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Existence and U-H-R Stability of Solutions to the Implicit Nonlinear FBVP in the Variable Order Settings

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Abstract: In this paper, the existence of the solution and its stability to the fractional boundary value problem (FBVP) were investigated for an implicit nonlinear fractional differential equation (VOFDE) of variable order. All existence criteria of the solutions in our establishments were derived via Krasnoselskii's fixed point theorem and in the sequel, and its Ulam–Hyers–Rassias (U-H-R) stability is checked. An illustrative example is presented at the end of this paper to validate our findings.

Keywords: variable-order operators; piecewise constant functions; Ulam–Hyers–Rassias stability; implicit problem; fixed point theorems

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1. Introduction

Fractional calculus has recently been discussed in various research works in multidisciplinary sciences due to its powerful applicability in modeling various scientific phenomena due to the property of the nonlocality and memory effect that some physical systems exhibit. Therefore, some interesting research works concerning the mathematical analysis and applications of fractional calculus have been discussed in [1–13]. The fractional calculus of variable order extends the theory of the constant order one. In such a direction, the order of a system, as a function in terms of independent or dependent variables, varies continuously to present a good description of the changes of memory property with space or time [14]. At first, Lorenzo et al. [15] considered fractional operators in the variable order settings to study the behaviors of a diffusion process. Later, other applications of variable-order spaces of fractional type have appeared in remarkable and interesting detail [16–18]. Such extensive and diverse applications immediately require a series of systematic studies on the qualitative specifications of solutions of VOFDEs such as existence–uniqueness–stability. Sun et al. [19] performed a comparative study on variable and constant order models to characterize the memory specification of given systems. Aguilar et al. [20] designed a nonlinear model of alcoholism in the context of VOFDEs and studied the solutions of such a system analytically and numerically. In 2021, Bouazza et al. [21] designed a multi-term VOFBVP and proved that under some conditions, there exists a unique solution for such a system. Li et al. [22], by defining a novel kernel function via polynomial form, studied a general structure of Atangana–Baleanu VOFBVPs. In [23], Derakhshan solved a Caputo linear time-fractional VOFDE arising in fluid mechanics and proved the existence, uniqueness and stability results. Recently, Refice et al. [24] focused on a Hadamard VOFBVP and derived solutions by means of Kuratowski's non-compactness measure. As can be seen, in recent years, limited contributions to the properties of solutions of fractional constant order BVPs have been conducted. However, the existence of solutions to FBVPs of variable order have rarely been studied (see [25–27]).

In [28], Benchohra et al. studied the existence and uniqueness of solutions for the following implicit nonlinear fractional differential equation in the framework of constant order:

$$\begin{cases} {}^c\mathfrak{D}^u_{0^+}x(t) = m(t, x(t), {}^c\mathfrak{D}^u_{0^+}x(t)), \;\; t \in \mathfrak{J} := [0, \Omega], \;\; 0 < \Omega < +\infty, \;\; 1 < u \le 2 \\ x(0) = x_0, \;\; x(\Omega) = x_1 \end{cases}$$

where a given function $m : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belongs to $C(\mathfrak{J} \times \mathbb{R}^2, \mathbb{R})$, x_0, $x_1 \in \mathbb{R}$, and the Caputo fractional derivative is denoted by ${}^c\mathfrak{D}^u_{0^+}$.

Inspired by all mentioned works in addition to [28], this article investigated some results about of possible solutions to the following FBVP for an implicit nonlinear VOFDE:

$$\begin{cases} \mathfrak{D}^{u(t)}_{0^+}x(t) = m(t, x(t), \mathfrak{D}^{u(t)}_{0^+}x(t)), \\ x(0) = 0, \;\; x(\Omega) = 0, \;\; t \in \mathfrak{J} := [0, \Omega], \end{cases} \tag{1}$$

where $0 < \Omega < +\infty$, $u(t) : \mathfrak{J} \to (1, 2]$, $m : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\mathfrak{I}^{u(t)}_{0^+}$, $\mathfrak{D}^{u(t)}_{0^+}$ are the Riemann–Liouville fractional (RLFr) integral and derivative in the context of variable order $u(t)$. To the best of our knowledge, it should be noted that due to the complexity of computations and partitioning the main time interval, very few papers can be found in the literature in which the authors studied existence theory and Ulam–Hyers-Rassias stability for nonlinear implicit boundary value problems in the fractional variable order settings. Since in variable order structures, the order of the existing boundary value problem as a function varies continuously on $\mathfrak{J}$, and the semigroup property does not hold for such variable order integration operators, thus to arrive at an accurate result, on the one hand, we have to define a partition of $\mathfrak{J}$ such that the order of the system is a piecewise constant function with respect to such a partition. All of these transformations are new for the given fractional implicit BVP (1) via the variable of order $u(t)$. On the other hand, we note that in spite of constant order FBVPs, to study the Ulam–Hyers–Rassias stability for this implicit nonlinear VOFDE, we define the finite number of continuous functions on each subinterval and investigated the stability condition for it by transforming a variable order problem into a constant order one on each $\mathfrak{J}_j$ which gives a novelty in such a level of arguments.

The plan of our paper is as follows. Some important notions that will be used later are presented in Section 2. Our main result is obtained in Section 3 via KFPTHm. The Ulam–Hyers–Rassias (U-H-R) stability of the proposed problem is discussed in Section 4. Our result is validated by giving an illustrative example in Section 5. This work is concluded in Section 6.

2. Essential Notions

Some important notions on variable order fractional calculus are presented in this section that will be subsequently utilized in our results.

By $\mathfrak{C}(\mathfrak{J}, \mathbb{R})$, we mean the Banach space of continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ via the norm:

$$\|x\| = \sup\{|x(t)| : \;\; t \in \mathfrak{J}\}.$$

Definition 1 ([29–31])**.** *Let $-\infty < c < d < +\infty$, and $u(t) : [c,d] \to (0,+\infty)$. The left Riemann–Liouville fractional (RLFr) integral in the context of variable order $u(t)$ for $h(t)$ is defined by*

$$\mathfrak{I}_{c^+}^{u(t)} h(t) = \int_c^t \frac{(t-w)^{u(t)-1}}{\Gamma(u(t))} h(w)dw, \; t > c, \tag{2}$$

where the Gamma function is denoted by $\Gamma(.)$.

Definition 2 ([29–31])**.** *Let $-\infty < c < d < +\infty$, $r \in \mathbb{N}$ and $u(t) : [c,d] \to (r-1,r)$. The left Riemann–Liouville fractional (RLFr) derivative in the context of variable order $u(t)$ for $h(t)$ is defined by*

$$\mathfrak{D}_{c^+}^{u(t)} h(t) = \left(\frac{d}{dt}\right)^r \mathfrak{I}_{c^+}^{r-u(t)} h(t) = \left(\frac{d}{dt}\right)^r \int_c^t \frac{(t-w)^{r-u(t)-1}}{\Gamma(r-u(t))} h(w)dw, \; t > c. \tag{3}$$

Obviously, if $u(t)$ is a constant function $u \in \mathbb{R}$, then the VORLFr derivative (3) and integral (2) are the usual RLFr derivative and integral, respectively, (see [29,30,32]). We know that there are some important properties for these operators as follows:

Lemma 1 ([32])**.** *Assume that $\delta > 0$. Then:*

$$\mathfrak{D}_{c^+}^{\delta} h = 0$$

has a unique solution:

$$h(t) = \omega_1 (t-c)^{\delta-1} + \omega_2 (t-c)^{\delta-2} + \ldots + \omega_r (t-c)^{\delta-r},$$

where $\omega_j \in \mathbb{R}$, $j = 1,2,\ldots,r$, and $r-1 < \delta \le r$.

Lemma 2 ([32])**.** *Let $c > 0$, $h \in L(c,d)$, $\mathfrak{D}_{c^+}^{\delta} h \in L(c,d)$. Then:*

$$\mathfrak{I}_{c^+}^{\delta} \mathfrak{D}_{c^+}^{\delta} h(t) = h(t) + \omega_1 (t-c)^{\delta-1} + \omega_2 (t-c)^{\delta-2} + \ldots + \omega_r (t-c)^{\delta-r},$$

where $\omega_j \in \mathbb{R}$, $j = 1,2,\ldots,r$, and $r-1 < \delta \le r$.

Lemma 3 ([32])**.** *Let $\delta > 0$. Then:*

$$\mathfrak{D}_{c^+}^{\delta} \mathfrak{I}_{c^+}^{\delta} h(t) = h(t).$$

Lemma 4 ([32])**.** *Let $\delta,\ \beta > 0$. Then:*

$$\mathfrak{I}_{c^+}^{\delta} \mathfrak{I}_{c^+}^{\beta} h(t) = \mathfrak{I}_{c^+}^{\beta} \mathfrak{I}_{c^+}^{\delta} h(t) = \mathfrak{I}_{c^+}^{\delta+\beta} h(t).$$

Remark 1 ([33–35])**.** *For general functions $u(t)$, $v(t)$, we notice that the semi-group property is not valid, meaning that:*

$$\mathfrak{I}_{c^+}^{u(t)} \mathfrak{I}_{c^+}^{v(t)} h(t) \neq \mathfrak{I}_{c^+}^{u(t)+v(t)} h(t).$$

Example 1. *Let:*

$$u(t) = \frac{t^2}{3}, \; t \in [0,4], \quad v(t) = \begin{cases} 3, & t \in [0,1] \\ 2, & t \in]1,4]. \end{cases} \quad h(t) = 2, \; t \in [0,4].$$

Then:

$$\mathfrak{I}_{0^+}^{u(t)} \mathfrak{I}_{0^+}^{v(t)} h(t) \;=\; \int_0^t \frac{(t-w)^{u(t)-1}}{\Gamma(u(t))} \int_0^w \frac{(w-\tau)^{v(w)-1}}{\Gamma(v(w))} h(\tau) d\tau dw$$

$$= \int_0^t \frac{(t-w)^{u(t)-1}}{\Gamma(u(t))}[\int_0^1 \frac{(w-\tau)^2}{\Gamma(3)}2d\tau + \int_1^w \frac{(w-\tau)}{\Gamma(2)}2d\tau]dw$$

$$= \int_0^t \frac{(t-w)^{u(t)-1}}{\Gamma(u(t))}[\frac{(w-1)^3}{3} + 2w - 1]dw,$$

and:

$$\mathfrak{I}_{0^+}^{u(t)+v(t)}h(t) = \int_0^t \frac{(t-w)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))}h(w)dw.$$

It is clear that:

$$\begin{aligned} \mathfrak{I}_{0^+}^{u(t)}\mathfrak{I}_{0^+}^{v(t)}h(t)|_{t=3} &= \int_0^3 \frac{(3-w)^2}{\Gamma(3)}[\frac{(w-1)^3}{3} + 2w - 1]dw \\ &= \frac{1}{2}\int_0^3 (\frac{w^5}{3} - 3w^4 + 12w^3 - \frac{85}{3}w^2 + 35w - 12)dw \\ &= \frac{21}{10}, \end{aligned}$$

and:

$$\begin{aligned} \mathfrak{I}_{0^+}^{u(t)+v(t)}h(t)|_{t=3} &= \int_0^3 \frac{(3-w)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))}h(w)dw \\ &= \int_0^1 \frac{(3-w)^5}{\Gamma(6)}2dw + \int_1^3 \frac{(3-w)^4}{\Gamma(5)}2dw \\ &= \frac{1}{60}\int_0^1 (-w^5 + 15w^4 - 90w^3 + 270w^2 - 405w + 243)dw \\ &+ \frac{1}{12}\int_1^3 (w^4 - 12w^3 + 54w^2 - 108w + 81)dw \\ &= \frac{665}{360} + \frac{32}{60} = \frac{857}{360}. \end{aligned}$$

Therefore, we obtain:

$$\mathfrak{I}_{0^+}^{u(t)}\mathfrak{I}_{0^+}^{v(t)}h(t)|_{t=3} \neq I_{0^+}^{u(t)+v(t)}h(t)|_{t=3}.$$

Lemma 5 ([36])**.** *Let $u : \mathfrak{J} \to (1,2]$ be a continuous function. Then, for:*

$$y \in \mathfrak{C}_\zeta(\mathfrak{J},\mathbb{R}) = \{y(t) \in \mathfrak{C}(\mathfrak{J},\mathbb{R}),\ t^\zeta y(t) \in \mathfrak{C}(\mathfrak{J},\mathbb{R})\},\ (0 \leq \zeta \leq \min_{t\in\mathfrak{J}} |u(t)|),$$

the variable order (VO) fractional integral $\mathfrak{I}_{0^+}^{u(t)}y(t)$ exists for all points on $\mathfrak{J}$.

Lemma 6 ([36])**.** *Let $u : \mathfrak{J} \to (1,2]$ be a continuous function. Then, $\mathfrak{I}_{0^+}^{u(t)}y(t) \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ for each $y \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$.*

Definition 3 ([37–39])**.** *The set $\mathfrak{J}$ in $\mathbb{R}$ is called a generalized interval (G-interval) if it is either a standard interval, a point $\{c_1\}$, or the empty set $\emptyset$:*

Definition 4 ([37–39])**.** *If $\mathfrak{J}$ is a G-interval, then the finite set $\mathcal{P}$ of G-intervals belonging to $\mathfrak{J}$ is a partition of $\mathfrak{J}$ whenever each $x \in \mathfrak{J}$ lies in exactly one of the G-intervals.*

In the following, let E be a Banach space.

Definition 5 ([37–39]). *Assume that $\mathfrak{J}$ is a G-interval, $g : \mathfrak{J} \to \mathbb{R}$ is a mapping, and $\mathcal{P}$ is a partition of $\mathfrak{J}$. Then, g is a piecewise constant by terms of $\mathcal{P}$ if for every $E \in \mathcal{P}$, g is constant on E.*

Theorem 1 ([32]). *(Krasnoselskii's fixed point theorem) Assume that S is a closed, convex, bounded subset of E and suppose that W_1 and W_2 are operators on S satisfying the following conditions:*
(i) $W_1(S) + W_2(S) \subset S$;
(ii) W_1 is continuous on S and $W_1(S)$ is relatively compact in E;
(iii) W_2 is a strict contraction on S, that is; $\exists\, k \in [0, 1)$ s.t.:

$$\|W_2(x) - W_2(y)\| \le k\|x - y\|$$

for every $x, y \in S$.
Then, there exists $x \in S$ such that $W_1(x) + W_2(x) = x$.

Definition 6 ([40]). *(Ulam–Hyers–Rassias stability) The equation of (1) is U-H-R stable with respect to $\varphi \in \mathfrak{C}(\mathfrak{J}, \mathbb{R}_+)$ if there exists $a_m > 0$ such that for all $\epsilon > 0$ and for all $z \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ satisfying:*

$$|\mathfrak{D}_{0^+}^{u(t)} z(t) - m(t, z(t), \mathfrak{I}_{0^+}^{u(t)} z(t))| \le \epsilon\varphi(t),\ t \in \mathfrak{J},$$

there is $x \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ as a solution of Equation (1) with:

$$|z(t) - x(t)| \le a_m \epsilon\varphi(t),\ t \in \mathfrak{J}.$$

3. Existence of Solutions

Let us present some assumptions as follows.

Hypothesis 1 (H1). *Let $r \in \mathbb{N}$, $\mathcal{P} = \{\mathfrak{J}_1 := [0, \Omega_1], \mathfrak{J}_2 := (\Omega_1, \Omega_2], \mathfrak{J}_3 := (T_2, \Omega_3], ...\mathfrak{J}_r := (\Omega_{r-1}, \Omega]\}$ be a partition of $\mathfrak{J}$, and $u(t) : \mathfrak{J} \to (1, 2]$ be a piecewise constant mapping with respect to $\mathcal{P}$, meaning that:*

$$u(t) = \sum_{j=1}^{r} u_j \mathfrak{J}_j(t) = \begin{cases} u_1, & if\ t \in \mathfrak{J}_1, \\ u_2, & if\ t \in \mathfrak{J}_2, \\ \cdot \\ \cdot \\ \cdot \\ u_r, & if\ t \in \mathfrak{J}_r, \end{cases}$$

in which $1 < u_j \le 2$ belong to $\mathbb{R}$, and $\mathfrak{J}_j$ is the indicator of $\mathfrak{J}_j := (\Omega_{j-1}, \Omega_j]$, $j = 1, 2, ..., r$, ($\Omega_0 = 0$, $\Omega_r = T$) such that:

$$\mathfrak{J}_j(t) = \begin{cases} 1, & for\ t \in \mathfrak{J}_j, \\ 0, & elsewhere. \end{cases}$$

Hypothesis 2 (H2). *Let $t^\zeta m : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function $(0 \le \zeta \le \min_{t \in \mathfrak{J}} |(u(t))|)$ and there exist constants $K, L > 0$ such that $t^\zeta |m(t, y_1, z_1) - m(t, y_2, z_2)| \le K|y_1 - y_2| + L|z_1 - z_2|$, for any $y_j, z_j \in \mathbb{R}$ and $t \in \mathfrak{J}$.*

By $E_j = \mathfrak{C}(\mathfrak{J}_j, \mathbb{R})$, we mean the Banach space of continuous functions from $\mathfrak{J}_j$ into $\mathbb{R}$ with the norm:

$$\|x\|_{E_j} = \sup_{t \in \mathfrak{J}_j} |x(t)|,$$

where $j \in \{1, 2, ..., r\}$.
Let us first analyze the FBVP (1) to obtain our main results.

By (3), FDE of FBVP (1) can be written as

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-w)^{1-u(t)}}{\Gamma(2-u(t))}x(w)dw = m(t, x(t), \mathfrak{D}_{0^+}^{u(t)}x(t)), \quad t \in \mathfrak{J}. \tag{4}$$

According to $(H1)$, the Equation (4) on $\mathfrak{J}_j$ can be represented by

$$\frac{d^2}{dt^2}\Big(\int_0^{\Omega_1} \frac{(t-w)^{1-u_1}}{\Gamma(2-u_1)}x(w)dw + \ldots + \int_{\Omega_{j-1}}^t \frac{(t-w)^{1-u_j}}{\Gamma(2-u_j)}x(w)dw\Big) = m(t, x(t), \mathfrak{D}_{0^+}^{u_j}x(t)), \quad t \in \mathfrak{J}_j, \tag{5}$$

for , $j = 1, 2, \ldots, r$. The solution of the supposed FBVP (1) is presented due to its essential role in our results as follows:

Definition 7. *We say that the FBVP (1) has a solution if there exists $x_j \in \mathfrak{C}([0, \Omega_j], \mathbb{R})$ satisfying Equation (5) and $x_j(0) = 0 = x_j(\Omega_j)$.*

From the above, the FDE of FBVP (1) can be given as the FDE (4), which it can be formulated on $\mathfrak{J}_j, j \in \{1, 2, \ldots, r\}$ as (5). For $0 \le t \le \Omega_{j-1}$, we set $x(t) \equiv 0$. Then, (5) is given as follows:

$$\mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t) = m(t, x(t), \mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t)), \quad t \in \mathfrak{J}_j.$$

Let us now consider the following equivalent standard FBVP:

$$\begin{cases} \mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t) = m(t, x(t), \mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t)), \\ x(\Omega_{j-1}) = 0, x(\Omega_j) = 0, \quad t \in \mathfrak{J}_j. \end{cases} \tag{6}$$

To prove the existence of solutions for the equivalent standard FBVP (6), an auxiliary lemma is presented by follows:

Lemma 7. *The function $x \in E_j$ is a solution to the equivalent standard FBVP (6) if and only if it satisfies:*

$$x(t) = -(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}\mathfrak{I}_{\Omega_{j-1}^+}^{u_j}y(\Omega_j)) + \mathfrak{I}_{\Omega_{j-1}^+}^{u_j}y(t), \quad t \in \mathfrak{J}_j, \tag{7}$$

where:

$$y(t) \;=\; m\Big(t, -(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}\mathfrak{I}_{\Omega_{j-1}^+}^{u_j}y(\Omega_j)) + \mathfrak{I}_{\Omega_{j-1}^+}^{u_j}y(t), y(t)\Big), \quad t \in \mathfrak{J}_j.$$

Proof. Let $x \in E_j$ be a solution to the equivalent standard FBVP (6). Now, we take $\mathfrak{D}_{\Omega_{j-1}^+}^{u_j}x(t) = y(t)$ and apply $\mathfrak{I}_{\Omega_{j-1}^+}^{u_j}$ to both sides of the FDE of the equivalent standard FBVP (6). By Lemma 2, we have:

$$x(t) = \omega_1(t-\Omega_{j-1})^{u_j-1} + \omega_2(t-\Omega_{j-1})^{u_j-2} + \mathfrak{I}_{\Omega_{j-1}^+}^{u_j}y(t), \quad t \in \mathfrak{J}_j.$$

By $x(\Omega_{j-1}) = 0$ and by the given assumption for the function m, we obtain $\omega_2 = 0$.

Assume that $x(t)$ satisfies $x(\Omega_j) = 0$. Thus, we obtain $\omega_1 = -(\Omega_j - \Omega_{j-1})^{1-u_j}$ $\mathfrak{I}_{\Omega_{j-1}^+}^{u_j}y(\Omega_j)$. Then, we have:

$$x(t) \;=\; -(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}\mathfrak{I}_{\Omega_{j-1}^+}^{u_j}y(\Omega_j) + \mathfrak{I}_{\Omega_{j-1}^+}^{u_j}y(t), \quad t \in \mathfrak{J}_j,$$

where:

$$y(t) = m\Big(t, -(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(t), y(t)\Big), \quad t \in \mathfrak{I}_j.$$

Conversely, assume that $x \in E_j$ satisfies the integral Equation (7). Then, according to the continuity of $t^\zeta m$ and Lemma 3, x is a solution to the equivalent standard FBVP (6) and the proof is completed. □

Our existence result is derived with the help of Theorem 1.

Theorem 2. *Suppose that* $(H1)$ *and* $(H2)$ *hold, and:*

$$\frac{(\Omega_j - \Omega_{j-1})^{u_j-1}({\Omega_j}^{1-\zeta} - {\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)}\Big(\frac{K(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + \frac{L}{2}\Big) < \frac{1}{4}. \tag{8}$$

Then, the FBVP (1) has a solution on E.

Proof. In the first step, we convert the equivalent standard FBVP (6) to a fixed point problem. Consider the following operators:

$$W_1, W_2 : E_j \to E_j$$

defined by

$$W_1y(t) = -(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(\Omega_j), \quad W_2y(t) = \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}y(t), \tag{9}$$

where:

$$y(t) = m(t, x(t), y(t)).$$

It follows from the properties of fractional operators and in view of the continuity of $t^\zeta m$, that the operators $W_1, W_2 : E_j \to E_j$ given by (9) are well defined. Let:

$$R_j \geq \frac{\frac{2m^*(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j)}}{1 - \frac{2(\Omega_j-\Omega_{j-1})^{u_j-1}({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)}\Big(\frac{2K(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + L\Big)},$$

where:

$$m^* = \sup_{t\in\mathfrak{I}_j}|m(t,0,0)|.$$

We consider the set:

$$B_{R_j} = \{x \in E_j, \ \|x\|_{E_j} \leq R_j\}.$$

Obviously, B_{R_j} is nonempty, bounded, convex and closed.

Let us prove that W_1 and W_2 satisfy the assumptions of Theorem 1. The proof is divided into four steps:

Step 1: $W_1(B_{R_j}) + W_2(B_{R_j}) \subseteq (B_{R_j})$.
Let $y \in B_{R_j}$. We show that $W_1(y) + W_2(y) \in B_{R_j}$.
For $t \in \mathfrak{I}_j$, we have:

$$|(W_1y)(t) + (W_2y)(t)| \leq \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}$$

$$
\begin{aligned}
& \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} \Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} y_r(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w), y(w)\Big)\Big| dw \\
+ & \frac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t - w)^{u_j-1} \Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w), y(w)\Big)\Big| dw \\
\leq & \frac{2}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} \Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w), y(w)\Big)\Big| dw \\
\leq & \frac{2}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} \Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w), y(w)\Big) \\
- & m(w,0,0)\Big| dw + \frac{2}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} |m(w,0,0)| dw \\
\leq & \frac{2(\Omega_j - \Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{\Omega_j} w^{-\zeta} (K| - (\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w)| \\
+ & L|y(w)|) dw + \frac{2m^*(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j)} \\
\leq & \frac{2(\Omega_j - \Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{\Omega_j} w^{-\zeta} \Big(K(|\Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j))| + |\Im^{u_j}_{\Omega^+_{j-1}} y(w)|) + L|y(w)|\Big) dw + \frac{2m^*(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j)} \\
\leq & \frac{2(\Omega_j - \Omega_{j-1})^{u_j-1}(\Omega_j{}^{1-\zeta} - \Omega_{j-1}{}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)} \Big(2K\|\Im^{u_j}_{\Omega^+_{j-1}} y\|_{E_j} + L\|y\|_{E_j}\Big) + \frac{2m^*(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j)} \\
\leq & \frac{2(\Omega_j - \Omega_{j-1})^{u_j-1}(\Omega_j{}^{1-\zeta} - \Omega_{j-1}{}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)} \Big(\frac{2K(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + L\Big) R_j + \frac{2m^*(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j)} \\
\leq & R_j,
\end{aligned}
$$

which means that $W_1(B_{R_j}) + W_2(B_{R_j}) \subseteq B_{R_j}$.

Step 2: W_1 is continuous.
Let (y_r) be a sequence such that $y_r \to y$ in E_j. Then, for each $t \in \Im_j$, we obtain:

$$
\begin{aligned}
& |(W_1 y_r)(t) - (W_1 y)(t)| \leq \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\
& \int_{\Omega_{j-1}}^{t} (t - w)^{u_j-1} \Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} y_r(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y_r(w), y_r(w)\Big) \\
- & m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w), y(w)\Big)\Big| dw \\
\leq & \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t - w)^{u_j-1} w^{-\zeta} \\
\times & \Big(K(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\Im^{u_j}_{\Omega^+_{j-1}} |y_r(\Omega_j) - y(\Omega_j)| + K\Im^{u_j}_{\Omega^+_{j-1}} |y_r(w) - y(w)| + L|y_r(w) - y(w)|\Big) dw
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} w^{-\zeta} \Big(2K \Im^{u_j}_{\Omega^+_{j-1}} \|y_r - y\|_{E_j} + L\|y_r - y\|_{E_j} \Big) dw \\
&\leq \frac{({\Omega_j}^{1-\zeta} - \Omega^{1-\zeta}_{j-1})(\Omega_j - \Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)} \Big(\frac{2K(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} \|y_r - y\|_{E_j} + L\|y_r - y\|_{E_j} \Big) \\
&\leq \frac{({\Omega_j}^{1-\zeta} - \Omega^{1-\zeta}_{j-1})(\Omega_j - \Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)} \Big(\frac{2K(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + L \Big) \|y_r - y\|_{E_j}.
\end{aligned}$$

Thus:

$$\|(W_1 y_r) - (W_1 y)\|_{E_j} \to 0 \ \ as \ \ r \to \infty.$$

As a result, W_1 is continuous on E_j.

Step 3: $W_1(B_{R_j})$ is relatively compact.

Let us now prove that $W_1(B_{R_j})$ is relatively compact. Obviously, $W_1(B_{R_j})$ has the uniform boundedness, since by Step 2, $W_1(B_{R_j}) = \{W_1(x) : x \in B_{R_j}\} \subset W_1(B_{R_j}) + W_2(B_{R_j}) \subseteq B_{R_j}$. Thus, for each $x \in B_{R_j}$, we have $\|W_1(x)\|_{E_j} \leq R_j$ which means that $W_1(B_{R_j})$ is uniformly bounded. Lastly, it is necessary that we prove that $W_1(B_{R_j})$ is equicontinuous. For $t_1, t_2 \in \Im_j$ and $y \in B_{R_j}$, we estimate $(t_1 < t_2)$:

$$\begin{aligned}
&|(W_1 y)(t_2) - (W_1 y)(t_1)| \leq \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}}{\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j-1} - (t_1 - T_{j-1})^{u_j-1} \Big) \\
&\int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} \Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j} (w - \Omega_{j-1})^{u_j-1} \Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w), y(w) \Big) \Big| dw \\
&\leq \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}}{\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j-1} - (t_1 - \Omega_{j-1})^{u_j-1} \Big) \\
&\int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} \Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j} (w - \Omega_{j-1})^{u_j-1} \Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w), y(w) \Big) - m(w,0,0) \Big| dw \\
&+ \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}}{\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j-1} - (t_1 - \Omega_{j-1})^{u_j-1} \Big) \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} |m(w,0,0)| dw \\
&\leq \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}}{\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j-1} - (t_1 - \Omega_{j-1})^{u_j-1} \Big) \\
&\int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} w^{-\zeta} \Big(K(\Omega_j - \Omega_{j-1})^{1-u_j} (w - \Omega_{j-1})^{u_j-1} |\Im^{u_j}_{\Omega^+_{j-1}} y(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y(w)| + L|y(w)| \Big) dw \\
&+ \frac{(\Omega_j - \Omega_{j-1})^{1-u_j} m^*}{\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j-1} - (t_1 - \Omega_{j-1})^{u_j-1} \Big) \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} w^{-\zeta} dw \\
&\leq \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}}{\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j-1} - (t_1 - \Omega_{j-1})^{u_j-1} \Big) \\
&\int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} w^{-\zeta} \Big(2K \Im^{u_j}_{\Omega^+_{j-1}} \|y\|_{E_j} + L\|y\|_{E_j} \Big) dw
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{(\Omega_j - \Omega_{j-1})^{1-u_j} m^*}{\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j - 1} - (t_1 - \Omega_{j-1})^{u_j - 1} \Big) \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j - 1} w^{-\zeta} dw \\
&\leq \quad \frac{({\Omega_j}^{1-\zeta} - {\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j - 1} - (t_1 - \Omega_{j-1})^{u_j - 1} \Big) \Big(2K \Im^{u_j}_{\Omega^+_{j-1}} \|y\|_{E_j} + L \|y\|_{E_j} \Big) \\
&+ \quad \frac{({\Omega_j}^{1-\zeta} - {\Omega_{j-1}}^{1-\zeta}) m^*}{(1-\zeta)\Gamma(u_j)} \Big((t_2 - \Omega_{j-1})^{u_j - 1} - (t_1 - \Omega_{j-1})^{u_j - 1} \Big) \\
&\leq \quad \frac{({\Omega_j}^{1-\zeta} - {\Omega_{j-1}}^{1-\zeta})}{(1-\zeta)\Gamma(u_j)} \Big(2K \Im^{u_j}_{\Omega^+_{j-1}} \|y\|_{E_j} + L \|y\|_{E_j} + m^* \Big) \Big((t_2 - \Omega_{j-1})^{u_j - 1} - (t_1 - \Omega_{j-1})^{u_j - 1} \Big).
\end{aligned}
$$

Hence, $|(W_1 y)(t_2) - (W_1 y)(t_1)| \to 0$ as $|t_2 - t_1| \to 0$. It implies that $W_1(B_{R_j})$ is equicontinuous.

Step 4: W_2 is a strict contraction on B_{R_j}.
For each $y_1(t), y_2(t) \in B_{R_i}$, we have:

$$
\begin{aligned}
& |(W_2 y_2)(t) - (W_2 y_1)(t)| \\
&\leq \quad \frac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j - 1} \Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j} (w - \Omega_{j-1})^{u_j - 1} \Im^{u_j}_{\Omega^+_{j-1}} y_2(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y_2(w), y_2(w) \Big) \\
&- \quad m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j} (w - \Omega_{j-1})^{u_j - 1} \Im^{u_j}_{\Omega^+_{j-1}} y_1(\Omega_j)) + \Im^{u_j}_{\Omega^+_{j-1}} y_1(w), y_1(w) \Big) \Big| dw \\
&\leq \quad \frac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j - 1} w^{-\zeta} \Big(K (\Omega_j - \Omega_{j-1})^{1-u_j} (w - \Omega_{j-1})^{u_j - 1} \Im^{u_j}_{\Omega^+_{j-1}} |y_2(\Omega_j) - y_1(\Omega_j)| \\
&+ \quad K \Im^{u_j}_{\Omega^+_{j-1}} |y_2(w) - y_1(w)| + L |y_2(w) - y_1(w)| \Big) dw \\
&\leq \quad \frac{1}{\Gamma(u_j)} \int_{\Omega_{j-1}}^{t} (t-w)^{u_j - 1} w^{-\zeta} \Big(2K \Im^{u_j}_{\Omega^+_{j-1}} \|y_2 - y_1\|_{E_j} + L \|y_2 - y_1\|_{E_j} \Big) dw \\
&\leq \quad \frac{({\Omega_j}^{1-\zeta} - \Omega_{j-1}^{1-\zeta})(\Omega_j - \Omega_{j-1})^{u_j - 1}}{(1-\zeta)\Gamma(u_j)} \Big(\frac{2K(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j + 1)} \|y_2 - y_1\|_{E_j} + L \|y_2 - y_1\|_{E_j} \Big) \\
&\leq \quad \frac{({\Omega_j}^{1-\zeta} - \Omega_{j-1}^{1-\zeta})(\Omega_j - \Omega_{j-1})^{u_j - 1}}{(1-\zeta)\Gamma(u_j)} \Big(\frac{2K(\Omega_j - \Omega_{j-1})^{u_j}}{\Gamma(u_j + 1)} + L \Big) \|y_2 - y_1\|_{E_j}.
\end{aligned}
$$

Consequently, by (8), W_2 is a strict contraction. Hence, by Krasnoselskii's fixed point theorem, there exists $\widetilde{x}_j \in B_{R_j}$ such that $W_1(x) + W_2(x) = x$, which is the solutions of the equivalent standard problem (6).
We let:

$$
x_j = \begin{cases} 0, & t \in [0, \Omega_{j-1}], \\ \widetilde{x}_j, & t \in \mathfrak{J}_j. \end{cases} \tag{10}
$$

On the other side, it is known that $x_j \in \mathfrak{C}([0, \Omega_j], \mathbb{R})$ given by (10) satisfies:

$$
\frac{d^2}{dt^2} \Big(\int_0^{\Omega_1} \frac{(t-w)^{1-u_1}}{\Gamma(2-u_1)} x_j(w) dw + \cdots + \int_{\Omega_{j-1}}^{t} \frac{(t-w)^{1-u_j}}{\Gamma(2-u_j)} x_j(w) dw \Big) = m(w, x_j(w), \mathfrak{D}^{u_j}_{0^+} x_j(w)),
$$

for $t \in \mathfrak{J}_j$, which indicates that x_j will be a solution to Equation (5) equipped with $x_j(0) = 0$, $x_j(\Omega_j) = \tilde{x}_j(\Omega_j) = 0$. Then:

$$x(t) = \begin{cases} x_1(t), \quad t \in \mathfrak{J}_1, \\ x_2(t) = \begin{cases} 0, & t \in \mathfrak{J}_1, \\ \tilde{x}_2, & t \in \mathfrak{J}_2 \end{cases} \\ \cdot \\ \cdot \\ \cdot \\ x_r(t) = \begin{cases} 0, & t \in [0, \Omega_{j-1}], \\ \tilde{x}_j, & t \in \mathfrak{J}_j \end{cases} \end{cases}.$$

is the solution for the main variable order FBVP (1) and the proof is completed. □

4. U-H-R Stability

U-H-R stability of solutions to every fractional FBVP is considered as an important criterion to study the behaviors of a given system. We here investigated a general form of such a notion in the sense of Ulam–Hyers-Rassias.

Theorem 3. *Consider (H1), (H2), (8) and let:*

Hypothesis 3 (H3). *$\varphi \in \mathfrak{C}(\mathfrak{J}_j, \mathbb{R}_+)$ is increasing and there exists $\lambda_\varphi > 0$ such that for all $t \in \mathfrak{J}_j$, we obtain:*

$$\mathfrak{I}^{u_j}_{\Omega_{j-1}^+}\varphi(t) \le \lambda_{\varphi(t)}\varphi(t).$$

Then, the given implicit nonlinear VOFBVP (1) is U-H-R stable with respect to φ.

Proof. Assume that $z \in \mathfrak{C}(\mathfrak{J}_j, \mathbb{R})$ is a solution of the inequality:

$$|\mathfrak{D}^{u_j}_{\Omega_{j-1}^+}z(t) - m(t, z(t), \mathfrak{D}^{u_j}_{\Omega_{j-1}^+}z(t))| \le \epsilon\varphi(t), t \in \mathfrak{J}_j. \tag{11}$$

For any $j \in \{1, 2, \dots, n\}$, we define the functions $z_1(t) \equiv z(t), t \in [1, \Omega_1]$ and for $j = 2, 3, \dots, n:$

$$z_j(t) = \begin{cases} 0, & t \in [0, \Omega_{j-1}], \\ z(t), & t \in \mathfrak{J}_j. \end{cases}$$

By considering $\mathfrak{I}^{u_j}_{\Omega_{j-1}^+}$ on both sides of the inequality (11), we obtain for $t \in \mathfrak{J}_j$:

$$\begin{aligned} \Big| z_j(t) &+ \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}(t - \Omega_{j-1})^{u_j - 1}}{\Gamma(u_j)} \\ &\int_{\Omega_{j-1}}^{\Omega_j} (\Omega_i - w)^{u_j - 1} m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w - \Omega_{j-1})^{u_j - 1}\mathfrak{I}^{u_j}_{\Omega_{j-1}^+} z_j(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega_{j-1}^+} z_j(w), z_j(w)\Big)dw \\ &- \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t} (t - w)^{u_j - 1} m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w - \Omega_{j-1})^{u_j - 1}\mathfrak{I}^{u_j}_{\Omega_{j-1}^+} z_j(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega_{j-1}^+} z_j(w), z_j(w)\Big)dw\Big| \\ &\le \epsilon\int_{\Omega_{j-1}}^{t} \frac{(t - w)^{u(j) - 1}}{\Gamma(u(j))}\varphi(w)dw \\ &\le \lambda_{\varphi(t)}\epsilon\varphi(t). \end{aligned}$$

In accordance with the above argument, VOFBVP (1) has a solution y which is defined by $y(t) = y_j(t)$ for $t \in \mathfrak{J}_j,\ j = 1, 2, ..., n$, where:

$$y_j(t) = \begin{cases} 0, & t \in [0, \Omega_{j-1}], \\ \widetilde{y}_j, & t \in \mathfrak{J}_j, \end{cases}$$

and $\widetilde{y}_j \in E_i$ is a solution of FBVP (6). By Lemma 7, the integral equation:

$$(\widetilde{y}_j)(t) = -\frac{(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}$$

$$\int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_J}_{\Omega^+_{j-1}}\widetilde{y}_j(\Omega_J)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_j(w), \widetilde{y}_j(w)\Big)dw$$

$$+ \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_j(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_j(w), \widetilde{y}_j(w)\Big)dw$$

holds. Then, for each $t \in \mathfrak{J}_j$, we obtain:

$$\begin{aligned}
& |z(t) - y(t)| = |z(t) - y_i(t)| = |z_i(t) - \widetilde{y}_i(t)| \\
= & \Big|(z_j)(t) + \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\
& \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(w), \widetilde{y}_i(w)\Big)dw \\
- & \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(w), \widetilde{y}_i(w)\Big)dw\Big| \\
\leq & \Big|(z_j)(t) + \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\
& \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1} m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}z_j(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}z_j(w), z_j(w)\Big)dw \\
- & \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1} m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}z_j(\Omega_j)) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}z_j(w), z_j(w)\Big)dw\Big| \\
+ & \frac{(\Omega_j - \Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\
& \int_{\Omega_{j-1}}^{\Omega_j} (\Omega_j - w)^{u_j-1}\Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}z_j(\Omega_j) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}z_j(w), z_j(w)\Big)dw \\
- & m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(\Omega_j) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(w), \widetilde{y}_i(w)\Big)dw\Big| \\
+ & \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t} (t-w)^{u_j-1}\Big| m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}z_j(\Omega_j) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}z_j(w), z_j(w)\Big)dw \\
- & m\Big(w, -(\Omega_j - \Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(\Omega_j) + \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\widetilde{y}_i(w), \widetilde{y}_i(w)\Big)dw\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{(\Omega_j-\Omega_{j-1})^{1-u_j}(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)} \\
&\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}w^{-\zeta}\Big(K\Big[(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|(z_j(\Omega_j)-\widetilde{y}_i(\Omega_j)| \\
&+ \mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|z_j(w)-\widetilde{y}_i(w)|\Big] + L|z_j(w)-\widetilde{y}_i(w)|\Big)dw \\
&+ \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}w^{-\zeta}\Big(K(\Omega_j-\Omega_{j-1})^{1-u_j}(w-\Omega_{j-1})^{u_j-1}\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|z_j(\Omega_j)-\widetilde{y}_i(\Omega_j)| \\
&+ \mathfrak{I}^{u_j}_{T^+_{j-1}}|z_j(w)-\widetilde{y}_i(w)| + L|z_j(w)-\widetilde{y}_i(w)|\Big)dw \\
&\leq \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{\Omega_j}(\Omega_j-w)^{u_j-1}w^{-\zeta}\Big(K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|z_j(\Omega_j)-\widetilde{y}_i(\Omega_j)| + K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|z_j(w)-\widetilde{y}_i(w)| \\
&+ L|z_j(w)-\widetilde{y}_i(w)|\Big)dw + \frac{1}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}w^{-\zeta}\Big(K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|z_j(\Omega_j)-\widetilde{y}_i(\Omega_j)| \\
&+ K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}|z_j(w)-\widetilde{y}_i(w)| + L|z_j(w)-\widetilde{y}_i(w)|\Big)dw + \frac{m^*}{\Gamma(u_j)}\int_{\Omega_{j-1}}^{t}(t-w)^{u_j-1}dw \\
&\leq \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{(\Omega_j-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}\Big(2K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\|z_j-\widetilde{y}_i\|_{E_j} + L\|z_j-\widetilde{y}_i\|_{E_j}\Big)\int_{\Omega_{j-1}}^{\Omega_j}w^{-\zeta}dw \\
&+ \frac{(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}\Big(2K\mathfrak{I}^{u_j}_{\Omega^+_{j-1}}\|z_j-\widetilde{y}_i\|_{E_j} + L\|z_j-\widetilde{y}_i\|_{E_j}\Big)\int_{\Omega_{j-1}}^{t}w^{-\zeta}dw \\
&\leq \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)}\Big(2K\frac{(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)}\|z_j-\widetilde{y}_i\|_{E_j} + L\|z_j-\widetilde{y}_i\|_{E_j}\Big) \\
&+ \frac{(t-\Omega_{j-1})^{u_j-1}}{\Gamma(u_j)}\Big(2K\frac{(t^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j}}{(1-\zeta)\Gamma(u_j+1)}\|z_j-\widetilde{y}_i\|_{E_j} + L\|z_j-\widetilde{y}_i\|_{E_j}\Big) \\
&\leq \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{2({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)}\Big(2K\frac{(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + L\Big)\|z_j-\widetilde{y}_i\|_{E_j}. \\
&\leq \lambda_{\varphi(t)}\epsilon\varphi(t) + \mu\|z-y\|,
\end{aligned}
$$

where:

$$\mu = \max_{i=1,2,\dots,n}\frac{2({\Omega_j}^{1-\zeta}-{\Omega_{j-1}}^{1-\zeta})(\Omega_j-\Omega_{j-1})^{u_j-1}}{(1-\zeta)\Gamma(u_j)}\Big(2K\frac{(\Omega_j-\Omega_{j-1})^{u_j}}{\Gamma(u_j+1)} + L\Big).$$

Then:

$$\|z-y\|(1-\mu) \leq \lambda_{\varphi(t)}\epsilon\varphi(t).$$

It gives, for each $t \in \mathfrak{J}$, that:

$$|z(t)-y(t)| \leq \|z-y\| \leq \frac{\lambda_{\varphi(t)}}{1-\mu}\epsilon\varphi(t) := a_m\epsilon\varphi(t).$$

Then, the given implicit nonlinear VOFBVP (1) is U-H-R stable with respect to φ. $\square$

5. Numerical Example

Example 2. *Let us consider the implicit nonlinear VOFBVP by assuming $\Omega = 2$, as follows:*

$$\begin{cases} \mathfrak{D}_{0^+}^{u(t)} x(t) = \Big(\dfrac{|x^{\frac{1}{2}}(t)|}{10} + \dfrac{2}{15} |\mathfrak{D}_{0^+}^{u(t)} x(t)| + \dfrac{1}{3} \Big) t^{-\frac{1}{4}}, \quad t \in \mathfrak{J} := [0,2], \\ x(0) = 0, \quad x(2) = 0. \end{cases} \tag{12}$$

Let:

$$m(t,y,z) = (\frac{1}{10} y^{\frac{1}{2}} + \frac{2}{15} z + \frac{1}{3}) t^{-\frac{1}{4}}, \ (t,y,z) \in [0,2] \times [1,+\infty) \times [1,+\infty),$$

and:

$$u(t) = \begin{cases} \dfrac{8}{5}, \quad t \in \mathfrak{J}_1 := [0,1], \\ \dfrac{9}{5}, \quad t \in \mathfrak{J}_2 :=]1,2]. \end{cases} \tag{13}$$

Thus, we obtain:

$$\begin{aligned} t^{\frac{1}{4}} |m(t,y_1,z_1) - m(t,y_2,z_2)| &= |\frac{1}{10}(y_1^{\frac{1}{2}} - y_2^{\frac{1}{2}}) + \frac{2}{15}(z_1 - z_2)| \\ &\leq \frac{1}{10}|y_1 - y_2| + \frac{2}{15}|z_1 - z_2|. \end{aligned}$$

Therefore, (H2) holds with $\zeta = \frac{1}{4}$, $K = \frac{1}{10}$ and $L = \frac{2}{15}$.
By (13), the implicit nonlinear VOFBVP (12) is divided into two expressions as follows:

$$\begin{cases} \mathfrak{D}_{0^+}^{\frac{8}{5}} x(t) = (\frac{1}{10}|x^{\frac{1}{2}}(t)| + \frac{2}{15}|\mathfrak{D}_{0^+}^{\frac{8}{5}} x(t)| + \frac{1}{3}) t^{-\frac{1}{4}}, \quad t \in \mathfrak{J}_1, \\ \mathfrak{D}_{1^+}^{\frac{9}{5}} x(t) = (\frac{1}{10}|x^{\frac{1}{2}}(t)| + \frac{2}{15}|\mathfrak{D}_{1^+}^{\frac{9}{5}} x(t)| + \frac{1}{3}) t^{-\frac{1}{4}}, \quad t \in \mathfrak{J}_2. \end{cases}$$

For $t \in \mathfrak{J}_1$, the implicit nonlinear VOFBVP (12) corresponds to the following FBVP:

$$\begin{cases} \mathfrak{D}_{0^+}^{\frac{8}{5}} x(t) = (\frac{1}{10}|x^{\frac{1}{2}}(t)| + \frac{2}{15}|\mathfrak{D}_{0^+}^{\frac{8}{5}} x(t)| + \frac{1}{3}) t^{-\frac{1}{4}}, \quad t \in \mathfrak{J}_1, \\ x(0) = 0, \quad x(1) = 0. \end{cases} \tag{14}$$

We can immediately check that (8) holds:

$$\frac{({\Omega_1}^{1-\zeta} - \Omega_0^{1-\zeta})(\Omega_1 - \Omega_0)^{u_1 - 1}}{(1-\zeta)\Gamma(u_1)} \Big(\frac{2K(\Omega_1 - \Omega_0)^{u_1}}{\Gamma(u_1+1)} + L \Big) = \frac{1}{\frac{3\Gamma(\frac{8}{5})}{4}} (\frac{\frac{1}{5}}{\Gamma(\frac{13}{5})} + \frac{2}{15}) \simeq 0.4076 < 1.$$

Let $\varphi(t) = t^{\frac{1}{2}}$. Then:

$$\begin{aligned} I_{0^+}^{u_1} \varphi(t) &= \frac{1}{\Gamma(\frac{8}{5})} \int_0^t (t-w)^{\frac{3}{5}} w^{\frac{1}{2}} dw \\ &\leq \frac{1}{\Gamma(\frac{8}{5})} \int_0^t (t-w)^{\frac{3}{5}} dw \end{aligned}$$

$$\leq \frac{5}{8\Gamma(\frac{8}{5})}\varphi(t) := \lambda_{\varphi(t)}\varphi(t).$$

Hence, (H3) holds with $\varphi(t) = t^{\frac{1}{2}}$ *and* $\lambda_{\varphi(t)} = \dfrac{5}{8\Gamma(\frac{8}{5})}$.

By Theorem 2, the equivalent standard implicit nonlinear FBVP (14) has a solution $x_1 \in E_1$, *and from Theorem 3, the same FBVP (14) is U-H-R stable.*

For $t \in \mathfrak{J}_2$, *the implicit nonlinear VOFBVP (12) can be converted to the equivalent standard implicit nonlinear FBVP as follows:*

$$\begin{cases} \mathfrak{D}_{1^+}^{\frac{9}{5}}x(t) = (\frac{1}{10}|x^{\frac{1}{2}}(t)| + \frac{2}{15}|\mathfrak{D}_{1^+}^{\frac{9}{5}}x(t)| + \frac{1}{3})t^{-\frac{1}{4}}, \quad t \in \mathfrak{J}_2, \\ x(1) = 0, \ x(2) = 0. \end{cases} \tag{15}$$

We simply see that:

$$\frac{({\Omega_2}^{1-\zeta} - \Omega_1^{1-\zeta})(\Omega_2 - \Omega_1)^{u_2-1}}{(1-\zeta)\Gamma(u_2)}\Big(\frac{2K(\Omega_2-\Omega_1)^{u_2}}{\Gamma(u_2+1)} + L\Big) = \frac{2^{\frac{3}{4}} - 1}{\frac{3\Gamma(\frac{9}{5})}{4}}\big(\frac{\frac{1}{5}}{\Gamma(2.8)} + \frac{2}{15}\big) \simeq 0.2465 < 1.$$

Thus, the condition (8) is satisfied. In addition:

$$\begin{aligned} \mathfrak{I}_{1^+}^{u_2}\varphi(t) &= \frac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-w)^{\frac{4}{5}} w^{\frac{1}{2}} dw \\ &\leq \frac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-w)^{\frac{4}{5}} dw \\ &\leq \frac{5}{9\Gamma(\frac{9}{5})}\varphi(t) := \lambda_{\varphi(t)}\varphi(t). \end{aligned}$$

Hence, (H3) fulfills with $\varphi(t) = t^{\frac{1}{2}}$ *and* $\lambda_{\varphi(t)} = \dfrac{5}{9\Gamma(\frac{9}{5})}$.

By Theorem 2, the equivalent standard implicit nonlinear FBVP (15) has a solution $\widetilde{x}_2 \in E_2$, *and from Theorem 3, the same implicit nonlinear FBVP (15) is U-H-R stable.*

Clearly, we have:

$$x_2(t) = \begin{cases} 0, & t \in \mathfrak{J}_1 \\ \widetilde{x}_2(t), & t \in \mathfrak{J}_2. \end{cases}$$

Accordingly, by Definition 7, the solution of the implicit nonlinear VOFBVP (12) admits a form as

$$x(t) = \begin{cases} x_1(t), & t \in \mathfrak{J}_1, \\ x_2(t) = \begin{cases} 0, & t \in \mathfrak{J}_1, \\ \widetilde{x}_2(t), & t \in \mathfrak{J}_2, \end{cases} \end{cases}$$

and, by Theorem 3, the implicit nonlinear VOFBVP (12) is U-H-R stable with respect to φ.

6. Conclusions

In this paper, we considered a nonlinear implicit fractional boundary value problem in the variable order settings and studied some qualitative aspects of possible solutions of this system. To prove our main results, through an example, we first showed that the semi-group property was not valid for Riemann–Liouville fractional variable order integrals, and to solve this problem, we defined an arbitrary partition on $\mathfrak{J}$ such that the variable order $u(t)$ was piecewise constant function. Then, we transformed the variable order implicit FBVP (1) into a constant order implicit FBVP (6), and by obtaining equivalent integral equations, we proved the existence of solutions by means of the Krasnoselskii's fixed point theorem. After that, we investigated the Ulam–Hyers–Rassias stability for the mentioned VOFBVP (1). Lastly, to validate our findings, we provided an example to show the applicability of results. All in all, our results can be further extended in future research works to study various classes of implicit nonlinear fractional differential equations in the variable order settings via singular and nonsingular operators.

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Abbreviations

The following abbreviations are used in this manuscript:

FDE	Fractional Differential Equation
VOFBVP	Variable Order Fractional Boundary Value Problem
U-H-R	Ulam–Hyers–Rassias

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