

## Article

# Generalizations of Hardy Type Inequalities by Abel–Gontscharoff’s Interpolating Polynomial

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**Abstract:** In this paper, we extend Hardy’s type inequalities to convex functions of higher order. Upper bounds for the generalized Hardy’s inequality are given with some applications.

**Keywords:** inequalities; Hardy type inequalities; Abel–Gontscharoff interpolating polynomial; Green function; Chebyshev functional; Grüss type inequalities; Ostrowski type inequalities; convex function; kernel; upper bounds



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## 1. Introduction and Preliminaries

Let  $(\Sigma_1, \Omega_1, \mu_1)$  and  $(\Sigma_2, \Omega_2, \mu_2)$  be measure spaces with positive  $\sigma$ -finite measures. For a measurable function  $f: \Omega_2 \rightarrow \mathbb{R}$ , let  $A_k$  denote the linear operator

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x, t) f(t) d\mu_2(t), \quad (1)$$

where  $k: \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$  is measurable and non-negative kernel with

$$0 < K(x) := \int_{\Omega_2} k(x, t) d\mu_2(t), \quad x \in \Omega_1. \quad (2)$$

The following result was given in [1] (see also [2]), where  $u$  is a positive function on  $\Omega_1$ .

**Theorem 1.** Let  $u$  be a weight function,  $k(x, y) \geq 0$ . Assume that  $\frac{k(x, y)}{K(x)} u(x)$  is locally integrable on  $\Omega_1$  for each fixed  $y \in \Omega_2$ . Define  $v$  by

$$v(y) := \int_{\Omega_1} \frac{k(x, y)}{K(x)} u(x) d\mu_1(x) < \infty. \quad (3)$$

If  $\phi$  is a convex function on the interval  $I \subseteq \mathbb{R}$ , then the inequality

$$\int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \leq \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) \quad (4)$$

holds for all measurable functions  $f: \Omega_2 \rightarrow \mathbb{R}$ , such that  $\text{Im} f \subseteq I$ , where  $A_k$  is defined by (1) and (2).

Inequality (4) is generalization of Hardy's inequality. G. H. Hardy [3] stated and proved that the inequality

$$\int_0^{\infty} \left( \frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_0^{\infty} f^p(x) dx, \quad p > 1, \quad (5)$$

holds for all  $f$  non-negative functions such that  $f \in L^p(\mathbb{R}_+)$  and  $\mathbb{R}_+ = (0, \infty)$ . The constant  $\left(\frac{p}{p-1}\right)^p$  is sharp. More details about Hardy's inequality can be found in [4,5].

Inequality (5) can be interpreted as the Hardy operator  $H : Hf(x) := \frac{1}{x} \int_0^x f(t) dt$ , maps  $L^p$  into  $L^p$  with the operator norm  $p' = \frac{p}{p-1}$ .

In this paper, we consider the difference of both sides of the generalized Hardy's inequality

$$\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$$

and obtain new inequalities that hold for  $n$ -convex functions.

Now, we recall  $n$ -convex functions. There are two parallel notations. First, is given by E. Hopf in 1926 and second by T. Popoviciu in 1934. E. Hopf defined that the function  $f$  is  $n$ -convex if difference  $[x_0, \dots, x_{n+1}, f]$  is nonnegative. The ordinary convex function is 1-convex. For more details see [6]. In the second definition  $f : [\alpha, \beta] \rightarrow \mathbb{R}$  is  $n$ -convex  $n \geq 0$ , if its  $n$ -th order divided differences  $[x_0, \dots, x_n; f]$  are nonnegative for all choices of  $(n+1)$  distinct points  $x_i \in [\alpha, \beta]$ . By second definition 0-convex function is nonnegative, 1-convex function is non-decreasing and 2-convex function is convex in the usual sense. If an  $n$ -convex function is  $n$  times differentiable, then  $\phi^{(n)} \geq 0$ . (see [7]).

An important role in the paper will be played by Abel–Gontscharoff interpolation, which was first studied by Whittaker [8], and later by Gontscharoff [9] and Davis [10]. The Abel–Gontscharoff interpolation for two points and the remainder in the integral form is given in the following theorem (for more details see [11]).

**Theorem 2.** Let  $n, m \in \mathbb{N}$ ,  $n \geq 2$ ,  $0 \leq m \leq n-1$  and  $\phi \in C^n([\alpha, \beta])$ . Then

$$\phi(u) = Q_{n-1}(\alpha, \beta, \phi, u) + R(\phi, u),$$

where  $Q_{n-1}$  is the Abel–Gontscharoff interpolating polynomial for two-points of degree  $n-1$ , i.e.,

$$Q_{n-1}(\alpha, \beta, \phi, u) = \sum_{s=0}^m \frac{(u-\alpha)^s}{s!} \phi^{(s)}(\alpha) + \sum_{r=0}^{n-m-2} \left[ \sum_{s=0}^r \frac{(u-\alpha)^{m+1+s} (\alpha-\beta)^{r-s}}{(m+1+s)!(r-s)!} \right] \phi^{(m+1+r)}(\beta)$$

and the remainder is given by

$$R(\phi, u) = \int_{\alpha}^{\beta} G_{mn}(u, t) \phi^{(n)}(t) dt,$$

where  $G_{mn}(u, t)$  is Green's function defined by

$$G_{mn}(u, t) = \frac{1}{(n-1)!} \begin{cases} \sum_{s=0}^m \binom{n-1}{s} (u-\alpha)^s (\alpha-t)^{n-s-1}, & \alpha \leq t \leq u; \\ - \sum_{s=m+1}^{n-1} \binom{n-1}{s} (u-\alpha)^s (\alpha-t)^{n-s-1}, & u \leq t \leq \beta. \end{cases} \quad (6)$$

**Remark 1.** For  $\alpha \leq t, u \leq \beta$  the following inequalities hold

$$\begin{aligned} (-1)^{n-m-1} \frac{\partial^s G_{mn}(u, t)}{\partial u^s} &\geq 0, \quad 0 \leq s \leq m, \\ (-1)^{n-s} \frac{\partial^s G_{mn}(u, t)}{\partial u^s} &\geq 0, \quad m+1 \leq s \leq n-1. \end{aligned}$$

## 2. Generalizations of Hardy's Inequality

Our first result is an identity related to generalized Hardy's inequality. We apply interpolation by the Abel–Gontscharoff polynomial and get the following result.

**Theorem 3.** Let  $(\Sigma_1, \Omega_1, \mu_1)$  and  $(\Sigma_2, \Omega_2, \mu_2)$  be measure spaces with positive  $\sigma$ -finite measures. Let  $u : \Omega_1 \rightarrow \mathbb{R}$ , be a weight function and  $v$  is defined by (3). Let  $A_k f(x), K(x)$  be defined by (1) and (2) respectively, for a measurable function  $f : \Omega_2 \rightarrow [\alpha, \beta]$  and let  $n, m \in \mathbb{N}, n \geq 2, 0 \leq m \leq n-1, \phi \in C^n([\alpha, \beta])$  and  $G_{mn}$  be defined by (6). Then

$$\begin{aligned} &\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &= \sum_{s=1}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x)d\mu_1(x) \right) \\ &+ \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s} (\beta - \alpha)^{r-s} \phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y)d\mu_2(y) \right. \\ &\quad \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x)d\mu_1(x) \right) \\ &+ \int_{\alpha}^{\beta} \left( \int_{\Omega_2} G_{mn}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{mn}(A_k f(x), t)u(x)d\mu_1(x) \right) \phi^{(n)}(t)dt. \end{aligned} \quad (7)$$

**Proof.** Using Theorem 2 we can represent every function  $\phi \in C^n([\alpha, \beta])$  in the form

$$\begin{aligned} \phi(u) &= \sum_{s=0}^m \frac{(u - \alpha)^s}{s!} \phi^{(s)}(\alpha) \\ &+ \sum_{r=0}^{n-m-2} \left[ \sum_{s=0}^r \frac{(u - \alpha)^{m+1+s} (-1)^{r-s} (\beta - \alpha)^{r-s}}{(m+1+s)!(r-s)!} \right] \phi^{(m+1+r)}(\beta) \\ &+ \int_{\alpha}^{\beta} G_{mn}(u, t) \phi^{(n)}(t)dt. \end{aligned} \quad (8)$$

By an easy calculation, applying (8) in  $\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$ , we get

$$\begin{aligned} &\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &= \sum_{s=0}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x)d\mu_1(x) \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s}(\beta-\alpha)^{r-s}\phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y) d\mu_2(y) \right. \\
& \quad \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x) d\mu_1(x) \right) \\
& + \int_{\alpha}^{\beta} \left( \int_{\Omega_2} G_{mn}(f(y), t) v(y) d\mu_2(y) - \int_{\Omega_1} G_{mn}(A_k f(x), t) u(x) d\mu_1(x) \right) \phi^{(n)}(t) dt.
\end{aligned}$$

Since

$$\begin{aligned}
& \int_{\Omega_2} v(y) d\mu_2(y) - \int_{\Omega_1} u(x) d\mu_1(x) \\
& = \int_{\Omega_2} \left( \int_{\Omega_1} \frac{k(x, y)}{K(x)} u(x) d\mu_1(x) \right) d\mu_2(y) - \int_{\Omega_1} u(x) d\mu_1(x) \\
& = \int_{\Omega_1} \frac{u(x)}{K(x)} \left( \int_{\Omega_2} k(x, y) d\mu_2(y) \right) d\mu_1(x) - \int_{\Omega_1} u(x) d\mu_1(x) \\
& = \int_{\Omega_1} u(x) d\mu_1(x) - \int_{\Omega_1} u(x) d\mu_1(x) = 0
\end{aligned}$$

the summand for  $s = 0$  in the first sum on the right hand side is equal to zero, so (7) follows.  $\square$

We continue with the following result.

**Theorem 4.** Let all the assumptions of Theorem 3 hold, let  $\phi$  be  $n$ -convex on  $[\alpha, \beta]$  and

$$\int_{\Omega_1} G_{mn}(A_k f(x), t) u(x) d\mu_1(x) \leq \int_{\Omega_2} G_{mn}(f(y), t) v(y) d\mu_2(y), \quad t \in [\alpha, \beta]. \quad (9)$$

Then

$$\begin{aligned}
& \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \\
& \geq \sum_{s=1}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x) d\mu_1(x) \right) \\
& + \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s}(\beta-\alpha)^{r-s}\phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y) d\mu_2(y) \right. \\
& \quad \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x) d\mu_1(x) \right).
\end{aligned} \quad (10)$$

If the reverse inequality in (9) holds, then the reverse inequality in (10) holds.

**Proof.** We assumed that  $\phi \in C^n([\alpha, \beta])$  is  $n$ -convex, so  $\phi^{(n)} \geq 0$  on  $[\alpha, \beta]$ . We apply Theorem 3 and (10).  $\square$

**Remark 2.** Notice that for  $n = 2$  and  $0 \leq m \leq 1$  the function  $G_{mn}(\cdot, t)$ ,  $t \in [\alpha, \beta]$  is convex on  $[\alpha, \beta]$ . Therefore the assumption (9) is satisfied and then the inequality (10) holds. For an arbitrary  $n \geq 3$  and  $0 \leq m \leq 1$ , we use Remark 1, i.e., we consider the following inequality:

$$(-1)^{n-2} \frac{\partial^2 G_{mn}(u, t)}{\partial u^2} \geq 0.$$

We conclude that the convexity of  $G_{mn}(\cdot, t)$  depends of a parity of  $n$ . If  $n$  is even, then  $\frac{\partial^2 G_{mn}(u, t)}{\partial u^2} \geq 0$ , i.e.,  $G_{mn}(\cdot, t)$  is convex and assumption (9) is satisfied. Also, the inequality (10) holds. For odd  $n$  we get the reverse inequality. For all other choices, the following generalization holds.

**Theorem 5.** Suppose that all assumptions of Theorem 1 hold. Additionally, let  $n, m \in \mathbb{N}$ ,  $n \geq 3$ ,  $2 \leq m \leq n - 1$  and  $\phi \in C^n([\alpha, \beta])$  be  $n$ -convex.

- (i) If  $n - m$  is odd, then the inequality (10) holds.
- (ii) If  $n - m$  is even, then the reverse inequality in (10) holds.

**Proof.**

- (i) By Remark 1, the following inequality holds

$$(-1)^{n-m-1} \frac{\partial^2 G_{mn}(u, t)}{\partial u^2} \geq 0, \quad \alpha \leq u, t \leq \beta.$$

In case  $n - m$  is odd ( $n - m - 1$  is even), we have

$$\frac{\partial^2 G_{mn}(u, t)}{\partial u^2} \geq 0,$$

i.e.,  $G_{mn}(\cdot, t)$ ,  $t \in [\alpha, \beta]$ , is convex on  $[\alpha, \beta]$ . Then by Theorem 1 we have

$$\int_{\Omega_1} u(x) G_{mn}(A_k f(x), t) d\mu_1(x) \leq \int_{\Omega_2} v(y) G_{mn}(f(y), t) d\mu_2(y),$$

i.e., the assumption (9) is satisfied. By applying Theorem 4 we get (10).

- (ii) Similarly, if  $n - m$  is even, then  $G_{mn}(\cdot, t)$ ,  $t \in [\alpha, \beta]$  is concave on  $[\alpha, \beta]$ , so the reversed inequality in (9) holds and, hence, in (10) as well.

□

**Theorem 6.** Suppose that all assumptions of Theorem 1 hold and let  $n, m \in \mathbb{N}$ ,  $n \geq 2$ ,  $0 \leq m \leq n - 1$ ,  $\phi \in C^n([\alpha, \beta])$  be  $n$ -convex and  $F : [\alpha, \beta] \rightarrow \mathbb{R}$  be defined by

$$\begin{aligned} F(t) = & \sum_{s=2}^m \frac{\phi^{(s)}(\alpha)}{s!} (t - \alpha)^s \\ & + \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s} (\beta - \alpha)^{r-s}}{(m+1+s)!(r-s)!} \phi^{(m+1+r)}(\beta) (t - \alpha)^{m+1+s}. \end{aligned} \quad (11)$$

- (i) If (10) holds and  $F$  is convex, then the inequality (4) holds.
- (ii) If the reverse of (10) holds and  $F$  is concave, then the reverse inequality (4) holds.

**Proof.**

- (i) Let (10) holds. If  $F$  is convex, then by Theorem 1 we have

$$\int_{\Omega_2} v(y) F(f(y)) d\mu_2(y) - \int_{\Omega_1} u(x) F(A_k f(x)) d\mu_1(x) \geq 0$$

which, changing the order of summation, can be written in form

$$\begin{aligned} & \sum_{s=1}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x) d\mu_1(x) \right) + \\ & \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s} (\beta - \alpha)^{r-s} \phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y) d\mu_2(y) \right. \\ & \quad \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x) d\mu_1(x) \right) \\ & \geq 0. \end{aligned}$$

We conclude that the right-hand side of (10) is nonnegative and the inequality (4) follows.

(ii) Similar to (i) case.

□

**Remark 3.** Note that the function  $t \mapsto (t - \alpha)^p$  is convex on  $[\alpha, \beta]$  for each  $p = 2, \dots, n - 1$ , i.e.,

$$\int_{\Omega_2} v(y) (f(y) - \alpha)^p d\mu_2(y) - \int_{\Omega_1} u(x) (A_k f(x) - \alpha)^p d\mu_1(x) \geq 0,$$

for each  $p = 2, \dots, n - 1$ .

- (i) If (10) holds,  $\phi^{(s)}(\alpha) \geq 0$  for  $s = 0, \dots, m$  and  $(-1)^{r-s} \phi^{(m+1+r)}(\beta) \geq 0$  for  $s = 0, \dots, r$  and  $r = 0, \dots, n - m - 2$  then the right hand side of (10) is non-negative, i.e., the inequality (4) holds.
- (ii) If the reverse of (10) holds,  $\phi^{(s)}(\alpha) \leq 0$  for  $s = 0, \dots, m$  and  $(-1)^{r-s} \phi^{(m+1+r)}(\beta) \leq 0$  for  $s = 0, \dots, r$  and  $r = 0, \dots, n - m - 2$ , then the right hand side of (10) is negative, i.e., the reverse inequality in (4) holds.

### 3. Upper Bound for Generalized Hardy's Inequality

The following estimations for Hardy's difference is given in the previous section, under special conditions in Theorem 6 and Remark 3.

$$\begin{aligned} & \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \\ & \geq \sum_{s=1}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x) d\mu_1(x) \right) \\ & + \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s} (\beta - \alpha)^{r-s} \phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y) d\mu_2(y) \right. \\ & \quad \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x) d\mu_1(x) \right) \\ & \geq 0 \end{aligned}$$

In this section, we present upper bounds for obtained generalization. We recall recent results related to the Chebyshev functional. For two Lebesgue integrable functions  $g, h : [a, b] \rightarrow \mathbb{R}$  we consider the Chebyshev functional.

$$T(g, h) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(t)h(t)dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(t)dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t)dt.$$

With  $\|\cdot\|_p$ ,  $1 \leq p \leq \infty$ , we denote the usual Lebesgue norms on space  $L_p[a, b]$ . In [12] authors proved the following theorems.

**Theorem 7.** Let  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  be a Lebesgue integrable function and  $h : [\alpha, \beta] \rightarrow \mathbb{R}$  be an absolutely continuous function with  $(\cdot - \alpha)(\beta - \cdot)[h']^2 \in L[\alpha, \beta]$ . Then we have the inequality

$$|T(g, h)| \leq \frac{1}{\sqrt{2}} [T(g, g)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left( \int_{\alpha}^{\beta} (x - \alpha)(\beta - x)[h'(x)]^2 dx \right)^{\frac{1}{2}}. \quad (12)$$

The constant  $\frac{1}{\sqrt{2}}$  in (12) is the best possible.

**Theorem 8.** Assume that  $h : [\alpha, \beta] \rightarrow \mathbb{R}$  is monotonic nondecreasing on  $[\alpha, \beta]$  and  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  is absolutely continuous with  $g' \in L_{\infty}[\alpha, \beta]$ . Then we have the inequality

$$|T(g, h)| \leq \frac{1}{2(\beta - \alpha)} \|g'\|_{\infty} \int_{\alpha}^{\beta} (x - \alpha)(\beta - x) dh(x). \quad (13)$$

The constant  $\frac{1}{2}$  in (13) is the best possible.

Under assumptions of Theorem 3 we define the function  $\mathcal{L} : [\alpha, \beta] \rightarrow \mathbb{R}$  by

$$\mathcal{L}(t) = \int_{\Omega_2} v(y) G_{mn}(f(y), t) d\mu_2(y) - \int_{\Omega_1} u(x) G_{mn}(A_k f(x), t) d\mu_1(x). \quad (14)$$

The Chebyshev functional is defined by

$$T(\mathcal{L}, \mathcal{L}) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{L}^2(t) dt - \left( \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{L}(t) dt \right)^2.$$

**Theorem 9.** Suppose that all the assumptions of Theorem 3 hold. Also, let  $(\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L_1[\alpha, \beta]$  and  $\mathcal{L}$  be defined as in (14). Then the following identity holds:

$$\begin{aligned} & \int_{\Omega_2} \phi(f(y))v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x) d\mu_1(x) \\ &= \sum_{s=1}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x) d\mu_1(x) \right) \\ &+ \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s} (\beta - \alpha)^{r-s} \phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y) d\mu_2(y) \right. \\ &\quad \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x) d\mu_1(x) \right) \\ &+ \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{L}(t) dt + R(\alpha, \beta; \phi), \end{aligned} \quad (15)$$

where the remainder  $R(\alpha, \beta; \phi)$  satisfies the estimation

$$|R(\alpha, \beta; \phi)| \leq \sqrt{\frac{\beta - \alpha}{2}} [T(\mathcal{L}, \mathcal{L})]^{\frac{1}{2}} \left| \int_{\alpha}^{\beta} (t - \alpha)(\beta - t) [\phi^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}.$$

**Proof.** Applying Theorem 7 for  $g \rightarrow \mathcal{L}$  and  $h \rightarrow \phi^{(n)}$  we get

$$\begin{aligned} & \left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{L}(t) \phi^{(n)}(t) dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{L}(t) dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi^{(n)}(t) dt \right| \\ & \leq \frac{1}{\sqrt{2}} [T(\mathcal{L}, \mathcal{L})]^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left| \int_{\alpha}^{\beta} (t - \alpha)(\beta - t) [\phi^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \end{aligned}$$

Therefore, we have

$$\int_{\alpha}^{\beta} \mathcal{L}(t) \phi^{(n)}(t) dt = \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{L}(t) dt + R(\alpha, \beta; \phi),$$

where the remainder  $R(\alpha, \beta; \phi)$  satisfies the estimation. Now from the identity (7) we obtain (15).  $\square$

The following Grüss type inequality also holds.

**Theorem 10.** Suppose that all the assumptions of Theorem 3 hold. Let  $\phi^{(n+1)} \geq 0$  on  $[\alpha, \beta]$  and  $\mathcal{L}$  be defined as in (14). Then the identity (15) holds and the remainder  $R(\phi; a, b)$  satisfies the bound

$$|R(\alpha, \beta; \phi)| \leq \|\mathcal{L}'\|_{\infty} \left\{ \frac{\phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta) - \phi^{(n-2)}(\alpha)}{\beta - \alpha} \right\}. \quad (16)$$

**Proof.** By applying Theorem 8 for  $g \rightarrow \mathcal{L}$  and  $h \rightarrow \phi^{(n)}$  we obtain

$$\begin{aligned} & \left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{L}(t) \phi^{(n)}(t) dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{L}(t) dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi^{(n)}(t) dt \right| \quad (17) \\ & \leq \frac{1}{2(\beta - \alpha)} \|\mathcal{L}'\|_{\infty} \int_{\alpha}^{\beta} (t - \alpha)(\beta - t) \phi^{(n+1)}(t) dt. \end{aligned}$$

Since

$$\begin{aligned} & \int_{\alpha}^{\beta} (t - \alpha)(\beta - t) \phi^{(n+1)}(t) dt = \int_{\alpha}^{\beta} [2t - (\alpha + \beta)] \phi^{(n)}(t) dt \\ & = (\beta - \alpha) [\phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha)] - 2(\phi^{(n-2)}(\beta) - \phi^{(n-2)}(\alpha)), \end{aligned}$$

using the identities (7) and (17) we deduce (16).  $\square$

We continue with the following result that is an upper bound for generalized Hardy's inequality.

**Theorem 11.** Suppose that all the assumptions of Theorem 3 hold. Let  $(p, q)$  be a pair of conjugate exponents, that is  $1 \leq p, q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then



$$\begin{aligned}
& \left| \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \right. \\
& - \sum_{s=1}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x)d\mu_1(x) \right) \\
& - \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s}(\beta-\alpha)^{r-s}\phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y)d\mu_2(y) \right. \\
& \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x)d\mu_1(x) \right) \Bigg| \\
& \leq \|\phi^{(n)}\|_p \left( \int_{\alpha}^{\beta} \left| \int_{\Omega_2} v(y)G_{mn}(f(y),t)d\mu_2(y) - \int_{\Omega_1} u(x)G_{mn}(A_k f(x),t)d\mu_1(x) \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{18}$$

The constant on the right-hand side of (18) is sharp for  $1 < p \leq \infty$  and the best possible for  $p = 1$ .

**Proof.** We apply the Hölder inequality to the identity (7) and get

$$\begin{aligned}
& \left| \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \right. \\
& - \sum_{s=1}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x)d\mu_1(x) \right) \\
& - \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s}(\beta-\alpha)^{r-s}\phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y)d\mu_2(y) \right. \\
& \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x)d\mu_1(x) \right) \Bigg| \\
& = \left| \int_{\alpha}^{\beta} \left( \int_{\Omega_2} v(y)G_{mn}(f(y),t)d\mu_2(y) - \int_{\Omega_1} u(x)G_{mn}(A_k f(x),t)d\mu_1(x) \right) \phi^{(n)}(t)dt \right| \\
& \leq \|\phi^{(n)}\|_p \left( \int_{\alpha}^{\beta} |\mathcal{F}(t)|^q dt \right)^{\frac{1}{q}}
\end{aligned} \tag{19}$$

where  $\mathcal{F}(t)$  is defined as in (14).

The proof of the sharpness is analog to one in proof of Theorem 11 in [13].  $\square$

We continue with a particular case of Green's function  $G_{mn}(u, t)$  defined by (6). For  $n = 2, m = 1$ , we have

$$G_{12}(u, t) = \begin{cases} u - t, & \alpha \leq t \leq u \\ 0, & u \leq t \leq \beta' \end{cases} \tag{20}$$

If we choose  $n = 2$  and  $m = 1$  in Theorem 11, we get the following corollary.

**Corollary 1.** Let  $\phi \in C^2([\alpha, \beta])$  and  $(p, q)$  be a pair of conjugate exponents, that is  $1 \leq p, q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\left| \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \right| \quad (21)$$

$$\leq \|\phi''\|_p \left( \int_{\alpha}^{\beta} \left| \int_{\Omega_2} v(y)G_{12}(f(y),t)d\mu_2(y) - \int_{\Omega_1} u(x)G_{12}(A_k f(x),t)d\mu_1(x) \right|^q dt \right)^{\frac{1}{q}}.$$

The constant on the right hand side of (21) is sharp for  $1 < p \leq \infty$  and the best possible for  $p = 1$ .

**Remark 4.** If we additionally suppose that  $\phi$  is convex, then the difference  $\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$  is non-negative and we have

$$0 \leq \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \quad (22)$$

$$\leq \|\phi''\|_p \left( \int_{\alpha}^{\beta} \left| \int_{\Omega_2} v(y)G_{12}(f(y),t)d\mu_2(y) - \int_{\Omega_1} u(x)G_{12}(A_k f(x),t)d\mu_1(x) \right|^q dt \right)^{\frac{1}{q}}.$$

In sequel we consider some particular cases of this result.

**Example 1.** Let  $\Omega_1 = \Omega_2 = (0, b)$ ,  $0 < b \leq \infty$ , replace  $d\mu_1(x)$  and  $d\mu_2(y)$  by the Lebesgue measures  $dx$  and  $dy$ , respectively, and let  $k(x, y) = 0$  for  $x < y \leq b$ . Then  $A_k$  coincides with the Hardy operator  $H_k$  defined by

$$H_k : H_k f(x) := \frac{1}{K(x)} \int_0^x f(t)k(x, t) dt, \quad (23)$$

where

$$K(x) := \int_0^x k(x, t) dt < \infty.$$

If also  $u(x)$  is replaced by  $u(x)/x$  and  $v(x)$  by  $v(x)/x$ , then

$$0 \leq \int_0^b v(y)\phi(f(y))\frac{dy}{y} - \int_0^b u(x)\phi(H_k f(x))\frac{dx}{x}$$

$$\leq \|\phi''\|_p \left( \int_{\alpha}^{\beta} \left| \int_0^b v(y)G_{12}(f(y),t)\frac{dy}{y} - \int_0^b u(x)G_{12}(H_k f(x),t)\frac{dx}{x} \right|^q dt \right)^{\frac{1}{q}}.$$

**Example 2.** By arguing as in Example 1 but  $\Omega_1 = \Omega_2 = (b, \infty)$ ,  $0 \leq b < \infty$  and with kernels such that  $k(x, y) = 0$  for  $b \leq y < x$  we obtain the following result

$$\begin{aligned}
0 &\leq \int_b^\infty \phi(f(y))v(y) \frac{dy}{y} - \int_b^\infty \phi(H_{\bar{k}}f(x))u(x) \frac{dx}{x} \\
&\leq \|\phi''\|_p \left( \int_\alpha^\beta \left| \int_b^\infty v(y)G_{12}(f(y),t) \frac{dy}{y} - \int_b^\infty u(x)G_{12}(H_{\bar{k}}f(x),t) \frac{dx}{x} \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned} \quad (24)$$

where the dual Hardy operator  $H_{\bar{k}}f$  is defined by

$$H_{\bar{k}}f(x) := \frac{1}{\bar{K}(x)} \int_x^\infty k(x,y)f(y)dy, \quad (25)$$

where  $\bar{K}(x) = \int_x^\infty k(x,y)dy < \infty$ .

We continue with the following Example.

**Example 3.** Let  $\Omega_1 = \Omega_2 = (0, \infty)$  and  $k(x, y) = 1, 0 \leq y \leq x, k(x, y) = 0, y > x, d\mu_1(x) = dx, d\mu_2(y) = dy$  and  $u(x) = \frac{1}{x}$  (so that  $v(y) = \frac{1}{y}$ ) we obtain the following result

$$\begin{aligned}
0 &\leq \int_0^\infty \phi(f(y)) \frac{dy}{y} - \int_0^\infty \phi(A_k f(x)) \frac{dx}{x} \\
&\leq \|\phi''\|_p \left( \int_\alpha^\beta \left| \int_0^\infty G_{12}(f(y),t) \frac{dy}{y} - \int_0^\infty G_{12}(A_k f(x),t) \frac{dx}{x} \right|^q dt \right)^{\frac{1}{q}}
\end{aligned}$$

where  $A_k$  is defined by

$$A_k f(x) = \frac{1}{x} \int_0^x f(y)dy.$$

**Example 4.** By arguing as in Example 3 but only with  $\phi(x) = x^p, \prod_{i=1}^k (p+1-i) \geq 0$  we obtain the following result

$$\begin{aligned}
0 &\leq \int_0^\infty f^p(x) \frac{dx}{x} - \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) dt \right)^p \frac{dx}{x} \\
&\leq \|\phi''\|_p \left( \int_\alpha^\beta \left| \int_0^\infty G_{12}(f(y),t) \frac{dy}{y} - \int_0^\infty G_{12}(A_k f(x)f(x),t) \frac{dx}{x} \right|^q dt \right)^{\frac{1}{q}}
\end{aligned}$$

We continue with the result that involves Hardy–Hilbert’s inequality.

If  $p > 1$  and  $f$  is a non-negative function such that  $f \in L^p(\mathbb{R}_+)$ , then

$$\int_0^\infty \left( \int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy \leq \left( \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y) dy. \quad (26)$$

Inequality (26) is sometimes called Hilbert’s inequality even if Hilbert himself only considered the case  $p = 2$ .

**Example 5.** Let  $\Omega_1 = \Omega_2 = (0, \infty)$ , replace  $d\mu_1(x)$  and  $d\mu_2(y)$  by the Lebesgue measures  $dx$  and  $dy$ , respectively. Let  $k(x, y) = \frac{(\frac{y}{x})^{-1/p}}{x+y}$ ,  $p > 1$  and  $u(x) = \frac{1}{x}$ . Then  $K(x) = K = \frac{\pi}{\sin(\pi/p)}$  and  $v(y) = \frac{1}{y}$ . Let  $\phi(u) = u^p$ ,  $\prod_{i=1}^k (p-i+1) \geq 0$ , replace  $f(y)$  with  $f(y)y^{\frac{1}{p}}$  then the following result follows

$$\begin{aligned} 0 &\leq \int_0^\infty f^p(y) dy - K^{-p} \int_0^\infty \left( \int_0^\infty \frac{f(y)}{x+y} dy \right)^p dx \\ &\leq \|\phi''\|_p \left( \int_\alpha^\beta \left| \int_0^\infty G_{12} \left( f(y)y^{\frac{1}{p}}, t \right) \frac{dy}{y} - \int_0^\infty G_{12}(A_k f(x), t) \frac{dx}{x} \right|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

where

$$A_k f(x) = \frac{\sin(\pi/p)}{\pi} \int_0^\infty \frac{f(y)}{x+y} x^{\frac{1}{p}} dy.$$

We also mention Pólya–Knopp’s inequality,

$$\int_0^\infty \exp \left( \frac{1}{x} \int_0^x \ln f(t) dt \right) dx < e \int_0^\infty f(x) dx, \quad (27)$$

for positive functions  $f \in L^1(\mathbb{R}_+)$ . Pólya–Knopp’s inequality may be considered as a limiting case of Hardy’s inequality since (27) can be obtained from (5) by rewriting it with the function  $f$  replaced with  $f^{\frac{1}{p}}$  and then by letting  $p \rightarrow \infty$ .

**Example 6.** By applying (22) with  $\phi(x) = e^x$ , and  $f$  replaced by  $\ln f^p$ ,  $p > 0$  we obtain that

$$\begin{aligned} 0 &\leq \int_{\Omega_2} f^p(y) v(y) d\mu_2(y) - \int_{\Omega_1} \left[ \exp \left( \frac{1}{K(x)} \int_{\Omega_2} k(x, y) \ln f(y) d\mu_2(y) \right) \right]^p u(x) d\mu_1(x) \\ &\leq \|\phi''\|_p \left( \int_\alpha^\beta \left| \int_{\Omega_2} v(y) G_{12}(\ln f^p(y), t) d\mu_2(y) - \int_{\Omega_1} u(x) G_{12}(A_k f(x), t) d\mu_1(x) \right|^q dt \right)^{\frac{1}{q}} \end{aligned} \quad (28)$$

where  $k(x, y)$ ,  $K(x)$ ,  $u(x)$  and  $v(y)$  are defined as in Theorem 1 and

$$A_k f(x) = \frac{p}{K(x)} \int_{\Omega_2} k(x, y) \ln f(y) d\mu_2(y).$$

At the end, we give interesting application.

Using (10), under the assumptions of Theorem 4, we define the linear functional  $A : C^n([\alpha, \beta]) \rightarrow \mathbb{R}$  by

$$\begin{aligned}
A(\phi) &= \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
&\quad - \sum_{s=1}^m \frac{\phi^{(s)}(\alpha)}{s!} \left( \int_{\Omega_2} (f(y) - \alpha)^s v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x) - \alpha)^s u(x)d\mu_1(x) \right) \\
&\quad - \sum_{r=0}^{n-m-2} \sum_{s=0}^r \frac{(-1)^{r-s}(\beta-\alpha)^{r-s}\phi^{(m+1+r)}(\beta)}{(m+1+s)!(r-s)!} \left( \int_{\Omega_2} (f(y) - \alpha)^{m+1+s} v(y)d\mu_2(y) \right. \\
&\quad \left. - \int_{\Omega_1} (A_k f(x) - \alpha)^{m+1+s} u(x)d\mu_1(x) \right).
\end{aligned}$$

If  $\phi \in C^n([\alpha, \beta])$  is  $n$ -convex, then  $A(\phi) \geq 0$  by Theorem 4. Using the positivity and the linearity of functional  $A$  we can get corresponding mean-value theorems. We may also obtain new classes of exponentially convex functions and get new means of the Cauchy type applying the same method as given in [14–21].

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## References

- Krulić, K.; Pečarić, J.; Persson, L.-E. Some new Hardy-type inequalities with general kernels. *Math. Inequal. Appl.* **2009**, *12*, 473–485.
- Krulić Himmelreich, K.; Pečarić, J.; Pokaz, D. *Inequalities of Hardy and Jensen*; Element: Zagreb, Croatia, 2013.
- Hardy, G.H. Notes on some points in the integral calculus LX: An inequality between integrals. *Messenger Math.* **1925**, *54*, 150–156.
- Kufner, A.; Maligranda, L.; Persson, L.-E. The prehistory of the Hardy inequality. *Am. Math. Mon.* **2006**, *113*, 715–732. [\[CrossRef\]](#)
- Kufner, A.; Maligranda, L.; Persson, L.-E. *The Hardy Inequality. About Its History and Some Related Results*; Vydavatelský Servis Publishing House: Pilsen, Czech Republic, 2007.
- Kuczma, M. *An Introduction to the Theory of Functional Equations and Inequalities*; Birkhauser: Basel, Switzerland; Boston, MA, USA; Berlin, Germany, 2009.
- Pečarić, J.E.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press: San Diego, CA, USA, 1992.
- Whittaker, J.M. *Interpolation Function Theory*; Cambridge University Press: Cambridge, UK, 1935.
- Gontscharoff, V.L. *Theory of Interpolation and Approximation of Functions*; Gostekhizdat: Moscow, Russia, 1954.
- Davis, P.J. *Interpolation and Approximation*; Blaisdell: Boston, MA, USA, 1961.
- Agarwal, R.P.; Wong, P.J.Y. *Error Inequalities in Polynomial Interpolation and Their Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993.
- Cerone, P.; Dragomir, S.S. Some new Ostrowski-type bounds for the Chebyshev functional and applications. *J. Math. Inequal.* **2014**, *8*, 159–170. [\[CrossRef\]](#)
- Bradanović, S.I.; Pečarić, J. Generalizations of Sherman's inequality by Taylor's formula. *J. Inequalities Spec. Funct.* **2017**, *8*, 18–30.
- Agarwal, R.P.; Gala, S.; Ragusa, M.A. A regulatory criterion in weak spaces to Boussinesq equations. *Mathematics* **2020**, *8*, 920 [\[CrossRef\]](#)
- Agarwal, R.P.; Ivelić Bradanović, S.; Pečarić, J. Generalizations of Sherman's inequality by Lidstone's interpolation polynomial. *J. Inequal. Appl.* **2016**, *6*, 1–18.
- Iqbal, W.; Noor, M.A.; Noor, K.I.; Safdar, F. Generalized Petrović inequalities for coordinated exponentially  $m$ -convex functions. *Int. J. Anal. Appl.* **2021**, *19*, 296–318.
- Jakšetić, J.; Pečarić, J. Exponential convexity method. *J. Convex Anal.* **2013**, *20*, 181–197.
- Krulić Himmelreich, K.; Pečarić, J. Some new Hardy-type inequalities with general kernels II. *Math. Inequal. Appl.* **2016**, *19*, 73–84.
- Krulić Himmelreich, K.; Pečarić, J.; Praljak, M. Generalizations of Hardy type inequalities via new Green functions. submitted.

- 
20. Krulić Himmelreich, K. Generalizations of Hardy type inequalities by Taylor's formula. submitted.
  21. Krulić Himmelreich, K.; Pečarić, J.; Praljak, M. Hardy type inequalities involving Lidstone interpolation polynomials. submitted.