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Production/Inventory Policies for a Two-Echelon System with Credit Period Incentives

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Abstract: Trade credit is a crucial source of capital particularly for small businesses with limited financing opportunities. Inventory models considering trade credit financing have been widely studied. However, while there is extensive research on the single-vendor single-buyer inventory model allowing delays in payments, the systems where the vendor supplies to more than one buyer have received less attention. In this paper, we analyze a two-echelon inventory system where a single vendor supplies an item to two buyers who face a constant deterministic demand. The vendor produces the items at a finite rate and offers the buyers a delay payment period. That is, the buyers can delay the payment for the purchased items until the end of the credit period. Therefore, during such a period, the buyers sell the items and use the sales revenue to earn interest. At the end of the credit period, the buyers should pay the purchasing cost to the vendor for which external funding may be necessary. It is widely accepted that, in general, centralized policies reduce the total cost of the supply chain. Therefore, we first deal with an integrated model assuming that the vendor and the buyers make decisions jointly. However, in some cases, the buyers are not willing to collaborate, and the management of the supply chain has to be carried out in a decentralized manner. Hence, we also address the problem under a non-cooperative setting. Numerical examples are presented to illustrate both models. Additionally, we perform a computational experiment to compare both strategies, and a sensitivity analysis of the parameters is also carried out. From the results, we derived that, in general, it was more profitable to follow the integrated policy excepting when the replenishment costs for the buyers were high. Finally, in order to validate the computational results, a statistical analysis is performed.

Keywords: integrated inventory model; non-integrated inventory model; trade credit

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1. Introduction

In business transactions, buyers usually are not forced to make the payment to the vendor when they receive the items. The vendor frequently allows the buyers a credit period for settling the amount owed instead. Indeed, trade credit is an increasingly general payment agreement between vendors and buyers who consider that it is an important source of external financing. During the credit period, the buyers earn interest on the revenue of items sold. At the end of the permissible delay period, the buyers should settle the debts for which external funding may be necessary. In recent decades, efforts have been carried out to develop inventory models considering trade credit financing.

However, many researchers have addressed the problem only from the perspective of the buyer in order to compute replenishment policies that minimize the total cost at that installation, without taking into account the cost at the vendor. It is well known that, usually, the coordination among the supply chain members provides a lower cost than if a decentralized strategy is followed. Hence, we can also find studies dealing with the integrated single-vendor single-buyer inventory model allowing delay in payments. Next, we present the most relevant works about these topics.

The idea of studying inventory models under trade credit financing was initiated by Haley and Higgins [1]. Then, Goyal [2] developed an economic order quantity (EOQ) model assuming that the vendor allows the buyer a predefined period to settle the account. Another approach for determining the economic order quantity in Goyal's [2] model was also proposed by Chung [3].

Teng [4] improved Goyal's [2] model by including the difference between the selling price and the purchase cost, and derived a closed-form solution to the inventory problem with a permissible delay in payments. Chung et al. [5] demonstrated that the proofs of the solution procedure of Teng [4] had shortcomings. Thus, they corrected those proofs and completed the results proposed by Teng [4]. From the notable Goyal's [2] work, several researchers have widely studied different inventory models under a permissible delay in payments. For example, Hwang and Shinn [6] presented a model to optimize both the price and the lot size.

Shah [7] and Aggarwal and Jaggi [8] extended the analysis to consider deteriorating items. Jamal et al. [9] studied the inventory model with a delay in payments and allowing shortages. Chang et al. [10], Chung and Liao [11] and Chung et al. [12] dealt with deteriorating items but the trade credit depended on the order quantity. The economic production quantity (EPQ) model with trade credit was addressed by Chung and Huang [13], Huang [14], and Hu and Liu [15]. Huang [16] presented an inventory model assuming that the buyer also offers a credit period to the customers.

Jaggi et al. [17] also assumed a two-level trade credit where demand was credit-linked. Annadurai and Uthayakumar [18] extended the Jaggi et al. [17] model by considering deteriorating items. Other interesting contributions to the inventory models under permissible delay in payments are given in [19–25]. Seifert et al. [26] provide an excellent review of inventory lot-size models under trade credits.

The above mentioned papers addressed the problem only from the side of the buyer in order to compute replenishment policies that minimize the total cost at that installation. However, the total cost of the system is usually reduced when the decisions in the supply chain are coordinated. Walmart was one of the first companies to cooperate with the suppliers, and the benefits increased considerably. The idea of minimizing the total cost in a single-vendor single-buyer inventory model was analyzed by Goyal [27]. Then, Banerjee [28] extended Goyal's model [27] to consider a finite production rate.

Afterward, many researchers, such as Lu [29], Hill [30,31], and Abdul-Jalbar et al. [32–36] analyzed different integrated inventory systems under various shipment policies. The reader is referred to Ben-Daya et al. [37] and Glock [38] for reviews of related works. Nevertheless, these studies assume that the buyers pay as soon as the items are received. Abad and Jaggi [39] incorporated the effect of trade credit on the integrated single-vendor single-buyer inventory model. Subsequently, numerous works dealing with the impact of the delay in payments on integrated inventory models under different conditions have been developed by different authors (see [40–63]).

The previous studies focused on the single-vendor single-buyer system. Clearly, vendors usually supply to multiple buyers in practice. However, the simplest supply chain consisting of a single vendor and two buyers is widely analyzed in the literature because it may offer insights to develop possible strategies for the general single-vendor multi-buyer problem. Indeed, we can find relevant studies in the literature dealing with the single-vendor two-buyer system, such as Siajedi et al. [64,65], Rad et al. [66], Jia et al. [67], Abdul-Jalbar et al. [68,69], among others.

More precisely, Jia et al. [67] showed how a single-vendor multi-buyer system can be easily addressed as a single-vendor two-buyer model by applying a clustering approach. Accordingly, the set of buyers can be partitioned into two groups, and those assigned to the same cluster are supplied at a common time; hence, the replenishment policies are easier to implement.

It is remarkable that most of the above articles considering two buyers do not allow for a delay in payments. In this paper, we study the inventory problem for the single-vendor two-buyer system under a trade credit policy.

Table 1 summarizes some of the references related to trade credit in two-echelon supply chains. To the best of our knowledge, few papers focused on the single-vendor two-buyer system under a permissible delay in payments. In particular, Giri and Maiti [70] and Hosseini-Motlagh et al. [71] studied a supply chain with trade credit where one vendor supplies the same item to two competing buyers and the demand is sensitive to the trade credit period. Abdul-Jalbar et al. [69] dealt with an inventory system where the vendor supplies a product to two non-competing buyers considering constant demand and instantaneous deliveries. Unlike previous models, in this work, we consider two non-competing buyers and a finite production rate at the vendor.

Table 1. Summary of works related to trade credit in two-echelon supply chains.

Paper	Supply Chain Structure	Model	Levels of Trade Credit	Demand	Credit Period	Vendor Production Rate
Abad and Jaggi [39]	single-vendor single-buyer	cooperative and non-cooperative	one	price dependent	decision variable	infinite
Jaber and Osman [40]	single-vendor single-buyer	integrated and non-integrated	one	constant	decision variable	infinite
Yang and Wee [41]	single-vendor single-buyer	cooperative	one	price dependent	fixed parameter	finite
Chen and Kang [42,43]	single-vendor single-buyer	integrated and non-integrated	one	constant	fixed parameter	finite
Ho et al. [44]	single-vendor single-buyer	integrated	one	price dependent	fixed parameter	finite
Ouyang et al. [45]	single-vendor single-buyer	integrated	one	price dependent	order-size dependent	finite
Ouyang et al. [46]	single-vendor single-buyer	integrated	one	constant	order-size dependent	finite
Chang et al. [47]	single-vendor single-buyer	integrated	one	price dependent	fixed parameter	infinite
Huang et al. [48], Su [52]	single-vendor single-buyer	integrated	one	constant	fixed parameter	finite
Ho [49]	single-vendor single-buyer	integrated	two	price and credit dependent	fixed parameters	finite
Teng et al. [50]	single-vendor single-buyer	integrated and non-cooperative	one	constant	fixed parameter	finite
Duan et al. [51]	single-vendor single-buyer	centralized and decentralized	one	constant	decision variable	infinite
Chern et al. [53]	single-vendor single-buyer	non-cooperative	one	credit dependent	decision variable	finite
Giri and Sharma [54]	single-vendor single-buyer	integrated	one	constant	decision variable	finite
Giri and Maiti [70]	single-vendor two competing buyers	integrated	two	credit dependent	fixed parameter for the vendor and decision variables for the buyers	finite
Hosseini-Motlagh et al. [71]	single-vendor two competing buyers	centralized and decentralized	one	credit dependent	decision variable	infinite
Abdul-Jalbar et al. [69]	single-vendor two-buyers	integrated and non-integrated	one	constant	fixed parameter	infinite
This work	single-vendor two-buyers	integrated and non-integrated	one	constant	fixed parameter	finite

From the supply chain literature, it is well known that the total inventory cost of the system is generally lower when the replenishment policies are determined through an integrated model. Concretely, in general, the total cost of the vendor decreases while the total costs for the buyers increase. Normally, the cost decrease at the vendor is greater than the cost increase at the buyers. Therefore, in most cases, the integrated policies are beneficial if the goal is to minimize the total cost of the system.

Evidently, when all the supply chain members belong to the same company, the integrated model can be easily applied. In the case that the buyers belong to different companies, the vendor can convince them to follow a centralized policy if the profits are adequately shared. Several researchers have proposed mechanisms for sharing the profits. See for example, Goyal [27], Wu and Ouyang [72], Ouyang et al. [73], Ho et al. [44] and Sajadieh et al. [74], among others.

However, the buyers may not be willing to cooperate even though the vendor shares the profits. For example, the buyers may not be inclined to share information necessary to apply the integrated model or the buyers may be more influential than the vendor. In this case each buyer could prefer to follow the replenishment policy that minimizes the total cost without taking into account the total cost of the vendor.

The main aim of this paper is to address the inventory problem related to the single-vendor two-buyer system with a permissible delay in payments, studying the cases of independence and coordination among the vendor and the buyers. We analyze both situations and state the problem under an integrated and a non-integrated perspective.

The rest of the paper is organized as follows. In the next section, we introduce the notation and assumptions required to formulate the problem under both scenarios. In Section 3, we analyze the problem assuming that the vendor and the buyers make decisions cooperatively to minimize the joint total cost of the system. In Section 4, we focus on the case where the vendor and the buyers make decisions in order to minimize their costs individually. Section 5 provides numerical examples that are solved using both models. In order to compare both strategies, we perform a computational experiment. The results and the statistical analysis are reported in Section 6. Finally, our concluding remarks and proposals for forthcoming research are presented.

2. Assumptions and Notation

A two-echelon inventory system consisting of one vendor and two buyers is considered. A single item is produced at a finite rate by the vendor that supplies both buyers allowing a credit period. The demand rate at each buyer is constant over time and shortages are not allowed.

There is a unit holding cost per unit of time at each buyer and at the vendor. We consider that the holding cost at the vendor is less than the holding cost at each buyer. This assumption is normal and logical because the vendor is generally more specialized than the buyers in terms of facilities and personnel. The average annual cost to carry a unit of inventory at the vendor is, thus, less than the corresponding costs at the buyers. Each time a batch is produced by the vendor a fixed setup cost is charged. The buyers incur a fixed delivery cost when they place an order.

During the credit period, the buyers sell the products and use the incomes to earn interest. At the end of the delay period, the buyers pay the purchasing cost to the vendor for which external funding may be necessary. In addition, since the payment is not received until the end of the credit period, the vendor incurs an opportunity cost. Under these assumptions, we first study the problem considering a centralized setting where the vendor and the buyers make decisions jointly in order to minimize the cost of the whole system.

Then, the problem is addressed from a decentralized standpoint in which the vendor and the buyers make decisions independently. To be more precise, we assume that the buyers compute first their individual replenishment policies, and then the vendor determines the optimal strategy under the decisions made by the buyers.

Throughout the paper, we use the following notation.

d_j	demand rate at buyer j per unit of time, $j = 1, 2$
D	sum of demand rate at both buyers, $D = d_1 + d_2$
P	finite production at the vendor per unit of time, $P > D$
h_0	vendor's unit holding cost per unit of time
h_j	buyer j 's unit holding cost per unit of time, $j = 1, 2$ ($h_0 < h_j, \forall j$)
k_0	vendor's setup cost per production run
k_j	buyer j 's replenishment cost per order, $j = 1, 2$
p_0	price that any buyer pays for an item to the vendor
p_j	buyer j 's selling price per unit, $j = 1, 2$, ($p_j > p_0, \forall j$)
M	length of the trade credit period offered by the vendor to the buyers
I_{e_j}	buyer j 's interest rate earned per monetary unit and per unit of time, $j = 1, 2$
I_{c_j}	buyer j 's interest rate charged per monetary unit and per unit of time, $j = 1, 2$
I_0	interest rate per monetary unit and per unit of time for determining the vendor's opportunity cost due to the delayed payments
t_0	time between two consecutive setups at the vendor in the integrated model
t_j	buyer j 's replenishment cycle time, $j = 1, 2$
q_j	lot size for buyer j , $j = 1, 2$, ($q_j = d_j t_j$)
$\overline{D_0}$	demand vector at the vendor in the non-integrated model
$\overline{T_0}$	vector containing the time instants where the buyers place an order to the vendor in the non-integrated model
$\overline{Q_0}$	production quantities vector at the vendor in the non-integrated model
C_0	total cost per unit of time incurred by the vendor
C_j	total cost per unit of time incurred by buyer j , $j = 1, 2$
$C_{Integrated}$	total cost per unit of time for the system in the integrated model
$C_{Non-integrated}$	total cost per unit of time for the system in the non-integrated model

3. Mathematical Formulation for the Integrated Model

In the integrated model, the vendor and the buyers make decisions jointly in order to minimize the cost of the whole system. From the multi-echelon inventory literature, it is well known that the optimal policy for the single-vendor multi-buyer systems can be very complex and, even if it could be computed efficiently, it might be very difficult to apply in practice (Schwarz [75]). Thus, many authors have analyzed such systems considering a simpler class of strategies, such as the single-cycle policies.

These policies are very efficient in many situations and have clear managerial advantages. Hence, we state the integrated model in terms of the single-cycle policies. That is, for each buyer j , $j = 1, 2$, the constraint $t_0 = n_j t_j$, $n_j \in \mathbb{N}$ must be fulfilled, where t_0 is the inventory cycle of the vendor, t_j is the replenishment cycle time of buyer j , and n_j represents the number of orders that buyer j places to the vendor during t_0 . Next, we focus on computing the total cost per unit of time for the system, which consists of the total cost per unit of time for the buyers and for the vendor.

3.1. Total Cost per Unit of Time for Buyer j

The total cost for buyer j includes the replenishment and holding costs as well as the interest payable and the interest earned.

The replenishment cost and the holding cost for buyer j per unit of time are easily computed as k_j/t_j and $h_j q_j/2 = h_j d_j t_j/2$, respectively.

In order to compute the interest payable and the interest earned, we should distinguish between the case $t_j < M$ and $t_j \geq M$. In the former case, there are still items in stock when buyer j must pay the purchasing cost to the vendor. In contrast, in the latter case, the inventory is completely depleted before the delayed payment due date. The following result establishes the total cost per unit time for any buyer.

Proposition 1. The total cost per unit time for buyer j , $j = 1, 2$, is

$$C_j(t_j) = \begin{cases} \frac{k_j}{t_j} + \frac{h_j d_j t_j}{2} - I_{e_j} p_j d_j (M - \frac{t_j}{2}), & \text{if } t_j < M \\ \frac{k_j}{t_j} + \frac{h_j d_j t_j}{2} - \frac{I_{e_j} p_j d_j M^2}{2t_j} + I_{c_j} p_0 d_j \frac{(t_j - M)^2}{2t_j}, & \text{if } t_j \geq M \end{cases} \quad (1)$$

Proof. (i) If $t_j < M$, then the buyer has sold all the items before the payment due date and is able to settle the account without paying any interest charges. Moreover, during the period $[0, t_j]$, the buyer uses the sales revenue to earn interest at a rate of I_{e_j} . In addition, during the period $[t_j, M]$ the buyer earns interest on the total revenue, $p_j d_j t_j$. Therefore, the interest earned per unit of time by buyer j is computed as follows

$$\frac{I_{e_j} p_j \int_0^{t_j} d_j t \, dt + I_{e_j} p_j d_j t_j (M - t_j)}{t_j} = \frac{I_{e_j} p_j \frac{d_j t_j^2}{2} + I_{e_j} p_j d_j t_j (M - t_j)}{t_j} = I_{e_j} p_j d_j (M - \frac{t_j}{2})$$

Consequently, summing the holding and the replenishment costs, the total cost per unit of time for buyer j is

$$C_j = \frac{k_j}{t_j} + \frac{h_j d_j t_j}{2} - I_{e_j} p_j d_j (M - \frac{t_j}{2})$$

(ii) If $t_j \geq M$, then during the period $[0, M]$ the buyer uses the incomes to earn interest at a rate of I_{e_j} . Hence, the interest earned per unit of time by buyer j is

$$\frac{I_{e_j} p_j \int_0^M d_j t \, dt}{t_j} = I_{e_j} p_j d_j \frac{M^2}{2t_j}$$

On the other hand, at time M , the buyer has some inventory available. Therefore, these items still in stock have to be financed at interest rate of I_{c_j} . Accordingly, the interest payable per unit of time by buyer j during the period $[M, t_j]$ is

$$\frac{I_{c_j} p_0 \int_M^{t_j} d_j (t_j - t) \, dt}{t_j} = I_{c_j} p_0 d_j \frac{(t_j - M)^2}{2t_j}$$

Thus, the total cost per unit of time for buyer j is

$$C_j = \frac{k_j}{t_j} + \frac{h_j d_j t_j}{2} - \frac{I_{e_j} p_j d_j M^2}{2t_j} + I_{c_j} p_0 d_j \frac{(t_j - M)^2}{2t_j}$$

□

3.2. Total Cost per Unit of Time for the Vendor

It is worth noting that the average inventory at the vendor can be computed as the average total inventory less the average inventory at the buyers. As depicted in Figure 1, when production starts, the total inventory at the system is at a minimum, and the inventory at the vendor is equal to zero. In addition, the inventory at each buyer is just enough to satisfy the demands until the next shipment. The following result determines the total cost per unit of time for the vendor.

Figure 1. Inventory fluctuations at the buyers and at the vendor. The dotted line represents the inventory fluctuations for the total system.

Proposition 2. The total cost per unit time for the vendor is

$$C_0(t_0, t_1, t_2) = \frac{k_0}{t_0} + h_0 \left[\frac{D(d_1t_1 + d_2t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0D}{2} - \frac{(d_1t_1 + d_2t_2)}{2} \right] + I_0 p_0 MD \quad (2)$$

Proof. Since buyer j orders $q_j = d_j t_j$ units, for $j = 1, 2$, the time required to produce the quantities ordered by both buyers is $(q_1 + q_2)/P = (d_1 t_1 + d_2 t_2)/P$. Hence, the inventory level at buyer j should be $d_j(d_1 t_1 + d_2 t_2)/P$ when production starts, and the inventory level at the whole system should be $D(d_1 t_1 + d_2 t_2)/P$. Then, the vendor continues to produce all the units demanded by both buyers during a cycle t_0 , which are $t_0 D$ units. Therefore, the production lasts for $t_0 D/P$ units of time. During this period, the stock in the total system increases at a rate of $P - D$.

As shown in Figure 1, the total inventory is at the maximum at instant $t_0 D/P$, that is, when the vendor has produced all the required units. Hence, the maximum value for the total inventory at the system is $D(d_1 t_1 + d_2 t_2)/P + (1 - \frac{D}{P})t_0 D$.

Accordingly, the average total inventory is given by

$$\frac{1}{t_0} \left[\frac{D(d_1 t_1 + d_2 t_2)}{P} t_0 + t_0 \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} \right] = \frac{D(d_1 t_1 + d_2 t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} \quad (3)$$

Once the average total inventory has been computed, the average inventory at the vendor is easily calculated by subtracting the average inventory at the buyers to give

$$\frac{D(d_1 t_1 + d_2 t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} - \frac{(d_1 t_1 + d_2 t_2)}{2}$$

Therefore, the holding cost per unit of time for the vendor can be expressed as

$$h_0 \left[\frac{D(d_1 t_1 + d_2 t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} - \frac{(d_1 t_1 + d_2 t_2)}{2} \right]$$

Since the vendor offers a fixed credit period M to the buyers, they will not pay to the vendor the purchasing cost until the end of M . The vendor, throughout the inventory cycle t_0 , has sold $d_j t_0$ units to buyer j , ($j = 1, 2$). If delay in payments was not allowed, the vendor would have $p_0 t_0 (d_1 + d_2) = p_0 t_0 D$ monetary units. This amount would have generated an interest of $I_0 p_0 t_0 D M$ in the credit period M . Therefore, during the period t_0 , the opportunity cost per unit of time of the vendor is $I_0 p_0 D t_0 M / t_0 = I_0 p_0 D M$.

Finally, since the setup cost per unit of time for the vendor is k_0 / t_0 , the total cost per unit of time for the vendor is

$$C_0 = \frac{k_0}{t_0} + h_0 \left[\frac{D(d_1 t_1 + d_2 t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} - \frac{(d_1 t_1 + d_2 t_2)}{2} \right] + I_0 p_0 D M$$

□

3.3. Total Cost per Unit of Time for the Joint System

Once the costs for the vendor and for the buyers have been computed by Propositions 1 and 2, the total cost per unit of time for the system can be formally expressed as follows

$$C_{\text{Integrated}}(t_0, t_1, t_2) =$$

$$= \begin{cases} \frac{k_0}{t_0} + h_0 \left[\frac{D(d_1 t_1 + d_2 t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} - \frac{(d_1 t_1 + d_2 t_2)}{2} \right] + I_0 p_0 D M \\ + \frac{k_1}{t_1} + \frac{h_1 d_1 t_1}{2} - I_{e1} p_1 d_1 \left(M - \frac{t_1}{2}\right) + \frac{k_2}{t_2} + \frac{h_2 d_2 t_2}{2} - I_{e2} p_2 d_2 \left(M - \frac{t_2}{2}\right), & \text{if } t_1, t_2 \leq M \\ \\ \frac{k_0}{t_0} + h_0 \left[\frac{D(d_1 t_1 + d_2 t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} - \frac{(d_1 t_1 + d_2 t_2)}{2} \right] + I_0 p_0 D M \\ + \frac{k_1}{t_1} + \frac{h_1 d_1 t_1}{2} - I_{e1} p_1 d_1 \left(M - \frac{t_1}{2}\right) + \frac{k_2}{t_2} + \frac{h_2 d_2 t_2}{2} - \frac{I_{e2} p_2 d_2 M^2}{2t_2} + I_{c2} p_0 d_2 \frac{(t_2 - M)^2}{2t_2}, & \text{if } t_1 \leq M \leq t_2 \\ \\ \frac{k_0}{t_0} + h_0 \left[\frac{D(d_1 t_1 + d_2 t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} - \frac{(d_1 t_1 + d_2 t_2)}{2} \right] + I_0 p_0 D M \\ + \frac{k_1}{t_1} + \frac{h_1 d_1 t_1}{2} - \frac{I_{e1} p_1 d_1 M^2}{2t_1} + I_{c1} p_0 d_1 \frac{(t_1 - M)^2}{2t_1} + \frac{k_2}{t_2} + \frac{h_2 d_2 t_2}{2} - I_{e2} p_2 d_2 \left(M - \frac{t_2}{2}\right), & \text{if } t_2 \leq M \leq t_1 \\ \\ \frac{k_0}{t_0} + h_0 \left[\frac{D(d_1 t_1 + d_2 t_2)}{P} + \left(1 - \frac{D}{P}\right) \frac{t_0 D}{2} - \frac{(d_1 t_1 + d_2 t_2)}{2} \right] + I_0 p_0 D M \\ + \frac{k_1}{t_1} + \frac{h_1 d_1 t_1}{2} - \frac{I_{e1} p_1 d_1 M^2}{2t_1} + I_{c1} p_0 d_1 \frac{(t_1 - M)^2}{2t_1} + \frac{k_2}{t_2} + \frac{h_2 d_2 t_2}{2} - \frac{I_{e2} p_2 d_2 M^2}{2t_2} + I_{c2} p_0 d_2 \frac{(t_2 - M)^2}{2t_2}, & \text{if } M \leq t_1, t_2 \end{cases}$$

Consequently, the optimal single-cycle policy may be found by solving the following mixed integer non-linear programming problem (MINLP).

$$\begin{aligned}
& \min C_{Integrated}(t_0, t_1, t_2) \\
& \text{s.t.} \\
& (d_1 t_1 + d_2 t_2) / P \leq t_1 \\
& (d_1 t_1 + d_2 t_2) / P \leq t_2 \\
& t_0 = n_1 t_1 \\
& t_0 = n_2 t_2 \\
& t_1, t_2 \in \mathcal{R}_+ \\
& n_1, n_2 \in \mathcal{N}
\end{aligned} \tag{4}$$

The first two constraints guarantee that the solutions are feasible, that is, the vendor delivers the orders on time. At the beginning of each cycle, the vendor supplies both buyers, and thus it must be ensured that the time required to produce $d_1 t_1 + d_2 t_2$ units of item is smaller than the replenishment cycle time at the buyers. In addition, the problem is formulated in terms of single-cycle policies, that is, we assumed that buyer j places n_j orders during t_0 . These restrictions are also considered in the model.

Therefore, the problem is formulated as a MINLP problem that can be solved using available software with mixed integer non-linear programming libraries. In particular, we developed a C++ program that uses LINDO API to calculate the optimal solution for each part of the cost function. The solution with the minimum cost associated is the optimal single-cycle policy for the problem.

4. Mathematical Formulation for the Non-Integrated Model

In this section, we analyze the problem assuming that the buyers are not willing to cooperate. For example, they may not be agree to share their private information especially in regard to the cost structure and the market demand. Therefore, a decentralized decision-making is proposed where each buyer is interested in minimizing their own cost independently. Accordingly, under this non-cooperative environment, first the replenishment cycle times at the buyers are computed, and then the shipment schedule at the vendor is determined.

4.1. Optimal Replenishment Policies for the Buyers

From (1), the total cost for buyer j with $j = 1, 2$ can be expressed as follows

$$C_j(t_j) = \begin{cases} \frac{k_j}{t_j} + \frac{(h_j + I_e p_j) d_j t_j}{2} - I_e p_j d_j M, & \text{if } t_j < M \\ \frac{k_j}{t_j} + \frac{(h_j + I_c p_0) d_j t_j}{2} + \frac{(I_c p_0 - I_e p_j) d_j M^2}{2 t_j} - I_c p_0 d_j M, & \text{if } t_j \geq M \end{cases} \tag{5}$$

Minimizing $C_j(t_j)$ with respect to t_j , it follows that the optimal replenishment cycle time at buyer j is given by

$$t_j^* = \begin{cases} \left[\frac{2k_j}{d_j(h_j + I_e p_j)} \right]^{\frac{1}{2}}, & \text{if } 2k_j < \eta_j \\ M, & \text{if } 2k_j = \eta_j \\ \left[\frac{2k_j + d_j M^2 (I_c p_0 - I_e p_j)}{d_j(h_j + I_c p_0)} \right]^{\frac{1}{2}}, & \text{if } 2k_j > \eta_j \end{cases} \tag{6}$$

where $\eta_j = d_j M^2 (h_j + I_e p_j)$.

Note that if $2k_j > \eta_j$, then $2k_j + d_j M^2 (I_c p_0 - I_e p_j) > 0$ since $2k_j + d_j M^2 (I_c p_0 - I_e p_j) > d_j M^2 (h_j + I_c p_0) > 0$.

The economic order quantity for buyer j , is $q_j^* = d_j t_j^*$, $j = 1, 2$, and the minimum cost for buyer j , is

$$C_j^* = C_j(t_j^*) = \begin{cases} \left[2k_j d_j (h_j + I_{c_j} p_j) \right]^{\frac{1}{2}} - I_{c_j} p_j d_j M, & \text{if } 2k_j < \eta_j \\ d_j h_j M, & \text{if } 2k_j = \eta_j \\ \left[[2k_j + d_j M^2 (I_{c_j} p_0 - I_{e_j} p_j)] d_j (h_j + I_{c_j} p_0) \right]^{\frac{1}{2}} - I_{c_j} p_0 d_j M, & \text{if } 2k_j > \eta_j \end{cases} \quad (7)$$

4.2. Production and Shipment Schedule for the Vendor

Once the buyers have determined their optimal replenishment cycle times, the next step consists of computing the production and shipment schedule that minimizes the total cost of the vendor. Since the buyers order their individually optimal replenishment quantities, discrete and unequally spaced depletions of the vendor's inventory may occur. Therefore, for the vendor an inventory problem with time-varying demand has to be solved.

In this case, the Wagner and Whitin [76] algorithm is the most used procedure for deriving the optimal solution, although Wagelmans et al. [77], Federgruen and Tzur [78] and Aggarwal and Park [79] have proposed more efficient approaches. However, these procedures cannot directly be used because they consider infinite production rate and we are assuming that the vendor produces the items at a finite rate. Nevertheless, Hill [80] showed that a dynamic lot-sizing problem with finite production rate can be restated as a dynamic lot-sizing problem with infinite production rate. Therefore, we first apply the Hill's [80] method to obtain the new reformulated problem and then the Wagelmans et al.'s [77] approach is used.

First, the demand vector at the vendor, $\overline{D}_0$ has to be determined. Since the optimal replenishment cycle times at the buyers are real values it cannot be assured that both buyers order simultaneously at any time instant. Consequently, the dimension of the demand vector at the vendor might not be finite. For that reason, we propose to use the approach presented in Abdul-Jalbar [81] and round up the real replenishment cycle times to the first two non-zero decimals.

Using the rounding replenishment intervals it is possible to find a point or time instant where the buyers order simultaneously. This value is the planning horizon for the vendor. Next, the demand vector $\overline{D}_0$ and the time instants vector $\overline{T}_0$ can be easily obtained. After that, we apply the Hill's [80] approach to reformulate the problem as a dynamic lot-sizing problem with infinite production rate.

Finally, the Wagelmans et al.'s [77] technique is used to obtain the order quantities vector at the vendor $\overline{Q}_0$ and the holding and setup costs per unit of time for the vendor. In this case, the vendor also incurs an opportunity cost per unit of time of $I_0 p_0 D M$, so this quantity should be added to the holding and setup costs to obtain the total cost per unit of time for the vendor. Finally, adding the total cost per unit of time for the vendor and the buyers, the cost per unit of time for the whole system is obtained.

The solution procedure can be summarized as follows (Algorithm 1):

Algorithm 1 Solution procedure**Step 1**

Compute t_j^* , $j = 1, 2$ using (6).

Step 2

Round up t_j^* values to the first two non-zero decimals to obtain t_j , $j = 1, 2$.

Step 3

Determine the planning horizon for the vendor, the time instants vector $\overline{T}_0$ and the demand vector $\overline{D}_0$.

Step 4

Apply the Hill's [80] procedure to transform the dynamic lot-sizing problem with finite production rate into another one with infinite production rate.

Step 5

Apply the Wagelmans et al.'s [77] technique to calculate the production quantities at the vendor, $\overline{Q}_0$.

Step 6

Compute the total cost of the vendor.

Step 7

Compute the total cost of the system.

5. Illustrative Examples

In this section, the solution approaches developed in Sections 4 and 5 are illustrated. We solve three examples using both the integrated and the non-integrated procedures.

5.1. Example 1

We consider a system with one vendor and two buyers with the parameters given in Table 2.

Table 2. Input data for Example 1.

	d_j	D	P	h_0	h_j	k_0	k_j	I_{e_j}	I_{c_j}	I_0	p_0	p_j	M
Vendor	-	500	2500	5	-	60	-	-	-	0.02	11	-	0.02
Buyer 1	260	-	-	-	12	-	66	0.02	0.05	-	-	24	-
Buyer 2	240	-	-	-	10	-	80	0.02	0.05	-	-	20	-

5.1.1. Solution for the Integrated Model

In this case, problem (4) is solved and the solutions obtained for each part of the cost function are the following:

Case 1: $t_1, t_2 \leq M$

The time between two consecutive setups at the vendor is $t_0 = 0.24$ and the replenishment cycle times at the buyers are $t_1 = 0.02$ ($n_1 = 12$), $t_2 = 0.02$ ($n_2 = 12$). The cost incurred is 7830.192 per unit of time.

Case 2: $t_1 \leq M \leq t_2$

The time between two consecutive setups at the vendor is $t_0 = 0.28$ and the replenishment cycle times at the buyers are $t_1 = 0.02$ ($n_1 = 14$), $t_2 = 0.28$ ($n_2 = 1$). The cost incurred is 4355.417 per unit of time.

Case 3: $t_2 \leq M \leq t_1$

The time between two consecutive setups at the vendor is $t_0 = 0.24$ and the replenishment cycle times at the buyers are $t_1 = 0.24$ ($n_1 = 1$), $t_2 = 0.02$ ($n_2 = 12$). The cost incurred is 5078.155 per unit of time.

Case 4: $M \leq t_1, t_2$

The time between two consecutive setups at the vendor is $t_0 = 0.255$ and the replenishment cycle times at the buyers are $t_1 = 0.255$ ($n_1 = 1$), $t_2 = 0.255$ ($n_2 = 1$). The cost incurred is 1607.189 per unit of time.

Taking the above results into account, the solution obtained in Case 4 is the optimal single-cycle policy for the problem. Specifically, for this policy the costs per unit of time incurred by the vendor, buyer 1 and buyer 2 are 300.684, 672.538 and 633.967, respectively.

5.1.2. Solution for the Non-Integrated Model

First, using (6) the following optimal replenishment cycle times at the buyers are obtained: $t_1^* = 0.204$ and $t_2^* = 0.253$.

Then, rounding these values we get that buyer 1 orders every $t_1 = 0.20$ and buyer 2 every $t_2 = 0.25$. Specifically, the buyers order 52 and 60 units, respectively. Notice that both buyers order simultaneously every 1 units of time. In this period, buyer 1 orders 5 times and buyer 2 orders 4 times. In addition, the costs per unit of time incurred by buyer 1 and buyer 2 are 653.458 and 633.889, respectively.

The time instants where at least one of the buyers places an order to the vendor are

$$\overline{T}_0 = (0.0, 0.2, 0.25, 0.4, 0.5, 0.6, 0.75, 0.8)$$

Then, it is straightforward to obtain the demand vector at the vendor.

$$\overline{D}_0 = (112, 52, 60, 52, 60, 52, 60, 52)$$

Once $\overline{T}_0$ and $\overline{D}_0$ have been computed, we apply the Hill's [80] approach to transform this dynamic lot-sizing problem where the production rate is finite into another one with infinite production rate. Next, the inventory problem can be solved using the Wagelmans et al.'s [77] technique. Proceeding in this way, the following solution is obtained

$$\overline{Q}_0 = (112, 164, 0, 0, 112, 0, 112, 0)$$

and the sum of the holding and setup costs incurred by the vendor is 350.496 per unit of time. Since the opportunity cost incurred per unit of time by the vendor is 2.2, the total cost per unit of time for the vendor is 352.696. Finally, adding the cost for each buyer, we obtain that the total cost of the system is 1640.043 per unit of time. Table 3 summarizes the costs incurred by each member of the system for the integrated and the non-integrated models.

Table 3. Costs for each member of the system.

	Non-Integrated Model	Integrated Model	Integrated Model with Cost Sharing
Cost of the vendor	352.696	300.684	345.631
Cost of buyer 1	653.458	672.538	640.367
Cost of buyer 2	633.889	633.967	621.191
Total cost of the system	1640.043	1607.189	1607.189

5.1.3. Cost Sharing

In this example, the total cost of the system is lower if the integrated policy is applied. To be precise, for the vendor is more profitable to follow the integrated policy. However, the costs for the buyers are lower without integration. This result is in line with related studies, which have shown that the the cost of the vendor and the cost of the whole system usually decrease when the decisions in the supply chain are coordinated, while the cost of the buyers increase.

Therefore, in order to benefit not only the vendor but also the buyers and to encourage the buyers to collaborate, many researchers (Wu and Ouyang [72], Ouyang et al. [73], Ho et al. [44], among others) have proposed to apply the compensation method suggested by Goyal [27] for sharing the costs between the vendor and the buyer. Goyal [27]'s method to allocate the total cost of the system between the buyer and the vendor can be easily extended to the two buyers case as follows:

$$\text{Cost assigned to buyer } j = \frac{C_{\text{buyer}j(\text{Non-integrated})}}{C_{\text{Non-integrated}}} C_{\text{Integrated}}$$

$$\text{Cost assigned to the vendor} = \frac{C_{\text{vendor}(\text{Non-integrated})}}{C_{\text{Non-integrated}}} C_{\text{Integrated}}$$

where $C_{\text{buyer}j(\text{Non-integrated})}$ and $C_{\text{vendor}(\text{Non-integrated})}$ denote the total cost for buyer j , with $j = 1, 2$, and for the vendor in the non-integrated model, respectively.

The allocated costs for the buyers and the vendor are also shown in Table 3. The vendor should compensate buyer 1 with 32.171 and buyer 2 with 12.776.

The results obtained for this example are aligned with those obtained in other studies in the literature that also evaluated the benefits of integration in supply chains. Integration and collaboration are currently considered as crucial factors in managing supply chains.

5.2. Example 2

Now, we consider the parameters given in Table 4.

Table 4. Input data for Example 2.

	d_j	D	P	h_0	h_j	k_0	k_j	I_{e_j}	I_{c_j}	I_0	p_0	p_j	M
Vendor	-	124	240	37	-	24	-	-	-	0.03	9	-	0.08
Buyer 1	75	-	-	-	127	-	35	0.05	0.1	-	-	29	-
Buyer 2	49	-	-	-	50	-	86	0.05	0.1	-	-	36	-

5.2.1. Solution for the Integrated Model

Solving problem (4), the optimal single-cycle policy for the system is given by $t_0 = 0.2316$, $t_1 = 0.0772$ ($n_1 = 3$), and $t_2 = 0.2316$ ($n_2 = 1$). Note that $t_1 < M < t_2$. The total cost of the system is 1846.204 per unit of time. Concretely, in this policy, the costs per unit of time incurred by the vendor, buyer 1, and buyer 2 are 373.664, 816.430, and 656.110, respectively.

5.2.2. Solution for the Non-Integrated Model

In a similar way to Example 1, we obtained that buyer 1 orders 6 units every $t_1 = 0.08$ and buyer 2 orders 12.74 every $t_2 = 0.26$. Hence, from (5), the costs per unit of time incurred by buyer 1 and buyer 2 are 814.150 and 650.931, respectively. Since, the time instants where at least one of the buyers places an order to the vendor are

$$\overline{T_0} = (0.00, 0.08, 0.16, 0.24, 0.26, 0.32, 0.40, 0.48, 0.52, 0.56, 0.64, 0.72, 0.78, 0.80, 0.88, 0.96)$$

it follows that the demand vector at the vendor is

$$\overline{D_0} = (18.74, 6, 6, 6, 12.74, 6, 6, 6, 12.74, 6, 6, 6, 12.74, 6, 6, 6)$$

Applying the Hill's [80] approach and using Wagelmans et al.'s [77] technique to solve the dynamic lot-sizing problem, the following solution is obtained

$$\overline{Q_0} = (24.74, 0, 24.74, 0, 0, 12, 0, 30.74, 0, 0, 0, 24.74, 0, 0, 12, 0)$$

and the total cost per unit of time incurred by the vendor is 316.659. Thus, adding the cost for each buyer, the total cost of the system is 1781.740 per unit of time. Therefore, in this case, in contrast to the previous example, the total cost of the system is lower when the non-integrated policy is applied.

5.3. Example 3

In this example, the parameters considered for the single-vendor two-buyer system are given in Table 5.

Table 5. Input data for Example 3.

	d_j	D	P	h_0	h_j	k_0	k_j	I_{e_j}	I_{c_j}	I_0	p_0	p_j	M
Vendor	-	172	471	73	-	97	-	-	-	0.03	27	-	0.06
Buyer 1	72	-	-	-	168	-	23	0.15	0.20	-	-	37	-
Buyer 2	100	-	-	-	171	-	13	0.15	0.20	-	-	52	-

5.3.1. Solution for the Integrated Model

For this example, the optimal single-cycle policy for the system is given by $t_0 = 0.1668$, $t_1 = 0.0556$ ($n_1 = 3$) and $t_2 = 0.0417$ ($n_2 = 4$). Hence, in this case $t_1, t_2 < M$. The total cost incurred by the vendor, buyer 1, and buyer 2 are 1174.218, 737.070, and 637.748, respectively. Therefore, the total cost of the system is 2549.036 per unit of time.

5.3.2. Solution for the Non-Integrated Model

Now, buyer 1 orders 4.32 units every $t_1 = 0.06$ and buyer 2 orders 3 every $t_2 = 0.03$, from it follows that $\overline{T_0} = (0.00, 0.03)$ and $\overline{D_0} = (7.32, 3)$. In addition, from (5), the costs per unit of time incurred by buyer 1 and buyer 2 are 734.225 and 654.733, respectively. In this case, the best policy for the vendor consists of producing all the units at the beginning of the cycle, that is, $\overline{Q_0} = (10.32, 0)$ and the total cost per unit of time is 1792.109. Therefore, the total cost of the system is 3181.067 per unit of time, which is greater than the total cost of the single-cycle policy for the integrated model.

In order to convince the buyers to follow the single-cycle policy, the total cost of the system can be shared between the buyers and the vendor. Hence, in a similar way to Example 1, the allocated costs of the buyers and the vendor can be computed. Concretely, the allocated costs of the vendor, buyer 1, and buyer 2 are 1436.043, 588.345, and 524.648, respectively. That is, the vendor should compensate buyer 1 with 148.725 and buyer 2 with 113.100.

6. Computational Experiment

There are many references in the literature that show that frequently integrated policies are more profitable from the total system perspective than the non-integrated strategies. However, in most cases, the studies have focused on different scenarios of the single-vendor single-buyer problem. For the single-vendor two-buyers systems analyzed in this paper, an extensive numerical experiment was performed to check whether, in general, the integrated policies continue being dominant. In order to derive some managerial insights we also performed a sensitivity study to evaluate the effects of the parameters on the total costs. Finally, we conducted a statistical analysis to validate the conclusions obtained.

6.1. Comparison between Integrated and Non-Integrated Policies

In this section, we present the results of the computational experiment performed to compare the policies provided by both the integrated and the non-integrated models. Specifically, 1000 instances were randomly generated with the model parameters taken from the uniform distributions shown in Table 6. Hereafter, these uniform distributions are referred to as Initial Distributions (ID).

For each instance, both the integrated and the non-integrated policies were obtained, and the results confirm that, in most of the examples, it was more profitable to apply the integrated policy. To be precise, the outcomes showed in first row of Table 7 reveal that, when parameters are taken from the initial uniform distributions, in 73.5% of the generated instances, the integrated policies outperformed the non-integrated strategies.

Table 6. Uniform distributions from which the model parameters were chosen.

Parameter	Uniform Distribution
d_j, h_0, k_0, k_j	$U[1, 100]$
h_j	$U[h_0, 100 + h_0]$
p_0	$U[1, 30]$
p_j	$U[p_0, 30 + p_0]$
I_{e_j}, I_0	$U[0.02, 0.05]$
I_{c_j}	$U[0.05, 1]$
M	$U[0.01, 0.1]$
P	$U[100, 500] + D$

This conclusion is in line with previous research, which showed that coordination within the supply chain plays a relevant role in minimizing the total costs. However, although, in general, integration strategies provide lower costs compared with non-integration policies, in some cases, the opposite occurs. The following gaps have been computed to evaluate the cost differences

If $C_{Integrated} < C_{Non-integrated}$, then we calculate

$$Gap_1 = Gap_{(C_{Non-integrated} - C_{Integrated})} = \frac{C_{Non-integrated} - C_{Integrated}}{C_{Integrated}} \times 100.$$

In the opposite case, we compute

$$Gap_2 = Gap_{(C_{Integrated} - C_{Non-integrated})} = \frac{C_{Integrated} - C_{Non-integrated}}{C_{Non-integrated}} \times 100.$$

In particular, we obtained that, on average, Gap_1 is equal to 4.05 and Gap_2 is equal to 4.67. Therefore, for the instances generated in this initial experiment, we can conclude that there was no significant difference between both gaps. The most important conclusion of this numerical study is that, in general, integration policies provide more profits than the non-integrated strategies. Nevertheless, unlike the single-vendor single-buyer system where the integrated model always dominates the non-integrated one, in the single-vendor two-buyer system, it may be more effective to follow the independent policies. In the next section, we explore under which conditions the non-integrated model provides more effective solutions.

Table 7. Comparison between the integrated and the non-integrated policies. In columns 3–4, the number of instances where the integrated strategies outperform the non-integrated policies, and vice versa, are shown.

	Uniform Distribution	$C_{Integrated} \leq C_{Non-integrated}$	$C_{Integrated} > C_{Non-integrated}$	Infeasible Solutions	$\overline{Gap}_1$	$\overline{Gap}_2$
	ID	735	241	24	4.04	4.67
d_j	$U[1, 1000]$	259	337	404	6.58	3.27
	$U[4500, 5500]$	292	11	697	18.12	2.76
	$U[9000, 10,000]$	263	17	720	17.76	0.95
k_j	$U[1, 1000]$	246	722	32	0.74	6.34
	$U[4500, 5500]$	56	944	0	0.25	5.63
	$U[9000, 10,000]$	54	946	0	0.29	5.74
k_0	$U[1, 1000]$	922	54	24	34.83	2.50
	$U[4500, 5500]$	975	1	24	253.50	0.05
	$U[9000, 10,000]$	976	0	24	395.61	-

Table 7. Cont.

	Uniform Distribution	$C_{Non-integrated}^{Integrated} \leq$	$C_{Non-integrated}^{Integrated} >$	Infeasible Solutions	$\overline{Gap}_1$	$\overline{Gap}_2$
	ID	735	241	24	4.04	4.67
h_j	$U[1, 1000]$	751	201	48	5.54	1.93
	$U[4500, 5500]$	971	11	18	14.77	3.18
	$U[9000, 10,000]$	974	6	20	15.93	5.35
h_0, h_j	$h_0 \sim U[1, 1000]$ $h_1 = 1100$ $h_2 = 1200$	686	288	26	4.56	4.17
	$h_0 \sim U[4500, 5500]$ $h_1 = 5600$ $h_2 = 5700$	671	306	23	3.23	6.33
	$h_0 \sim U[9000, 10,000]$ $h_1 = 10,100$ $h_2 = 10,200$	679	298	23	3.88	6.55
p_j	$U[10, 100]$	733	243	24	4.03	4.67
	$U[450, 550]$	724	251	25	4.46	4.70
	$U[900, 1000]$	719	253	28	5.23	4.78
p_0, p_j	$p_0 \sim U[10, 100]$ $p_1 = 150$ $p_2 = 200$	728	255	17	4.34	4.13
	$p_0 \sim U[450, 550]$ $p_1 = 600$ $p_2 = 650$	760	230	10	5.57	2.44
	$p_0 \sim U[900, 1000]$ $p_1 = 1050$ $p_2 = 1100$	775	213	12	6.35	1.91
M	$U[0.02, 0.15]$	736	239	25	4.04	4.70
	$U[0.15, 0.3]$	726	246	28	3.95	4.67
I_0	$U[0.05, 0.1]$	735	241	24	4.03	4.66
	$U[0.1, 0.2]$	735	241	24	4.02	4.65
I_{c_j}	$U[0.1, 0.25]$	723	250	27	3.94	4.66
	$U[0.25, 0.5]$	728	245	27	4.01	4.61
I_{e_j}	$U[0.05, 0.1]$	733	243	24	4.06	4.66
	$U[0.1, 0.2]$	732	242	26	4.02	4.69
P	$U[1000, 5000]$	836	164	0	5.18	6.41
	$U[10,000, 20,000]$	842	158	0	5.48	6.57

6.2. Sensitivity Analysis

We performed a sensitivity analysis to assess the impact of the model parameters on the total costs provided by both models and to derive some managerial implications and insights. Concretely, we generated new problem sets varying the uniform distribution from which one of the model parameters for both buyers or for the vendor is chosen, while all other parameters are taken from the Initial Distributions given in Table 6. For example, we obtained the first new problem set taking demand values d_1 and d_2 from $U[1, 1000]$, while the rest of parameters were chosen from the Initial Distributions. Only in some cases was it necessary to vary some parameters at the same time due to the restrictions of the problem.

Thus, when h_0 is modified, h_1 and h_2 also have to be chosen from other uniform distributions to guarantee that h_0 is lower than h_1 and h_2 . In the same way, if we change p_0 , then p_1 and p_2 also have to be adjusted since they must be greater than p_0 . Following this idea, 31 problem sets were created, and, for each one, 1000 instances were generated. For each instance, the non-integrated and the integrated policies were determined, and the results are reported in Table 7. The second column of this table gives the new uniform distribution from which the parameter, indicated in the first column, was chosen. In the next two columns, we give the number of cases where integrated policies outperformed the non-integrated ones and vice versa, respectively.

For some instances, the non-integrated procedure cannot compute a feasible solution. Column five shows the number of instances for which the latter procedure yields infeasible solutions. This is due to the fact that the production rate is insufficient to meet the demand at both buyers at some time instances. In the non-integrated model, the buyers independently determine their replenishment cycle times, and then the vendor calculates the best shipment schedule to satisfy the demands. Since the buyers obtain their replenishment cycle times regardless of the production rate, in some cases, the vendor might not be able to meet the demand at specific time instants.

The last two columns contain the average values $\overline{Gap}_1$ and $\overline{Gap}_2$, respectively. The first row contains the outcomes achieved when the parameters are chosen from the uniform distributions given in Table 6. The second row shows the results obtained when d_1 and d_2 are chosen from $U[1, 1000]$ and all other parameters are taken from the Initial Distributions. Similarly, the rest of rows present the results yielded when the corresponding parameter is selected from the uniform distribution shown in the second column.

From the results in Table 7, the following conclusions can be drawn:

- (i) In most instances, the integrated strategies outperformed the non-integrated policies. Only when the replenishment costs for the buyers were chosen from the uniform distributions $U[1, 1000]$, $U[4500, 5500]$ or $U[9000, 10,000]$, the non-integrated policies provided lower costs than the integrated strategies in most instances.
- (ii) It is important to remark that, under integration, the total cost of the buyers increased. In contrast, the total cost of the vendor may decrease. Therefore, in some instances, as for example when the replenishment costs at the buyers are considerably high, the total cost at the buyers may increase in an amount that is greater than the savings at the vendor. Hence, it is better to follow the non-integrated policy.
- (iii) Conversely, if the setup cost of the vendor is taking from $U[1, 1000]$, $U[4500, 5500]$, or $U[9000, 10,000]$, the integrated model provided more profitable solutions in almost all instances. In addition, for these problem sets, $\overline{Gap}_1$ considerably increased. In particular, as k_0 increased so did the profits of using the integrated policies.
- (iv) Finally, as shown in the fifth column of Table 7, for some instances, the non-integrated procedure is not able to compute a feasible solution since the production rate is insufficient to meet the demand at both buyers at some time instants. Specifically, when the demand rate at the buyers increased, so did the number of infeasible non-integrated solutions.

In order to avoid infeasible non-integrated solutions in these problem sets, we, again, executed the computational experiment, and, for each instance, the production rate was set equal to the minimum value necessary to ensure the feasibility of the non-integrated solution. Accordingly, once the replenishment cycle times at the buyers, t_1 and t_2 , are independently determined, the production rate is computed as $P = (d_1 t_1 + d_2 t_2) / t_{min}$, where $t_{min} = \min\{t_1, t_2\}$.

The new results are given in Table 8, and they show that integrated policies continue being more efficient than the non-integrated strategies in most cases. In addition, it is worth noting that $\overline{Gap}_2$ is tighter than $\overline{Gap}_1$. That is, when the non-integrated policies outperformed the integrated ones, the differences between the costs were smaller than in the opposite case.

Table 8. Comparison between the integrated and the non-integrated policies when the production rate was sufficient to ensure the feasibility of the non-integrated solutions.

	<i>Uniform Distribution</i>	$C_{Integrated} \leq$ $C_{Non-integrated}$	$C_{Integrated} >$ $C_{Non-integrated}$	$\overline{Gap}_1$	$\overline{Gap}_2$
d_j	$U[1, 1000]$	507	493	9.34	3.8
	$U[4500, 5500]$	716	284	12.8	4.26
	$U[9000, 10,000]$	562	438	14.03	4.38

6.3. Statistical Study

In order to support the conclusions derived from the computational experiment, we performed a statistical analysis using the software IBM® SPSS® version 25 (IBM Corporation, Armonk, NY, USA). The analysis is divided into two parts. In the former, we focused on analyzing the effect of the model parameters on the total costs. In the latter part, we assessed whether there was a significant difference between the costs obtained through the integrated and the non-integrated models.

Since the parameters d_j , k_j , k_0 , h_j , h_0 , p_j and p_0 were chosen from three different uniform distributions, they were factors with three levels. Therefore, a one-way analysis of variance (ANOVA), followed by the Tukey's pairwise comparison test, was conducted to highlight the effects of these factors on the total costs. Prior to the analysis, the assumptions of normality and homoscedasticity were tested.

In some cases, the homogeneity of the variances assumption was not achieved and the Welch's test, followed by the Games–Howell's pairwise comparison test, were used to check for significant differences in the total costs. Otherwise, since the factors M , I_0 , I_{c_j} , I_{e_j} and P only vary at two levels, differences in the total costs were examined with t -tests or Mann–Whitney tests.

Finally, in order to verify that there were significant differences between the total costs obtained through the integrated and the non-integrated models, the paired t -tests or Wilcoxon tests were performed. For all statistical tests, $p < 0.05$ was considered statistically significant.

6.3.1. Effect of Model Parameters on Integrated and Non-Integrated Costs

Table 9 shows the p -values obtained from ANOVA or Welch's tests. We concluded that d_j , k_j , k_0 , h_j , h_0 , and p_0 had a significant effect on both costs ($p < 0.05$). In addition, for each significant parameter, Tukey or Games–Howell's tests were used to determine the significant difference between pairs of mean costs. The results proved that the three pairwise were significantly different ($p < 0.05$, for each pair). That is, the mean costs were significantly different depending on in which interval the parameter varied. In particular, when d_j , k_j , k_0 , h_j , h_0 , or p_0 increased, so did both costs. In contrast, the outcome related to p_j allow us to conclude that changes analyzed on p_j did not considerably affect both costs ($p > 0.05$).

Results regarding the parameters M , I_0 , I_{c_j} , I_{e_j} , and P are given in Table 10. Specifically, this table presents p -values from t -tests or Mann–Whitney tests. As shown, these factors did not have a significant impact on the total costs ($p > 0.05$). For example, focusing on M , we can conclude that, if the credit period is chosen from $U[0.15, 0.3]$ instead of from $U[0.02, 0.15]$, the mean costs obtained are not significantly different ($p > 0.05$). Similar conclusions can be deduced for I_0 , I_{c_j} , I_{e_j} , and P . That is, at least for the considered levels, significant differences in the mean costs cannot be proved.

6.3.2. Integrated versus Non-Integrated Costs

Finally, 32 paired t -tests or Wilcoxon tests were performed to compare the costs obtained from the integrated and the non-integrated models along the 32 generated problem sets. In Table 11, we present the mean values and the standard deviations (SD) for the total costs as well as the p -values, from which we concluded that there were significant differences between the mean costs obtained with both models ($p < 0.05$). These results corroborate that, in general, the integrated model outperformed the non-integrated procedure, except for the case where the replenishment costs for the buyers were high.

Table 9. The effects of the parameters d_j , k_j , k_0 , h_j , h_0 , p_j , and p_0 on the integrated and the non-integrated costs.

Factor	Levels	Effect on Integrated Cost	Effect on Non-Integrated Cost
		<i>p</i> -Value	<i>p</i> -Value
d_j	$U[1, 1000]$	0.000	0.000
	$U[4500, 5500]$		
	$U[9000, 10,000]$		
k_j	$U[1, 1000]$	0.000	0.000
	$U[4500, 5500]$		
	$U[9000, 10,000]$		
k_0	$U[1, 1000]$	0.000	0.000
	$U[4500, 5500]$		
	$U[9000, 10,000]$		
h_j	$U[1, 1000]$	0.000	0.000
	$U[4500, 5500]$		
	$U[9000, 10,000]$		
h_0	$U[1, 1000]$	0.000	0.000
	$U[4500, 5500]$		
	$U[9000, 10,000]$		
p_j	$U[10, 100]$	0.236	0.339
	$U[450, 550]$		
	$U[900, 1000]$		
p_0	$U[10, 100]$	0.000	0.000
	$U[450, 550]$		
	$U[900, 1000]$		

Table 10. The effects of the parameters M , I_0 , I_{c_j} , I_{e_j} , and P on the integrated and the non-integrated costs.

Factor	Levels	Effect on Integrated Cost	Effect on Non-Integrated Cost
		<i>p</i> -Value	<i>p</i> -Value
M	$U[0.02, 0.15]$	0.537	0.502
	$U[0.15, 0.3]$		
I_0	$U[0.05, 1]$	0.832	0.837
	$U[0.1, 0.2]$		
I_{c_j}	$U[0.1, 0.25]$	0.742	0.714
	$U[0.25, 0.5]$		
I_{e_j}	$U[0.05, 0.1]$	0.949	0.942
	$U[0.1, 0.2]$		
P	$U[1000, 5000]$	0.720	0.884
	$U[10,000, 20,000]$		

Table 11. The mean values and standard deviations for the integrated and the non-integrated costs for each generated problem set. Significant differences were evaluated by paired *t*-tests or Wilcoxon tests.

Uniform Distribution		Integrated Cost	Non-Integrated Cost	<i>p</i>
		Mean ± SD	Mean ± SD	
ID		1695.52 ± 666.90	1728.34 ± 685.33	0.000
d_j	$U[1, 1000]$	5233.38 ± 2105.68	5269.27 ± 2126.07	0.005
	$U[4500, 5500]$	18,746.64 ± 5145.45	21,725.78 ± 5653.19	0.000
	$U[9000, 10,000]$	26,363.09 ± 7673.26	30,202.26 ± 7974.33	0.000
k_j	$U[1, 1000]$	4607.62 ± 1873.58	4438.53 ± 1853.41	0.000
	$U[4500, 5500]$	14,888.13 ± 4766.94	14,189.67 ± 4675.63	0.000
	$U[9000, 10,000]$	20,516.96 ± 6550.73	19,533.29 ± 6425.80	0.000
k_0	$U[1, 1000]$	2759.42 ± 1178.70	3698.00 ± 2177.10	0.000
	$U[4500, 5500]$	6598.38 ± 2506.88	21,963.40 ± 15,906.47	0.000
	$U[9000, 10,000]$	8674.82 ± 3332.85	40,062.14 ± 30,407.03	0.000
h_j	$U[1, 1000]$	3215.44 ± 1322.54	3325.19 ± 1384.68	0.000
	$U[4500, 5500]$	9547.09 ± 3377.75	10,822.13 ± 3901.90	0.000
	$U[9000, 10,000]$	12,986.18 ± 4619.68	14,874.36 ± 5529.23	0.000
h_0, h_j	$h_0 \sim U[1, 1000], h_1 = 1100, h_2 = 1200$	5647.98 ± 1879.34	5737.16 ± 1892.54	0.000
	$h_0 \sim U[4500, 5500], h_1 = 5600, h_2 = 5700$	13,844.95 ± 4426.82	13,934.96 ± 4560.50	0.000
	$h_0 \sim U[9000, 10,000], h_1 = 10,100, h_2 = 10,200$	18,689.65 ± 5974.78	18,936.30 ± 6326.91	0.000
p_j	$U[10, 100]$	1693.30 ± 666.33	1725.66 ± 684.78	0.000
	$U[450, 550]$	1670.05 ± 659.50	1704.61 ± 680.62	0.000
	$U[900, 1000]$	1642.44 ± 656.82	1680.34 ± 675.79	0.000
p_0, p_j	$p_0 \sim U[10, 100], p_1 = 150, p_2 = 200$	1744.84 ± 673.13	1781.631 ± 688.22	0.000
	$p_0 \sim U[450, 550], p_1 = 600, p_2 = 650$	2093.99 ± 788.06	2158.46 ± 796.56	0.000
	$p_0 \sim U[900, 1000], p_1 = 1050, p_2 = 1100$	2288.25 ± 875.41	2375.33 ± 902.77	0.000
M	$U[0.02, 0.15]$	1687.85 ± 665.52	1720.89 ± 684.95	0.000
	$U[0.15, 0.3]$	1669.27 ± 663.43	1700.08 ± 682.83	0.000
I_0	$U[1, 1000]$	1698.98 ± 667.79	1731.81 ± 686.25	0.000
	$U[1, 1000]$	1705.41 ± 669.81	1738.22 ± 688.32	0.000
I_{c_j}	$U[4500, 5500]$	1682.50 ± 666.91	1713.31 ± 686.38	0.000
	$U[9000, 10,000]$	1692.45 ± 666.43	1724.71 ± 685.33	0.000
I_{e_j}	$U[1, 1000]$	1693.53 ± 666.35	1726.23 ± 684.94	0.000
	$U[4500, 5500]$	1691.58 ± 664.99	1723.98 ± 683.59	0.000
P	$U[1000, 5000]$	1635.15 ± 613.34	1695.27 ± 656.72	0.000
	$U[10,000, 20,000]$	1625.38 ± 605.18	1691.01 ± 653.38	0.000

6.4. Managerial Implications

The results obtained in the previous sections provide some guidelines that could help inventory managers to make decisions about whether or not collaborate with other members of the supply chain, or about the impact of the model parameters on the total costs of both models.

The most important finding revealed by the sensitivity analysis is that the integrated policy provided lower costs for the system compared with the non-integrated strategy for most of the parameter settings, and thus managers should attempt to establish an integrated decision support system. However, in general, the integrated model reduced the total cost of the vendor but increased the total cost of the buyers and, consequently, the latter may not agree to cooperate. If this is the case, the vendor should be willing to apply a profit sharing approach to convince the buyers to follow the integrated strategy.

Another useful implication of this work is that the replenishment costs at the buyers is a key factor, since the profit improvements generated by the integrated policy decrease as such costs increase. Even the non-integrated policy could become more profitable than the integrated one if the replenishment costs at the buyers are considerably high. This effect makes sense since the integrated model computes single-cycle policies, which restrict the time interval between replenishments at the buyers to be equal to or smaller than the time between two consecutive setups at the vendor.

Accordingly, the integrated policy incurs high costs at the buyers due to a low replenishment interval. Hence, the inventory managers should worry about the replenishment costs at the buyers and try to reduce them to make the integrated policy more effective.

It is also remarkable that, in order to apply the integrated model, it is necessary that the buyers share private information related to their costs and demand. Under some circumstances, for example, if the buyers are more powerful than the vendor in the supply chain, they may not be inclined to share such information and may persist in applying their individual optimal policies. The sensitivity analysis showed that, for most parameter settings, if the non-integrated policy was followed instead of the integrated one, the total cost of the system increased on average by 4%.

Nevertheless, as d_j or h_j increased, so did the gap between costs, which could rise up to 18%. The gap increase was even worse if the vendor's setup cost increased excessively. If k_0 was very high, the total cost of the non-integrated policy could be much higher than the cost of the integrated one. Therefore, in this case, it is particularly important that the decision makers reach agreements to manage the supply chain under the integrated perspective.

7. Conclusions

We investigated the one-vendor two-buyer inventory system where the vendor produces a single product and supplies items to both buyers allowing a credit period. This case is the simplest within the two-echelon systems but it is commonly used in practice. Even when there are multiple buyers, they are frequently clustered into two groups to obtain replenishment policies that are easier to implement. With this consideration, the inventory problem with multiple buyers can be simplified to the single-vendor two-buyer system.

We first formulated the problem under an integrated perspective where the vendor and the buyers make decisions jointly with the goal of minimizing the cost of the whole system. Next, we also tackled the problem under a non-cooperative environment, that is, assuming that the buyers are not willing to cooperate and that they prefer to minimize their total costs without taking into account the total cost of the vendor.

The formulation of the integrated model was established in terms of single-cycle policies. Regarding the non-integrated model, first the replenishment cycle times at the buyers were computed, and then the shipment schedule at the vendor was determined.

The solutions obtained by both procedures were compared through a computational study, and the results showed that, generally, the total cost of the system was lower for the integrated model. This conclusion is aligned with those obtained in other studies in the literature, which also confirmed the benefits of integration in supply chains.

Our results support the current trend toward collaboration between members of the supply chain, since companies realize that, in most cases, cooperation between suppliers and vendors is crucial for success. However, our research also revealed certain situations where it could be better to follow a non-integrated policy. Therefore, the findings of this work are valuable for inventory managers because they might help them to choose an adequate replenishment policy according to the circumstances.

In particular, from the sensitivity analysis, some useful managerial implications are derived. For example, the results showed that the benefits of applying an integrated policy instead of a non-integrated one were huge when the setup cost of the vendor increased considerably. In contrast, the profit improvements generated by the integrated policy decreased as the replenishment costs at the buyers increased. The non-integrated policy could even dominate the integrated one if the replenishment costs at the buyers became very high. The explanation for this behavior is that we confined ourselves to use the single-cycle policies to state the integrated model.

In such policies, the time interval between replenishments at the buyers are forced to be equal to or smaller than the time interval between two consecutive setups at the vendor, which may yield high costs at the buyers. Generally, the single-cycle policies are very efficient and have clear managerial advantages. However, under certain scenarios, the performance of these policies could become worse. For example, when considerably

high replenishment costs are combined with relatively low demands, the application of these policies could be less efficient.

In further research, it would be worthwhile to study a more general class of integrated policies that allow the buyers to order at time intervals greater than the time between two consecutive setups at the vendor.

In addition, the deterministic nature of the production and the demand rate are also limitations of the present model. These features could also be relaxed in future extensions of the model. Finally, another direction for forthcoming research could be to analyze the system considering the credit period as a decision variable, or under two-level trade credit where the buyers also offer a credit period to the final customers.

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