



Article On a Multivariate Analog of the Zolotarev Problem

Yury Khokhlov ^{1,2,*} and Victor Korolev ^{1,2,3}

- ¹ Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, 119991 Moscow, Russia; vkorolev@cs.msu.ru
- Moscow Center for Fundamental and Applied Mathematics, 119991 Moscow, Russia
- ³ Federal Research Center "Informatics and Control" of the Russian Academy of Sciences, 119333 Moscow, Russia
- * Correspondence: yskhokhlov@yandex.ru

Abstract: A generalized multivariate problem due to V. M. Zolotarev is considered. Some related results on geometric random sums and (multivariate) geometric stable distributions are extended to a more general case of "anisotropic" random summation where sums of independent random vectors with multivariate random index having a special multivariate geometric distribution are considered. Anisotropic-geometric stable distributions are introduced. It is demonstrated that these distributions are coordinate-wise scale mixtures of elliptically contoured stable distributions with the Marshall–Olkin mixing distributions. The corresponding "anisotropic" analogs of multivariate Laplace, Linnik and Mittag–Leffler distributions are introduced. Some relations between these distributions are presented.

Keywords: characterization problems; multivariate geometric random sums; multivariate anisotropic geometric stable distributions; anisotropic multivariate Laplace distribution; anisotropic multivariate Linnik distribution; anisotropic multivariate Mittag–Leffler distribution



1.1. Notation and Preliminaries

Assume that all the random variables and random vectors are defined on one and the same probability space $(\Omega, \mathcal{A}, \mathsf{P})$. Let $d \in \mathbb{N}$. The distribution of a random variable Y or a d-variate random vector Y with respect to the measure P will be denoted $\mathcal{L}(Y)$ and $\mathcal{L}(Y)$, respectively. The weak convergence, the coincidence of distributions and the convergence in probability with respect to a specified probability measure will be denoted by the symbols \Longrightarrow , $\frac{d}{=}$ and \xrightarrow{P} , respectively. The product of *independent* random elements will be denoted by the symbol \circ . The symbol \odot denotes the operation of coordinate-wise multiplication of independent random vectors. The vector with all zero coordinates will be denoted **1**: $\mathbf{1} = (1, \ldots, 1)$.

A univariate random variable with the standard normal distribution function $\Phi(x)$ will be denoted *X*,

$$\mathsf{P}(X < x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^{2}/2} dz, \quad x \in \mathbb{R}$$

Let Σ be a positive definite $(d \times d)$ -matrix. The normal distribution in \mathbb{R}^d with zero vector of expectations and covariance matrix Σ will be denoted \mathfrak{N}_{Σ} . This distribution is defined by its density

$$\phi(\mathbf{x}) = rac{\exp\{-rac{1}{2}\mathbf{x}^{ op}\Sigma^{-1}\mathbf{x}\}}{(2\pi)^{r/2}|\Sigma|^{1/2}}, \quad \mathbf{x} \in \mathbb{R}^d.$$



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Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). The characteristic function $f^{(X)}(t)$ of a random vector X such that $\mathcal{L}(X) = \mathfrak{N}_{\Sigma}$ has the form

$$f^{(X)}(t) \equiv \mathsf{E}\exp\{it^{\top}X\} = \exp\{-\frac{1}{2}t^{\top}\Sigma t\}, \ t \in \mathbb{R}^d.$$

Let *E* be a random variable with the standard exponential distribution: $P(E < x) = [1 - e^{-x}]\mathbf{1}(x \ge 0)$. The characteristic function of the r.v. *E* has the form

$$f^{(E)}(t) = \mathsf{E}e^{itE} = \frac{1}{1-it}, \quad t \in \mathbb{R}.$$
 (1)

Let $\gamma > 0$. The distribution of the random variable W_{γ} :

$$\mathsf{P}(W_{\gamma} < x) = \left[1 - e^{-x^{\gamma}}\right] \mathbf{1}(x \ge 0),$$

is called the *Weibull distribution* with shape parameter γ . It is obvious that $W_1 \stackrel{d}{=} E$. It is easy to see that $E^{1/\gamma} \stackrel{d}{=} W_{\gamma}$.

Recall that the distribution of a *d*-variate random vector *S* is called *stable*, if for any $a, b \in \mathbb{R}$ there exist $c \in \mathbb{R}$ and $d \in \mathbb{R}^r$ such that $aS_1 + bS_2 \stackrel{d}{=} cS + d$, where S_1 and S_2 are independent and $S_1 \stackrel{d}{=} S_2 \stackrel{d}{=} S$. In what follows, we will concentrate our attention on a special sub-class of stable distributions called *strictly stable*. This sub-class is characterized by that in the definition given above d = 0.

In the univariate case, the characteristic function g(t) of a strictly stable random variable can be represented in several equivalent forms (see, e.g., [1,2]). For our further constructions the most convenient form is

$$g(t) = \exp\{-|t|^{\alpha} + i\theta w(t,\alpha)\}, \quad t \in \mathbb{R},$$
(2)

where

$$w(t,\alpha) = \begin{cases} \tan\frac{\pi\alpha}{2} \cdot |t|^{\alpha} \operatorname{sign} t, & \alpha \neq 1, \\ -\frac{2}{\pi} \cdot t \log|t|, & \alpha = 1. \end{cases}$$
(3)

Here, $\alpha \in (0, 2]$ is the *characteristic exponent*, $\theta \in [-1, 1]$ is the *skewness* parameter. Representation (2) leads to a more general representation by introducing a scale parameter additionally. Any random variable with characteristic function (2) will be denoted $S(\alpha, \theta)$ and the characteristic function (2) itself will be written as $g_{\alpha,\theta}(t)$. The distribution function corresponding to the characteristic function $g_{\alpha,\theta}(t)$ will be denoted $G_{\alpha,\theta}(x)$. For definiteness, S(1, 1) = 1.

From (2) it follows that the characteristic function of a symmetric ($\theta = 0$) strictly stable distribution has the form

$$g_{\alpha,0}(t) = e^{-|t|^{\alpha}}, \quad t \in \mathbb{R}.$$
(4)

From (4) it is easy to see that $S(2,0) \stackrel{d}{=} \sqrt{2}X$.

Univariate stable distributions are popular examples of heavy-tailed distributions. Their moments of orders $\delta \ge \alpha$ do not exist (the only exception is the normal law corresponding to $\alpha = 2$). Stable laws and only they can be limit distributions for sums of a non-random number of independent identically distributed random variables with infinite variance under linear normalization.

Let $0 < \alpha \leq 1$. By $S(\alpha, 1)$ we will denote a positive random variable with the one-sided stable distribution corresponding to the characteristic function $g_{\alpha,1}(t)$, $t \in \mathbb{R}$. The Laplace–Stieltjes transform $\psi_{\alpha,1}^{(S)}(s)$ of the random variable $S(\alpha, 1)$ has the form

$$\psi_{\alpha,1}^{(S)}(s) \equiv \mathsf{E}\exp\{-sS(\alpha,1)\} = e^{-s^{\alpha}}, s > 0.$$

The moments of orders $\delta \ge \alpha$ of the random variable $S(\alpha, 1)$ are infinite. For more details see [2,3].

Now turn to the multivariate case. By \mathbb{Q}^d we denote the unit sphere: $\mathbb{Q}^r = \{u : ||u|| = 1\}$. Let μ be a finite ('spectral') measure on \mathbb{Q}^d . It is known that the characteristic function of a strictly stable random vector *S* has the form

$$\mathsf{E}\exp\{i\mathbf{t}^{\mathsf{T}}\mathbf{S}\} = \exp\left\{-\int_{\mathbb{Q}^r} \left(|\mathbf{t}^{\mathsf{T}}\mathbf{s}|^{\alpha} + iw(\mathbf{t}^{\mathsf{T}}\mathbf{s}, \alpha)\right)\mu(d\mathbf{s})\right\}, \quad \mathbf{t} \in \mathbb{R}^d,$$
(5)

with $w(\cdot, \alpha)$ defined in (3), see [4–7]. A *d*-variate random vector with the characteristic function (5) will be denoted $S(\alpha, \mu)$.

As is known, a random vector *S* has a strictly stable distribution with some characteristic exponent α if and only if for any $u \in \mathbb{R}^d$ the random variable $u^{\top}S$ (the projection of *S*) has the univariate strictly stable distribution with the same characteristic exponent α and some skewness parameter $\theta(u)$ up to a scale coefficient $\gamma(u)$:

$$\boldsymbol{u}^{\mathsf{T}}\boldsymbol{S}(\boldsymbol{\alpha},\boldsymbol{\mu}) \stackrel{d}{=} \gamma(\boldsymbol{u})\boldsymbol{S}\big(\boldsymbol{\alpha},\boldsymbol{\theta}(\boldsymbol{u})\big),\tag{6}$$

see [8]. Moreover, the projection parameter functions are related with the spectral measure μ as

$$\left(\gamma(\boldsymbol{u})\right)^{\alpha} = \int_{\mathbb{Q}^r} |\boldsymbol{u}^{\mathsf{T}}\boldsymbol{s}|^{\alpha} \mu(d\boldsymbol{s}), \quad \boldsymbol{u} \in \mathbb{R}^d,$$
(7)

$$\theta(\boldsymbol{u})(\gamma(\boldsymbol{u}))^{\alpha} = \int_{\mathbb{Q}^r} |\boldsymbol{u}^{\mathsf{T}}\boldsymbol{s}|^{\alpha} \operatorname{sign}(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{s})\mu(d\boldsymbol{s}), \tag{8}$$

see [6–8]. Conversely, the spectral measure μ is uniquely determined by the projection parameter functions $\gamma(u)$ and $\theta(u)$. However, there is no simple formula for this [7].

A *d*-variate analog of a one-sided univariate strictly stable random variable $S(\alpha, 1)$ is the random vector $S(\alpha, \mu^+)$ where $0 < \alpha \le 1$ and μ^+ is a finite measure concentrated on the set $\mathbb{Q}^d_+ = \{ u = (u_1, \dots, u_r)^\top \in \mathbb{Q}^d : u_i \ge 0, i = 1, \dots, d \}.$

Let Σ be a symmetric positive definite $(d \times d)$ -matrix, $\alpha \in (0, 2]$. If the characteristic function of a strictly stable random vector $S(\alpha, \mu)$ has the form

$$\mathsf{E}\exp\{it^{\mathsf{T}}S_{\alpha,\mu}\} = \exp\{-(t^{\mathsf{T}}\Sigma t)^{\alpha/2}\}, \quad t \in \mathbb{R}^d,$$
(9)

then the random vector $S(\alpha, \mu)$ is said to have the (centered) *elliptically contoured* stable distribution with characteristic exponent α . In this case for better vividness we will use the special notation $S(\alpha, \mu) \equiv S(\alpha, \Sigma)$.

The paper is organized as follows. In Section 1.2, a detailed description of the univariate Zolotarev problem is given as well as of some related results. Examples of distributions related with the univariate Zolotarev problem are presented. In Section 2, a multivariate analog of the Zolotarev problem is considered. In Section 2.1, the notion of a general multivariate geometric sum is introduced. For this purpose we first give the definitions of a multivariate Bernoulli scheme and related multivariate geometric distribution. It should be noted that the multivariate geometric distribution can be defined in several different ways, however, the asymptotic behavior of the corresponding distributions in limit theorems is the same. The properties of the multivariate geometric distribution are discussed as well as its relation with the Marshall–Olkin distribution within a special model. In Section 2.2 a multivariate version of the Zolotarev problem and the implied problems for general multivariate geometric sums are considered. Contrary to expectations, the limit distributions appearing within the model under consideration are not necessarily multivariate geometric stable. In particular, it is shown here that the Marshall–Olkin distribution is limiting in the general scheme of multivariate geometric summation, but, in general, is not multivariate geometric stable. In Section 2.3, the notion of an anisotropic multivariate geometric stable distribution is introduced. These distributions can be regarded as limit analogs of multivariate geometric stable distributions. It is shown that a rather wide class of limit distributions for multivariate geometric sums possesses the property of anisotropic geometric stability. The structure of anisotropic multivariate geometric stable distributions is described. In Section 2.4, some examples of these limit distributions are considered. In particular, anisotropic multivariate Linnik and Mittag–Leffler distributions are introduced and some of their properties are discussed.

1.2. Univariate Zolotarev Problem and Related Distributions

In the 1960s and 1980s, the topics related to the so-called characterization problems became very popular in probability theory and mathematical statistics. Many excellent results were obtained yielding, in particular, new statistical techniques. The importance of these problems was acknowledged by the publication of the book [9].

In the beginning of the 1980s V. M. Zolotarev put forward the problem of description of all the r.v.s Y such that for any $p \in (0, 1)$ there exists a r.v. X_p providing the validity of equality

$$Y \stackrel{d}{=} \varepsilon_p \cdot Y + X_p,\tag{10}$$

with the r.v.'s Y, ε_p , X_p being independent and the r.v. ε_p having the Bernoulli distribution with parameter 1 - p.

Initially it seemed that this is just one more special characterization problem. This problem was solved in 1984 in the paper [10]. It turned out that it is closely tied with generalizations of classical limit theorems to the case of geometric summation. In particular, in the paper [10] it was demonstrated that the Zolotarev problem is equivalent to the problem of description of all r.v.s *Y* such that for any $p \in (0, 1)$ the representation

$$Y = \sum_{j=1}^{N_p} X_p^{(j)},$$
(11)

holds with the r.v.s N_p , $X_p^{(j)}$, $j \ge 1$, being independent, $X_p^{(j)}$, $j \ge 1$ are identically distributed and the r.v. N_p has the geometric distribution with parameter p. These r.v.s Y were called *geometrically infinitely divisible*. Thus, the Zolotarev problem was reduced to the description of the class of geometrically infinitely divisible distributions.

The problems of this type themselves are interesting. However, they find numerous applications in many applied problems, for example, in financial and insurance mathematics, reliability and queueing theory, etc. (see, e.g., [11]). Below we will discuss one of these problems considered by Kovalenko [12].

The solution of the Zolotarev problem is given by the following theorem following theorem proved in [10].

Theorem 1. A function f(t) is the characteristic function of a geometrically infinitely divisible distribution if and only if it can be represented as

$$f(t) = \frac{1}{1 - \ln g(t)},$$
(12)

where g(t) is an infinitely divisible characteristic function.

By analogy with problems of "conventional" summation, in [10] the following important notion was introduced as well. Later this notion was successfully used in many problems.

Definition 1. A r.v. Y is said to have a geometrically strictly stable distribution, if for any $p \in (0, 1)$ there exists a constant c(p) > 0 such that

$$Y \stackrel{d}{=} c(p) \cdot \sum_{j=1}^{N_p} Y_j, \tag{13}$$

where the r.v.s $Y, Y_1, Y_2, ...$ are independent and identically distributed and the r.v. N_p is independent of $Y, Y_1, Y_2, ...$ and has the geometric distribution with parameter p.

The following theorem was proved in [10].

Theorem 2. A function f(t) is the characteristic function of a geometrically strictly stable distribution if and only if it can be represented as

$$f(t) = \frac{1}{1 - \ln g_{\alpha,\theta}(t)},\tag{14}$$

where $g_{\alpha,\theta}(t)$ is a strictly stable characteristic function with some characteristic exponent $\alpha \in (0, 2]$.

By the Fubini theorem (or the formula of total expectation) and (1) it is easy to see that the characteristic function (14) corresponds to the r.v.

$$Z \stackrel{d}{=} E^{1/\alpha} \circ S(\alpha, \theta) \stackrel{d}{=} W_{\alpha} \circ S(\alpha, \theta), \tag{15}$$

that is, any geometrically strictly stable distribution is a scale mixture of a strictly stable law, the mixing distribution being Weibull.

To trace the relation of geometrically strictly stable distributions with random summation, we will use the following auxiliary result proved in [13,14]. Consider a sequence of r.v.s S_1, S_2, \ldots Let N_1, N_2, \ldots be natural-valued r.v.s such that for every $n \in \mathbb{N}$ the r.v. N_n is independent of the sequence S_1, S_2, \ldots In the following statement the convergence is meant as $n \to \infty$.

Lemma 1. Assume that there exist an infinitely increasing (convergent to zero) sequence of positive numbers $\{b_n\}_{n>1}$ and a r.v. S such that

$$b_n^{-1}S_n \Longrightarrow S.$$
 (16)

If there exist an infinitely increasing (convergent to zero) sequence of positive numbers $\{d_n\}_{n>1}$ and a r.v. N such that

$$d_n^{-1}b_{N_n} \Longrightarrow N, \tag{17}$$

then

$$d_n^{-1}S_{N_n} \Longrightarrow N \circ S, \tag{18}$$

where the r.v.s on the right-hand side of (18) are independent. If, in addition, $N_n \longrightarrow \infty$ in probability and the family of scale mixtures of the distribution function of the r.v. S is identifiable, then condition (17) is not only sufficient for (18), but is necessary as well.

This lemma is actually a generalization and sharpening of the famous Gnedenko– Fahim transfer theorem proved in [15] for random sums and the Dobrushin lemma proved in [16] for power-type normalizing functions to arbitrary general random sequences with independent random indices.

Univariate geometric distributions possess the following well-known property.

Lemma 2. Let $\lambda > 0$, $p \in (0,1)$ so that $\lambda p < 1$. If the r.v. N_p^* has the geometric distribution with parameter λp , then $p \cdot N_p^* \Longrightarrow E(\lambda)$ as $p \to 0$, where the r.v. $E(\lambda)$ has the exponential distribution with parameter λ .

One of most important results concerning geometrically strictly stable distributions is the following theorem.

Theorem 3. A univariate probability distribution is geometrically strictly stable if and only if it is limiting for a geometric random sum of independent identically distributed r.v.s as the parameter p of the random index tends to zero.

We supply this result by a sketch of the proof. *The 'if' part* directly follows from Lemmas 1 and 2, Theorem 2 and (15). To prove *the 'only if' part* consider a geometrically strictly stable distribution corresponding to the characteristic function (14) with some $\alpha \in (0,2]$ and $\theta \in [-1, 1]$. Choose a distribution function *F* from the domain of attraction of the strictly stable distribution $G_{\alpha,\theta}(x)$ and consider independent identically distributed r.v.s X_1, X_2, \ldots with the common distribution function *F*. For $n \ge 1$ denote $S_n = X_1 + \ldots + X_n$. Since *F* belongs to the domain of attraction of the strictly stable distribution $G_{\alpha,\theta}(x)$, there exists a sequence $\{b_n\}_{n\ge 1}$ of positive numbers such that (16) holds with $S \stackrel{d}{=} S(\alpha, \theta)$. Moreover, in [17] it was shown that b_n can be chosen as $b_n = n^{1/\alpha}L(n)$, $n \ge 1$, where L(x) is a slowly varying function: for any y > 0

$$\lim_{x \to \infty} \frac{L(xy)}{L(x)} = 1 \tag{19}$$

(also see [18]). Let $p \in (0,1)$ and N_p^* be a r.v. having the geometric distribution with parameter p. For simplicity, without loss of generality, let $p = p_n = \frac{1}{n}$ and $N_n = N_{1/n}^*$, $n \ge 1$. Assume that for each $n \ge 1$ the r.v.s N_n, X_1, X_2, \ldots are independent. Consider the limit behavior of the r.v.s $b_n^{-1}b_{N_n}$. We have

$$\frac{b_{N_n}}{b_n} = \left(\frac{N_n}{n}\right)^{1/\alpha} + \left(\frac{N_n}{n}\right)^{1/\alpha} \left(\frac{L(N_n)}{L(n)} - 1\right).$$
(20)

Consider the second term on the right-hand side of (20). Let ϵ be an arbitrary small positive number $0 < \epsilon < \frac{1}{2}$, $M_1 = M_1(\epsilon) = \ln \frac{1}{1-\epsilon}$, $M_2 = M_2(\epsilon) = \ln \frac{1}{\epsilon}$. By virtue of Lemma 2 there exists an $n_0 = n_0(\epsilon)$ such that

$$\mathsf{P}\Big(\frac{N_n}{n} \notin [M_1, M_2]\Big) \le 4\epsilon \tag{21}$$

for all $n \ge n_0$. Let $\sigma > 0$. From (21) it follows that for $n \ge n_0$ we have

$$P\left(\left|\left(\frac{N_n}{n}\right)^{1/\alpha}\left(\frac{L(N_n)}{L(n)}-1\right)\right| > \sigma\right) =$$

$$= P\left(\left(\frac{N_n}{n}\right)^{1/\alpha}\left|\frac{L(N_n)}{L(n)}-1\right| > \sigma; \frac{N_n}{n} \in [M_1, M_2]\right) +$$

$$+ P\left(\left(\frac{N_n}{n}\right)^{1/\alpha}\left|\frac{L(N_n)}{L(n)}-1\right| > \sigma; \frac{N_n}{n} \notin [M_1, M_2]\right) \leq$$

$$\leq P\left(\left(\frac{N_n}{n}\right)^{1/\alpha}\left|\frac{L(N_n)}{L(n)}-1\right| > \sigma; \frac{N_n}{n} \in [M_1, M_2]\right) + 4\epsilon \leq$$

$$\leq P\left(\left|\frac{L(N_n)}{L(n)}-1\right| > \frac{\sigma}{M_2^{1/\alpha}}\right) + 4\epsilon \leq$$

$$\leq P\left(\left|\sup_{M_1 \leq x \leq M_2}\left|\frac{L(nx)}{L(n)}-1\right| > \frac{\sigma}{M_2^{1/\alpha}}\right) + 4\epsilon.$$
(22)

According to Theorem 1.1 in [19], convergence (19) is uniform in every closed segment of values of *y*. Therefore, an $n_1 = n_1(\epsilon, \sigma)$ can be found such that for all $n \ge n_1$ we have

$$\sup_{M_1 \leq x \leq M_2} \left| \frac{L(nx)}{L(n)} - 1 \right| < \frac{\sigma}{M_2^{1/\alpha}},$$

so that for these *n* the first term on the right-hand side of (22) equals zero. Thus,

$$\left(\frac{N_n}{n}\right)^{1/\alpha} \left(\frac{L(N_n)}{L(n)} - 1\right) \stackrel{P}{\longrightarrow} 0$$

as $n \to \infty$. By virtue of (22) and the Slutsky lemma (see [20]) this means that the asymptotic behavior of $b_n^{-1}b_{N_n}$ as $n \to \infty$ coincides with that of $(N_n/n)^{1/\alpha}$, that is, $b_n^{-1}b_{N_n} \Longrightarrow W_{\alpha}$. Now the reference to lemma 1 with $d_n = b_n$, $N \stackrel{d}{=} W_{\alpha}$ and (15) completes the proof. An alternative proof of this result can be found in [21].

Based on the 'if and only if' character of the result presented in Theorem 3, it became conventional to *define* geometric strictly distribution as weak limits for the distributions of geometric sums of independent identically distributed r.v.s.

Well-known examples of geometrically strictly stable distributions are exponential distribution with parameter $\lambda > 0$ corresponding to the case $\alpha = 1$, $\theta = 1$, whose Laplace–Stieltjes transform has the form

$$\psi^{(E)}_{\lambda}(s) = \mathsf{E} e^{-sZ} = rac{\lambda}{\lambda+s}, \ s \geq 0;$$

the Linnik distribution with parameters $\lambda > 0$ and $0 < \alpha \le 2$ defined by the characteristic function

$$f^{(L)}_{\alpha}(t) = rac{1}{1+\lambda|t|^{lpha}}, \ t\in\mathbb{R}^1,$$

with the Laplace distribution defined by the Lebesgue density

$$\ell(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R},$$
(23)

being a particular case corresponding to $\alpha = 2$ (see, e.g., [22]), and the Mittag–Leffler distribution defined by its Laplace–Stieltjes transform

$$\psi_{\delta}^{(M)}(s) = \frac{\lambda}{\lambda + s^{\delta}}, \quad s \ge 0.$$
(24)

The numbers $\lambda > 0$ $\delta \in (0, 1]$ are the parameters of this distribution. If $\delta = 1$, we arrive at the exponential distribution. The r.v.s with Laplace–Stieltjes transform (24) will be denoted M_{δ} . As far ago as in 1965, it was shown that the distributions with the Laplace–Stieltjes transform (24) and only they can be limiting for the distributions of geometric sums of independent identically distributed nonnegative r.v.s (see [12]). As this is so, from Theorem 3 it follows that these distributions are geometrically strictly stable. Moreover, from (15) it follows that

$$M_{\delta} \stackrel{d}{=} E^{1/\delta} \circ S(\delta, 1). \tag{25}$$

For more details and the history of the Mittag–Leffler distribution see [22]. In what follows, r.v.s with the Linnik distribution and Laplace distribution will be denoted L_{α} and L_2 , respectively.

As it has been already mentioned, geometric strictly stable distributions appear in limit theorems for random sums of independent identically distributed r.v.s in which the number of summands has the geometric distribution and is independent of the summands. We recall some theorems of this type.

Consider a sequence $\{X_j\}_{j\geq 1}$ of identically distributed r.v.s and the integer-valued r.v. N_p having the geometric distribution with parameter $p \in (0, 1)$. Assume that all these r.v.s are jointly independent.

Theorem 4. Assume that the r.v.s X_i have finite expectation $\mathsf{E}X_i = a \in (0, \infty)$. Then

$$\frac{p}{a} \cdot \sum_{j=1}^{N_p} X_j \Longrightarrow E$$

as $p \rightarrow 0$.

This theorem is a 'geometric' analog of the law of large numbers and is often called the Rényi theorem, see [23].

The following result (a 'light' version of the result of [12]) can be regarded as a generalization of the Rényi theorem.

Theorem 5. Let the common distribution of nonnegative r.v.s X_j belong to the domain of normal attraction of a one-sided strictly stable distribution with characteristic exponent $0 < \delta \leq 1$. Then

$$p^{1/\delta} \cdot \sum_{j=1}^{N_p} X_j \Longrightarrow M_{\delta}$$

as $p \rightarrow 0$.

Theorem 6. Let the common distribution of r.v.s X_j belong to the domain of normal attraction of a symmetric strictly stable distribution with characteristic exponent $0 < \alpha \le 2$. Then

$$p^{1/\alpha} \cdot \sum_{j=1}^{N_p} X_j \Longrightarrow L_\alpha$$

as $p \rightarrow 0$.

As a corollary of this result we obtain the following 'geometric' version of the central limit theorem.

Theorem 7. Assume that $EX_j = 0$ and $DX_j = \sigma^2 \in (0, \infty)$, $j \ge 1$. Then

$$p^{1/2} \cdot \sum_{j=1}^{N_p} X_j \Longrightarrow L_2$$

as $p \rightarrow 0$.

In the present paper we consider a multivariate version of the Zolotarev problem generalizing some results of [10]. An 'isotropic' multivariate generalization of these results to the case of geometric random sums of random vectors was considered in [21,24]. In that case all the coordinates of the vectors *are summed up to one and the same* geometrically distributed r.v. resulting in random scalar scaling of the multivariate stable distribution in the limit geometrically stable law. Here, we extend these results to a more general case of "anisotropic" random summation where sums of independent random vectors with multivariate random index having a special multivariate geometric distribution are considered resulting in that in each coordinate of the random vectors *the summation is performed up to a separate random index*. Anisotropic-geometric stable distributions are introduced. It is demonstrated that these distributions are coordinate-wise scale mixtures of elliptically contoured stable distributions with the Marshall–Olkin mixing distributions. The corresponding "anisotropic" analogs of multivariate Laplace, Linnik and Mittag–Leffler distributions are introduced. Some relations between these distributions are presented.

2. A multivariate Analog of the Zolotarev Problem

2.1. Multivariate Geometric Distribution

Several versions of a multivariate geometric distribution are known, see, e.g., [25]. Here, we will keep to the definition used in [26]. First, recall the definition of the multivariate Bernoulli distribution and the multivariate Bernoulli scheme. Let \Im be the set of *d*-variate indices $i = (i_1, \ldots, i_d)$, where each i_k takes one of two values 0 or 1, $k = 1, \ldots, d$.

By \bar{i} we will mean the *d*-variate index obtained by replacing the coordinates of i by their binary counterparts: $\bar{i} = (\bar{i}_1, ..., \bar{i}_d)$, $\bar{i}_k = 1 - i_k$, k = 1, ..., d. Let $\Im_k = \{i \in \Im : i_k = 1\}$, k = 1, ..., d.

Definition 2. A random vector $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_d)$ is said to have the multivariate Bernoulli distribution, if it takes values in the set \Im and

$$\mathsf{P}(\vec{\varepsilon}=i)=p_i.$$

The set of numbers $Q = \{p_i, i \in \Im\}$ *is called the parameter(s) of this distribution.*

To emphasize the dependence of the multivariate Bernoulli distribution on the parameters Q we will sometimes use the notation $\vec{\epsilon} = \vec{\epsilon}(Q)$.

Definition 3. A multivariate Bernoulli scheme is a sequence of independent random vectors ${\vec{\epsilon}_j = (\epsilon_{j,1}, \dots, \epsilon_{j,d})}_{j \ge 1}$, each of which has the same multivariate Bernoulli distribution.

Now define the multivariate geometric distribution. Let $\{\vec{\varepsilon}_j\}_{j\geq 1}$ be a multivariate Bernoulli scheme with parameters Q and infinite number of trials. For each k = 1, ..., d define the r.v.

$$N_k = \inf\{j \ge 1: \ \varepsilon_{j,k} = 1\}. \tag{26}$$

Definition 4. The random vector $\vec{N}_Q = (N_1, ..., N_d)$ whose components are defined in accordance with (26) is said to have the multivariate geometric distribution with parameters Q.

To provide that the r.v. N_k is finite and positive, the random vector $\vec{\epsilon}_1$ must satisfy the condition $0 < P(\epsilon_{1,k} = 1) < 1$. If this condition is satisfied, then the corresponding multivariate geometric distribution will be called admissible. Everywhere in what follows we will consider only admissible multivariate geometric distributions.

If a random vector $\vec{\epsilon}$ has the multivariate Bernoulli distribution, then, to avoid double superscripts, by $\vec{\delta}$ we will denote its binary counterpart: $\vec{\delta} = \vec{\epsilon} = \mathbf{1} - \vec{\epsilon}$.

In what follows we will use the following result.

Lemma 3. A random vector $\vec{N}_Q = (N_1, ..., N_d)$ has the multivariate geometric distribution with parameters Q if and only if it can be represented as

$$\vec{N} = \mathbf{1} + \sum_{j=1}^{\infty} \vec{\delta}_1 \odot \ldots \odot \vec{\delta}_j , \qquad (27)$$

where $\{\vec{\delta}_j\}_{j\geq 1}$ is the binary counterpart of the Bernoulli scheme $\{\vec{\varepsilon}_j\}_{j\geq 1}$ with parameters Q.

Proof. Let a multivariate Bernoulli scheme $\{\vec{e}_j\}_{j\geq 1}$, the corresponding random vector $\vec{N}_Q = (N_1, \ldots, N_d)$ with the multivariate geometric distribution and the binary counterpart $\{\vec{\delta}_j\}_{j\geq 1}$ of the Bernoulli scheme $\{\vec{e}_j\}_{j\geq 1}$ be defined on the same probability space. To prove the lemma it suffices to show that each component of the random vector on the left-hand side of (27) coincides with the corresponding component of the random vector on the right-hand side. It is easy to see that $N_k = n$ if and only if for the *k*th component of the multivariate Bernoulli scheme the success (i.e., one) for the first time occurs in the *n*th trial.

As this is so, in the binary counterpart of the Bernoulli scheme the failure (i.e., zero) for the first time occurs in the *n*th trial. However, this means that all the terms beginning from the number *n* vanish. Therefore, the sum will again be equal to *n*. \Box

The explicit expression for the multivariate geometric distribution is rather cumbersome, but it is rather easy to obtain the following recurrent formula for its Laplace transform.

Theorem 8. Let the random vector $\vec{N} = (N_1, ..., N_d)$ have the multivariate geometric distribution with parameters $Q = \{p_i, i \in \Im\}$. Then for any vector $\mathbf{t} = (t_1, ..., t_d)$ with nonnegative coordinates we have

$$\varphi_{\vec{N}}(t) := \mathsf{E}\Big(e^{-t_1N_1} \cdot \ldots \cdot e^{-t_dN_d}\Big) = \frac{e^{-(t_1+\ldots+t_d)}}{1-e^{-(t_1+\ldots+t_d)} \cdot p_0} \cdot \sum_{i \neq 0} p_i \cdot \varphi_{\vec{N}}(\bar{i} \odot t).$$
(28)

The proof can be found, e.g., in [26]. Note that under the sum sign there are the Laplace transforms of random vectors with (multivariate) geometric distributions of dimensionality less than *d*. It is well known that for d = 1 we have

$$\varphi_N(t) = \frac{p \cdot e^{-t}}{1 - (1 - p)e^{-t}},$$

so that it is possible to recursively calculate the Laplace transform for the multivariate geometric distribution of an arbitrary dimensionality.

There is another version of the definition of the multivariate geometric distribution yielding the same result. Let for any $i \in \mathfrak{I}$, $i \neq 0$, V_i be the r.v. with the univariate geometric distribution with parameter p_i . Let, moreover, the r.v.s $\{V_i\}_{i\in\mathfrak{I}}$ be independent and $\sum_{i\neq 0} p_i < 1$. Let $p_0 = 1 - \sum_{i\neq 0} p_i$. For each $k = 1, \ldots, d$ define the r.v.

$$M_k := \min_{i \in \mathfrak{I}_k} V_i \,. \tag{29}$$

In [27], the vector $\vec{M}_i = (M_1, ..., M_d)$ was said to have the multivariate geometric distribution with parameters $Q = \{p_i, i \in \mathcal{I}\}.$

Generally speaking, these two definitions are different. However, within the framework of the model considered in this paper, as $p \rightarrow 0$, they are equivalent with the accuracy up to p^2 .

Recall the definition of the Marshall–Olkin distribution which is a version of the definition of a multivariate exponential distribution [28]. Let for each $i \in \mathfrak{I}, i \neq \mathbf{0}$, E_i be an exponentially distributed r.v. with the parameter $\lambda_i \geq 0$. Assume that the r.v.s $\{E_i\}_{i\in\mathfrak{I}}$ are independent. For each $k = 1, \ldots, d$ define the r.v.

$$Z_k := \min_{i:i=1} E_i. \tag{30}$$

Definition 5. The distribution of the random vector $\mathbf{Z} = (Z_1, ..., Z_d)$, whose components are determined in accordance with (30), is called the Marshall–Olkin distribution with parameters $\Lambda = \{\lambda_i : i \in \mathfrak{I}, i \neq \mathbf{0}\}.$

This definition is a complete 'continuous' analog of the definition of the multivariate geometric distribution given by Equation (29). This relation is supported by the following multivariate analog of Lemma 2 (see, e.g., [27]).

Theorem 9. Let the random vector $\vec{N} = (N_1, ..., N_d)$ have the multivariate geometric distribution with parameters Q. Moreover, let $\mathsf{P}(\vec{\varepsilon} = \mathbf{0}) = 1 - p$, $\mathsf{P}(\vec{\varepsilon} = \mathbf{i}) = p \cdot \lambda_i$, $\mathbf{i} \neq \mathbf{0}$. Then, as $p \to 0$, the random vector $p \cdot \vec{N}$ has an asymptotically multivariate Marshall–Olkin distribution with parameters $\Lambda = \{\lambda_i, \mathbf{i} \in \mathfrak{I}, \mathbf{i} \neq \mathbf{0}\}$.

By unifying the results of Theorems 8 and 9 we can obtain the following useful result that makes it possible to calculate the Laplace transform of the Marshall–Olkin distribution of an arbitrary dimensionality.

Theorem 10. Let the random vector $\mathbf{Z} = (Z_1, ..., Z_d)$ have the Marshall–Olkin distribution with parameters $\Lambda = \{\lambda_i, i \neq \mathbf{0}\}$ such that $\sum_{i \neq 0} \lambda_i = 1$. Then

$$\varphi_{\mathbf{Z}}(t) := E\left(e^{-t_1N_1} \cdot \ldots \cdot e^{-t_dN_d}\right) = \frac{1}{1 + (t_1 + \ldots + t_d)} \cdot \sum_{i \neq \mathbf{0}} \lambda_i \cdot \varphi_{\vec{N}}(\vec{i} \odot t).$$
(31)

The condition $\sum_{i\neq 0} \lambda_i = 1$ is not restrictive and is used only to simplify the formulations.

2.2. A Multivariate Analog of the Zolotarev Problem

By analogy with the univariate case, consider the following problem. Let the random vector $\vec{\delta}(Q)$ be the binary counterpart to a random vector $\vec{\epsilon}(Q)$ that has the multivariate Bernoulli distribution with parameters Q. Consider the problem of description of the set of random vectors Y that satisfy the condition: for any admissible set of parameters Q there exists a random vector $\vec{X}(Q)$ such that

$$\mathbf{Y} \stackrel{a}{=} \vec{\delta}(Q) \odot \mathbf{Y} + \vec{X}(Q), \tag{32}$$

where the random vectors $\boldsymbol{Y}, \boldsymbol{\delta}(Q)$ and $\boldsymbol{X}(Q)$ are independent.

Having compared (32) with (10) we can conclude that this problem can be regarded as a multivariate analog of the Zolotarev problem.

Recursively applying relation (32) we arrive at the following representation of the random vector Y:

$$\mathbf{Y} \stackrel{d}{=} \vec{X}_1(Q) + \sum_{j=1}^{\infty} \vec{\delta}_1(Q) \odot \ldots \odot \vec{\delta}_j(Q) \odot \vec{X}_{j+1}(Q), \tag{33}$$

where $\{\vec{\delta}_j(Q)\}_{j\geq 1}$ is a multivariate Bernoulli scheme that is the binary counterpart of the Bernoulli scheme with parameters Q and $\vec{X}_1(Q), \vec{X}_2(Q), \ldots$ are independent identically distributed random vectors independent of the Bernoulli scheme $\{\vec{\delta}_j(Q)\}_{j\geq 1}$. It should be noted that in the case under consideration the number of summands in the sum on the right-hand side of (33) is a. s. finite.

The following technical result is important for further considerations.

Lemma 4. Let $\{\vec{\varepsilon}_j\}_{j\geq 1}$ be a multivariate Bernoulli scheme with parameters Q, $\vec{N}_Q = (N_1, \ldots, N_d)$ be the random vector with the corresponding multivariate geometric distribution, $\{\vec{X}_j = (X_{j,1}, \ldots, X_{j,d})\}_{j\geq 1}\}$ be a sequence of independent identically distributed random vectors independent of the Bernoulli scheme under consideration. Then

$$\vec{X}_1 + \sum_{j=1}^{\infty} \vec{\delta}_1 \odot \ldots \odot \vec{\delta}_j \odot \vec{X}_{j+1} \stackrel{d}{=} \left(\sum_{j=1}^{N_1} X_{j,1}, \ldots, \sum_{j=1}^{N_d} X_{j,d} \right), \tag{34}$$

where $\{\vec{\delta}_i\}_{i\geq 1}$ is the binary counterpart of the Bernoulli scheme $\{\vec{\varepsilon}_i\}_{i\geq 1}$.

Proof. Let a multivariate Bernoulli scheme $\{\vec{\epsilon}_j\}_{j\geq 1}$, the corresponding random vector $\vec{N}_Q = (N_1, \ldots, N_d)$ with the multivariate geometric distribution, the binary counterpart $\{\vec{\delta}_j\}_{j\geq 1}$ of the Bernoulli scheme $\{\vec{\epsilon}_j\}_{j\geq 1}$ and a sequence of independent identically distributed random vectors $\{\vec{X}_j = (X_{j,1}, \ldots, X_{j,d})\}_{j\geq 1}$ independent of $\{\vec{\epsilon}_j\}_{j\geq 1}, \{\vec{\delta}_j\}_{j\geq 1}$ and \vec{N}_Q be defined on the same probability space.

To prove the lemma it suffices to show that each component of the random vector on the left-hand side of (34) coincides with the corresponding component of the random vector on the right-hand side. It is easy to see that $N_k = n$ if and only if for the *k*th component of the multivariate Bernoulli scheme the success (i.e., one) for the first time occurs in the *n*th trial. As this is so, in the binary counterpart of the Bernoulli scheme the failure (i.e., zero) for the first time occurs in the *n*th trial. This means that all the terms beginning from the number *n* vanish. The rest of the summands on the left-hand side have coefficients equal to one. However, in this case both on the right-hand and left-hand sides we obtain the quantity $X_{1,k} + \ldots + X_{n,k}$.

From this lemma it follows that the solution of the multivariate Zolotarev problem is a multivariate random sum of independent identically distributed random vectors with multivariate random index that is independent of summands and has the multivariate geometric distribution. It is natural to say that this random sum has a multivariate geometric infinitely divisible distribution. Moreover, since each *k*th coordinate is summed up to its own geometrically distributed r.v. N_k , it is natural to call this distribution *anisotropic*. This circumstance distinguishes the case under consideration from the case considered in preceding works (see, e.g., [21,29]) where the geometrically distributed index was one and the same for all coordinates and was actually univariate ($p_0 + p_1 = 1$).

Let us introduce the notion of an anisotropic geometrically stable distribution. Everywhere in what follows it is assumed that the set Q of parameters belongs to some family Q of admissible parameters.

Definition 6. The distribution of a random vector $\mathbf{Y} = (Y_1, \dots, Y_d)$ is called anisotropic multivariate geometric strictly stable, if

$$\mathbf{Y} = (Y_1, \dots, Y_d) \stackrel{d}{=} c(Q) \cdot \left(\sum_{j=1}^{N_1} Y_{j,1}, \dots, \sum_{j=1}^{N_d} Y_{j,d}\right),$$
(35)

the random vectors \mathbf{Y} , $\{\mathbf{Y}_j = (Y_{j,1}, \ldots, Y_{j,d})\}_{j \ge 1}$ are independent and identically distributed, the random vector $\mathbf{N} = (N_1, \ldots, N_d)$ has an arbitrary multivariate geometric distribution with parameters $Q \in \mathcal{P}$ and is independent of $\mathbf{Y}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots$, and c(Q) is a positive number.

By virtue of Lemma 4, an equivalent definition of an anisotropic multivariate geometric strictly stable distribution can be given. Let $Y, Y_1, Y_2, ...$ be a sequence of independent identically distributed random vectors, $\{\vec{\varepsilon}_j\}_{j\geq}$ be a multivariate Bernoulli scheme with parmeters Q independent of $Y, Y_1, Y_2, ...$, and $\{\delta_j\}_{j\geq 1}$ be the binary counterpart to $\{\vec{\varepsilon}_j\}_{j\geq}$.

Lemma 5. The distribution of a random vector **Y** is anisotropic multivariate geometric strictly stable if and only if

$$\mathbf{Y} \stackrel{d}{=} c(Q) \cdot \left(\mathbf{Y}_1 + \sum_{j=1}^{\infty} \vec{\delta}_1 \odot \ldots \odot \vec{\delta}_j \odot \mathbf{Y}_{j+1} \right).$$
(36)

From Lemma 5 we obtain the following result.

Theorem 11. *The distribution of a random vector* **Y** *is anisotropic multivariate geometric strictly stable if and only if*

$$\mathbf{Y} \stackrel{d}{=} c(Q) \cdot \mathbf{Y}_1 + \vec{\delta}(Q) \odot \mathbf{Y}_2, \tag{37}$$

where the random vectors $\mathbf{Y}, \mathbf{Y}_1, \mathbf{Y}_2$ are independent and identically distributed, $\vec{\delta}(Q) = \mathbf{1} - \vec{\epsilon}(Q)$ and the random vector $\vec{\epsilon}(Q)$ is independent of $\mathbf{Y}, \mathbf{Y}_1, \mathbf{Y}_2$ and has the multivariate Bernoulli distribution with parameters Q, whereas c(Q) is a positive number.

Our nearest aim is to describe the set of all anisotropic strictly stable distributions. If, in order to do so, we use all admissible multivariate geometric distributions, then the only possible solution will be trivial, namely, the solution will be reduced to the only distribution degenerate in **0**. Therefore, the set of admissible multivariate geometric distributions should be narrowed to some sub-family \mathcal{P} .

As it has already been mentioned, in many papers the case was considered where $p_1 = p \in (0, 1)$, $p_0 = 1 - p$. This corresponds to the case where random sums of random vectors are considered with the univariate random index having the univariate geometric distribution. Actually, this is a complete analog of the univariate case. In this setting the main result is the following theorem.

Theorem 12. A function f(t), $t \in \mathbb{R}^d$, is the characteristic function of the (isotropic) multivariate strictly stable distribution if and only if

$$f(\boldsymbol{t}) = \frac{1}{1 - \ln g(\boldsymbol{t})},\tag{38}$$

where g(t) is the characteristic function of a 'conventional' multivariate strictly stable distribution (see the Section 1).

It can be easily seen that this result can be extended to the case where $p_i = p \in (0, 1)$, $p_0 = 1 - p$. Here, $i \in \mathfrak{I}$ and $i \neq 0$, $i \neq 1$. The result has exactly the same form but holds in some subspace of \mathbb{R}^d corresponding to the structure of the index *i*.

2.3. Anisotropic Multivariate Geometric Stable Distributions

Consider another possible case. Let for all $i \in \mathfrak{I}$, $i \neq 0$, the numbers $\lambda_i \geq 0$ be given such that $\sum_{i\neq 0} \lambda_i = 1$. Define the parameter set Q in the following way: $p_0 = 1 - p$, and $p_i = p \cdot \lambda_i$, $0 for <math>i \neq 0$. In this case the random vector $\vec{\epsilon}(Q)$ has the multivariate Bernoulli distribution with $Q \in \mathcal{P}$. Everywhere below we will consider only this case and will write that we deal with the parameter set Q = Q(p). The task is to describe the class of anisotropic multivariate geometric stable distributions within this setting.

If we consider a projection of this random vector onto a coordinate for which the univariate distribution is not degenerate in zero, we obtain the classical univariate setting where, as is known, for the geometric stable distribution

$$c(Q(p)) = p^{1/\alpha},\tag{39}$$

with α being the characteristic exponent of the corresponding univariate strictly stable distribution.

It is easy to see that any anisotropic multivariate geometric strictly stable distribution can be limiting as $p \rightarrow 0$ within the setting described above. The natural question asizes, whether anisotropic multivariate geometric strictly stable distributions exhaust the class of possible limit laws for multivariate geometric random sums of independent identically distributed random vectors. The answer is 'no', that is, the class of these limit laws is wider than the class of distributions satisfying Definition 6. We will show this by the example of the Marshall–Olkin distribution. Namely, we will demonstrate that this distribution is anisotropic multivariate geometric stable only in one particular case, whereas it can be limiting for multivariate geometric random sums of independent identically distributed random vectors (see Theorem 9).

So, let the Marshall–Olkin distribution be anisotropic multivariate geometric strictly stable within the setting described above. The univariate marginals of this distributions are exponential distributions that are geometrically strictly stable with c((Q(p)) = p. It is required to verify the validity of relation (37) which, in the case under consideration, has the form

$$\mathbf{Z} \stackrel{a}{=} p \cdot \mathbf{Z}_1 + \vec{\delta}(p) \odot \mathbf{Z}_2,\tag{40}$$

where the random vectors \mathbf{Z} , \mathbf{Z}_1 and \mathbf{Z}_2 are independent and have one and the same Marshall–Olkin distribution with parameters $\Lambda = \{\lambda_i, i \neq 0\}$ with $\sum_{i\neq 0} \lambda_i = 1$,

 $\vec{\delta}(p) = \mathbf{1} - \vec{\epsilon}(p)$ and the random vector $\vec{\epsilon}(p)$ has the multivariate Bernoulli distribution with parameters Q = Q(p). The condition that the sum of parameters equals 1 is not a restriction and is used only to simplify formulations.

Write (40) in terms of the Laplace transform:

$$\varphi_{\mathbf{Z}}(t) = \varphi_{\mathbf{Z}}(pt) \cdot \Big[\sum_{i} p_{i} \varphi_{\mathbf{Z}}(\bar{i} \odot t)\Big].$$
(41)

Using relation (31) for the Laplace transform of the Marshall–Olkin distribution, we can rewrite the right-hand side of (41) as

$$\varphi_{\mathbf{Z}}(pt) \cdot \left[p_{\mathbf{0}} \cdot \varphi_{\mathbf{Z}}(t) + \sum_{i \neq \mathbf{0}} p_{i} \cdot \varphi_{\mathbf{Z}}(\overline{i} \odot t) \right] =$$

$$=\varphi_{\mathbf{Z}}(p\mathbf{t})\cdot\left[(1-p)\varphi_{\mathbf{Z}}(\mathbf{t})+\left(1+\sum_{k}t_{k}\right)\varphi_{\mathbf{Z}}(\mathbf{t})\right]=\varphi_{\mathbf{Z}}(p\mathbf{t})\cdot\left[(1-p)+\left(1+\sum_{k}t_{k}\right)\right]\cdot\varphi_{\mathbf{Z}}(\mathbf{t}).$$

Hence, if (41) holds, then

$$\varphi_{\mathbf{Z}}(p\mathbf{t}) = \frac{1}{1 - p + (1 + \sum_{k} t_k)}$$

or

$$\varphi_{\mathbf{Z}}(t) = \frac{1}{1 + \sum_{k} t_{k}}.$$

However, this is valid only for the random vector \mathbf{Z} of the form

$$\mathbf{Z} = E \cdot \mathbf{1},$$

where the r.v. *E* has the standard exponential distribution.

As we have already mentioned, the Marshall–Olkin distribution can be limiting for anisotropic multivariate geometric random sums with Q = Q(p). However, it will far not always be anisotropic geometric strictly stable within the same setting. So, the conventional definition of stability appears to be very strong in the problem under consideration and brings us back to the isotropic setting of geometric summation with a univariate index.

For the sake of proving limit theorems, we will loosen condition (37) and require it to hold only for *p* small enough. Moreover, we will assume that it holds asymptotically as $p \rightarrow 0$.

Definition 7. A probability distribution F in R^d is called anisotropic asymptotically geometric strictly stable, if

$$c(Q) \cdot Y_1 + \delta(Q) \cdot Y_2 \Longrightarrow Y_1 \tag{42}$$

as $p \to 0$ for independent random vectors Y_1 and Y_2 with distribution F, where c(Q) = c(Q(p)) > 0 is a constant and the random vector $\delta(Q) = \delta(Q(p))$ is a binary counterpart to the random vector with the multivariate Bernoulli distribution with parameters Q(p) independent of Y_1 and Y_2 .

Let us make sure that within the model Q = Q(p) the Marshall–Olkin distribution with parameters Λ is anisotropic asymptotically geometric strictly stable.

Theorem 13. Let the random vectors Z_1 and Z_2 be independent and have one and the same Marshall–Olkin distribution with parameters Λ . Then, as $p \to 0$,

$$p \cdot \mathbf{Z}_1 + \vec{\delta}(Q(p)) \cdot \mathbf{Z}_2 \Longrightarrow \mathbf{Z}_2,$$
(43)

that is, this distribution is anisotropic asymptotically geometric strictly stable within the model Q = Q(p).

Proof. Rewriting the left-hand side of (43) in terms of Laplace transforms and repeatedly applying formula (31), we obtain

$$\begin{split} \varphi_{\mathsf{Z}_{1}}(pt) \cdot \mathsf{E}(\varphi_{\mathsf{Z}_{2}}(\vec{\delta} \odot t)) &= \varphi_{\mathsf{Z}_{1}}(pt) \cdot \left[(1-p) \cdot \varphi_{\mathsf{Z}_{2}}(t) + p \cdot \sum_{i \neq 0} \lambda_{i} \cdot \varphi_{\mathsf{Z}_{2}}(\vec{i} \odot t)\right] = \\ &= \varphi_{\mathsf{Z}_{1}}(pt) \cdot \left[(1-p) \cdot \varphi_{\mathsf{Z}_{2}}(t) + p \cdot \left(1 + \sum_{k} t_{k}\right) \cdot \varphi_{\mathsf{Z}_{2}}(t)\right] = \\ &= \varphi_{\mathsf{Z}_{2}}(t) \cdot \varphi_{\mathsf{Z}_{1}}(pt) \left[1 + p \sum_{k} t_{k}\right] = \varphi_{\mathsf{Z}_{2}}(t) \cdot \left[\sum_{i \neq 0} \lambda_{i} \cdot \varphi_{\mathsf{Z}_{2}}(\vec{i} \odot t)\right] \rightarrow \\ &\to \varphi_{\mathsf{Z}_{2}}(t) \cdot \sum_{i \neq 0} \lambda_{i} = \varphi_{\mathsf{Z}_{2}}(t) \end{split}$$

as $p \to 0$ yielding the derived result. \Box

This makes it possible to obtain the following useful result. Let $\mathbf{Z} = (Z_1, ..., Z_d)$ be a random vector with the Marshall–Olkin distribution with parameters $\Lambda = \{\lambda_i, i \in \mathfrak{I}, i \neq \mathbf{0}\}$. Let $a \in \mathbb{R}, \mathbf{Y} = (Y_1, ..., Y_d)$. Denote $\mathbf{Y}^a = (Y_1^a, ..., Y_d^a)$.

Theorem 14. Within the model Q = Q(p) described above, any random vector **Y** that admits the representation

$$\mathbf{Y} \stackrel{a}{=} \mathbf{Z}^{1/\alpha} \odot \mathbf{S}(\alpha, \mu), \tag{44}$$

with the random vector $S(\alpha, \mu)$ having the multivariate strictly stable distribution with characteristic exponent α and spectral measure μ and being independent of \mathbf{Z} , has the anisotropic asymptotically geometric strictly stable distribution.

Proof. It suffices to verify relation (42). Let the random vectors Z_1 and Z_2 be independent and have one and the same Marshall–Olkin distribution with parameters $\Lambda = \{\lambda_i, i \neq 0\}$ such that $\sum_{i \neq 0} \lambda_i = 1$, $\vec{\delta}(p)$ be a binary counterpart to the random vector with the multivariate Bernoulli distribution with parameters Q = Q(p), the random vectors $S_1(\alpha, \mu)$ and $S_2(\alpha, \mu)$ have one and the same multivariate strictly stable distribution with characteristic exponent α . Assume that all the random vectors are independent. Let

$$Y_1 = \mathbf{Z}_1^{1/\alpha} \odot S_1(\alpha, \mu), \quad Y_2 = \mathbf{Z}_2^{1/\alpha} \odot S_2(\alpha, \mu).$$

Then

$$c(p) \cdot \mathbf{Y}_1 + \vec{\delta}(p) \odot \mathbf{Y}_2 = c(p) \cdot \mathbf{Z}_1^{1/\alpha} \odot \mathbf{S}_1(\alpha, \mu) + \vec{\delta}(p) \odot \mathbf{Z}_2^{1/\alpha} \odot \mathbf{S}_2(\alpha, \mu)$$
$$\stackrel{d}{=} (p \cdot \mathbf{Z}_1 + \vec{\delta}(p) \odot \mathbf{Z}_2)^{1/\alpha} \odot \mathbf{S}_1(\alpha, \mu) \Longrightarrow \mathbf{Z}_1^{1/\alpha} \odot \mathbf{S}_1(\alpha, \mu) = \mathbf{Y}_1.$$

The last relation holds by virtue of Theorem 13. \Box

Based on this theorem, it is useful to introduce an anisotropic analog of the Weibull distribution.

Definition 8. Let $\alpha > 0$, the random vector \mathbf{Z} have the Marshall–Olkin distribution with parameters $\Lambda = \{\lambda_{\mathbf{i}}, \mathbf{i} \in \mathfrak{I}, \mathbf{i} \neq \mathbf{0}\}$. The random vector $\mathbf{W}_{\alpha,\Lambda}$ is said to have the anisotropic multivariate Weibull distribution with parameters α, Λ , if it can be represented as

$$W_{\alpha,\Lambda} = (\mathbf{Z})^{1/\alpha}.$$
(45)

In these terms, Theorem 14 can be re-formulated in the following way.

$$\boldsymbol{Y} \stackrel{d}{=} \boldsymbol{W}_{\boldsymbol{\alpha},\boldsymbol{\Lambda}} \odot \boldsymbol{S}(\boldsymbol{\alpha},\boldsymbol{\mu}), \tag{46}$$

where the random vector $S(\alpha, \mu)$ has a multivariate strictly stable distribution with characteristic exponent α and is independent of $W_{\alpha,\Lambda}$, has the anisotropic asymptotically geometric strictly stable distribution.

2.4. Anisotropic Multivariate Analogs of the Mittag–Leffler and Linnik Distributions

Theorems 14 and 15 provide the possibility to suggest several anisotropic generalizations of the multivariate Mittag–Leffler and Linnik distributions. For the definitions of 'conventional' multivariate Mittag–Leffler and Linnik distributions, their history and properties see, e.g., [30].

First consider the generalizations of the Mittag–Leffler distribution. As the base for these we consider relation (25). In that relation both multipliers can be replaced by their (anisotropic) multivariate analogs.

Let $\alpha \in (0, 1)$. If in (25) the exponentially distributed r.v. *E* is replaced by the random vector **Z** with the Marshall–Olkin distribution with parameters Λ , then we obtain the random vector

$$\overline{M}_{\alpha,\Lambda} = S(\alpha, 1) \circ Z^{1/\alpha},\tag{47}$$

whose distribution is the scale mixture of the Marshall–Olkin distribution with the mixing distribution being univariate one-sided strictly stable.

Definition 9. The distribution of the random vector $\overline{M}_{\alpha,\Lambda}$ defined by (47) is called anisotropic multivariate Mittag–Leffler distribution of the first kind with parameters α and Λ .

It is easily seen that the univariate marginal distributions of the so defined anisotropic Mittag–Leffler distribution of the first kind are univariate Mittag–Leffler distributions differing, possibly, by their scale parameters. Consider the following analog of the multiplication theorem for stable distributions.

Theorem 16. Let $0 < \alpha' < 1$, $0 < \alpha \leq 1$, $\Upsilon_{\alpha'\alpha,\Lambda}$ be a random vector having anisotropic multivariate asymptotically geometric strictly stable distribution with parameters $\alpha'\alpha$ and Λ that admits representation (43). Then

$$\boldsymbol{Y}_{\alpha'\alpha,\Lambda} \stackrel{d}{=} \left(\overline{\boldsymbol{M}}_{\alpha',\Lambda}\right)^{1/\alpha'} \odot \boldsymbol{S}(\alpha,\mu).$$

This means that every anisotropic multivariate asymptotically geometric strictly stable distribution is an 'anisotropic' scale mixture of the multivariate strictly stable distribution with greater parameter.

To prove this theorem use Theorem 14 and multiplication theorem for multivariate stable laws (see [30]) and obtain

$$\begin{aligned} Y_{\alpha'\alpha} &= \mathbf{Z}^{1/\alpha'\alpha} \odot \mathbf{S}(\alpha'\alpha,\mu) \stackrel{d}{=} \mathbf{Z}^{1/\alpha'\alpha} \odot \left(S^{1/\alpha}(\alpha',1) \circ \mathbf{S}(\alpha,\mu) \right) \stackrel{d}{=} \\ &\stackrel{d}{=} \left(S(\alpha',1) \circ \mathbf{Z}^{1/\alpha'} \right)^{1/\alpha} \odot \mathbf{S}(\alpha'\alpha,\mu) \stackrel{d}{=} \left(\overline{\mathbf{M}}_{\alpha,\Lambda} \right)^{1/\alpha} \odot \mathbf{S}(\alpha,\mu). \end{aligned}$$

Using the multiplication theorem for one-sided strictly stable distributions we can obtain a similar recursive mixture representation for the anisotropic multivariate Mittag– Leffler distributions of the first kind themselves. **Theorem 17.** Let $0 < \alpha' < 1$, $0 < \alpha \leq 1$, $\overline{M}_{\alpha,\Lambda}$ be a random vector having anisotropic multivariate Mittag–Leffler distribution of the first kind parameters α and Λ . Then

$$\overline{\boldsymbol{M}}_{\alpha'\alpha,\Lambda} \stackrel{d}{=} S(\alpha,1) \circ \left(\overline{\boldsymbol{M}}_{\alpha'\Lambda}\right)^{1/\alpha}$$

Proof. Using Definition 9 and multiplication theorem for univariate stable distributions (see Theorem 3.3.1 in [2]) we obtain

$$\overline{\boldsymbol{M}}_{\alpha'\alpha,\Lambda} \stackrel{d}{=} S(\alpha'\alpha,1) \circ \mathbf{Z}^{1/\alpha'\alpha} \stackrel{d}{=} S(\alpha',1)^{1/\alpha} \circ S(\alpha,1) \circ \mathbf{Z}^{1/\alpha'\alpha} \stackrel{d}{=} \\ \stackrel{d}{=} S(\alpha,1) \circ \left(S(\alpha',1) \circ \mathbf{Z}^{1/\alpha'}\right)^{1/\alpha} \stackrel{d}{=} S(\alpha,1) \circ \left(\overline{\boldsymbol{M}}_{\alpha'\Lambda}\right)^{1/\alpha}$$

yielding the desired result. \Box

Consider the second version of the anisotropic generalization of the multivariate Mittag–Leffler distribution.

Definition 10. Let $\alpha \in (0, 1)$, the random vector **Z** have the Marshall–Olkin distribution with parameters Λ , $S(\alpha, \mu^+)$ be a random vector with the one-sided multivariate strictly stable distribution. The distribution of the random vector

$$\widetilde{M}_{\alpha,\Lambda} \stackrel{d}{=} \mathbf{Z}^{1/\alpha} \odot \mathbf{S}(\alpha,\mu^+)$$

is called anisotropic multivariate Mittag-Leffler distribution of the second kind.

The anisotropic multivariate Mittag–Leffler distributions of the first and second kind are related by the following theorem.

Theorem 18. *Let* $0 < \alpha' < 1$, $0 < \alpha \le 1$. *Then*

$$\widetilde{M}_{\alpha'\alpha,\Lambda} \stackrel{d}{=} \overline{M}_{\alpha',\Lambda}^{1/\alpha} \odot S(\alpha,\mu^+).$$

Proof. Using Definition 10 and the multiplication theorem for multivariate one-sided strictly stable distributions (see [30]) we obtain

$$\widetilde{M}_{\alpha'\alpha,\Lambda} \stackrel{d}{=} \mathbf{Z}^{1/\alpha'\alpha} \odot \mathbf{S}(\alpha'\alpha,\mu^+) \stackrel{d}{=} \mathbf{Z}^{1/\alpha'\alpha} \odot \left(S^{1/\alpha}(\alpha',1) \circ \mathbf{S}(\alpha,\mu^+) \right) \stackrel{d}{=} \\ \stackrel{d}{=} \left(S(\alpha',1) \circ \mathbf{Z}^{1/\alpha'} \right)^{1/\alpha} \odot \mathbf{S}(\alpha,\mu^+) \stackrel{d}{=} \overline{M}_{\alpha',\Lambda}^{1/\alpha} \odot \mathbf{S}(\alpha,\mu^+)$$

yielding the desired result. \Box

Now turn to the Linnik distribution. Let $\alpha \in (0, 2]$. In [31,32], it was demonstrated that

$$L_{\alpha} \stackrel{d}{=} E^{1/\alpha} \circ S(\alpha, 0). \tag{48}$$

The multivariate Linnik distribution was introduced in [33] where it was proved that the function

$$\mathbf{f}_{\alpha,\Sigma}^{(L)}(t) = \left[1 + (t^{\top} \Sigma t)^{\alpha/2}\right]^{-1}, \quad t \in \mathbb{R}^d, \ \alpha \in (0,2),$$

$$\tag{49}$$

is the characteristic function of a *d*-variate probability distribution, where Σ is a positive definite $(d \times d)$ -matrix. It is not difficult to make sure that the characteristic function (49) corresponds to the random vector

$$L_{\alpha} \stackrel{a}{=} E^{1/\alpha} \circ S(\alpha, \Sigma). \tag{50}$$

Relations (48) and (50) can be the starting point for the anisotropic generalization of the Linnik distribution.

First, by analogy with Definition 9 replace the exponentially distributed r.v. *E* in (48) by a random vector having the Marshall–Olkin distribution with parameters Λ .

Definition 11. Let $\alpha \in (0, 2]$, the random vector **Z** have the Marshall–Olkin distribution with parameters Λ . The distribution of the random vector

$$\overline{L}_{\alpha,\Lambda} = S(\alpha,0) \circ \mathbf{Z}^{1/\alpha}$$

is called the anisotropic multivariate Linnik distribution of the first kind.

Using the univariate multiplication theorem we obtain

Theorem 19. Let $\alpha \in (0, 2]$, the random vector **Z** have the Marshall–Olkin distribution with parameters Λ , X be a r.v. with the standard normal distribution. Then

$$\overline{L}_{\alpha,\Lambda} \stackrel{d}{=} X \circ \left(2\overline{M}_{\alpha/2,\Lambda} \right)^{1/2}$$

As is known, univariate Linnik distributions are normal scale mixtures with the mixing univariate Mittag–Leffler distributions. Therefore, in other words, Theorem 11 means that any anisotropic multivariate Linnik distribution of the first kind is a multivariate distribution whose all univariate marginals are normal scale mixtures, moreover, they are univariate Linnik distributions differing, possibly, by their scale parameters.

To prove Theorem 11, use Definition 9 and the univariate multiplication theorem and obtain

$$\overline{L}_{\alpha,\Lambda} \stackrel{d}{=} \sqrt{2}S(\alpha,0) \circ \mathbf{Z}^{1/\alpha} \stackrel{d}{=} \sqrt{2}S(\frac{\alpha}{2},1) \circ X \circ \mathbf{Z}^{1/\alpha} \stackrel{d}{=} \\ \stackrel{d}{=} \left(2S(\frac{\alpha}{2},1) \circ \mathbf{Z}^{2/\alpha}\right)^{1/2} \circ X \stackrel{d}{=} X \circ \left(2\overline{\mathbf{M}}_{\alpha/2,\Lambda}\right)^{1/2}.$$

A more general version of Theorem 11 is the following statement.

Theorem 20. *Let* $0 < \alpha' < 1, 0 < \alpha \le 1$ *. Then*

$$\overline{L}_{\alpha'\alpha,\Lambda} \stackrel{d}{=} S(\alpha,0) \circ \overline{M}_{\alpha',\Lambda}^{1/\alpha}.$$

Definition 12. Let $\alpha \in (0,2]$, the random vector **Z** have the Marshall–Olkin distribution with parameters Λ . Σ be a positive definite $(d \times d)$ -matrix, $S(\alpha, \Sigma)$ be the elliptically contoured multivariate stable distribution with characteristic exponent α . The distribution of the random vector

$$\widetilde{L}_{\alpha,\Lambda} = \mathbf{Z}^{1/\alpha} \odot S(\alpha, \Sigma)$$

is called the anisotropic multivariate Linnik distribution of the second kind.

Definition 9, 12 and multiplication theorem in [30] imply the following result.

Theorem 21. *Let* $0 < \alpha' < 1$, $0 < \alpha \le 1$. *Then*

$$\widetilde{L}_{\alpha'\alpha,\Lambda} \stackrel{d}{=} \overline{M}_{\alpha',\Lambda}^{1/\alpha} \odot S(\alpha,\Sigma).$$

Using the multivariate multiplication theorem in [30] we obtain the following important corollary of Theorem 22.

Theorem 22. *Let* $0 < \alpha \leq 2$ *. Then*

$$\widetilde{L}_{\alpha,\Lambda} \stackrel{d}{=} \left(2\overline{M}_{\alpha/2,\Lambda}\right)^{1/2} \odot X_{\Sigma}.$$

In other words, the anisotropic multivariate Linnik distribution of the second kind is an anisotropically scale mixed multivariate normal distribution.

The two last theorems are proved by the reasoning similar to that used to prove Theorems 19 and 20.

With $\alpha = 2$ Definitions 11 and 12 describe the anisotropic multivariate Laplace distributions of the first and second kinds, respectively.

It should be noted that all the distributions considered in this section are anisotropic multivariate asymptotically geometric strictly stable distributions.

3. Conclusions

In this paper, the multivariate analog of the Zolotarev characterization problem is considered as well as related limit theorems for general multivariate geometric random sums. In the preceding studies, as a rule, this problem was considered either for the case where the summation index is univariate [21], or for the case where a strong additional restriction of the independence of coordinates was imposed [34,35]. The general case of the multivariate summation index with the multivariate geometric distribution is for the first time considered in this paper. A notion of a general multivariate geometric stability is introduced and it is shown that, in general, the limit distributions appearing in the model under consideration do not possess this property. Consequently, a limit analog of this property, the anisotropic multivariate geometric stability, is introduced. It is demonstrated that all the the anisotropic multivariate geometric stable distributions are limiting in the problem under consideration. Their structure and relation with the Marshall–Olkin distribution are discussed. Important special cases of anisotropic multivariate geometric stable distributions, for example, anisotropic multivariate Linnik and Mittag–Leffler distributions are considered and some of their properties are discussed.

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