mathematics

MDPI

Article

Inquiry and Modeling for Teaching Mathematics in Interdisciplinary Contexts: How Are They Interrelated?

Gemma Sala Sebastià *, Berta Barquero * and Vicenç Font *

Departament d'Educació Lingüística i Literària i de Didàctica de les CCEE i de la Matemàtica, Faculty of Education, Campus Mundet, Universitat de Barcelona, Passeig de la Vall d'Hebron, 171, 08035 Barcelona, Spain
* Correspondence: gsala@ub.edu (G.S.S.); bbarquero@ub.edu (B.B.); vfont@ub.edu (V.F.)

Abstract: International research has pointed out the importance of integrating mathematical modeling and inquiry processes into the teaching and learning of mathematics. This paper aims to present an integrative model that enables analyzing the characteristics inquiry and modeling processes share in the same model with a view to using them when designing and implementing interdisciplinary teaching sequences. After presenting a synthesis of the literature review, our theoretical approach to inquiry and modeling for the analysis of an interdisciplinary teaching sequence will be introduced. We focus here on the case of an inquiry situation in an archaeological context where mathematics and history are interrelated. It was implemented at secondary school level with students aged 13–14. We use this particular case study to analyze the appearance of both processes, in order to look for coincidences, concatenations and synergies. The main result is an integrative model for the joint analysis of both processes.

Keywords: inquiry; mathematical modeling; integrative model; interdisciplinary contexts

Citation: Sala Sebastià, G.; Barquero, B.; Font, V. Inquiry and Modeling for Teaching Mathematics in Interdisciplinary Contexts: How Are They Interrelated? *Mathematics* **2021**, *9*, 1714. https://doi.org/10.3390/math9151714

Academic Editor: Jay Jahangiri

Received: 9 June 2021
Accepted: 16 July 2021
Published: 21 July 2021

1. Introduction

Current trends in the teaching of mathematics reveal that the extensive mathematical education community has reached a consensus on how to improve the teaching of the subject. One such trend emphasizes the importance of problem solving and modeling (see, for instance, [1,2]). To "know mathematics" is supposed to include the competence of using mathematics in and applying it to real-life extra-mathematical situations. In some countries, this recent trend has led to competence-based curricula including inquiry and modeling competences. This is the case, for instance, of the modeling competency, as explained by Blum [1]. A wide range of international studies, such as the TIMSS study [3] or the Programme for International Students Assessment (PISA) [4,5], provide significant recommendations with regard to integrating inquiry and modeling competences in school curricula.

A variety of approaches and research projects related to how to foster inquiry-based learning (IBL) in mathematics education also attach considerable importance to problem solving and modeling processes. For instance, the PRIMAS international project aimed at implementing inquiry-based methodologies in mathematics and science classrooms as well as in teacher education [6,7]. As highlighted by Maass and Engeln ([7], p. 3), the teaching of modeling within this project's framework equates to adopting an inquiry approach in realistic contexts.

Several research studies on the likely relationship between mathematical modeling and inquiry—for instance, Niss [8], (focuses on the connections between the processes of pre-mathematization and inquiry; or Stillman and Brown [9], who analyze the notion of "implemented anticipation" [8] in the modeling cycle, are regarded as closely related processes that should be combined. Some authors even consider modeling as a particular way to approach inquiry-based teaching and learning of mathematics, as stated in Artigue and Blomhoj [10]. They believe that the similarity with the modeling cycles is striking and state the following: "we see a trans-disciplinary structure of a dynamical inquiry process behind both processes". In line with what they say, we assume that working with modeling in mathematics, as in other subjects, can thus lead to gaining a valuable insight into inquiry as a general process using different implementations in diverse disciplines and contexts.

For the past few years, our research team has been working on the implementation of inquiry projects in interdisciplinary contexts, involving the subjects of mathematics and history. Through the analysis of different case studies, it was observed that the students often developed mathematical modeling cycles at specific moments during their inquiry process [11–13]. Consequently, one open question that emerged from these works is about how the students developed inquiry together with mathematical modeling.

However, these commonalities between modeling and inquiry have not been examined in depth nor have they been clearly conceptualised. This led us to gain a deeper understanding of how both processes, inquiry and modeling, are interrelated. This paper aims at contributing to the understanding of the relationships between both processes, especially when used in an interdisciplinary context, such as archaeology, to develop inquiry and mathematical modeling. With this purpose in mind, the following objectives of this article were established: (1) to analyze the presence of the processes of inquiry and mathematical modeling in the implementation of an interdisciplinary teaching sequence, aiming to look for the coincidences, concatenations and synergies between both processes; and (2) to elaborate an integrated proposal for the joint analysis of both processes in the implementation of teaching sequences in interdisciplinary contexts.

To accomplish these objectives, we start by presenting a review of some dominant conceptualizations of modeling and inquiry. Then, we make our theoretical position on inquiry and mathematical modeling explicit. Next, the implementation of an interdisciplinary teaching sequence is analyzed from each of these two processes. Finally, as one of the main results of this research, we present a proposal of a model that integrates both processes for the analysis of interdisciplinary teaching sequences that aims to promote inquiry and modeling. Some final considerations derived from the analysis of the implementation and potentialities of the integrative model are highlighted in the last section. We hope that this integrative model will become a useful tool to, on the one hand, describe the possible learning paths of students and, on the other hand, to design teaching sequences to promote mathematical modeling and inquiry in a dialectical relationship. Finally, based on the literature review we have done, this integrated view of the inquiry and mathematical modeling is represented as a new contribution to the research field of mathematical modelling in mathematics education.

2. Literature Review

2.1. Mathematical Modeling and the Teaching of Mathematics

Over the last decades, the research field known as "Modeling and Applications" has brought together a range of approaches [14] that share the common objective of fostering modeling activity in the teaching of mathematics at different educational levels and countries (e.g., [15]) and in teacher education. These developments and their impact are described, for example, in the 14th ICMI Study [16], the special issues of ZDM in 2006 (38(2) and 38(3)) and ZDM in 2018 (50(1) and 50(2)) as well as in the biannual International Conferences on the Teaching of Mathematical Modeling and Applications (ICTMA).

In this line of research, modeling has been conceptualized in various ways. One of the most widely accepted descriptions is the so-called "modeling cycle" of which several versions exist ([17–19] among others).

This approach describes the modeling cycle as a cyclical process that can be broken down into different sub-processes (see Figure 1), as proposed by Blomhøj ([20], p. 148) and represents the modeling process behind the construction of a mathematical model. This does not imply that the sub-processes need to be followed sequentially, but rather that they can be used as a framework to analyze the overall dynamic process. The author describes the modeling process consisting of the following six sub-processes [18,20]:

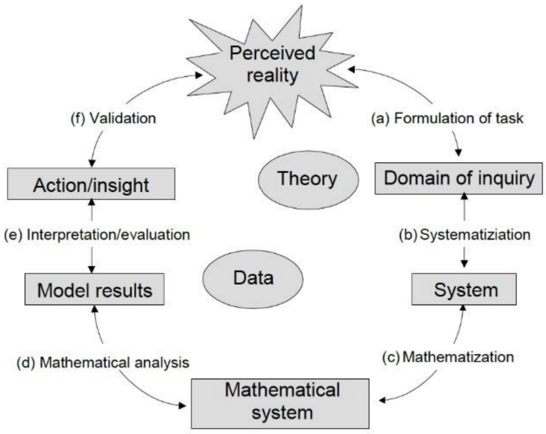


Figure 1. A mathematical modeling process description with six sub-processes ([20], p. 148).

(a) Formulation of a task (more or less explicit) that guides you to identify the characteristics of the perceived reality that is to be modelled; (b) Selection of the relevant objects, relations etc. from the resulting domain of inquiry, and idealisation of these in order to make possible a mathematical representation; (c) Translation of these objects and relations from their initial mode of appearance to mathematics; (d) Use of mathematical methods to achieve mathematical results and conclusions; (e) Interpretation of these as results and conclusions regarding the initiating domain of inquiry; (f) Evaluation of the validity of the model by comparison with data (observed or predicted) and/or with knowledge (theoretically based or shared/personal experience based) ([20], p. 148).

According to the author, in Figure 1, theoretical knowledge and empirical data (represented in the diagram by two ellipses) concerning the domain of inquiry are the basis for all the sub-processes into which modeling has been decomposed. The term "theory" refers to the knowledge of the "domain of inquiry" used in the modeling process. As far as the "data" are concerned, they exist prior to the modeling process and may hence be used to support the processes of systematization and mathematization and, eventually, as a basis for validating the model. In addition, relevant data often have to be collected as part of the modeling process.

2.2. Inquiry-Based Teaching and Learning

The word "inquiry" is used with different meanings. Since 1938, in the field of education, John Dewey [21] described inquiry as "the controlled or directed transformation of an indeterminate situation into one that is as determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole" (Ibid, p. 108). The educational potential of inquiry and its impact on curricula achieved further recognition when it was included in the National Science Education Standards [22]. Educational practices based on the Inquiry-Based Learning (IBL) approach have been the subject of countless studies. Some of this research has produced relevant results from the perspectives of both science education and mathematics.

Among them, as mentioned by Dorier and Maass [23], worthy of note is the research conducted by Artigue and Baptiste [24] as part of the Fibonacci Project, seeking to promote and study the teaching of mathematics using an Inquiry Based Mathematical Education (IBME) approach, as well its relationships with Inquiry Based Science Education (IBSE). Inquiry-based mathematics education (IBME) refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer those questions (like carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, and making guesses and generalizations), interpret and evaluate their solutions, and communicate and discuss them effectively [23].

Moreover, Artigue et al. [25] proposed the use of a diagram to illustrate the inquiry process, a legacy of the model proposed in science (see Figure 2). As the authors explain, "science and mathematics share the dominant mode of knowledge building through inquiry" (p. 9). The sequence in this Figure 2 shows that an inquiry process may start with an initial question. Some preliminary exploration may uncover aspects that remind the students of existing ideas, which then lead to possible explanations. To falsify (or not) the hypothesis, data related to the problem studied need to be collected. As shown by the sequence "prediction–plan and conduct investigation–interpret data", the result of the analysis can be used as evidence to check against the predicted result. If more than one hypothesis is formulated, as is desirable, the sequence must be repeated several times. Based on these results, a conclusion can be drawn providing a solution to the initial problem.

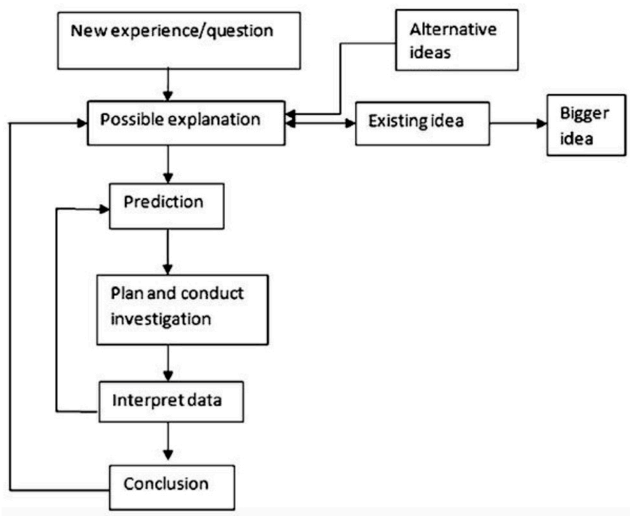


Figure 2. A route diagram aiming at understanding the inquiry process, as proposed in the framework of the Fibonacci Project ([26], p. 5).

Harlen [27] reflected on the similarities and differences between classroom experiences that foster understanding in science and mathematics. On the one hand, it is important in both subjects to establish a question or problem to solve, to engage in collaborative work, to enter discussions and dialogue, to consider alternative approaches, to build critical thinking, and to reflect on learning and to communicate. On the other hand, the author highlights differences in how questions can be addressed, how solutions can be sought, how they can be validated, and the nature of the explanations. In addition, an important part of IBME is that of transforming a problem into mathematically based questions through a process of mathematical modeling. Furthermore, Artigue and Blomhøj [10] have highlighted the multiple commonalities between the inquiry and modeling research field. Despite the previous remarks, it remains unclear when, how and why modeling and inquiry interact. One of the aims of our paper is to examine this in greater depth.

Artigue and Blomhøj [10], in their conceptualization of inquiry-based mathematics education (IBME), analyze how inquiry-based education, which originated in science education, has migrated to mathematics education. The authors examine different approaches and highlight two aspects of mathematical modeling research. Firstly, inquiry and modeling both seek to link the situations under study with the building of knowledge applicable to personal, educational and social contexts. Secondly, modeling and inquiry conceptualizations present many similarities. Artigue and Blomhøj [10] refer to the description of inquiry by Harlen [26] in the Fibonacci Project (Figure 2), and to the modeling cycle presented in Blomhøj [20] (Figure 1) to highlight their commonalities.

The modeling process can be described as a cyclic process where reflections along the process can lead to changes in previous sub-processes and thereby initiate new loops in the modeling cycle. The similarity with the diagram [referring to Harlen [26] and Blomhøj [20]] is striking. We see a trans-disciplinary structure of a dynamical inquiry process behind both processes.

According to Artigue and Baptiste [24], mathematical inquiry is often motivated by questions arising from the natural world or the world around us (as in scientific inquiry). One of the main ambitions of mathematics is to contribute to the understanding of the natural, social and cultural world, and to enable human beings to act on this world. Moreover, mathematics, as a science, also creates its own objects and its own reality, and the questions posed by these objects have promoted its development. The nature of the questions obviously influences the inquiry process. When the questions come from an external source (e.g., daily life, natural phenomena or, as in our work, human artefacts) the transformation of these questions into accessible issues for mathematical work is an important part of the inquiry process, which may involve a modelling process. However, it should be noted that in mathematics education, the term modelling is used in a restricted sense: it refers to a process of mathematization and construction of mathematical models.

With the same aim of describing the inquiry process, in her doctoral thesis, Sala Sebastià [28] studied the competence of inquiry by designing and implementing various teaching sequences for mathematics in interdisciplinary contexts. These teaching sequences were set in historical contexts where the students had to address a range of alternative hypotheses until they determined the most plausible one by taking into account not only historical aspects, but also mathematical ones by developing mathematical modeling processes. One of the derived results was the proposal of a specific model to conceptualize inquiry processes which are presented in Section 3 (in Figure 3). Briefly summarized, this inquiry cycle starts from a problematic and authentic situation—understood in the same sense as Vos [29] defines 'authenticity' for tasks in mathematics education. From this starting point, the students are expected to develop each of the sub-processes progressing (and/or receding) throughout the cycle. When they come to the validation of their results (sub-process 6 in Figure 3), they can return (dotted arrow) to formulate new hypotheses—in case the results cannot be validated—or move on to communicate the results obtained and the answers found to the initial questions.

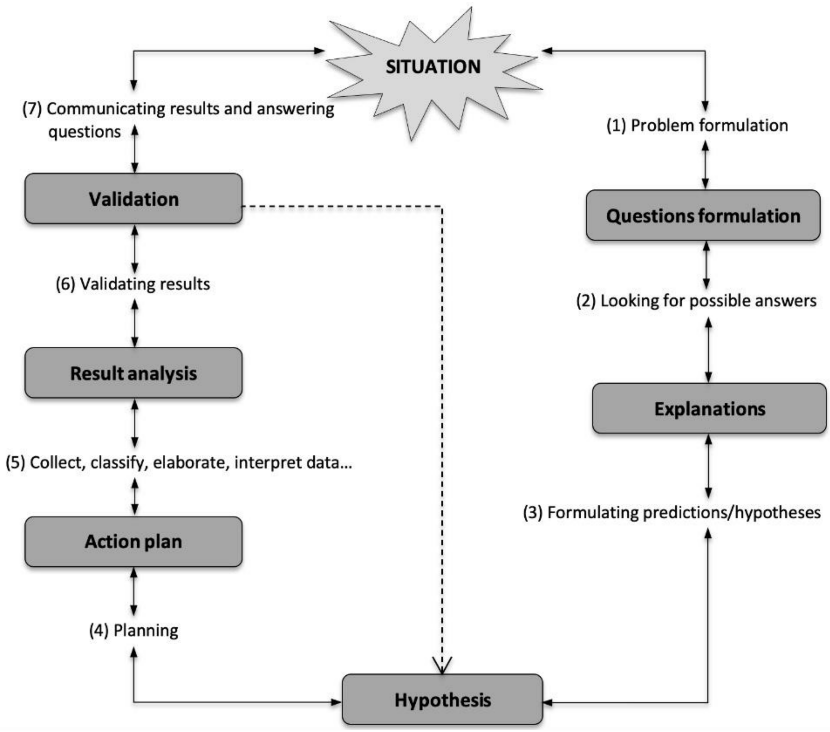


Figure 3. Inquiry process model ([28], p. 67).

2.3. Historical Contexts to Foster Inquiry

As history and social sciences are important contexts in the daily lives of students, it seems obvious that these subjects are present when we are talking about an interdisciplinary approach in education and, in particular, in mathematics education. Before providing more details about the design and implementation of the teaching sequence analyzed in this paper, it is important to clarify that it does not refer to the domain of history of mathematics in mathematics education, (see, for instance, [30,31]) nor to the domain of ethnomathematics and its place in the history and pedagogy of mathematics [32]. It is based on the use of mathematics to provide answers to some questions appearing in historical contexts, specifically involving archaeological and real historical data.

Traditionally, history has been taught as a static knowledge, mostly presented as a set of already finished contents that students have to memorize. However, some innovative trends in the Didactics of History strive for rethinking historical contents and their organization in school curricula to present their rationale. According to Dean [33], history is a discipline that consists of posing questions about the sources considering the context of the epoch and using the imagination to comprehend the facts of the past with empathy. Historians build a view of the past based on a complex interaction between two "narratives": the first one is the narrative of the past, based on all the objective information from the past which historians can study; the second one includes their wisdom, life experiences, perceptions, curiosity and their ability to imagine. The result of the interaction between two narratives is a creative construction, based on facts and developed by means of informed imagination. To study history, according to this author, students should have access to the first narrative, but their second narrative is usually highly restricted due to their young age and lack of experience. Teachers should therefore focus on providing students with a second narrative that enables them to learn history. Along the same lines, Prats [34] encourages students to "learn history by doing history", gaining insight into the historical method so as to learn biology, physics, etc.

In this work, students could learn mathematics and history by doing inquiry as archaeologists. Following the work of the museum archaeologists' team, and interacting with them, students could know the methods experts use to formulate and validate their hypothesis and how experts in the field use to interpret and give sense to the results thanks constant feedback and dialogue between history and mathematics.

3. Our Framework of Modeling and Inquiry

In this paper, we assume the modeling cycle put forward in Blomhøj [20] (as explained in Section 2.1 and in Figure 1). We have chosen this conceptualization of mathematical modeling (among other possible options) as it explicitly includes the domain of inquiry in the modeling process, in particular, between sub-process (a) "task formulation" and sub-process (b) "systematization".

With respect to the analysis of the inquiry process, we use the inquiry cycle proposed in Sala Sebastià [28] (Figure 3), which is described as a cycle of seven sub-processes that are supposed to be developed by the students when involved in an inquiry into an authentic problem and situation.

The sub-processes into which inquiry is decomposed and analytically described are the following:

(1) Problem formulation, which consist in turning the initial problematic situation into particular questions to address, and thus converting it into the main object and aim of the inquiry.
(2) Looking for possible answers, intentionally showing a critical attitude, of doubt, and of contrast to the information that emerges from the context and from the inquiry work in progress.
(3) Formulating hypotheses and/or predictions, making predictions and/or hypotheses based on different variables identified in the problematic situation, with the aim of contrasting and validating them.

(4) Planning and organizing the work of the inquiry community (students and teachers), proposing flexible and adaptable plans for the development of the inquiry work.
(5) Collecting, classifying, elaborating and interpreting data, when looking for, collecting, recording and selecting data useful for the inquiry from all possible sources. When the information is analyzed and evaluated, connections are created between that information and the previously formulated hypotheses with the help of the necessary theoretical tools chosen.
(6) Validating results, when validating the results by contrasting them with the initially formulated hypotheses and analyzing those findings critically. In the case of non-acceptance of the initial hypotheses, alternative or new hypotheses are here expected to be formulated (shown in Figure 3 with a dashed arrow going from validation to hypothesis).
(7) Communicating results and answering questions, when one elaborates research reports, arguing the results and answers to the questions addressed. Moreover, this is when any other question raised throughout the inquiry work carried out may be answered, either individually or in groups, and when its impact and limitations are observed.

4. Research Methodology and Methods of Analysis

The methodology adopted in this research was structured into the following steps. We first conducted a literature review on modeling and inquiry to later, based on this review, adopt a theoretical approach on how to interpret and conceptualize inquiry and mathematical modeling. This positioning has allowed us to have a priori defined categories for data analysis (see Table 1), which correspond to the sub-processes into which inquiry and modeling were decomposed. Using these categories, an implementation of an interdisciplinary teaching sequence at secondary school was then analyzed based on the categories (sub-processes) for modeling and inquiry respectively. This analysis allows us to determine when both processes, modeling and inquiry, occur simultaneously, when they are concatenated, or when synergies between both arise. The term "synergy" is here used in the same sense as the one defined by Maracci et al. [35] as "the interaction of two or more agents or forces so that their combined effect is greater than the sum of their individual effects".

Table 1. List of sub-processes of inquiry and of mathematical modeling.

Sub-Processes of Inquiry from Figure 3		Sub-Processes of Mathematical Modeling from Figure 1	
I1	Problem formulation	Ma	Task formulation
I2	Looking for possible answers	Mb	Systematization
I3	Formulating hypotheses	Mc	Mathematization
I4	Planning	Md	Mathematical analysis
I5	Collecting, classifying and interpreting data	Me	Interpretation/Evaluation
I6	Validating results	Mf	Validation
I7	Communicating results and answering questions		

Design and Implementation of an Interdisciplinary Teaching Sequence

With regard to this last step, this paper focuses on analyzing one implementation of a teaching sequence with secondary school students with the aim of developing a complete inquiry process in an interdisciplinary context combining mathematics with history. This teaching sequence is referred to as "*What lies behind these ruins?*" (The worksheets used in the implementation are available at: https://ruinesdebaetulo.blogspot.com.es/ (accessed on 19 July 2021). It corresponds to the blog created for the implementation, where the option "Translate" can be used to select the language) (more details can be found [28]).

The design of the teaching sequence followed the theoretical conceptualization of an inquiry process, as shown in Figure 3, described in [28], and some of the main traits of the proposal of the *study and research paths* [36,37]. From this last proposal, we used the arborescence of questions and answers to trace the design and possible paths followed by the students. The starting point of this sequence was an initial generating question about hidden Roman ruins. The question was productive enough to make history and mathematics interact and was structured in terms of some derived questions that guided the inquiry process throughout its implementation. Its structure in term of questions and answers is used in the next section to analyze the implementation of this interdisciplinary project.

It was implemented for two weeks, during 10 sessions of approximately 4 h each, at the end of school year 2014–2015 with a group of 32 students (14 boys and 18 girls), in their first year of compulsory secondary education (12–13 years) at a secondary school in Badalona (Catalonia, Spain). The history and mathematics teachers guided the implementation of the project. The students worked in cooperative teams of inquiry. There were 10 teams or groups of 3–4 students (identified from 1 to 10). All of them were asked to write a report throughout their process of inquiry. The curricular objectives of the teaching sequence were to develop the mathematical and core competences corresponding to the first year of secondary education through key contents such as spatial reasoning, representation of figures, geometric relationships and transformations, etc. At a social level, the influence of Romanization in Catalonia was considered.

To perform the analysis, three group of students [Group 6, Group 8 and Group 10] were selected to analyze the inquiry and modeling processes they followed in depth. Following the objectives set out in this research, we selected these three groups after the implementation was completed. This decision was made based on the researchers' observations and after watching the audio and video recordings of the sessions. Said three groups were the most involved in the inquiry and developed the richest modeling processes. It is worth mentioning that during the implementation of the teaching sequence the first author of the paper acted as a non-participating observer. The observer regularly elaborated observation reports about the different discussions and activities posted on the blog created for the implementation. Apart from these observation reports, the data gathered concerned the students' reports, and the students' interactions in the classroom sessions. The teachers and the observer had permanent access to the students' reports, which were uploaded as a document in Google Drive. In summary, the data gathered and analyzed in this paper correspond to: (1) the students' reports in Google Drive, (2) the final report submitted by the working groups, (3) the "chronicles" written by one of the educators acting as an observer of the whole group of participants, and (4) audio and video recordings.

We used the decomposition of modeling and inquiry processes into sub-processes to analyze the four called type of evidence of the students' work in the interdisciplinary project ab out Roman ruins. Table 1 includes the sub-processes of the modeling cycle (from [20] as described in Figure 1) and of the inquiry cycle ([28] as described in Figure 3), and the acronyms that will be used throughout the analysis of the students' work. The column on the left corresponds to the inquiry process, and the one on the right to the modeling process.

The first author of the article did the preliminary analysis of all evidence collected in each session of implementation. This data come from the transcripts of the sessions' recordings, her fieldnotes during the implementation, and the students' productions, most of which were collected through the writing of students' reports. All this empirical data was firstly analyzed by the first authors who code the data and identified the evidence corresponding to the categorization presented in Table 1. Each category corresponds, respectively, to one of subprocess of inquiry and modeling, which are described in detail in the theoretical framework section. Following, this first analysis was triangulated with the other two authors of the paper, who analyzed again the data, complementing and validating the initial analysis. As the data collected was too big, we agreed to focus on the most active working groups. That is, our analysis focused on Group 6, Group 8 and Group 10 whose work we consider rich and representative enough of implementation. As an example of the kind of analysis that will be presented in Section 5, we illustrate the initial episode when the question that initiated the inquiry was presented to students as: *What lies behind these ruins?* After a group discussion, the students agreed that the *problem formulation* could be reformulated as: *What type of Roman building could the Roman ruins correspond to?* In this episode, the whole group agreed to reformulate the problem and task to be solved, which is the first sub-process of the cycles of modeling and inquiry. This sub-process took place and is indicated by the acronyms [I1] and [Ma].

The analysis of the presence of the different sub-processes (categories described in Table 1) was carried out following the timeline of the implementation of the whole sequence during 10 sessions. For each session, a summary of the students' activity is included, which allows us to infer the use of the sub-processes linked to inquiry and modeling.

5. Analysis of the Didactic Sequence Based on Inquiry and Modeling Processes

In the first session, the students were introduced to the initial problem, which was related to the discovery ten years ago of Roman ruins in the center of Badalona by the team of archaeologists of the Badalona Museum. According to archaeological research [38], these ruins belonged to an old building in the Roman city of *Baetulo*—the Roman name for Badalona. The activities were structured along certain questions that facilitated the students to progress within the cycle of inquiry. The students were asked to find out what type of public building the Roman ruins belonged to. The teachers suggested the main questions of the project described below.

The students worked with real data from the beginning of the project. For example, to introduce the project, the teachers showed the students how to locate and study the area of the city where the ruins were discovered by using Google Maps and Badalona's urban geo-portal. Managing these real local data was important for some teams in order to formulate their initial working hypotheses about what kind of building the ruins could correspond to.

The main initial question that initiated and motivated the inquiry was: *What lies behind these ruins?* After a group discussion, the students agreed that the *problem formulation* (and the *task formulation)* of the real situation had to be specified as: *What type of Roman building could the Roman ruins correspond to?* [I1] and [Ma].

Faced with this question, the students proposed that the ruins could correspond to, for instance, a theater, a circus, an amphitheater, a basilica, baths, a pantheon, a temple, etc. The students then focused on searching historical information to look for *possible answers* [I2] about the kind of roman construction this building could be. Different kinds of explanations [sub-process to justify the choice of building] could be provided, but the most satisfactory ones in the school context were related to taking into account the shape of the building. For example, the students deduced that "if the ruins correspond to an ellipse, the building could be an amphitheater; or, if it is a semicircle it could be a theater; or, if a section of its perimeter was rectangular, it could have been a circus, etc." [Mb]. The rest of the possible buildings that have a regular polygonal layout were rejected, as it turned out the wall was curvilinear.

To work on the justification and validation of these hypotheses, the students worked with a new set of questions and their corresponding answers, which meant going back a few steps in the cycle of inquiry. The teachers, following the students' comments and their firsts hypotheses, proposed working on these questions: *What geometrical shapes could fit into the building's partial wall discovered by the archaeologists (a 1.5-m-high curvilinear wall)?*

The students knew that the wall was curvilinear, as they refer to it using the archaeological report available on the blog. They thus *hypothesized* [I3] that the Roman wall had been part of a curved building [Mb], such as a (semi-circular) theater, an (elliptical) amphitheater or a circus (as a part of its semi-circular floor). Different questions arose that were further specified as follows: *How can we find out the original shape of the discovered wall?*

To validate their hypotheses, the students worked on defining an *action plan* [I4] with the help of the history and mathematics teachers to determine the original shape of the remnants of the curved wall. To do this, the students undertook a range of mathematically orientated actions. To find out whether the Roman wall was part of an ellipse or a circumference, in order to *look for possible answers* [I2]), the students went to a public square near their school where the teachers had drawn the shape of the Roman wall on the ground using the archaeological information available. In this activity, the students developed mathematical modeling because they tried to find a geometrical model that would fit a representation of the original wall. First of all, the students tried to graphically fit the drawn wall into an ellipse using a manual method. Two students of group 10 were then placed at the foci (determined by trial and error) of the possible ellipse, each holding the end of a rope. A ring, which could move along the rope, was used. A third student held on to the ring to keep the rope tight at all times while following the trajectory of the drawn wall. This process of *data collection* [I5], *mathematization* [Mc] *mathematical systematization* [Md] *analysis* and *interpretation* [Me] was repeated several times as the students changed the location of the foci of the ellipse to be able to follow the wall drawn on the ground and to try to *validate the model* [Mf] and [I6] *the results* geometrically. The students encountered numerous difficulties to fit the shape of the wall into an ellipse, basically because it was not easy to draw such a large ellipse. Considering the dimensions of the wall represented on the floor of the square, the radius of the ellipses had to be at least 16 m long. Moreover, it was not easy for the students to visualize the wall's curved shape, as it looked more like straight line than part of an ellipsis or a circumference. The students concluded that the wall could not possibly be part of an ellipse. A student from inquiry team 8 drew an ellipse on the floor and justified the following [Mb]: "If it really was an ellipse, it could only be the flatter part, the middle part, because if it was this part (pointing toward the vertex of an imaginary ellipse she was drawing with her finger), it wouldn't be curved enough". Another student added: "But . . . It could be a part of a big circumference, right? There were buildings that were round, don't you remember? We should try to find out if this shape has a center

Once the ellipse hypothesis was rejected, the students continued *looking for answers* [I2] and tried to fit the wall into a circumference by aligning its center and radius. To *collect new data, to mathematize, systematize, analyze* and *interpret them* [I5], [Mc], [Md] and [Me], the inquiry teams used different methods of construction. First, when the students remembered that all the points of a circumference are at the same distance from its center, they tried to find the center by testing possible centers through trial and error [Mc] and [Md]. As this was unsuccessful, they tried again by using plaster to draw some of the tangents at different points of the wall. They drew a perpendicular to these tangents (using a large ruler and a wooden square) to find the center at their intersection to then be able to determine the radius ([Mc] and [Md]). Teams 6 and 10 opted for a different strategy: they drew two bisections at three points of the arch of the wall using a rope and plaster to mark them. They found the center at the midpoint of the bisector and thus concluded that the wall belonged to a circular building ([Mc] and [Md]). The teachers encouraged the students to measure the radius—16 m—and to draw the complete semi-circumference on the square [I5]. The students decided to 'draw' it placing some classmates side-by-side at a distance of 16 m to the center of the circumference. The students realized that the perimeter of the building was enormous. A student said: "It was bigger than our school! And . . . how many people fit in it? A lot of people, right?" [Me]. At this stage of the activity, it is observed that all the inquiry teams completed a mathematical modeling cycle and they managed to develop the initial strategies to work on the *validation* of the initial hypothesis formulated ([Mf] and [I6]) testing the geometrical models that best fit the simulation of the Roman ruins and rejecting those that do not fit. The use the students gave to the tangible construction of the geometrical model was an important contribution in order to progress in the modeling and inquiry cycle. At this point of the inquiry, almost all the teams concluded that the wall fit into a circumference.

Given the fact that the question of having to discover what kind of curve the wall could fit into was complex, the historical context of the situation played a central role. The students *look for answers* [I2], *collect and interpret* [I5] data from the historical context, that is, from the information found about the reported shape of the Roman building, the information from the archaeological report, etc. This historical context allowed the students to formulate and work on the most plausible *hypotheses* [I3]. Indeed, the number of possible curves was limited to the forms used in public Roman constructions (an ellipse or circumference and their defining elements). Moreover, the students could select (*systematization* [Mb]) the most relevant and mathematizable [Mc] data based on limited information. Limiting the wide variety of possible models to fit the shape of the ruins, thanks to the historical context and the answers found, allowed the students to make the problem accessible to them and to put into practice some mathematical knowledge easily available to them.

Once the students *validated* [I6] and [Mf] that the wall fit into a circumference, two possible types of buildings were historically possible: a (semi-circular) theater or a circus (with a semi-circular floor layout on one side and a rectangular one on the other). To determine which of these two options the building could correspond to [I2], the students went back to the information provided by the historical data about the size and area [I5], which helped them discard one of the options.

At this point, students considered two new hypotheses (a theater or a circus), problematized by the following questions: In Roman times, what were theaters actually like? What about circuses? Do we have the necessary data to sketch their layout?

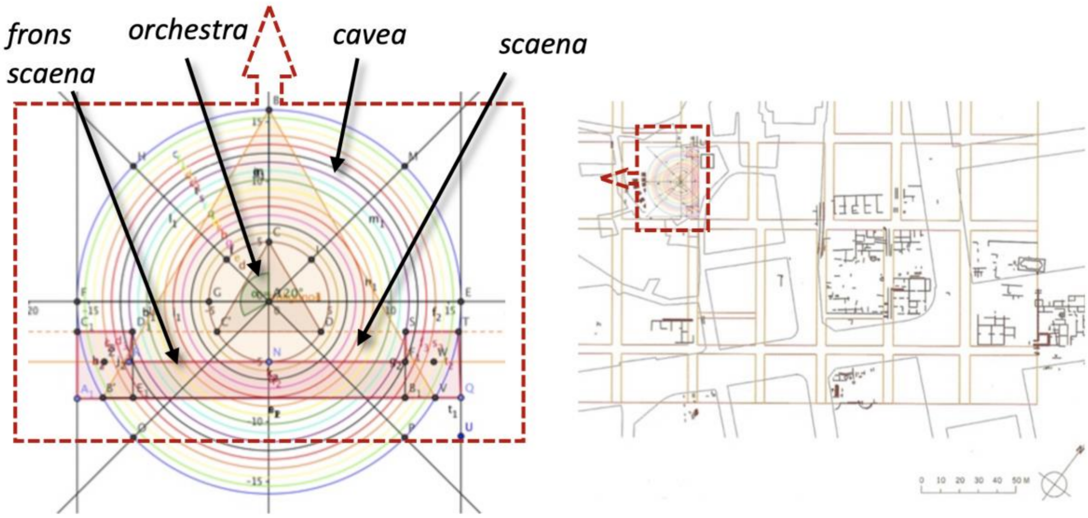
The students needed to know more about Roman architecture. The teachers decided to talk about the work "De Architectura" by Marcus Vitruvius (a free English version of "The Ten Books on Architecture" by Vitruvius is available online 19 July 2021: http://www.gutenberg.org/files/20239/20239-h/20239-h.htm#Page_137 but we use the Spanish version available on the blog (see endnote 1)) (c. 80-70 BCE-15 BCE) that describes the construction rules for a range of public Roman buildings, detailing their parts, proportions, etc. By doing so, the students could *look for possible answers* [I2].

After presenting this external resource, the students decided to try to *validate the hypothesis* of a theater. They planned the actions to be carried out and started by *systematizing* [Mb] and *mathematizing* [Mc] the data collected during the inquiry (data from the archaeological report, historical data, data from the site's location, data from the previous modeling cycle developed, etc.), *interpreting* them [Me] according to the rules of Vitruvius on theater building with the help of the teachers. At this stage, the students started a modeling cycle that consisted of building a geometrical model of a Roman theater by following the construction method recommended by Vitruvius.

The students had a Spanish translation of the facsimile in the blog of the project, where the method for the construction of a Roman theater model was explained. The students also had a document on the blog, prepared by the participant-researcher, which helped them to reproduce Vitruvius' method by using GeoGebra (considering it could be difficult for the students to set it up using Geogebra). Nevertheless, they quickly learned how to use the program. Moreover, they resorted to images of ancient Roman theaters they found on the Internet in order to understand and visualize their forms and likely sizes.

This whole process concluded with the construction of several geometric models of the *Baetulo* theater that each research team built (see Figure 4a). The models were adjusted and simulated using the information provided by Vitruvius' canon of proportions as well as the measurements the students had gathered and shared in the class with the other teams.

Regarding the work developed by Group 8, they distinguished the main parts of a Roman theater, which were: the *cavea* (tiered seating space for spectators), the *orchestra* (semicircular space for the musicians, between the bleachers and the stage), the *frons scaena* (decorated stage background to which actors had access) and the *stage* (rectangular space in front of the stage background). The *cavea*, which was shaped like a semicircle, was the grandstand where the spectators used to sit. It was enclosed by an external semicircular wall. Group 8 assumed that the fragment of the curvilinear wall found in the ruins was a part of this wall bordering the *cavea*.



(**a**) (**b**)

Figure 4. (**a**). Drawing of the theater model built by Group 8 using GeoGebra; (**b**). Model simulation proposed by Group 8 superimposing it onto the archaeological site map.

Group 8 followed the procedure described in Figure 4, when they built and simulated their model with GeoGebra. As shown in Figure 4, they distinguished the different parts to propose different models fitting each part. Firstly, group 8 estimated the radius of the semicircle to be 16 m, as was concluded in the first part of the activity carried out at the square near the school. The students drew the outer semicircular shape of a part of the theater. Then, following the classical rules of construction, which indicated to give to the *orchestra* part of the building a value of 5 m, they drew it. They subsequently simulated the *frons scaena* and the *scaena*, which corresponds to a polygon that closes the perimeter of the theater on the opposite side. They extended two straight lines tangent to the semicircle of the outer perimeter of the *cavea*, drawing a perpendicular to both, as shown in Figure 4. Next, they drew, between the external circumference (corresponding to the *cavea*) and the internal one (corresponding to the *orchestra*), other circumferences distanced at 0.74 m (equivalent to 2.5 feet), which corresponds to the classical proportions of construction. Group 8 correctly interpreted that each of these circumferences represented one tier (or terrace) in the *cavea* (or bleacher) where Roman citizens sat to watch the performance.

Once the model was built using GeoGebra, the teachers asked some new questions to help the students reject the *hypothesis* of the circus and *validate* the hypothesis of the theater: *Does the built model fit properly into the dimensional layout of the area where the ruins were found? How can we evaluate how well it fits?*

To answer these questions, the students superimposed a simulation of the theater they had drawn using GeoGebra onto the archaeological site plan (which showed the details and contours of the ruins) adapting the scale and checking whether the contours fit or not. Figure 4b shows the proposal of group 8. This group, like others in the class, pivoted their model until they thought the theater simulation was in an appropriate position according to what was shown on the real map. In other words, the students thought it necessary to *validate* ([Mf] and [I6]) the model simulation with respect to reality, in this case the real map.

The fact that the GeoGebra model fit the site plan *did not validate* (or disprove) the hypothesis of the circus. Since the process of fitting the model meant it had to be reduced or enlarged beyond its form, it was necessary to check if the resulting model continued to meet the characteristics of a Roman theater set out by Vitruvius, such as its location (theater exits had to be placed next to city squares) or (that had to continue to comply with several building rules after applying the scale of the map to the building). The students performed these checks appropriately, obtaining the answer to the questions proposed by the teachers and, consequently, validating the hypothesis that the wall might have belonged to a Roman theater.

At this point of the inquiry, the students were asked to compare their conclusions with the results obtained by the team of archaeologists of the local museum so as to *validate their answers and processes followed with external experts* [I6]. To this end, a meeting was scheduled with the main archaeologist, P. Padrós, who was responsible for the discovery of the real ruins. Each team prepared some questions to interview her. The students' questions were about the archaeologist's discoveries to check their own results. She answered all the questions and showed interest in the students' work.

She also offered new historical information on the theater in the city of *Baetulo*, such as the estimated population in the first century before the common era (BCE) when the theater was already in operation, the location of the city wall and its gates in relation to the theater, etc. One of the gates was located close by, supposedly so the theater could welcome people from neighboring towns.

Thanks to the archaeologist's new contributions and after sharing this information, the students were able to answer the inquiry's initial and main question. They argued that the Roman wall belonged to a Roman theater, which meant the inquiry could be considered as concluded. However, the archaeologist had sparked off new questions related to the details of her team's current research process and this led to a new cycle of inquiry. For example, the students wanted to calculate the capacity of their theater model to *re-evaluate it*, checking whether it coincided with that obtained by the archaeological experts. This means that the situation was again *problematized* ([I1] and [Ma]) and new questions were posed: *What was the theater's occupant capacity? How many people could it hold?*

A new cycle of inquiry was initiated to *look for possible answers* regarding the occupant capacity of the theater [I2]. *Baetulo's* population (approximate data provided by the archaeologist) was taken into account to formulate new *hypotheses* [I3] on the theater's maximum occupancy to meet the needs of the inhabitants of *Baetulo*. Historical data were collected to estimate the number of inhabitants out of the total population that would usually have gone to the theater: for example, what type of performances were planned, to which audience were the performances addressed—only to adults or to children, to what social classes, etc. In this way, a hypothesis was *formulated* only based, like in the first stage, on inquiry processes of historical content looking for information about other similar Roman theaters in Spain, such as Pollentia (in Mallorca) or theaters in Málaga.

To validate their hypotheses, the students devised an *action plan* [I4] based on completing the model built in the second stage of the previous cycle with a drawing of the essential details (such as stands, stairs, access, evacuation corridors, etc.) to calculate the number of seats. They continually took the proportions recommended by Vitruvius into account. Group 8, for instance, counted how many rows there were present in their model. They had considered a total of 15 rows (see Figure 4a). Then, using their model, they calculated the approximate length of each of the semicircles that represented a stand of the *cavea*. With this information, they estimated how many people could sit in the theater, assuming that two people could be seated in 1 square meter.

This meant a new model had to be prepared to *systematize* [Mb] the data, to *mathematize* [Mc] them, to analyze [Md] them, and obtain new results to be *validated* based on the real archaeological context. The students performed a graphical analysis of the resulting new *model* [Me], based on the determining elements of the theater's capacity (number of stands, distance between stands, space between spectators, location, number of stairs and entrances, etc.). Several groups made some adjustments with regard to the capacity. For instance, they took into account that it was likely that the first stand was reserved for senators and other important people in the city and that they were therefore supposed to have more space to sit there. Again, historical and archaeological data were enriched by mathematical data, and vice versa. Both kinds of data and models had to be combined to obtain relevant answers.

The model that emerged from this second modeling process based on the new archaeological and historical data and Vitruvius' rules enabled quantifying the theater's capacity. All the inquiry teams obtained similar results on the approximation of this capacity, which oscillated between 878 and 1100 spectators. This result, together with the new model, was *interpreted* and *validated* [Me] with respect to the real work of the experts, the archaeologists. The students, guided by their history teacher, concluded that the theater seemed to be able to hold more spectators than expected, taking into account *Baetulo*'s population at that time. This exceeded the students' initial hypothesis. However, taking into account the information provided by the archaeologist of the museum, this result was consistent with the fact that the theater welcomed inhabitants from neighboring towns. Finally, once all the inquiry teams had shared their work, the students drew up their inquiry report with the aim of *communicating their results* and providing their answers to the *questions* posed throughout the process [I7]. During this collective process, *new questions* emerged that could have led to new cycles of inquiry, accompanied by new modeling processes, but mainly due to time constraints, it could not be further extended. At the end of the implementation, for instance, some questions emerged about if the way to inquire into the Roman theater model here studied could help to model and explain other ruins found around the world.

6. Results and Discussion

6.1. Visualization of the Relationships between Inquiry and Modeling

The analysis described in the previous section has been outlined and represented using the online visual collaboration platform called Miro. It has allowed us to visualize the relationships between the different sub-processes of inquiry and modeling, as shown in Figure 5. This platform is an online resource that enabled us to cluster categories and to show the relationships established between them. The circles corresponding to the inquiry sub-processes were colored in yellow, and the ones for the modeling sub-processes in blue.

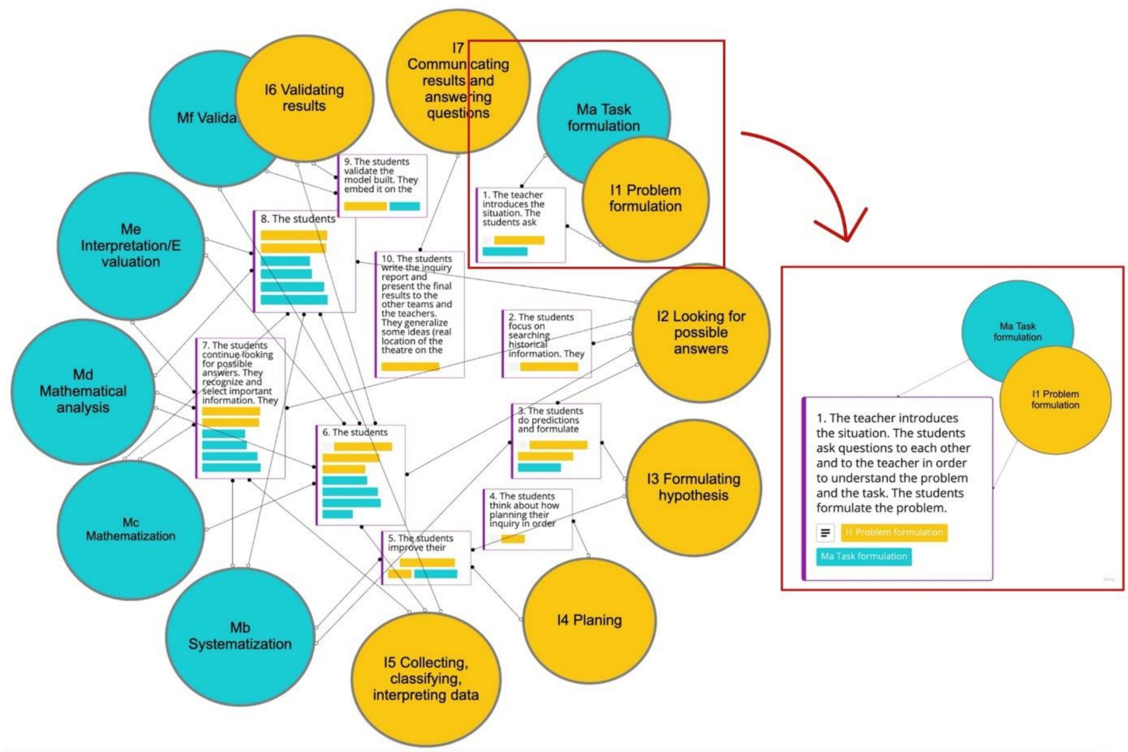


Figure 5. Representation of relationships between the sub-processes of inquiry and the sub-processes of mathematical modeling.

To facilitate the understanding of Figure 5, the cluster represented corresponds to the analysis of the first cycle of inquiry and modeling (explained in Section 5). This figure was elaborated from the synthesis of the different episodes described earlier, following the chronological thread of the implementation. The different episodes, numbered from 1 to 10, in accordance with the chronological order of the implementation, are represented in rectangles inside the figure. They are connected to the sub-processes of inquiry and modeling (as described in Table 1) in the same way as in the analysis in Section 5. By means of arrows, each circle (or sub-process) is connected to the different episodes where evidence has been found that the students accomplished this particular sub-process.

On the right of the cluster, an example is shown of how to zoom into the representation of the sub-processes [I1] and [Ma] of inquiry and modeling. This particular example corresponds to episode 1 when the teacher introduced the situation and to the main initial question that initiated the inquiry (*What lies behind these ruins?*). After a group discussion to understand the situation, the students reformulated the problem by specifying the questions to address.

6.2. Coincidences, Concatenation, and Synergies between the Two Processes

The first thing that can be observed in the analysis of the implementation is that there are some sub-processes that are common to both the inquiry and modeling process, such as *validating results*, as described by [I6] and [Mf], and the *problem formulation* or *task formulation*, corresponding to [I1] and [Ma]. Similarly, *systematization* [Mb] is a sub-process of modeling that appears together with that of *formulating hypotheses* [I3] during the implementation, which is the moment when the students defined the system and formulated the first hypotheses about it.

The subprocess of inquiry about *communicating results and answering questions* [I7] is not considered as such in the modeling cycle that we have used as reference for the analysis. In this regard, this sub-process seems independent of the modeling process, although we are aware that there are other conceptualizations of modeling proposed by other authors that take into account this important sub-process, such as the one put forward by Galbraith and Stillman [39].

The inquiry sub-process called *looking for possible answers* [I2] is present throughout the implementation, and overlaps some modeling subprocesses, such as *systematization* [Mb] and the *mathematical analysis* [Md]. On the one hand, this further confirms that students do not follow the sub-processes of the cycle sequentially, as has been commented by several authors (e.g., [12,13,20], among others). On the other hand, it shows the importance of this subprocess to ensure that the students' work is successful both in the inquiry and modeling activity.

When the students reached the point of *collecting, classifying and interpreting data* [I5] (inquiry sub-process) because they had some data to be *mathematized*, they began with a modeling process that allowed them to continue progressing in the inquiry. This modeling process is inserted between [I5] *about collecting, classifying and interpreting data* and [I6], the validation of the results. When this modeling cycle unfolded, the inquiry process and certain modeling sub-processes concatenated, as was the case of the *systematization* [Mb], *mathematization* [Mc], *mathematical analysis* [Md] and *interpretation* and *evaluation* [Me] of the model. This concatenation and, to some extent, complementarity, has allowed the emergence of a mathematical work in relation to, for instance, the properties of curves, measuring tools, polygons, etc. It provides more accurate tools to look for answers and to validate the results ([I6] and [Mf]).

The relationships between inquiry and modeling have already aroused the interest of various researchers, as observed in the literature review section. Artigue and Blomhøj [10], in their paper on conceptualizing inquiry-based education in mathematics when they address the modeling perspectives, comment that the modeling cycle and the inquiry cycle share certain similarities. In their words:

Working with modeling in mathematics and in other subjects can thereby lead to valuable understanding of inquiry as a more general process with different particular realizations in different disciplines and contexts [10].

This paper intends to delve deeper into how these two processes relate to each other. It shows the coincidences, concatenations and synergies between the two processes.

What has allowed the modeling and inquiry processes, beyond being linked or concatenated to enrich each other, producing certain synergies, is the key role of interdisciplinarity in the proposal of the teaching sequence.

History allowed the students to limit the range of possible hypotheses and models to be considered (of the possible curves that could fit the wall, as there were only two types of curves: a circumference and an ellipse). Limiting the hypotheses meant that the mathematical knowledge required of the students to be able to progress with regard to elaborating their answers was part of their Zone of Proximal Development (ZPD). During the implementation, the students had the chance to improve their knowledge of the properties of these curves, circumferences and ellipses. The fact that there were two types of possible curves (known by the students) that appeared as hypotheses to validate or reject was a key aspect for the students to develop complete modeling cycles, as the ones described in the analysis. It led the students to the construction and use of these models to fit geometrical forms.

Another aspect that favors the synergy between inquiry and modeling is based on the fact that history provides the rules of construction (the ones established by Vitruvius were used in the implementation). This allows the students to progress in the formulation of geometrical models and enables them to simulate them and validate them when comparing them with the real plan of the ruins. The use and role of technology might be emphasized, especially the use of GeoGebra. It had a significant role mainly to facilitate that students go ahead with the representation and simulation of the models proposed and their contrast against their data. The use of these technological tools allowed the students to reflect on their hypothesis, providing them tools to work with different representations of models, their experimentation (as is explained in Pedersen et al. [40], simulations and validation.

Last but not least, the interaction the students had with an expert of the team of archaeologists of the museum in Badalona responsible for the ruins found in the city let them obtain an external validation of their answers and of the whole process followed. It also facilitated the formulation of new questions, such as what the capacity of the Roman theater of *Baetulo* might be, thus allowing for the students to begin a new cycle in inquiry, as explained in Section 5.

7. Conclusions

To analyze inquiry processes and modeling processes we could use either one of the pre-existing analytical models for each type of process, which correspond to specific conceptualizations of what inquiry and modeling consist of, could be used. However, the models proposed for inquiry do not consider some important processes characteristic of mathematical modeling, such as the mathematization or the systematization processes. Consequently, none of these models allowed us to adopt an integrative approach concerning inquiry and modeling developed when students are involved in open-disciplinary and ground-breaking teaching projects. We thus aimed at building an integrative model, presented in this paper, that integrates all these aspects and that enables us to perform the joint analysis of inquiry and modeling within the empirical context of our implementation.

To do so, first, we adopted the proposal put forward by Blomhøj [20] to analyze the mathematical modeling processes and the model formulated by Sala Sebastià [28] for the inquiry processes. We then used these two models to analyze the presence of these processes in the implementation of an interdisciplinary teaching sequence, which was designed for students to develop inquiry. Our aim was also to look for coincidences, concatenations and synergies, established between both processes.

Based on this analysis, we obtained the relationships between the two processes shown in Figure 5. These relationships led us to propose a model (Figure 6) that integrates both processes here considered. The sub-processes corresponding to an inquiry cycle, in accordance with the model used by Sala Sebastià [28], are represented on the right in Figure 6. In the analysis of the teaching sequence, we observed that both processes begin with the *problematization* ([Ma] and [I1]) of a real-life situation, as we have seen in the implementation analyzed. After the emergence of some derived questions [I2], a process is initiated to look for answer to them—which could correspond to the so-called *domain of inquiry* in the Blomhøj [20] diagram (Figure 1). We consider that, at this point, both cycles could be connected.

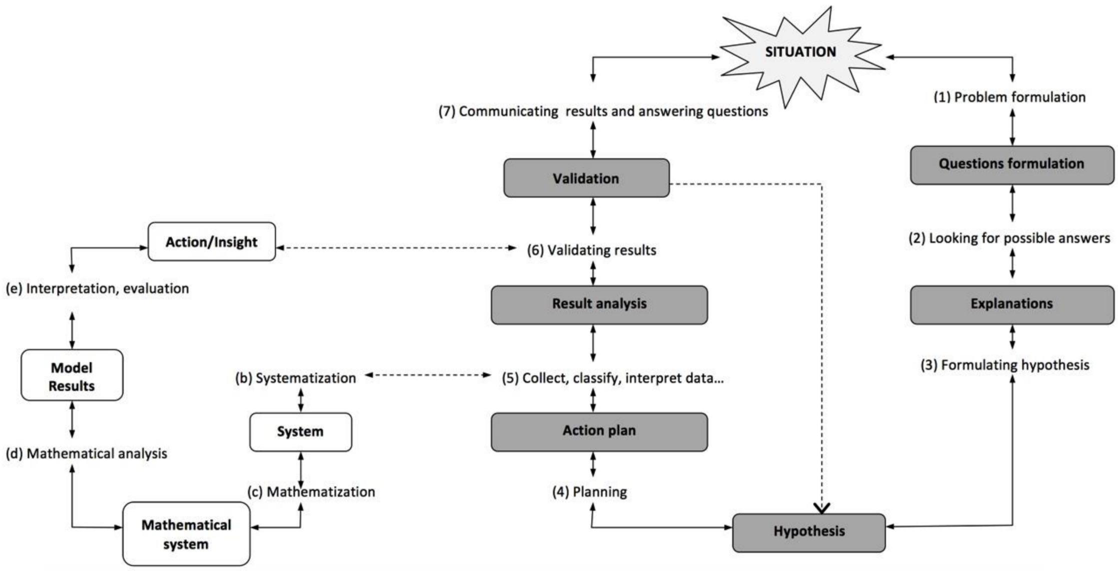


Figure 6. Integrative analytical model of the inquiry and modeling processes based on inquiry models by Sala Sebastià [28] Authors (2016) and modeling by Blomhøj (2004) [20].

Different *hypotheses* can be considered (sub-process 3 in Figures 3 and 6) and, in order to validate them—or reject them—, an action plan is drawn up (sub-process 4 in Figures 3 and 6) to *collect data* and *select relevant data* in order to *classify, organize and interpret* them (sub-process 5 in Figures 3 and 6). Modeling can come into play at this stage when looking for the validation of the hypotheses under consideration. The inquiry cycle could again be connected with a modeling cycle (on the left in Figure 6) in the 5 sub-processes, when the students carried out the inquiry and had to mathematize the data collected—as we have seen in the implementation analysis—. This connection is shown in Figure 6 by a horizontal dotted arrow. The left part of Figure 6 displays the cycle of modeling, adapted from the original model of Blomhøj [20].

The proposed integrative model (Figure 6) shows how the modeling cycle is initiated after the sub-process (5)—sub-process (b) on the left part in Figure 6 corresponding to the *systematization* of the data for its subsequent *mathematization* (sub-process c). Next come the sub-processes including the *mathematical analysis* (sub-process d), and the *interpretation* and/or *evaluation* of the resulting model. This sub-process is another shared characteristic (shown by another horizontal dotted arrow) between the two cycles. *Validation* is a common sub-process in the two cycles: it is sub-process f of the modeling (in Figure 1) and sub-process 6 of the inquiry (in Figure 3). In both processes, as observed in the analysis of the implementation, this step consists in validating the constructed model. Its validity is evaluated taking into account the inquiry context, data and/or theoretical knowledge and personal or shared experience. When the model is not validated and it needs to be either improved or rejected, or a different model needs to be built, the modeling cycle can start again from sub-process 5 based on the *collection of new data* or a previous sub-process (see the vertical dotted arrow inside the cycle of inquiry in Figure 6), with the *formulation of new hypotheses*. Once the validation stage of the model(s) is completed, the sub-process consisting in *communicating results* begins (sub-process 7 in Figure 6) during which questions from the initial problem are answered. At this stage, at the end of the cycle of inquiry, it is likely–and desirable–that new *questions* be generated in relation to the initial problem thus launching new cycles of inquiry.

We consider the integrative model presented in this section useful to analyze other teaching proposals originated in other kinds of interdisciplinary contexts for inquiry. It can also be used as a descriptive and analytical tool, and as a design-oriented tool to help provide a blended view of inquiry and mathematical modeling. In these terms, this work could constitute a significant contribution to the field.

Author Contributions: Conceptualization, G.S.S.; methodology, G.S.S., B.B. and V.F.; validation, G.S.S., B.B. and V.F.; formal analysis, G.S.S., B.B. and V.F.; resources, G.S.S.; data curation, G.S.S.; writing—original draft preparation, G.S.S., B.B. and V.F.; writing—review and editing, G.S.S., B.B. and V.F. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Spanish R&D Projects: PGC2018-098603-B-I00 (MCIU/AEI/FEDER, UE) and RTI2018-101153-A-C22 (MCIU/AEI/FEDER, UE).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Blum, W. Quality teaching of mathematical modelling: What do we know, what can we do? In Proceedings of the 12th International Congress on Mathematical Education, Seoul, Korea, 8–15 July 2012; Cho, S.J., Ed.; Springer International Publishing: Cham, Switzerland, 2015; pp. 73–96.
2. Burkhardt, H. Ways to teach modelling—A 50 year study. *ZDM Math. Educ.* **2018**, *50*, 61–75. [CrossRef]
3. Mullis, I.V.S.; Martin, M.O. (Eds.) *TIMSS Advanced 2015 Assessment Frameworks*; TIMSS & PIRLS International Study Center: Boston, MA, USA, 2014.
4. OECD. *PISA 2015 Assessment and Analytical Framework: Science, Reading, Mathematic and Financial Literacy*; OECD Publishing: Paris, France, 2016.
5. OECD. *PISA 2018 Assessment and Analytical Framework*; OECD Publishing: Paris, France, 2019. [CrossRef]
6. Maass, K.; Doorman, M. A model for a widespread implementation of inquiry-based learning. *ZDM Math. Educ.* **2013**, *45*, 887–899. [CrossRef]
7. Maass, K.; Engeln, K. Impact of professional development involving modelling on teachers and their teaching. *ZDM Math. Educ.* **2018**, *50*, 273–285. [CrossRef]
8. Niss, M. Modelling a crucial aspect of students' mathematical modelling. In *Modelling Students' Mathematical Modelling Competencies. ICTMA13*; Lesh, R., Galbraith, P.L., Haines, C.R., Hurford, A., Eds.; Springer: New York, NY, USA, 2010; pp. 43–59.
9. Stillman, G.; Brown, J.P. Evidence of implement anticipation in mathematising by beginning modellers. *Math. Educ. Res. J.* **2014**, *26*, 763–789. [CrossRef]
10. Artigue, M.; Blomhøj, M. Conceptualizing inquiry-based education in mathematics. *ZDM Math. Educ.* **2013**, *45*, 797–810. [CrossRef]
11. Sala Sebastià, G.; Barquero, B.; Font, V.; Giménez, J. A multidisciplinary approach to model some aspects of historical events. In Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education (CERME9), Prague, Czech Republic, 4–8 February 2015; Krainer, K., Vondrová, N., Eds.; Charles University in Prague, Faculty of Education and ERME: Prague, Czech Republic, 2016; pp. 923–929.
12. Sala Sebastià, G.; Font, V.; Giménez, J.; Barquero, B. Inquiry and modelling in an archaeological context. In *Mathematical Modelling and Applications: Crossing and Researching Boundaries in Mathematical Education*; Stillman, G., Blum, W., Kaiser, G., Eds.; Springer: Cham, Germany, 2017; pp. 325–335.
13. Sala Sebastià, G.; Font, V.; Barquero, B.; Giménez, J. Mathematical modelling in an archaeological context: Their complementarity as essential tool for inquiry. In Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10), Dublin, Ireland, 1–5 February 2017; Dooley, T., Gueudet, G., Eds.; DCU Institute of Education and ERME: Dublin, Ireland, 2017; pp. 1033–1040.
14. Kaiser, G.; Sriraman, B. A global survey of international perspectives on modelling in mathematics education. *ZDM Math. Educ.* **2006**, *38*, 302–310. [CrossRef]
15. De Sousa Lins Mutti, G.; Klüber, T.E. Adoption of Mathematical Modelling: What is seen in the research produced by the Mathematical Education community. *Bolema* **2021**, *35*, 129–157. [CrossRef]
16. Niss, M.; Blum, W.; Galbraith, P. Introduction. In *Modelling and Applications in Mathematics Education. The 14th ICMI Study*; Blum, W., Galbraith, P., Henn, H.-W., Niss, M., Eds.; Springer: New York, NY, USA, 2007; pp. 3–32.

17. Blum, W.; Leiß, D. How do students and teachers deal with modelling problems? In *Mathematical Modelling. Education, Engineering and Economics*; Haines, C., Galbraith, P., Blum, W., Khan, S., Eds.; Horwood: Chichester, UK, 2007; pp. 222–231.
18. Blomhøj, M.; Højgaard-Jensen, T. Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teach. Math. Its Appl.* **2003**, *22*, 123–129. [CrossRef]
19. Borromeo Ferri, R. Theoretical and empirical differentiations of phases in the modelling process. *ZDM Zentralblatt Didaktik Mathematik* **2006**, *38*, 86–95. [CrossRef]
20. Blomhøj, M. Mathematical modelling—A theory for practice. In *International Perspectives on Learning and Teaching Mathematics*; Clarke, B., Clark, D., Lambdin, D., Lester, F., Emanuelsson, G., Johanson, B., Walbym, A., Walby, K., Eds.; NCM, Gothenburg University: Gothenburg, Sweden, 2004; pp. 145–160.
21. Dewey, J. *Logic: The Theory of Inquiry*; Holt: New York, NY, USA, 1938.
22. NRC. *National Research Council. National Science Education Standards*; National Academy Press: Washington, DC, USA, 1996.
23. Dorier, J.-L.; Maass, K. Inquiry-based mathematics education. In *Encyclopedia of Mathematics Education*; Lerman, S., Ed.; Springer: Dordrecht, The Netherlands, 2014; pp. 300–304.
24. Artigue, M.; Baptiste, P. Inquiry in mathematics education. In *Resources for Implementing Inquiry in Science and Mathematics at School*; Carulla, S.B., Ed.; Fibonacci Project; 2012. Available online: https://www.fondation-lamap.org/sites/default/files/upload/media/minisites/action_internationale/inquiry_in_mathematics_education.pdf (accessed on 19 July 2021).
25. Artigue, M.; Dillon, J.; Harlen, W.; Léna, P. Learning through Inquiry. In *Resources for Implementing Inquiry in Science and Mathematics at School*; Carulla, S.B., Ed.; Fibonacci Project; 2012. Available online: https://www.maisons-pour-la-science.org/sites/default/files/upload/learning_through_inquiry.pdf (accessed on 19 July 2021).
26. Harlen, W. Inquiry in science education. In *Resources for Implementing Inquiry in Science and Mathematics at School*; Carulla, S.B., Ed.; Fibonacci Project; 2012. Available online: https://www.fondation-lamap.org/sites/default/files/upload/media/minisites/action_internationale/inquiry_in_science_education.pdf (accessed on 19 July 2021).
27. Harlen, W. The "tools" for enhancing inquiry in Science Education and inquiry in mathematics. In *Tools for Enhancing Inquiry in Science Education*; Carulla, S.B., Ed.; Fibonacci Project; 2012. Available online: https://www.fondation-lamap.org/sites/default/files/upload/media/minisites/action_internationale/1-tools_for_enhancing_inquiry_in_science_education.pdf (accessed on 19 July 2021).
28. Sala Sebastià, G. Competència D'indagació Matemàtica en Contextos Històrics a Primària i Secundària [Mathematical Inquiry Competence in Historical Contexts at Primary and Secondary School]. Ph.D. Thesis, Universitat de Barcelona, Barcelona, Spain, 2016. Available online: http://www.tdx.cat/handle/10803/388035 (accessed on 19 July 2021).
29. Vos. What is 'Authentic' in the Teaching and Learning of Mathematical Modelling? In *Trends in Teaching and Learning of Mathematical Modelling*; Kaiser, G., Blum, W., Borromeo, R., Stillman, G., Eds.; Springer: New York, NY, USA, 2011; pp. 717–722.
30. Barbin, E.; Tzanakis, C. History of Mathematics and Education. In *Encyclopedia of Mathematics Education*; Lerman, S., Ed.; Springer: Dordrecht, The Netherlands, 2014.
31. Kjeldsen, T.H.; Jankvist, U.T. New Avenues for History in Mathematics Education: Mathematical Competencies and Anchoring. *Sci. Educ.* **2011**, *20*, 831–862. [CrossRef]
32. D'Ambrosio, U. Ethnomathematics and its Place in the History and Pedagogy of Mathematics. *Learn. Math.* **1985**, *5*, 44–48.
33. Dean, J. *Teaching History at Key Stage 2*; Chris Kington Publishing: Cambridge, UK, 1995.
34. Prats, J. *Teaching History: Notes for a Renewal Didactic*; Junta de Extremadura; Consejería de Educación, Ciencia y Tecnología: Mérida, Spain, 2001.
35. Maracci, M.; Cazes, C.; Vandebrouck, F.; Mariotti, M.A. Synergies between theoretical approaches to mathematics education with technology: A case study through a cross-analysis methodology. *Educ. Stud. Math.* **2013**, *84*, 461–485. [CrossRef]
36. Bosch, M. Study and Research Paths: A model for inquiry. In *International Congress of Mathematicians*; Sirakov, B., de Souza, P.N., Viana, M., Eds.; World Scientific Publishing Co. Pte. Ltd.: Rio de Janeiro, Brazil, 2018; Volume 3, pp. 4001–4022.
37. Barquero, B.; Bosch, M.; Gascón, J. The ecological dimension in the teaching of mathematical modelling at university. *Rech. en Didactique des Mathématiques* **2013**, *33*, 3017–3338.
38. Padrós, P.; Moranta, L. La ciutat i la memòria: El teatre romà de Baetulo. [The city and the memory: The Baetulo's Roman theatre]. *Carrer dels Arbres 3a època* **2001**, *12*, 15–31.
39. Galbraith, P.; Stillman, G. A Framework for Identifying Student Blockages during Transitions in the Modelling Process. *ZDM Zentralblatt Didaktik Mathematik* **2006**, *38*, 143–162. [CrossRef]
40. Pedersen, M.K.; Bach, C.C.; Gregersen, R.M.; Højsted, I.H.; Jankvist, U.T. Mathematical Representation Competency in Relation to Use of Digital Technology and Task Design—A Literature Review. *Mathematics* **2021**, *9*, 444. [CrossRef]