\xi,q}^{(h,v)}(-t \mid \overrightarrow{a})dt$$

and we obtain

$$\mathcal{Z}_{q,\xi}^{(h,v)}(s \mid \overrightarrow{a}) = (-1)^v \prod_{j=1}^{v} a_j \sum_{k=0}^{v} \binom{v}{k} (\log q^h)^k \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-v-1} t^{v-k} \prod_{j=1}^{v} \frac{dt}{(\xi q^h)^{a_j} e^{-a_j t} - 1}$$

Hence, from the above, we define the zeta functions $\mathcal{Z}_{q,\xi}^{(h,v)}(s, \overrightarrow{a})$.

**Definition 1.** *Let $s, q \in \mathbb{C}$ with $|q| < 1$ and $\xi^r = 1$ with $\xi \neq 1$. We define*

$$\frac{\mathcal{Z}_{q,\xi}^{(h,v)}(s \mid \overrightarrow{a})}{\prod_{j=1}^{v} a_j} = \sum_{k=0}^{v} \binom{v}{k} (\log \frac{1}{q^h})^k \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-k-1} dt}{\prod_{j=1}^{v} \left(\xi q^h\right)^{a_j} e^{-a_j t} - 1}, \tag{19}$$

*where $\Re(s) > v+1$ and $\Re(a_j) > 0, 1 \leq j \leq v$.*

In [20], the authors gave another kind of the twisted Barnes zeta function.

By substituting $v = 1$ into (19), we have

$$\mathcal{Z}_{q,\xi}^{(h,1)}(s \mid a_1) = a_1^{1-s} \left( \sum_{n=1}^{\infty} \frac{(q^{a_1} \xi^{a_1})^n}{n^s} - \frac{\log q^{ha_1}}{s-1} \sum_{n=0}^{\infty} \frac{(q^{ha_1} \xi^{a_1})^n}{n^{s-1}} \right) = a_1^{1-s} \zeta_{q,\xi}^{(h)}(s)$$

(*cf.* [9]).

From (19), we find that

$$\frac{\mathcal{Z}_{q,\xi}^{(h,v)}(s,\overrightarrow{a})}{\prod_{j=1}^{v}a_j}=\binom{v}{0}\sum_{n_1,\cdots,n_v=1}^{\infty}\frac{\left(\xi q^h\right)^{a_1n_1+\cdots+a_vn_v}}{(a_1n_1+\cdots+a_vn_v)^s}$$
$$+\frac{\binom{v}{1}\log(\frac{1}{q})}{s-1}\sum_{n_1,\cdots,n_v=1}^{\infty}\frac{\left(\xi q^h\right)^{a_1n_1+\cdots+a_vn_v}}{(a_1n_1+\cdots+a_vn_v)^{s-1}}$$
$$+\cdots+\frac{\binom{v}{v}\log(\frac{1}{q^h})^v}{(s-1)(s-2)\cdots(s-v)}\sum_{n_1,\cdots,n_v=1}^{\infty}\frac{\left(\xi q^h\right)^{a_1n_1+\cdots+a_vn_v}}{(a_1n_1+\cdots+a_vn_v)^{s-v}}.$$

From the above, we arrive at the following main theorem

**Theorem 2.** *[Explicit formula]. Let $s,q\in\mathbb{C}$, $\overrightarrow{a}=(a_1,\cdots,a_v)$, with $|q|<1$, $\Re(s)>v$, $v\in\mathbb{N}$ and $\xi^r=1$ with $\xi\neq 1$. Then, we have*

$$\mathcal{Z}_{q,\xi}^{(h,v)}(s\mid\overrightarrow{a})=\prod_{j=1}^{v}a_j\sum_{k=0}^{v}\binom{v}{k}\left(\log\frac{1}{q^h}\right)^k\zeta_{q,\xi}^{(h,v)}(s-k\mid\overrightarrow{a})\prod_{j=0}^{k}\frac{s}{s-j},\tag{20}$$

*where*

$$\zeta_{q,\xi}^{(h,v)}(s\mid\overrightarrow{a})=\sum_{n_1,\cdots,n_v=1}^{\infty}\frac{\left(\xi q^h\right)^{a_1n_1+\cdots+a_vn_v}}{(a_1n_1+\cdots+a_vn_v)^s}.\tag{21}$$

We now give some special values of the function $\zeta_{q,\xi}^{(h,v)}(s,\overrightarrow{a})$ as follows:

**Corollary 3.** *Let $\Re(s)>v$. Then, we have*

$$\zeta_{q,\xi}^{(h,v)}(s,x\mid\overrightarrow{a})=\sum_{n_1,\cdots,n_v=1}^{\infty}\frac{\left(\xi q^h\right)^{a_1n_1+\cdots+a_vn_v}}{(x+a_1n_1+\cdots+a_vn_v)^s}.$$

*(i) Difference equation of the function $\zeta_{q,\xi}^{(h,v)}(s,x\mid\overrightarrow{a})$:*

$$\zeta_{q,\xi}^{(h,v)}(s,x+1\mid\overrightarrow{a})=\zeta_{q,\xi}^{(h,v)}(s,x\mid\overrightarrow{a})-\frac{1}{x^s},\tag{22}$$

*(ii) Distribution relation of the function $\zeta_{q,\xi}^{(h,v)}(s,x\mid\overrightarrow{a})$:*

$$b^s\zeta_{q,\xi}^{(h,v)}(s,x\mid\overrightarrow{a})=\sum_{j=0}^{b-1}\left(\xi q^h\right)^j\zeta_{q^b,\xi^b}^{(h,v)}(s,x+\frac{j}{b}\mid\overrightarrow{a}).\tag{23}$$

*(iii) For real number $y$,*

$$b^s\zeta_{q,\xi}^{(h,v)}(s,\{by\}\mid\overrightarrow{a})=\sum_{j=0}^{b-1}\left(\xi q^h\right)^j\mathfrak{z}_{q^b,\xi^b}^{(h,v)}(s,\left\{y+\frac{j}{b}\right\}\mid\overrightarrow{a}),$$

*where*

$$\mathfrak{z}_{q,\xi}^{(h,v)}(s,x\mid\overrightarrow{a})=\begin{cases}\zeta_{q,\xi}^{(h,v)}(s,x\mid\overrightarrow{a})\ \text{if}\ \Re(x)>0\\[2ex] \zeta_{q,\xi}^{(h,v)}(s\mid\overrightarrow{a})\ \text{if}\ x=0.\end{cases}\tag{24}$$

We note that it is easy to prove Corollary 3 from the definition $\zeta_{q,\xi}^{(h,v)}(s,x,\overrightarrow{a})$. Observe that if $v=1$ and $a=1$, then (24) reduces to (15) and (16), that is,

$$\mathfrak{Z}_{q,\xi}^{(h,1)}(s,x\mid 1)=\begin{cases}\zeta_{q,\xi}^{(h)}(s,x) \text{ if } \Re(x)>0\\ \\ \zeta_{q,\xi}^{(h,v)}(s) \text{ if } x=0.\end{cases}$$

**Remark 2.** *We give comments on some special cases.*

- *Substituting $a_1=\cdots=a_v=1$ into (20), for $s,\xi,q\in\mathbb{C}$ with $Re(s)>v$, $\mid q\mid<1$ and $\xi^r=1$, $r\in\mathbb{Z}$, Theorem 2 recovers the results of [10] (p. 488).*
- *If $v=1$, then Theorem 2 reduces to*

$$\mathcal{Z}_{q,\xi}^{(h,1)}(s\mid\overrightarrow{a})=a_1^{1-s}\zeta_{q^{a_1},\xi^{a_1}}^{(h)}(s),$$

*where*

$$\zeta_{q,\xi}^{(h)}(s)=\sum_{n=1}^{\infty}\frac{q^{nh}\xi^n}{n^s}-\frac{\log q^h}{s-1}\sum_{n=1}^{\infty}\frac{q^{nh}\xi^n}{n^{s-1}}, \tag{25}$$

- *In [29], Simsek defined and studied the $(\xi,q)$-Hurwitz zeta function which is related to the function $\zeta_{q,\xi}^{(h)}(s)$:*

$$^*\zeta_{\xi,q}(s,z)=\frac{1-q}{\log q}\zeta_{q,\xi}^{(2)}(s,z).$$

- *In addition, if $\xi=1$, then $\zeta_{q,\xi}^{(h)}(s)$ is reduced to $\zeta_q^{(h)}(s)$ (cf. [16]).*
- *If $v=1$, and $q\to1$, then Theorem 2 reduces to the twisted zeta functions which interpolate the twisted Bernoulli numbers:*

$$\zeta_\xi(s)=\sum_{n=1}^{\infty}\frac{\xi^n}{n^s},$$

*where $\xi^r=1$, $r\in\mathbb{Z}$.*
- *The Lerch transcendent $\Phi(z,s,a)$ (cf. e.g., [32] p. 121, [21]) is the analytic continuation of the series*

$$\begin{aligned}\Phi(z,s,a)&=\frac{1}{a^s}+\frac{z}{(a+1)^s}+\frac{z}{(a+2)^s}+\cdots\\&=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s},\end{aligned}$$

*The above series converges for ($a\in\mathbb{C}\diagdown\mathbb{Z}_0^-$, $s\in\mathbb{C}$ when $|z|<1$; $\Re(s)>1$ when $|z|=1$), where*

$$\mathbb{Z}_0^-=\mathbb{Z}^-\cup\{0\},\ \mathbb{Z}^-=\{-1,-2,-3,\cdots\}.$$

*Hence, the function $\zeta_{q,\xi}^{(h)}(s)$ is related to $\Phi(z,s,a)$ as follows:*

$$\zeta_{q,\xi}^{(h)}(s)=\Phi(q^h\xi,s,1)-\frac{\log q^h}{s-1}\Phi(q^h\xi,s-1,1).$$

By substituting $s=1-n$, $n\in\mathbb{N}$ into (19) and using Cauchy's residue theorem for the Hankel contour, we obtain the following theorem:

**Theorem 3.** *[Values at negative integers]. Let $n\in\mathbb{N}$ and $\overrightarrow{a}=(a_1,\cdots,a_v)$. Then, we have*

$$\mathcal{Z}_{q,\xi}^{(h,v)}(1-n\mid\overrightarrow{a})=\frac{n!}{(n+v)!}(-1)^vB_{n+v,\xi}^{(h,v)}(q\mid\overrightarrow{a}).$$

From the above theorem, we arrive at the following corollary:

**Corollary 4.** *Let $n, v \in \mathbb{N}$. Thus, we have*

$$\begin{aligned} B_{n+v,\xi}^{(h,v)}(q \mid \overrightarrow{a}) &= \tfrac{(n+v)!}{n!} \prod_{j=1}^{v} (-a_j)^v \sum_{k=0}^{v} \binom{v}{k} \left(\log \frac{1}{q^h}\right)^k \\ &\quad \times \zeta_{q,\xi}^{(h,v)}(1-n-k \mid \overrightarrow{a}) \prod_{j=0}^{k} \frac{1-n}{1-n-j}. \end{aligned}$$

By substituting $v = 2$ into Theorem 2, we have

$$\frac{\mathcal{Z}_{q,\xi}^{(h,2)}(s \mid \overrightarrow{a})}{\prod_{j=1}^{2} a_j} = \sum_{k=0}^{2} \binom{2}{k} \left(\log \frac{1}{q^h}\right)^k \zeta_{q,\xi}^{(h,2)}(s-k \mid \overrightarrow{a}) \prod_{j=0}^{k} \frac{s}{s-j}.$$

Thus, we find that

$$\begin{aligned} \mathcal{Z}_{q,\xi}^{(h,2)}(s \mid \overrightarrow{a}) &= a_1 a_2 \zeta_{q,\xi}^{(2)}(s \mid \overrightarrow{a}) + 2a_1 a_2 \tfrac{s}{s-1} \left(\log \tfrac{1}{q^h}\right) \zeta_{q,\xi}^{(h,2)}(s-1 \mid \overrightarrow{a}) \\ &\quad + a_1 a_2 \tfrac{s^2}{(s-1)(s-2)} \left(\log \tfrac{1}{q^h}\right)^2 \zeta_{q,\xi}^{(h,2)}(s-2 \mid \overrightarrow{a}). \end{aligned}$$

## 3. Twisted $(h, q)$-Dedekind Type Sums

In this section, we define twisted $(h, q)$-Dedekind type sums. We state and prove their reciprocity law. For more details on the elliptic analogue for the Dedekind reciprocity laws, see [6].

Let us recall the Apostol–Dedekind sums $s_n(h, k)$:

$$s_n(h,k) = \sum_{a=1}^{k-1} \frac{a}{k} \overline{B}_n\left(\frac{ha}{k}\right), \tag{26}$$

where $h$ and $k$ are coprime integers with $k > 0$, $n$ is a positive integer, and $\overline{B}_n(x)$ is the $n$th Bernoulli function, which is defined as follows:

$$\overline{B}_n(x) = B_n(x - [x]_G), \text{ if } n > 1,$$

and

$$\overline{B}_1(x) = \begin{cases} B_1(x - [x]_G), & \text{if } x \notin \mathbb{Z} \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

where $B_n(x)$ is the Bernoulli polynomial.

For even values of $n$, the sums $s_n(h, k)$ are relatively uninteresting. However, for odd values of $n$, these sums have a reciprocity law, first proved by Apostol [2]:

$$\begin{aligned} &(n+1)(hk^n s_n(h,k) + kh^n s_n(k,h)) \\ &= nB_{n+1} + \sum_{j=1}^{n+1} \binom{n+1}{j} (-1)^j B_j B_{n+1-j} h^j k^{n+1-j}, \end{aligned} \tag{27}$$

where $(h, k) = 1$. If $n = 1$, then the Apostol–Dedekind sums reduce to the classical Dedekind sums.

We are ready to define the $(h, q)$-twisted Dedekind sums.

**Definition 2.** *Let $h$ and $k$ be coprime integers with $k > 0$. Then, we define*

$$S_{\xi,q,m}^{(h_1)}(h,k) = \sum_{j=1}^{k-1} \frac{j}{k} \left(\xi q^{h_1}\right)^j \overline{b}_{m,\xi}^{(h_1)}\left(\frac{hj}{k}, q\right), \tag{28}$$

*where $h_1$ is an integer number and*

$$\overline{b}_{m,\xi}^{(h_1)}(x,q) = b_{m,\xi}^{(h_1)}(x - [x]_G, q),$$

*where $b_{m,\xi}^{(h_1)}(x,q)$ is defined in (14).*

**Example 1.** *By substituting $\xi = 1$ into (28), we have*

$$S_{q,m}^{(h_1)}(h,k) = \sum_{j=1}^{k-1} \frac{j}{k} q^{h_1 j} \overline{b}_{m}^{(h_1)}\left(\frac{hj}{k}, q\right).$$

*by substituting $h_1 = 1$, $\xi = 1$ and $q \to 1$ into (28), (28) reduces to (26).*

By substituting $h = 1$ into (28) and using (13), we obtain

$$\begin{aligned} S_{\xi,q,m}^{(h_1)}(1,k) &= \sum_{j=1}^{k-1} \frac{j}{k} \left(\xi q^{h_1}\right)^j b_{m,\xi}^{(h_1)}\left(\frac{j}{k}, q\right) \\ &= \sum_{l=0}^{m} \binom{m}{l} \frac{b_{l,\xi}^{(h_1)}(q)}{k^{m-l+1}} \sum_{j=1}^{k-1} \left(\xi q^{h_1}\right)^j j^{m-l+1} \end{aligned}$$

By using the well-known alternating sums of powers of consecutive $(h,q)$-integers for $b_{m,\xi}^{(h_1)}(q)$,

$$\sum_{b=0}^{k-1} \xi^b q^{bh_1} b^{m-1} = \frac{\xi^k q^{kh_1} b_{m,\xi}^{(h_1)}(k,q) - b_{m,\xi}^{(h_1)}(0,q)}{m}$$

(cf. [10]), we then obtain the following Theorem.

**Theorem 4.** *[Explicit evaluation for $h = 1$]. Let $m$ and $k$ be positive integers. Then, we have*

$$S_{\xi,q,m}^{(h_1)}(1,k) = \sum_{l=0}^{m} \binom{m}{l} \frac{b_{l,\xi}^{(h_1)}(q)\left(\xi^k q^{h_1 k} b_{m-l+2,\xi}^{(h_1)}(k,q) - b_{m-l+2,\xi}^{(h_1)}(q)\right)}{(m-l+2)k^{m-l+1}}.$$

In [3], Ota showed how to prove Apostol's reciprocity law by using values at non-positive integers of Barnes' double zeta function. In this section, we give a generalization of Apostol's reciprocity law.

By substituting $v = 2$ and $\vec{a} = (k,h)$ into (21), and by using a method similar to that in , then we have

$$\begin{aligned} &\mathcal{Z}_{q,\xi}^{(h_1,2)}(s \mid (k,h)) = \\ &hk \sum_{\substack{n_1,n_2=0 \\ (n_1,n_2)\neq(0,0)}}^{\infty} \frac{\left(\xi q^{h_1}\right)^{kn_1+hn_2}}{(kn_1+hn_2)^s} - \frac{2hk \log q^{h_1}}{s-1} \sum_{\substack{n_1,n_2=0 \\ (n_1,n_2)\neq(0,0)}}^{\infty} \frac{\left(\xi q^{h_1}\right)^{kn_1+hn_2}}{(kn_1+hn_2)^{s-1}} \\ &+ \frac{hk\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{\substack{n_1,n_2=0 \\ (n_1,n_2)\neq(0,0)}}^{\infty} \frac{\left(\xi q^{h_1}\right)^{kn_1+hn_2}}{(kn_1+hn_2)^{s-2}}. \end{aligned}$$

Setting $n_1 = a + mh$ and $n_2 = b + nk$, where $a = 0, 1, 2, \cdots, h-1$, $b = 0, 1, 2, \cdots, k-1$ and $m, n = 0, 1, 2, \cdots, \infty$) in the above,

$$\begin{aligned}
&\mathcal{Z}_{q,\xi}^{(h_1,2)}(s \mid (h,k)) = \\
&hk \sum_{\substack{a \mod h \\ b \mod k}} \sum_{m,n=0}^{\infty *} \frac{\left(\xi q^{h_1}\right)^{ak+bh+hk(m+n)}}{(ak+bh+hk(m+n))^s} \\
&- \frac{2hk \log q^{h_1}}{s-1} \sum_{\substack{a \mod h \\ b \mod k}} \sum_{m,n=0}^{\infty *} \frac{\left(\xi q^{h_1}\right)^{ak+bh+hk(m+n)}}{(ak+bh+hk(m+n))^{s-1}} \\
&+ \frac{hk\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{\substack{a \mod h \\ b \mod k}} \sum_{m,n=0}^{\infty *} \frac{\left(\xi q^{h_1}\right)^{ak+bh+hk(m+n)}}{(ak+bh+hk(m+n))^{s-2}}.
\end{aligned}$$

where $\sum^*$ means that the above sum take pairs on non-negative integers $(m, n)$ with the exception of $(0, 0)$ when $ak + bh = 0$. Putting $m + n = j$ in the above, we then obtain

$$\begin{aligned}
&\mathcal{Z}_{q,\xi}^{(h_1,2)}(s \mid (h,k)) = \\
&hk \sum_{\substack{a \mod h \\ b \mod k}} \sum_{j=0}^{\infty *} \frac{(j+1)\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^s} \\
&- \frac{2hk \log q^{h_1}}{s-1} \sum_{\substack{a \mod h \\ b \mod k}} \sum_{j=0}^{\infty *} \frac{(j+1)\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^{s-1}} \\
&+ \frac{hk\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{\substack{a \mod h \\ b \mod k}} \sum_{j=0}^{\infty *} \frac{(j+1)\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^{s-2}}.
\end{aligned}$$

After some elementary calculations in the above, we easily see that

$$\begin{aligned}
&\zeta_{q,\xi}^{(h_1,2)}(s \mid (h,k)) = \\
&\sum_{\substack{a \mod h \\ b \mod k}} \sum_{j=0}^{\infty *} \frac{(ak+bh+hkj)\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^s} \\
&+ hk \sum_{\substack{a \mod h \\ b \mod k}} \left(1 - \frac{ak+bh}{hk}\right) \sum_{j=0}^{\infty *} \frac{\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^s} \\
&- \frac{2 \log q^{h_1}}{s-1} \sum_{\substack{a \mod h \\ b \mod k}} \sum_{j=0}^{\infty *} \frac{(ak+bh+hkj)\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^{s-1}} \\
&- \frac{2hk \log q^{h_1}}{s-1} \sum_{\substack{a \mod h \\ b \mod k}} \left(1 - \frac{ak+bh}{hk}\right) \sum_{j=0}^{\infty *} \frac{\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^{s-1}} \\
&+ \frac{\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{\substack{a \mod h \\ b \mod k}} \sum_{j=0}^{\infty *} \frac{(ak+bh+hkj)\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^{s-2}} \\
&+ \frac{hk\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{\substack{a \mod h \\ b \mod k}} \left(1 - \frac{ak+bh}{hk}\right) \sum_{j=0}^{\infty *} \frac{\left(\xi q^{h_1}\right)^{ak+bh+hkj}}{(ak+bh+hkj)^{s-2}}.
\end{aligned}$$

By using (22) in the above, we find that it is equal to

$$
\begin{aligned}
&\mathcal{Z}_{q,\xi}^{(h_1,2)}(s \mid (h,k)) = \\
&(hk)^{1-s} \sum_{\substack{a \bmod h \\ b \bmod k}} \left(\xi q^{h_1}\right)^{ak+bh} \mathfrak{z}_{q,\xi}\left(s-1, \left\{\frac{ak+bh}{hk}\right\}\right) \\
&-(hk)^{1-s} \sum_{(a,b)\in\mathcal{B}} \left(\xi q^{h_1}\right)^{ak+bh} \left(\frac{ak+bh}{hk}-1\right)^{1-s} \\
&+(hk)^{1-s} \sum_{\substack{a \bmod h \\ b \bmod k}} \left(\xi q^{h_1}\right)^{ak+bh} \left(1-\frac{ak+bh}{hk}\right) \mathfrak{z}_{q,\xi}\left(s, \left\{\frac{ak+bh}{hk}\right\}\right) \\
&-(hk)^{1-s} \sum_{(a,b)\in\mathcal{B}} \left(\xi q^{h_1}\right)^{ak+bh} \left(1-\frac{ak+bh}{hk}\right) \left(\frac{ak+bh}{hk}-1\right)^{-s} \\
&-\frac{2(hk)^{2-s}\log q^{h_1}}{s-1} \sum_{\substack{a \bmod h \\ b \bmod k}} \left(\xi q^{h_1}\right)^{ak+bh} \mathfrak{z}_{q,\xi}\left(s-2, \left\{\frac{ak+bh}{hk}\right\}\right) \\
&+\frac{2(hk)^{2-s}\log q^{h_1}}{s-1} \sum_{(a,b)\in\mathcal{B}} \left(\xi q^{h_1}\right)^{ak+bh} \left(\frac{ak+bh}{hk}-1\right)^{2-s} \\
&-\frac{2(hk)^{2-s}\log q^{h_1}}{s-1} \sum_{\substack{a \bmod h \\ b \bmod k}} \left(\xi q^{h_1}\right)^{ak+bh} \left(1-\frac{ak+bh}{hk}\right) \mathfrak{z}_{q,\xi}\left(s-1, \left\{\frac{ak+bh}{hk}\right\}\right) \\
&+\frac{2(hk)^{2-s}\log q^{h_1}}{s-1} \sum_{(a,b)\in\mathcal{B}} \left(\xi q^{h_1}\right)^{ak+bh} \left(1-\frac{ak+bh}{hk}\right) \left(\frac{ak+bh}{hk}-1\right)^{1-s} \\
&+\frac{(hk)^{3-s}\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{\substack{a \bmod h \\ b \bmod k}} \left(\xi q^{h_1}\right)^{ak+bh} \mathfrak{z}_{q,\xi}\left(s-3, \left\{\frac{ak+bh}{hk}\right\}\right) \\
&-\frac{(hk)^{3-s}\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{(a,b)\in\mathcal{B}} \left(\xi q^{h_1}\right)^{ak+bh} \left(\frac{ak+bh}{hk}-1\right)^{3-s} \\
&+\frac{(hk)^{3-s}\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{\substack{a \bmod h \\ b \bmod k}} \left(1-\frac{ak+bh}{hk}\right) \left(\xi q^{h_1}\right)^{ak+bh} \mathfrak{z}_{q,\xi}\left(s-2, \frac{ak+bh}{hk}\right) \\
&-\frac{(hk)^{3-s}\left(\log q^{h_1}\right)^2}{(s-1)(s-2)} \sum_{(a,b)\in\mathcal{B}} \left(\xi q^{h_1}\right)^{ak+bh} \left(1-\frac{ak+bh}{hk}\right) \left(\frac{ak+bh}{hk}-1\right)^{2-s},
\end{aligned}
$$

where

$$\mathcal{B} = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : 0 \le a \le h-1, 0 \le b \le k-1, ka+hb \ge kh\}.$$

Since $\{ka+hb : 0 \le a \le h-1, 0 \le b \le k-1, a, b \in \mathbb{Z}\}$ is a complete set of representatives modulo $hk$, by using (23), (15), and (16), we easily obtain

$$
\begin{aligned}
&\mathcal{Z}_{q,\xi}^{(h_1,2)}(s \mid (h,k)) = \\
&\left(\mathcal{G}_{q,\xi}^{(h_1)}(s-1) + hk\mathcal{G}_{q,\xi}^{(h_1)}(s)\right) - \tfrac{2\log q^{h_1}}{s-1}\left(\mathcal{G}_{q,\xi}^{(h_1)}(s-2) + hk\mathcal{G}_{q,\xi}^{(h_1)}(s-1)\right) \\
&+\tfrac{\left(\log q^{h_1}\right)^2}{(s-1)(s-2)}\left(\mathcal{G}_{q,\xi}^{(h_1)}(s-3) + hk\mathcal{G}_{q,\xi}^{(h_1)}(s-2)\right) \\
&-kh^{1-s}\sum_{a \mod h}\left(\xi q^{h_1}\right)^{ak}\mathcal{G}_{q,\xi}^{(h_1)}\left(s, \left\{\frac{ak}{h}\right\}\right) \\
&-hk^{1-s}\sum_{b \mod k}\left(\xi q^{h_1}\right)^{bh1}\mathcal{G}_{q,\xi}^{(h_1)}\left(s, \left\{\frac{bh}{k}\right\}\right) \\
&+\tfrac{2\log q^{h_1}}{s-1}kh^{2-s}\sum_{a \mod h}\left(\xi q^{h_1}\right)^{ak1}\mathcal{G}_{q,\xi}^{(h_1)}\left(s-1, \left\{\frac{ak}{h}\right\}\right) \\
&+\tfrac{2\log q^{h_1}}{s-1}hk^{2-s}\sum_{b \mod k}\left(\xi q^{h_1}\right)^{bh1}\mathcal{G}_{q,\xi}^{(h_1)}\left(s-1, \left\{\frac{bh}{k}\right\}\right) \\
&+\tfrac{\left(\log q^{h_1}\right)^2}{(s-1)(s-2)}kh^{3-s}\sum_{a \mod h}\left(\xi q^{h_1}\right)^{ak}\mathcal{G}_{q,\xi}^{(h_1)}\left(s-2, \left\{\frac{ak}{h}\right\}\right) \\
&+\tfrac{\left(\log q^{h_1}\right)^2}{(s-1)(s-2)}hk^{3-s}\sum_{b \mod k}\left(\xi q^{h_1}\right)^{bh}\mathcal{G}_{q,\xi}^{(h_1)}\left(s-2, \left\{\frac{bh}{k}\right\}\right).
\end{aligned}
$$

By using the above, we arrive at the following theorem.

**Theorem 5.** *Let $s, q \in \mathbb{C}$, with $\Re(s) > 4$ and $|q| < 1$ and $|\xi| < 1$. Then, we have*

$$
\begin{aligned}
&\mathcal{Z}_{q,\xi}^{(h_1,2)}(s \mid (h,k)) = \\
&\zeta_{q,\xi}^{(h_1)}(s-1) + hk\zeta_{q,\xi}^{(h)}(s) - \tfrac{\log q^{h_1}}{s-1}\zeta_{q,\xi}^{(h_1)}(s-2) + \tfrac{hk\log q^{h_1}}{s-1}\zeta_{q,\xi}^{(h_1)}(s-1) \\
&-kh^{1-s}\sum_{a \mod h}\left(\xi q^{h_1}\right)^{ak}\mathcal{G}_{q,\xi}^{(h_1)}\left(s, \left\{\frac{ak}{h}\right\}\right) \\
&-hk^{1-s}\sum_{b \mod k}\left(\xi q^{h_1}\right)^{bh}\mathcal{G}_{q,\xi}^{(h_1)}\left(s, \left\{\frac{bh}{k}\right\}\right) \\
&-\tfrac{2\log q^{h_1}}{s-1}kh^{2-s}\sum_{a \mod h}\left(\xi q^{h_1}\right)^{ak}\mathcal{G}_{q,\xi}^{(h_1)}\left(s-1, \left\{\frac{ak}{h}\right\}\right) \\
&-\tfrac{2\log q^{h_1}}{s-1}hk^{2-s}\sum_{b \mod k}\left(\xi q^{h_1}\right)^{bh}\mathcal{G}_{q,\xi}^{(h_1)}\left(s-1, \left\{\frac{bh}{k}\right\}\right) \\
&-\tfrac{(\log q)^2}{(s-1)(s-2)}kh^{3-s}\sum_{a \mod h}\left(\xi q^{h_1}\right)^{ak}\mathcal{G}_{q,\xi}^{(h_1)}\left(s-2, \left\{\frac{ak}{h}\right\}\right) \\
&-\tfrac{(\log q)^2}{(s-1)(s-2)}hk^{3-s}\sum_{b \mod k}\left(\xi q^{h_1}\right)^{bh}\mathcal{G}_{q,\xi}^{(h_1)}\left(s-2, \left\{\frac{bh}{k}\right\}\right)
\end{aligned}
$$

*where the function $\zeta_{q,\xi}(s)$ is defined in (25).*

By using Theorem 5, we prove the reciprocity law of the $(h,q)$-twisted Dedekind sums in the next theorem.

**Theorem 6.** *[Reciprocity law]. Let h and k be coprime integers with $k > 0$. Let $n \in \mathbb{N}$. Then, we have*

$$\begin{aligned}
&\left(kh^{n} S_{\xi,q,n}^{(h_1)}(k,h) + hk^{n} S_{\xi,q,n}^{(h_1)}(h,k)\right) + \frac{2\log q^{h_1}}{n+1}\left(kh^{n+1} S_{\xi,q,n+1}^{(h_1)}(k,h) + hk^{n+1} S_{\xi,q,n+1}^{(h_1)}(h,k)\right) \\
&\qquad + \frac{\left(\log q^{h_1}\right)^2}{(n+1)(n+2)}\left(kh^{n+2} S_{\xi,q,n+2}^{(h_1)}(k,h) + hk^{n+2} S_{\xi,q,n+2}^{(h_1)}(h,k)\right) \\
&= \frac{n\left(B_{n+2,\xi}^{(h_1,2)}(q|(h,k)) + B_{n+1,\xi}^{(h_1)}(q)\right)}{n+1} + hk\left(nB_{n,\xi}^{(h_1)}(q) + \log q^{\frac{h_1}{n+1}} B_{n+1,\xi}^{(h_1)}(q)\right) + \log q^{\frac{h_1}{n+2}} B_{n+2,\xi}^{(h_1)}(q),
\end{aligned}$$

*where $B_{n+2,\xi}^{(h_1,2)}(q \mid (h,k))$ is defined in (8).*

**Proof.** By substituting $s = 1 - n$, $(n \in \mathbb{Z}^{+})$ into Theorem 5 and using (17) and (18), after some calculations, we arrive at the desired result. □

**Remark 3.** *We point out how we recover, from our Theorem 6, some known results.*

- *By substituting $q \to 1$ and $\xi = 1$ into Theorem 6, we obtain the result of Ota ([3], [p. 8, Theorem B]):*

$$\begin{aligned}
& kh^{n} S_{\xi,q,n}^{(h_1)}(k,h) + hk^{n} S_{\xi,q,n}^{(h_1)}(h,k) \\
= \ & \frac{n}{n+1}\left(B_{n+2}^{(2)}((h,k)) + B_{n+1}\right) + hknB_{n}.
\end{aligned}$$

  *Consequently, Theorem 6 is a generalization of Ota's theorem ([3], [p. 8, Theorem B]).*
- *By substituting $h_1 = 1$ and $\xi = 1$, $q \to 1$ into Theorem 6, then we easily arrive at (27). Thus, Theorem 6 is also a generalization of Apostol's reciprocity theorem [2] for odd n.*
- *In [29], Simsek constructed p-adic $(\xi, q)$-Dedekind sums and Hardy–Berndt type sums. In the future, we will study the properties of the* ***twisted p-adic*** *Dedekind sums associated with our objects of study here.*

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