# On Sharp Oscillation Criteria for General Third-Order Delay Differential Equations 

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#### Abstract

In this paper, effective oscillation criteria for third-order delay differential equations of the form, $\left(r_{2}\left(r_{1} y^{\prime}\right)^{\prime}\right)^{\prime}(t)+q(t) y(\tau(t))=0$ ensuring that any nonoscillatory solution tends to zero asymptotically, are established. The results become sharp when applied to a Euler-type delay differential equation and, to the best of our knowledge, improve all existing results from the literature. Examples are provided to illustrate the importance of the main results.


Keywords: third-order differential equation; delay; property A; oscillation

## 1. Introduction

In this article, we consider linear third-order delay differential equations of the form

$$
\begin{equation*}
\left(r_{2}\left(r_{1} y^{\prime}\right)^{\prime}\right)^{\prime}(t)+q(t) y(\tau(t))=0, \quad t \geq t_{0} \tag{1}
\end{equation*}
$$

where $r_{1}, r_{2}, q, \tau \in \mathcal{C}(\mathcal{I}, \mathbb{R}), \mathcal{I}=\left[t_{0}, \infty\right) \subset \mathbb{R}, t_{0}>0$ is a fixed constant such that $r_{1}>0$, $r_{2}>0, q \geq 0$ does not vanish eventually, $\tau(t) \leq t$, and $\lim _{t \rightarrow \infty} \tau(t)=\infty$.

For any solution $y$ of (1), we denote the $i$ th quasi-derivative of $y$ as $L_{i} y$, that is,

$$
L_{0} y=y, \quad L_{1} y=r_{1} y^{\prime}, \quad L_{2} y=r_{2}\left(r_{1} y^{\prime}\right)^{\prime}, \quad L_{3} y=\left(r_{2}\left(r_{1} y^{\prime}\right)^{\prime}\right)^{\prime} \quad \text { on } \mathcal{I}
$$

and assume that

$$
\begin{equation*}
\int_{t_{0}}^{\infty} \frac{\mathrm{d} t}{r_{i}(t)}=\infty, \quad i=1,2 \tag{2}
\end{equation*}
$$

By a solution of Equation (1), we mean a nontrivial function $y$ with the property $L_{i} y \in \mathcal{C}^{1}\left(\left[T_{y}, \infty\right), \mathbb{R}\right)$ for $i=0,1,2$ and a certain $T_{y} \geq t_{0}$, which satisfies (1) on $\left[T_{y}, \infty\right)$. Our attention is restricted to proper solutions of (1), which exist on some half-line $\left[T_{y}, \infty\right)$ and satisfy the condition

$$
\sup \{|x(s)|: t \leq s<\infty\}>0 \quad \text { for any } t \geq T_{y}
$$

The oscillatory nature of the solutions is understood in the usual way, that is, a proper solution is termed oscillatory or nonoscillatory according to whether it does or does not have infinitely many zeros.

Following classical results of Kondrat'ev and Kiguradze, see, e.g., [1], we say that Equation (1) has property $A$ if any solution $y$ of (1) is either oscillatory or tends to zero as $t \rightarrow \infty$. By a proper modification of the well-known result of Kiguradze [1] (Lemma 1), one
can easily classify the possible nonoscillatory solutions of (1). As a matter of fact, assuming (2) shows that (1) has only two types of nonoscillatory, positive solutions

$$
\begin{array}{lll}
y \in \mathcal{N}_{0} & \Longleftrightarrow & y(t)>0, \\
L_{1} y(t)<0, & L_{2} y(t)>0 \\
y \in \mathcal{N}_{2} & \Longleftrightarrow & y(t)>0,
\end{array} L_{1} y(t)>0, \quad L_{2} y(t)>0,
$$

for $t$ large enough, see, e.g., [2] (Lemma 2) or [3] (Lemma 1). Solutions belonging to the class $\mathcal{N}_{0}$ are called Kneser solutions. Clearly, (1) has property A if $\mathcal{N}_{2}=\varnothing$ and any Kneser solution of (1) tends to zero asymptotically.

The oscillation theory of third-order differential equations with variable coefficients has been attracting considerable attention over the last decades, which is evidenced by a large number of published studies in the area, most of which have been collected and presented in the monographs [4,5].

In particular, various criteria for property A of (1) have been presented in the literature, see $[3,6-17]$ and the references cited therein. The methodology in these articles has been mainly based on the use of the so-called Riccati technique or suitable comparison principles with lower-order delay differential inequalities. In [3], the authors point out that the proofs essentially use the estimates relating a solution $y \in \mathcal{N}_{2}$ of (1) with its first and second quasi-derivatives and "despite the differences in the proofs of the cited works, the resulting criteria have in common that their strength depends on the sharpness of these estimates". Here, it is worth noting that in order to test the strength of the oscillation criteria derived by different methods, Euler-type differential equations are mostly used.

For our comparison purposes, let us consider a particular case of (1)—the third-order Euler differential equation with proportional delay of the form

$$
\begin{equation*}
\left(t^{\gamma}\left(t^{\alpha} y^{\prime}(t)\right)^{\prime}\right)^{\prime}+q_{0} t^{\alpha+\gamma-3} y(\tau t)=0 \tag{3}
\end{equation*}
$$

where $\tau \in(0,1], q_{0}>0, \alpha<1$, and $\gamma<1$. It is easy to verify by a direct substitution that (3) has a nonoscillatory solution $y=t^{\mu}$ belonging to the class $\mathcal{N}_{2}$, when $\mu \in(1-\alpha, 2-\alpha-\gamma)$ is a root of the characteristic equation

$$
c(\mu)=q_{0}
$$

where

$$
\begin{equation*}
c(\mu):=\mu(\mu+\alpha-1)(2-\mu-\alpha-\gamma) \tau^{-\mu} \tag{4}
\end{equation*}
$$

or equivalently, if

$$
\begin{equation*}
q_{0} \leq \max \{c(\mu): 1-\alpha<\mu<2-\alpha-\gamma\} . \tag{5}
\end{equation*}
$$

For a special case of (3) with $\alpha=\gamma=0$ and $\tau=1$, i.e., for the linear third-order Euler differential equation

$$
\begin{equation*}
y^{\prime \prime \prime}(t)+\frac{q_{0}}{t^{3}} y(t)=0 \tag{6}
\end{equation*}
$$

condition (5) for the existence of a solution from the class $\mathcal{N}_{2}$ reduces to

$$
q_{0} \leq \max \{\mu(\mu-1)(2-\mu): 1<\mu<2\}=\frac{2}{3 \sqrt{3}}
$$

which is sharp in the sense that if

$$
q_{0}>\frac{2}{3 \sqrt{3}}
$$

then $\mathcal{N}_{2}=\varnothing$.
We stress that there is no result so far in the literature on the property $A$ of (1), which would be sharp for (3). The main purpose of the paper is to positively answer this open problem. Following the direction initiated in [3], we present new asymptotic properties of solutions belonging to the class $\mathcal{N}_{2}$. Our approach differs from that applied in [3] and
allows us to relax the assumption of the monotonicity of the delay function $\tau(t)$, which is generally required in previous works. As a consequence, we establish efficient criteria for detecting property A for Equation (1), which are unimprovable in the sense that they give a necessary and sufficient condition for the delay Euler Equation (3) to have property A. Our motivation comes from the recent papers [18-20], where a similar technique leads to obtaining sharp oscillation results for second-order half-linear differential equations with deviating arguments. Such an idea was successfully adopted for the third-order Equation (1) with $r_{1}=r_{2}=1$ in a recent work [21]. However, it turns out that the general functions $r_{i}$ require a carefully modified the approach.

The organization of the paper is as follows. In Section 2, we introduce the basic notations and assumptions. In Section 3, we state the main results of the paper. In particular, we present a single condition criteria for property A of (1) in case when the functions $r_{1}$ and $r_{2}$ are of the same type (see Definition 1 and condition (15) below). In Section 4, we illustrate the importance of the main results by means of a couple of examples.

## 2. Preliminaries

In this section, we will introduce a set of assumptions and notation used in the paper. To start with, we define

$$
\begin{aligned}
R_{i}(t) & =\int_{t_{0}}^{t} \frac{\mathrm{~d} s}{r_{i}(s)}, \quad i=1,2 \\
R_{12}(t) & =\int_{t_{0}}^{t} \frac{R_{2}(s)}{r_{1}(s)} \mathrm{d} s
\end{aligned}
$$

and

$$
\begin{align*}
& \lambda_{*}:=\liminf _{t \rightarrow \infty} \frac{R_{12}(t)}{R_{12}(\tau(t))} \\
& \beta_{*}:=\liminf _{t \rightarrow \infty} R_{2}(t) R_{12}(\tau(t)) q(t) r_{2}(t),  \tag{7}\\
& k_{*}:=\liminf _{t \rightarrow \infty} \frac{R_{2}^{\beta_{*}(t) \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{*}}(s)}{r_{1}(s)} \mathrm{d} s}}{R_{12}(t)} \text { for } \beta_{*} \in(0,1) .
\end{align*}
$$

As the limit inferior triple $\lambda_{*}, \beta_{*}$, and $k_{*}$ is defined on an extended range of real $\mathbb{R} \cup\{\infty\}$, in our proofs, we will rather make use of real constants $\lambda \leq \lambda_{*}, \beta \leq \beta_{*}$ and $k \leq k_{*}$ defined by $(C)_{\lambda},(C)_{\beta}$, and $(C)_{k}$, respectively, for the particular cases that can occur depending on the delay function $\tau$.
$(C)_{\lambda}$ Since $R_{12}$ is increasing and $\tau(t) \leq t$, clearly $\lambda_{*} \geq 1$. Then, for
(a) $\lambda=1$ if $\lambda_{*}=1$;
(b) any $\lambda \in\left(1, \lambda_{*}\right)$ if $\lambda_{*} \in(1, \infty)$;
(c) any $\lambda \in(1, \infty)$ arbitrarily large if $\lambda_{*}=\infty$,
there exists $t_{\lambda} \geq t_{0}$ such that

$$
\begin{equation*}
\frac{R_{12}(t)}{R_{12}(\tau(t))} \geq \lambda, \quad t \geq t_{\lambda} \tag{8}
\end{equation*}
$$

$(C)_{\beta}$ For any $\beta \in\left(0, \beta_{*}\right)$, there exists $t_{\beta} \geq t_{0}$ such that

$$
\begin{equation*}
R_{2}(t) R_{12}(\tau(t)) q(t) r_{2}(t) \geq \beta, \quad t \geq t_{\beta} \tag{9}
\end{equation*}
$$

$(C)_{k}$ For $\beta_{*} \in(0,1)$, it follows from the increasing nature of $R_{2}$ that $k_{*} \geq 1$. Then, for
(a) $k=1$ if $k_{*}=1$;
(b) any $k \in\left(1, k_{*}\right)$ if $k_{*} \in(1, \infty)$;
(c) any $k \in(1, \infty)$ arbitrarily large if $k_{*}=\infty$,
there exists $t_{k} \geq t_{0}$, such that

$$
\begin{equation*}
\frac{R_{2}^{\beta}(t) \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta}(s)}{r_{1}(s)} \mathrm{d} s}{R_{12}(t)} \geq k, \quad t \geq t_{k} . \tag{10}
\end{equation*}
$$

For our purposes, we also need to define, for $\beta_{*} \in(0,1), \lambda_{*} \in[1, \infty), k_{*} \in[1, \infty)$ the following sequence $\left\{\beta_{n}\right\}_{n=0}$ (as far as it exists):

$$
\begin{align*}
& \beta_{0}=\beta_{*} \\
& \beta_{n}=\frac{\beta_{0} k_{n-1} \lambda_{*}^{1-1 / k_{n-1}}}{\left(1-\beta_{n-1}\right)}, \quad n \in \mathbb{N}, \tag{11}
\end{align*}
$$

where $k_{n}$ satisfies

$$
\begin{equation*}
k_{n}=\liminf _{t \rightarrow \infty} \frac{R_{2}^{\beta_{n}}(t) \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{n}}(s)}{r_{1}(s)} \mathrm{d} s}{R_{12}(t)}, \quad n \in \mathbb{N}_{0} \tag{12}
\end{equation*}
$$

Clearly, $\beta_{n+1}$ exists if $\beta_{i}<1$ and $k_{i} \in[1, \infty)$ for $i=0,1, \ldots, n$. In such a case, we have

$$
\frac{\beta_{1}}{\beta_{0}}=\frac{k_{0} \lambda_{*}^{1-1 / k_{0}}}{1-\beta_{0}}>1
$$

and

$$
\begin{aligned}
k_{1} & =\liminf _{t \rightarrow \infty} \frac{R_{2}^{\beta_{0}}(t) \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{0}}(s)}{r_{1}(s)} \mathrm{d} s}{R_{12}(t)}=\liminf _{t \rightarrow \infty} \frac{R_{2}^{\beta_{1}}(t) \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{0}-\left(\beta_{1}-\beta_{0}\right)}(s)}{r_{1}(s)} \mathrm{d} s}{R_{12}(t)} \\
& \geq \liminf _{t \rightarrow \infty} \frac{R_{2}^{\beta_{0}}(t) \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{0}}(s)}{r_{1}(s)} \mathrm{d} s}{R_{12}(t)}=k_{0},
\end{aligned}
$$

i.e.,

$$
k_{1} \geq k_{0}
$$

By induction on $n$, it is easy to show that

$$
\begin{equation*}
\frac{\beta_{n+1}}{\beta_{n}}=\ell_{n}>1 \tag{13}
\end{equation*}
$$

where

$$
\begin{align*}
& \ell_{0}:=\frac{k_{0} \lambda_{*}^{1-1 / k_{0}}}{1-\beta_{0}} \\
& \ell_{n}:=\frac{k_{n} \lambda_{*}^{1 / k_{n-1}-1 / k_{n}}\left(1-\beta_{n-1}\right)}{k_{n-1}\left(1-\beta_{n}\right)}, \quad n \in \mathbb{N} \tag{14}
\end{align*}
$$

with

$$
k_{n} \geq k_{n-1}
$$

It is useful to note that there are two situations when the impact of the delay would not influence the value of $\beta_{n}$ in the sequence (11): $\lambda_{*}=1$ or $k_{i}=1, i=0,1, \ldots, n$. Below, we point out that the second one cannot occur in a particular case, when coefficients $r_{1}$ and $r_{2}$ are of the same type, e.g., either $r_{i}=\mathrm{e}^{a_{i} t}$ or $r_{i}=t^{a_{i}}$ and likewise. With this aim, we use a concept of asymptotically similar functions.

Definition 1. We say that the functions $f$ and $g$ are asymptotically similar $(f \sim g)$ if there exists a positive constant $\ell$ such that

$$
\lim _{t \rightarrow \infty} \frac{f(t)}{g(t)}=\ell
$$

As a special case of (1), we will consider the case when

$$
\begin{equation*}
r_{1} R_{1} \sim r_{2} R_{2} \tag{15}
\end{equation*}
$$

Lemma 1. Assume (15). Then, for any $c \in(0,1)$,

$$
\begin{equation*}
R_{12}(t) \geq \frac{c}{1+\ell} R_{1}(t) R_{2}(t) \tag{16}
\end{equation*}
$$

eventually.
Proof. It follows from (15) that for any $\varepsilon>0$, we have

$$
\begin{equation*}
\frac{r_{1}(t) R_{1}(t)}{r_{2}(t) R_{2}(t)}<\ell+\varepsilon \tag{17}
\end{equation*}
$$

eventually. Integrating the identity

$$
\left(R_{1} R_{2}\right)^{\prime}(t)=\frac{1}{r_{1}(t)} R_{2}(t)+\frac{1}{r_{2}(t)} R_{1}(t)
$$

from $t_{0}$ to $t$ and using (17), we obtain

$$
R_{1}(t) R_{2}(t)-R_{1}\left(t_{0}\right) R_{2}\left(t_{0}\right)=R_{12}(t)+\int_{t_{0}}^{t} \frac{1}{r_{2}(s)} R_{1}(s) \mathrm{d} s<(1+\ell+\varepsilon) R_{12}(t)
$$

By virtue of (2), we conclude that (16) holds.
Now, we give an interesting property of the sequence $\left\{\beta_{n}\right\}$ under the similarity assumption (15).

Lemma 2. Let (15) hold, $\beta_{*}>0$ and $\beta_{i}<1, i=0,1, \ldots, n$. Then,

$$
k_{n} \geq \frac{\beta_{n} \ell}{1+\ell}+1>1, \quad n \in \mathbb{N}_{0}
$$

Proof. Using l'Hôspital's rule, it is easily seen that

$$
\begin{align*}
k_{n} & =\liminf _{t \rightarrow \infty} \frac{R_{2}^{\beta_{n}}(t) \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{n}}(s)}{r_{1}(s)} \mathrm{d} s}{R_{12}(t)} \\
& \geq \liminf _{t \rightarrow \infty} \frac{\beta_{n} R_{2}^{\beta_{n}-1}(t) \frac{1}{r_{2}(t)} \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{n}}(s)}{r_{1}(s)} \mathrm{d} s+R_{2}^{\beta_{n}}(t) \frac{R_{2}^{1-\beta_{n}}(t)}{r_{1}(t)}}{\frac{R_{2}(t)}{r_{1}(t)}}  \tag{18}\\
& =\beta_{n} \liminf _{t \rightarrow \infty} \frac{r_{1}(t)}{r_{2}(t)} \frac{\int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{n}}(s)}{r_{1}(s)} \mathrm{d} s}{R_{2}^{2-\beta_{n}}(t)}+1 .
\end{align*}
$$

Taking into account the fact that $R_{2}$ is increasing and (16) holds, we have, for any $c \in(0,1)$,

$$
\frac{r_{1}(t)}{r_{2}(t)} \frac{\int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{n}}(s)}{r_{1}(s)} \mathrm{d} s}{R_{2}^{2-\beta_{n}}(t)} \geq \frac{r_{1}(t)}{r_{2}(t)} \frac{R_{12}(t)}{R_{2}^{2}(t)} \geq c \frac{\ell}{1+\ell}>0
$$

The proof is complete.
Corollary 1. Let $r_{1}=r_{2}, \beta_{*}>0$ and $\beta_{i}<1, i=0,1, \ldots, n$. Then,

$$
k_{n}=\frac{2}{2-\beta_{n}}>1, \quad n \in \mathbb{N}_{0}
$$

Proof. It is simple to compute the limit (18) when $r_{1}=r_{2}$; hence, we omit the details.
For the sake of convenience, we assume here that all functional inequalities hold eventually, that is, they are satisfied for all $t$ that are large enough. As usual and without loss of generality, we can assume from now on that nonoscillatory solutions of (1) are eventually positive.

## 3. Main Results

### 3.1. Nonexistence of Solutions from the Class $\mathcal{N}_{2}$

In this section, we give a series of lemmas about the asymptotic properties of solutions belonging to the class $\mathcal{N}_{2}$, which will play a crucial role in proving our main oscillation results stated in Section 3.3.

Lemma 3. Assume $\beta_{*}>0$ and let $y$ be an eventually positive solution of (1) belonging to the class $\mathcal{N}_{2}$. Then, for at that is sufficiently large:
(i) $\lim _{t \rightarrow \infty} L_{2} y(t)=\lim _{t \rightarrow \infty} L_{1} y(t) / R_{2}(t)=\lim _{t \rightarrow \infty} y(t) / R_{12}(t)=0$;
(ii) $L_{1} y>R_{2} L_{2} y$ and $L_{1} y / R_{2}$ is decreasing;
(iii) $y>\left(R_{12} / R_{2}\right) L_{1} y$ and $y / R_{12}$ is decreasing.

Proof. Let $y \in \mathcal{N}_{2}$ and choose $t_{1} \geq t_{0}$ such that $y(\tau(t))>0$ and $\beta$ satisfies (9) for $t \geq t_{1}$.
(i) Since $L_{2} y$ is a positive decreasing function, clearly

$$
\lim _{t \rightarrow \infty} L_{2} y(t)=\xi \geq 0
$$

If $\xi>0$, then $L_{2} y(t) \geq \xi>0$ and so for any $\varepsilon \in(0,1)$, we have

$$
y(t) \geq \xi \int_{t_{1}}^{t} \frac{1}{r_{1}(u)} \int_{t_{1}}^{u} \frac{1}{r_{2}(s)} \mathrm{d} s \mathrm{~d} u \geq \tilde{\xi} R_{12}(t), \quad \tilde{\xi}:=\varepsilon \xi
$$

Using this in (1), we have

$$
L_{3} y(t) \geq q(t) y(\tau(t)) \geq \tilde{\xi} R_{12}(\tau(t)) q(t)
$$

Integrating from $t_{1}$ to $t$, we obtain

$$
L_{2} y(t) \geq \tilde{\xi} \int_{t_{1}}^{t} R_{12}(\tau(t)) q(s) \mathrm{d} s \geq \beta \tilde{\xi} \int_{t_{1}}^{t} \frac{1}{r_{2}(s) R_{2}(s)} \mathrm{d} s=\beta \tilde{\xi} \ln \frac{R_{2}(t)}{R_{2}\left(t_{1}\right)} \rightarrow \infty \quad \text { as } t \rightarrow \infty
$$

which is a contradiction. Hence, $\xi=0$. Applying l'Hôspital's rule, we see that (i) holds.
(ii) Again, using the fact that $L_{2} y$ is positive and decreasing, it follows that

$$
\begin{aligned}
L_{1} y(t) & =L_{1} y\left(t_{1}\right)+\int_{t_{1}}^{t} \frac{1}{r_{2}(s)} L_{2} y(s) \mathrm{d} s \\
& \geq L_{1} y\left(t_{1}\right)+L_{2} y(t) \int_{t_{1}}^{t} \frac{1}{r_{2}(s)} \mathrm{d} s \\
& =L_{1} y\left(t_{1}\right)+L_{2} y(t) R_{2}(t)-L_{2} y(t) \int_{t_{0}}^{t_{1}} \frac{1}{r_{2}(s)} \mathrm{d} s
\end{aligned}
$$

In view of (i), there is a $t_{2}>t_{1}$, such that

$$
L_{1} y\left(t_{1}\right)>L_{2} y(t) \int_{t_{0}}^{t_{1}} \frac{1}{r_{2}(s)} \mathrm{d} s, \quad t \geq t_{2}
$$

Thus,

$$
L_{1} y(t)>L_{2} y(t) R_{2}(t), \quad t \geq t_{2}
$$

and consequently,

$$
\left(\frac{L_{1} y}{R_{2}}\right)^{\prime}(t)=\frac{L_{2} y(t) R_{2}(t)-L_{1} y(t)}{R_{2}^{2}(t) r_{2}(t)}<0, \quad t \geq t_{2}
$$

which proves (ii).
(iii) In view of the fact that $L_{1} y / R_{2}$ is a decreasing function tending to zero, we have

$$
\begin{aligned}
y(t) & =y\left(t_{2}\right)+\int_{t_{2}}^{t} \frac{R_{2}(s)}{r_{1}(s)} \frac{L_{1} y(s)}{R_{2}(s)} \mathrm{d} s \geq y\left(t_{2}\right)+\frac{L_{1} y(t)}{R_{2}(t)} \int_{t_{1}}^{t} \frac{R_{2}(s)}{r_{1}(s)} \mathrm{d} s \\
& =y\left(t_{2}\right)+\frac{L_{1} y(t)}{R_{2}(t)} R_{12}(t)-\frac{L_{1} y(t)}{R_{2}(t)} \int_{t_{0}}^{t_{2}} \frac{R_{2}(s)}{r_{1}(s)} \mathrm{d} s>\frac{L_{1} y(t)}{R_{2}(t)} R_{12}(t)
\end{aligned}
$$

for $t \geq t_{3}$ for some $t_{3}>t_{2}$. Therefore,

$$
\left(\frac{y}{R_{12}}\right)^{\prime}(t)=\frac{L_{1} y(t) R_{12}(t)-y(t) R_{2}(t)}{R_{12}^{2}(t) r_{1}(t)}<0, \quad t \geq t_{3}
$$

which proves (iii). The proof is complete.
The next lemma provides some additional properties of solutions from the class $\mathcal{N}_{2}$.
Lemma 4. Assume $\beta_{*}>0$ and let $y$ be an eventually positive solution of (1) belonging to $\mathcal{N}_{2}$. Then, for $k$ defined by (10) and for a that is sufficiently large:
( $a_{0}$ ) $\left(1-\beta_{*}\right) L_{1} y>R_{2} L_{2} y$ and $L_{1} y / R_{2}^{1-\beta_{*}}$ decrease;
$\left(b_{0}\right) \lim _{t \rightarrow \infty} L_{1} y(t) / R_{2}^{1-\beta_{*}}(t)=0$;
(co) $y>k\left(R_{12} / R_{2}\right) L_{1} y$ and $y / R_{12}^{1 / k}$ decreases.
Proof. Let $y \in \mathcal{N}_{2}$ with $y(\tau(t))>0$ satisfy the conclusion of Lemma 3 for $t \geq t_{1} \geq t_{0}$ and choose fixed but arbitrarily large $\beta \in\left(\beta_{*} /\left(1+\beta_{*}\right), \beta_{*}\right)$ and $k \leq k_{*}$ satisfying (9) and (10), respectively, for $t \geq t_{1}$.

Since

$$
\frac{\beta}{1-\beta}>\beta_{*},
$$

there exist constants $c_{1} \in(0,1)$ and $c_{2}>0$ such that

$$
\begin{equation*}
\frac{c_{1} \beta}{1-\beta}>\beta_{*}+c_{2} . \tag{19}
\end{equation*}
$$

$\left(a_{0}\right)$ Define the function

$$
\begin{equation*}
z(t):=L_{1} y(t)-R_{2}(t) L_{2} y(t) \tag{20}
\end{equation*}
$$

which is clearly positive by (ii). Differentiating $z$ and using (1) and (9), we see that

$$
\begin{align*}
z^{\prime}(t) & =\left(L_{1} y(t)-R_{2}(t) L_{2} y(t)\right)^{\prime} \\
& =-R_{2}(t) L_{3} y(t) \\
& =R_{2}(t) q(t) y(\tau(t))  \tag{21}\\
& \geq \beta \frac{y(\tau(t))}{r_{2}(t) R_{12}(\tau(t))}
\end{align*}
$$

By virtue of (iii), we have

$$
z^{\prime}(t) \geq \beta \frac{y(t)}{r_{2}(t) R_{12}(t)} \geq \beta \frac{L_{1} y(t)}{r_{2}(t) R_{2}(t)}
$$

for $t \geq t_{2}$ for some $t_{2} \geq t_{1}$. Integrating from $t_{2}$ to $t$ and using the fact that $L_{1} y / R_{2}$ is decreasing and tends to zero asymptotically (see (i) and (ii)), there exists $t_{3} \geq t_{2}$ such that

$$
\begin{align*}
z(t) & \geq z\left(t_{2}\right)+\beta \int_{t_{2}}^{t} \frac{L_{1} y(s)}{r_{2}(s) R_{2}(s)} \mathrm{d} s \geq z\left(t_{2}\right)+\beta \frac{L_{1} y(t)}{R_{2}(t)} \int_{t_{2}}^{t} \frac{1}{r_{2}(s)} \mathrm{d} s  \tag{22}\\
& =z\left(t_{2}\right)+\beta L_{1} y(t)-\beta \frac{L_{1} y(t)}{R_{2}(t)} \int_{t_{0}}^{t_{2}} \frac{1}{r_{2}(s)} \mathrm{d} s>\beta L_{1} y(t), \quad t \geq t_{3}
\end{align*}
$$

Then,

$$
(1-\beta) L_{1} y(t)>R_{2}(t) L_{2} y(t)
$$

and

$$
\begin{equation*}
\left(\frac{L_{1} y}{R_{2}^{1-\beta}}\right)^{\prime}(t)=\frac{L_{2} y(t) R_{2}(t)-(1-\beta) L_{1} y(t) R_{2}(t)}{R_{2}^{2-\beta}(t) r_{2}(t)}<0, \quad t \geq t_{3} \tag{23}
\end{equation*}
$$

It follows directly from (23) and the fact that $L_{1} y$ is increasing that $\beta<1$. Using this in (22) and taking (19) into account, we find that there is $t_{4} \geq t_{3}$ such that

$$
\begin{aligned}
z(t) & \geq \beta \int_{t_{3}}^{t} \frac{L_{1} y(s)}{r_{2}(s) R_{2}(s)} \mathrm{d} s \\
& \geq \beta \frac{L_{1} y(t)}{R_{2}^{1-\beta}(t)} \int_{t_{3}}^{t} \frac{1}{r_{2}(s) R_{2}^{\beta}(s)} \mathrm{d} s \\
& \geq \frac{\beta}{1-\beta} \frac{L_{1} y(t)}{R_{2}^{1-\beta}(t)}\left(R_{2}^{1-\beta}(t)-R_{2}^{1-\beta}\left(t_{3}\right)\right) \\
& \geq \frac{c_{1} \beta}{1-\beta} L_{1} y(t) \\
& >\left(\beta_{*}+c_{2}\right) L_{1} y(t), \quad t \geq t_{4}
\end{aligned}
$$

which implies

$$
\left(1-\beta_{*}\right) L_{1} y(t)>\left(1-\beta_{*}-c_{2}\right) L_{1} y(t)>R_{2}(t) L_{2} y(t)
$$

and

$$
\begin{equation*}
\left(\frac{L_{1} y}{R_{2}^{1-\beta_{*}-c_{2}}}\right)^{\prime}(t)<0 \tag{24}
\end{equation*}
$$

The conclusion of this is in the following.
( $\mathrm{b}_{0}$ ) Clearly, (24) also implies that $L_{1} y / R_{2}^{1-\beta_{*}} \rightarrow 0$ as $t \rightarrow \infty$, since otherwise

$$
\begin{equation*}
\frac{L_{1} y(t)}{R_{2}^{1-\beta_{*}-c_{2}}(t)}=\frac{L_{1} y(t)}{R_{2}^{1-\beta_{*}}(t)} R_{2}^{c_{2}}(t) \rightarrow \infty \quad \text { as } t \rightarrow \infty \tag{25}
\end{equation*}
$$

which is a contradiction.
$\left(\mathrm{c}_{0}\right)$ Using the fact that by $\left(\mathrm{a}_{0}\right)$ and $\left(\mathrm{b}_{0}\right), L_{1} y / R_{2}^{1-\beta_{*}}$ is a decreasing function tending to zero, we have

$$
\begin{aligned}
y(t) & =y\left(t_{4}\right)+\int_{t_{4}}^{t} \frac{R_{2}^{1-\beta_{*}}(s)}{r_{1}(s)} \frac{L_{1} y(s)}{R_{2}^{1-\beta_{*}}(s)} \mathrm{d} s \\
& \geq y\left(t_{4}\right)+\frac{L_{1} y(t)}{R_{2}^{1-\beta_{*}}(t)} \int_{t_{4}}^{t} \frac{R_{2}^{1-\beta_{*}}(s)}{r_{1}(s)} \mathrm{d} s \\
& =y\left(t_{4}\right)+\frac{L_{1} y(t)}{R_{2}^{1-\beta_{*}}(t)} \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{*}}(s)}{r_{1}(s)} \mathrm{d} s-\frac{L_{1} y(t)}{R_{2}^{1-\beta_{*}(t)}} \int_{t_{0}}^{t_{4}} \frac{R_{2}^{1-\beta_{*}}(s)}{r_{1}(s)} \mathrm{d} s \\
& >\frac{L_{1} y(t)}{R_{2}^{1-\beta_{*}}(t)} \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{*}}(s)}{r_{1}(s)} \mathrm{d} s \\
& \geq k \frac{R_{12}(t)}{R_{2}(t)} L_{1} y(t), \quad t \geq t_{5}>t_{4} .
\end{aligned}
$$

Therefore,

$$
\left(\frac{y}{R_{12}^{1 / k}}\right)^{\prime}(t)=\frac{k L_{1} y(t) R_{12}(t)-y(t) R_{2}(t)}{k R_{12}^{1 / k+1}(t) r_{1}(t)}<0, \quad t \geq t_{5}
$$

The proof is complete.
Corollary 2. Assume $\beta_{*} \geq 1$. Then, $\mathcal{N}_{2}=\varnothing$.
Proof. This follows directly from $\left(a_{0}\right)$ and the fact that $L_{2} y$ is positive.
Corollary 3. Assume $\beta_{*}>0$ and $\lambda_{*}=\infty$. Then, $\mathcal{N}_{2}=\varnothing$.
Proof. Let $y \in \mathcal{N}_{2}$ with $y(\tau(t))>0$ satisfy conclusions of Lemma 4 for $t \geq t_{1}$ for some $t_{1} \geq t_{0}$ and choose fixed but arbitrarily large $\beta \leq \beta_{*}, k \leq k_{*}$ and $\lambda \leq \lambda_{*}$, satisfying (9), (10) and (8), respectively, for $t \geq t_{1}$. Using ( $c_{0}$ ) and the definition of $\lambda$ in (21), we have

$$
\begin{aligned}
z^{\prime}(t) & \geq \beta \frac{y(\tau(t))}{r_{2}(t) R_{12}^{1 / k}(\tau(t)) R_{12}^{1-1 / k}(\tau(t))} \\
& \geq \beta \frac{y(t)}{R_{12}^{1 / k}(t)} \frac{1}{r_{2}(t) R_{12}^{1-1 / k}(\tau(t))} \\
& \geq \beta k \frac{R_{12}^{1-1 / k}(t)}{R_{12}^{1-1 / k}(\tau(t))} \frac{1}{R_{2}(t) r_{2}(t)} L_{1} y(t) \\
& \geq \beta k \lambda^{1-1 / k} \frac{1}{R_{2}(t) r_{2}(t)} L_{1} y(t) .
\end{aligned}
$$

Integrating the latter inequality from $t_{2}$ to $t$ and using that $L_{1} y / R_{2}$ as a decreasing function tending to zero, we obtain

$$
\begin{equation*}
z(t)>\beta k \lambda^{1-1 / k} L_{1} y(t), \quad t>t_{2} \tag{26}
\end{equation*}
$$

i.e.,

$$
\left(1-\beta k \lambda^{1-1 / k}\right) L_{1} y(t)>R_{2}(t) L_{2} y(t), \quad t \geq t_{2}>t_{1}
$$

Since $\lambda$ can be arbitrarily large, we can set $\lambda>(1 / k \beta)^{k /(k-1)}$, which contradicts the positivity of $L_{2} y$. The proof is complete.

Corollary 4. Assume $\beta_{*}>0$ and $k_{*}=\infty$. Then, $\mathcal{N}_{2}=\varnothing$.
Proof. Using the fact that $k$ can be arbitrarily large, the proof follows the lines of Corollary 3, and so we omit it.

In what follows, we can assume without loss of generality that $\beta_{*}, k_{*}, \lambda_{*}$ are well defined, and $\beta_{*} \in(0,1), k_{*} \in[1, \infty)$, and $\lambda_{*} \in[1, \infty)$. Now, we will show how the results from Lemma 4 can be improved iteratively.

Lemma 5. Assume $\beta_{*}>0$ and let $y$ be an eventually positive solution of (1) belonging to $\mathcal{N}_{2}$. Then, for any $n \in \mathbb{N}_{0}, \beta_{n}$ and $k_{n}$ defined by (11) and (12), respectively, and for at that is sufficiently large:
( $a_{n}$ ) $\left(1-\beta_{n}\right) L_{1} y>R_{2} L_{2} y$ and $L_{1} y / R_{2}^{1-\beta_{n}}$ decrease;
( $b_{n}$ ) $\lim _{t \rightarrow \infty} L_{1} y(t) / R_{2}^{1-\beta_{n}}(t)=0$;
( $c_{n}$ ) $y>\varepsilon_{n} k_{n}\left(R_{12} / R_{2}\right) L_{1} y$ and $y / R_{12}^{1 /\left(\varepsilon_{n} k_{n}\right)}$ is decreasing for any $\varepsilon_{n} \in(0,1)$.
Proof. Let $y \in \mathcal{N}_{2}$ with $y(\tau(t))>0$ satisfy the conclusion of Lemma 3 for $t \geq t_{1} \geq t_{0}$ and choose fixed but arbitrarily large $\beta \leq \beta_{*}$ and $k \leq k_{*}$, satisfying (9) and (10), respectively, for $t \geq t_{1}$. We will proceed by induction on $n$. For $n=0$, the conclusion follows from Lemma 4 with $\varepsilon_{0}=k / k_{*}$. Next, assume that $\left(\mathrm{a}_{n}\right)-\left(\mathrm{c}_{n}\right)$ hold for $n \geq 1$ for $t \geq t_{n} \geq t_{1}$. We need to show that they each hold for $n+1$.
$\left(\mathrm{a}_{n+1}\right)$ Using ( $\left.\mathrm{c}_{n}\right)$ in (21), we obtain

$$
\begin{aligned}
z^{\prime}(t) & \geq \beta \frac{y(\tau(t))}{r_{2}(t) R_{12}^{1 /\left(\varepsilon_{n} k_{n}\right)}(\tau(t)) R_{12}^{1-1 /\left(\varepsilon_{n} k_{n}\right)}(\tau(t))} \\
& \geq \beta \frac{y(t)}{R_{12}^{1 /\left(\varepsilon_{n} k_{n}\right)}(t)} \frac{1}{r_{2}(t) R^{1-1 /\left(\varepsilon_{n} k_{n}\right)}(\tau(t))} \\
& \geq \beta \varepsilon_{n} k_{n} \frac{R_{12}^{1-k_{n} / \varepsilon_{n}}(t)}{R^{1-k_{n} / \varepsilon_{n}}(\tau(t))} \frac{1}{R_{2}(t) r_{2}(t)} L_{1} y(t) \\
& \geq \beta \varepsilon_{n} k_{n} \lambda^{1-1 /\left(\varepsilon_{n} k_{n}\right)} \frac{1}{R_{2}(t) r_{2}(t)} L_{1} y(t) .
\end{aligned}
$$

Integrating the above inequality from $t_{n}$ to $t$ and using $\left(\mathrm{a}_{n}\right)$ and $\left(\mathrm{b}_{n}\right)$,

$$
\begin{align*}
z(t) & =z\left(t_{n}\right)+\beta \varepsilon_{n} k_{n} \lambda^{1-1 /\left(\varepsilon_{n} k_{n}\right)} \int_{t_{n}}^{t} \frac{L_{1} y(s)}{R_{2}(s) r_{2}(s)} \mathrm{d} s \\
& \geq z\left(t_{n}\right)+\beta \varepsilon_{n} k_{n} \lambda^{1-1 /\left(\varepsilon_{n} k_{n}\right)} \frac{L_{1} y(t)}{R_{2}^{1-\beta_{n}}} \int_{t_{n}}^{t} \frac{1}{R_{2}^{\beta_{n}}(s) r_{2}(s)} \mathrm{d} s \\
& \geq z\left(t_{n}\right)+\frac{\beta \varepsilon_{n} k_{n} \lambda^{1-1 /\left(\varepsilon_{n} k_{n}\right)}}{1-\beta_{n}} \frac{L_{1} y(t)}{R_{2}^{1-\beta_{n}}(t)}\left(R_{2}^{1-\beta_{n}}(t)-R_{2}^{1-\beta_{n}}\left(t_{n}\right)\right)  \tag{27}\\
& >\frac{\beta \varepsilon_{n} k_{n} \lambda^{1-1 /\left(\varepsilon_{n} k_{n}\right)}}{1-\beta_{n}} L_{1} y(t)=\mu \beta_{n+1} L_{1} y(t), \quad t \geq t_{n}^{\prime} \geq t_{n}
\end{align*}
$$

where

$$
\mu:=\frac{\beta}{\beta_{*}} \varepsilon_{n} \frac{\lambda^{1-1 /\left(k_{n} \varepsilon_{n}\right)}}{\lambda_{*}^{1-1 / k_{n}}} \in(0,1)
$$

and

$$
\lim _{\substack{\lambda \rightarrow \lambda_{*} \\ \varepsilon_{n} \rightarrow 1 \\ \beta \rightarrow \beta_{*}}} \mu=1 .
$$

Choose $\mu$ such that

$$
\begin{equation*}
\mu>\frac{1}{1-\beta_{n}+\beta_{n+1}}=\frac{1}{1+\beta_{n}\left(\ell_{n}-1\right)}, \tag{28}
\end{equation*}
$$

where $\ell_{n}$ satisfies (14). Then,

$$
\frac{\mu \beta_{n+1}}{1-\mu \beta_{n+1}}>\frac{\beta_{n+1}}{\left(1+\beta_{n}\left(\ell_{n}-1\right)\right)\left(1-\frac{\ell_{n} \beta_{n}}{\left(1+\beta_{n}\left(\ell_{n}-1\right)\right)}\right)}=\frac{\beta_{n+1}}{1-\beta_{n}}
$$

and there exist two constants $c_{1} \in(0,1)$ and $c_{2}>0$ such that

$$
c_{1} \frac{\mu \beta_{n+1}\left(1-\beta_{n}\right)}{1-\mu \beta_{n+1}}>\beta_{n+1}+c_{2} .
$$

In view of the definition (20) of $z$, we see that

$$
\left(1-\mu \beta_{n+1}\right) L_{1} y(t)>L_{2} y(t) R_{2}(t)
$$

and

$$
\left(\frac{L_{1} y}{R_{2}^{1-\mu \beta_{n+1}}}\right)^{\prime}(t)<0, \quad t \geq t_{n}^{\prime}
$$

Using the above monotonicity in (27), we find that there exists $t_{n}^{\prime \prime} \geq t_{n}^{\prime}$ that is sufficiently large such that

$$
\begin{aligned}
z(t) & =z\left(t_{n}\right)+\beta \varepsilon_{n} k_{n} \lambda^{1-1 /\left(\varepsilon_{n} k_{n}\right)} \int_{t_{n}}^{t} \frac{L_{1} y(s)}{R_{2}(s) r_{2}(s)} \mathrm{d} s \\
& \geq \frac{\beta \varepsilon_{n} k_{n} \lambda^{1-1 /\left(\varepsilon_{n} k_{n}\right)}}{1-\mu \beta_{n+1}} \frac{L_{1} y(t)}{R_{2}^{1-\mu \beta_{n+1}(t)}}\left(R_{2}^{1-\mu \beta_{n+1}}(t)-R_{2}^{1-\mu \beta_{n+1}}\left(t_{n}\right)\right) \\
& \geq \frac{c_{1} \beta \varepsilon_{n} k_{n} \lambda^{1-1 /\left(\varepsilon_{n} k_{n}\right)}}{1-\mu \beta_{n+1}} L_{1} y(t) \\
& =c_{1} \mu \beta_{n+1} \frac{1-\beta_{n}}{1-\mu \beta_{n+1}} L_{1} y(t) \\
& >\left(\beta_{n+1}+c_{2}\right) L_{1} y(t), \quad t \geq t_{n}^{\prime \prime} .
\end{aligned}
$$

Then,

$$
\begin{equation*}
\left(1-\beta_{n+1}-c_{2}\right) L_{1} y(t)>R_{2}(t) L_{2} y(t) \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{L_{1} y}{R_{2}^{1-\beta_{n+1}-c_{2}}}\right)^{\prime}(t)<0 \tag{30}
\end{equation*}
$$

from which the conclusion follows.
$\left(\mathrm{b}_{n+1}\right)$ Clearly, (30) also implies that $L_{1} y / R_{2}^{1-\beta_{n+1}} \rightarrow 0$ as $t \rightarrow \infty$, since otherwise

$$
\begin{equation*}
\frac{L_{1} y(t)}{R_{2}^{1-\beta_{n+1}-c_{2}}(t)}=\frac{L_{1} y(t)}{R_{2}^{1-\beta_{n+1}}(t)} R_{2}^{c_{2}}(t) \rightarrow \infty \quad \text { as } t \rightarrow \infty, \tag{31}
\end{equation*}
$$

which is a contradiction.
$\left(\mathrm{c}_{n+1}\right)$ Using that by $\left(\mathrm{a}_{n+1}\right)$ and $\left(\mathrm{b}_{n+1}\right), L_{1} y / R_{2}^{1-\beta_{n+1}}$ is a decreasing function tending to zero, we have, for any $\varepsilon_{n+1} \in(0,1)$,

$$
\begin{aligned}
y(t) & =y\left(t_{n}^{\prime \prime}\right)+\int_{t_{n}^{\prime \prime}}^{t} \frac{R_{2}^{1-\beta_{n+1}}(s)}{r_{1}(s)} \frac{L_{1} y(s)}{R_{2}^{1-\beta_{n+1}}(s)} \mathrm{d} s \\
& \geq y\left(t_{n}^{\prime \prime}\right)+\frac{L_{1} y(t)}{R_{2}^{1-\beta_{n+1}}(t)} \int_{t_{n}^{\prime \prime}}^{t} \frac{R_{2}^{1-\beta_{n+1}}(s)}{r_{1}(s)} \mathrm{d} s \\
& =y\left(t_{n}^{\prime \prime}\right)+\frac{L_{1} y(t)}{R_{2}^{1-\beta_{n+1}}(t)} \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{n+1}}(s)}{r_{1}(s)} \mathrm{d} s-\frac{L_{1} y(t)}{R_{2}^{1-\beta_{n+1}(t)}} \int_{t_{0}}^{t_{n}^{\prime \prime}} \frac{R_{2}^{1-\beta_{n+1}(s)}}{r_{1}(s)} \mathrm{d} s \\
& >\frac{L_{1} y(t)}{R_{2}^{1-\beta_{n+1}}(t)} \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{n+1}}(s)}{r_{1}(s)} \mathrm{d} s \geq \varepsilon_{n+1} k_{n+1} \frac{R_{12}(t)}{R_{2}(t)} L_{1} y(t), \quad t \geq t_{n+1} \geq t_{n}^{\prime \prime} .
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\frac{y}{R_{12}^{1 / \varepsilon_{n+1} k_{n+1}}}\right)^{\prime}(t) & =\frac{\varepsilon_{n+1} k_{n+1} L_{1} y(t) R_{12}^{1 / \varepsilon_{n+1} k_{n+1}}(t)-y(t) R_{12}^{1 / \varepsilon_{n+1} k_{n+1}-1}(t) R_{2}(t)}{\varepsilon_{n+1} k_{n+1} R_{12}^{2 / \varepsilon_{n+1} k_{n+1}}(t) r_{1}(t)} \\
& =\frac{\varepsilon_{n+1} k_{n+1} L_{1} y(t) R_{12}(t)-y(t) R_{2}(t)}{R_{12}^{1 / \varepsilon_{n+1} k_{n+1}+1}(t) r_{1}(t)}<0 .
\end{aligned}
$$

The proof is complete.
Corollary 5. Assume that $\beta_{i}<1$ for $i=0,1, \ldots, n-1$ and $\beta_{n} \geq 1$. Then, $\mathcal{N}_{2}=\varnothing$.
In view of the above corollary and (13), the sequence $\left\{\beta_{n}\right\}$ defined by (11) is increasing and bounded from the above, i.e., there exists a limit

$$
\lim _{n \rightarrow \infty} \beta_{n}=\beta_{f} \in(0,1)
$$

satisfying the equation

$$
\begin{equation*}
\beta_{f}=\frac{\beta_{*} k_{f} \lambda_{*}^{1-1 / k_{f}}}{1-\beta_{f}} \tag{32}
\end{equation*}
$$

where

$$
k_{f}=\liminf _{t \rightarrow \infty} \frac{R_{2}^{\beta_{f}}(t) \int_{t_{0}}^{t} \frac{R_{2}^{1-\beta_{f}}(s)}{r_{1}(s)} \mathrm{d} s}{R_{12}(t)}
$$

Then, the following crucial result on the nonexistence of $\mathcal{N}_{2}$-type solutions is immediate.
Lemma 6. Assume $\lambda_{*}<\infty$ and (32) does not possess a root on ( 0,1 ). Then, $\mathcal{N}_{2}=\varnothing$.
Corollary 6. Assume $\lambda_{*}<\infty$. If

$$
\begin{equation*}
\beta_{*}>\max \left\{\frac{\beta_{f}\left(1-\beta_{f}\right) \lambda_{*}^{1 / k_{f}-1}}{k_{f}}: 0<\beta_{f}<1\right\} . \tag{33}
\end{equation*}
$$

then $\mathcal{N}_{2}=\varnothing$.

### 3.2. Convergence to Zero of Kneser Solutions

In this section, we state some results ensuring that any Kneser solution converges to zero asymptotically. We start by pointing out the useful fact that

$$
\begin{equation*}
\int_{t_{0}}^{\infty} q(s) \mathrm{d} s<\infty \tag{34}
\end{equation*}
$$

is necessary for the existence of an unbounded nonoscillatory solution. For the reader's convenience, we state its one-line proof.

Lemma 7. Assume

$$
\begin{equation*}
\int_{t_{0}}^{\infty} q(s) \mathrm{d} s=\infty . \tag{35}
\end{equation*}
$$

Then, (1) has property $A$.
Proof. Assume, on the contrary, that $y$ is a nonvanishing, nonoscillatory, positive solution of (1), i.e., $y(t) \geq \xi>0$ for $t \geq t_{1}$. Then, the integration of (1) from $t_{2}$ to $t$ yields

$$
\begin{equation*}
L_{2} y(t)=L_{2} y\left(t_{2}\right)-\int_{t_{2}}^{t} q(s) y(\tau(s)) \mathrm{d} s \leq L_{2} y\left(t_{2}\right)-\xi \int_{t_{2}}^{t} q(s) \mathrm{d} s \rightarrow-\infty \quad \text { as } t \rightarrow \infty \tag{36}
\end{equation*}
$$

which contradicts the positivity of $L_{2} y$.
Hence, we will assume (34). Next, we will distinguish between two cases:

$$
\begin{equation*}
\int_{t_{0}}^{\infty} \frac{1}{r_{2}(u)} \int_{u}^{\infty} q(s) \mathrm{d} s \mathrm{~d} u=\infty \tag{37}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{t_{0}}^{\infty} \frac{1}{r_{2}(u)} \int_{u}^{\infty} q(s) \mathrm{d} s \mathrm{~d} u<\infty . \tag{38}
\end{equation*}
$$

Lemma 8. Assume either (37) or

$$
\begin{equation*}
\int_{t_{0}}^{\infty} \frac{1}{r_{1}(t)} \int_{t}^{\infty} \frac{1}{r_{2}(s)} \int_{s}^{\infty} q(u) \mathrm{d} u \mathrm{~d} s \mathrm{~d} t=\infty \tag{39}
\end{equation*}
$$

If $y$ is a Kneser solution of (1), then $\lim _{t \rightarrow \infty} y(t)=0$.
Proof. Use $y(t) \in \mathcal{N}_{0}$ and choose $t_{1} \geq t_{0}$ such that $y(\tau(t))>0$ on $\left[t_{1}, \infty\right)$. Clearly, there exists a finite number $\xi$ such that $\lim _{t \rightarrow \infty} y(t)=\xi \geq 0$. Assume that $\xi>0$. Then, there exists $t_{2} \geq t_{1}$ such that $y(\tau(t)) \geq \xi$ for $t \geq t_{2}$.

If (37) holds, then by integrating (1) from $t$ to $\infty$, we obtain

$$
L_{2} y(t) \geq \int_{t}^{\infty} q(s) y(\tau(s)) \mathrm{d} s \geq \xi \int_{t}^{\infty} q(s) \mathrm{d} s,
$$

that is,

$$
\begin{equation*}
\left(L_{1} y(t)\right)^{\prime} \geq \frac{\xi}{r_{2}(t)} \int_{t}^{\infty} q(s) \mathrm{d} s \tag{40}
\end{equation*}
$$

Integrating (40) from $t_{2}$ to $t$, we obtain

$$
-L_{1} y(t) \leq-L_{1} y\left(t_{2}\right)-\int_{t_{2}}^{t} \frac{\xi}{r_{2}(u)} \int_{u}^{\infty} q(s) \mathrm{d} s \mathrm{~d} u \rightarrow-\infty \quad \text { as } t \rightarrow \infty
$$

which contradicts the positivity of $-L_{1} y$.

If (39) holds, then integration of (40) from $t$ to $\infty$ gives

$$
-y^{\prime}(t) \geq \frac{\xi}{r_{1}(t)} \int_{t}^{\infty} \frac{1}{r_{2}(u)} \int_{t}^{\infty} q(s) \mathrm{d} s \mathrm{~d} u
$$

and, consequently,

$$
\begin{equation*}
y(t) \leq y\left(t_{2}\right)-\int_{t_{2}}^{t} \frac{\xi}{r_{1}(x)} \int_{x}^{\infty} \frac{1}{r_{2}(u)} \int_{t}^{\infty} q(s) \mathrm{d} s \mathrm{~d} u \mathrm{~d} x \rightarrow \infty \quad \text { as } t \rightarrow \infty \tag{41}
\end{equation*}
$$

which contradicts the positivity of $y$. The proof is complete.
Using the positivity of $\beta_{*}$ which we always require in our results for the nonexistence of $\mathcal{N}_{2}$-type solutions, it is possible to simplify condition (39) or even omit it when $r_{1}$ and $r_{2}$ are of the same type. We will use this knowledge to formulate a single-condition criterion for property A of (1) in Section 3.3.

Lemma 9. Use (38) and assume $r_{1} R_{1} \sim r_{2} R_{2}$ and $\beta_{*}>0$. Then, (39) holds.
Proof. By interchanging the order of integration, we rewrite (39) as follows:

$$
\begin{aligned}
& \int_{t_{0}}^{\infty} \frac{1}{r_{1}(t)} \int_{t}^{\infty} \frac{1}{r_{2}(s)} \int_{s}^{\infty} q(u) \mathrm{d} u \mathrm{~d} s \mathrm{~d} t \\
& =\int_{t_{0}}^{\infty} \frac{1}{r_{1}(s)} \int_{s}^{\infty} q(u)\left(R_{2}(u)-R_{2}(s)\right) \mathrm{d} u \mathrm{~d} s \\
& =\int_{t_{0}}^{\infty} \frac{1}{r_{1}(s)} \int_{s}^{\infty} q(u) R_{2}(u) \mathrm{d} u \mathrm{~d} s-\int_{t_{0}}^{\infty} \frac{R_{2}(s)}{r_{1}(s)} \int_{s}^{\infty} q(u) \mathrm{d} u \mathrm{~d} s \\
& =\int_{t_{0}}^{\infty} q(s) R_{2}(s) R_{1}(s) \mathrm{d} s-\int_{t_{0}}^{\infty} q(s) R_{12}(s) \mathrm{d} s .
\end{aligned}
$$

Using $\lambda$ and $\beta$, satisfying (8) and (9), for $t \geq t_{1} \geq t_{0}$, we obtain

$$
\begin{aligned}
& \int_{t_{0}}^{\infty} \frac{1}{r_{1}(t)} \int_{t}^{\infty} \frac{1}{r_{2}(s)} \int_{s}^{\infty} q(u) \mathrm{d} u \mathrm{~d} s \mathrm{~d} t \\
& \geq \beta \int_{t_{1}}^{\infty} \frac{R_{2}(s) R_{1}(s)-R_{12}(s)}{r_{2}(s) R_{2}(s) R_{12}(\tau(s))} \mathrm{d} s \\
& =\beta \lambda \int_{t_{1}}^{\infty}\left(\frac{R_{1}(s)}{r_{2}(s) R_{12}(s)}-\frac{1}{r_{2}(s) R_{2}(s)}\right) \mathrm{d} s \\
& =\int_{t_{1}}^{\infty} \frac{1}{r_{2}(s) R_{2}(s)}\left(\frac{R_{1}(s) R_{2}(s)}{R_{12}(s)}-1\right) \mathrm{d} s .
\end{aligned}
$$

On the other hand, by using l'Hôspital's rule,

$$
\begin{aligned}
\liminf _{t \rightarrow \infty} \frac{R_{1}(t) R_{2}(t)}{R_{12}(t)} & \geq \liminf _{t \rightarrow \infty} \frac{\frac{1}{r_{1}(t)} R_{2}(t)+\frac{1}{r_{2}(t)} R_{1}(t)}{\frac{R_{2}(t)}{r_{1}(t)}} \\
& =1+\liminf _{t \rightarrow \infty} \frac{r_{1}(t) R_{1}(t)}{r_{2}(t) R_{2}(t)}=1+\ell
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{t_{1}}^{\infty} \frac{1}{r_{1}(t)} \int_{t}^{\infty} \frac{1}{r_{2}(s)} \int_{s}^{\infty} q(u) \mathrm{d} u \mathrm{~d} s \mathrm{~d} t & \geq \int_{t_{1}}^{\infty} \frac{\ell \mathrm{d} s}{r_{2}(s) R_{2}(s)} \\
& =\ell \lim _{t \rightarrow \infty} \ln \left(\frac{R(t)}{R\left(t_{1}\right)}\right) \rightarrow \infty \quad \text { as } t \rightarrow \infty
\end{aligned}
$$

In view of Lemma 8, the conclusion follows directly.

Corollary 7. Let (38) and assume $r_{1} R_{1} \sim r_{2} R_{2}$ and $\beta_{*}>0$. If $y$ is a Kneser solution of (1), then $\lim _{t \rightarrow \infty} y(t)=0$.

### 3.3. Property $A$ of (1)

Combining the results from previous two sections, we are prepared to state the main results of this paper in three cases: for general functions $r_{1}$ and $r_{2}$, for the same-type functions $r_{1}$ and $r_{2}$ satisfying (15), and for the same functions $r_{1}=r_{2}$, respectively.

Theorem 1. Assume $\beta_{*}>0, \lambda_{*}=\infty$, and either (37) or (39) holds. Then, (1) has property $A$.
Theorem 2. Assume $\lambda_{*}<\infty$, (33), and either (37) or (39) holds. Then (1) has property $A$.
Theorem 3. Assume $r_{1} R_{1} \sim r_{2} R_{2}$. If $\beta_{*}>0$ and $\lambda_{*}=\infty$, then (1) has property $A$.
Theorem 4. Assume $r_{1} R_{1} \sim r_{2} R_{2}$. If $\lambda_{*}<\infty$ and (33) hold, then (1) has property $A$.
Theorem 5. Assume $r_{1}=r_{2}$. If $\beta_{*}>0$ and $\lambda_{*}=\infty$, then (1) has property $A$.
Theorem 6. Assume $r_{1}=r_{2}$. If $\lambda_{*}<\infty$ and

$$
\begin{equation*}
\beta_{*}>\max \left\{\frac{\beta_{f}\left(1-\beta_{f}\right)\left(2-\beta_{f}\right) \lambda_{*}^{\beta_{f} / 2}}{2}: 0<\beta_{f}<1\right\} \tag{42}
\end{equation*}
$$

then (1) has property $A$.

## 4. Examples and Discussion

We illustrate the worth of the obtained results on the examples. Firstly and most importantly, we show that that condition (5) is necessary and sufficient for property A of the Euler equation (3).

Example 1. Let us consider the Euler Equation (3). Clearly, (15) holds and from straightforward computation, we see that

$$
\begin{aligned}
\lambda_{*} & =\tau^{\gamma+\alpha-2} \\
\beta_{*} & =\frac{q_{0} \tau^{2-\gamma-\alpha}}{(1-\gamma)^{2}(2-\gamma-\alpha)} \\
k_{f} & =\frac{(2-\gamma-\alpha)}{2-\beta_{f}(1-\gamma)-\gamma-\alpha}
\end{aligned}
$$

Consequently, condition (33), which in view of Theorem 4 ensures that (3) has property $A$, reduces to

$$
\begin{equation*}
q_{0}>\max \left\{\beta_{f}\left(1-\beta_{f}\right)\left(2-\beta_{f}(1-\gamma)-\gamma-\alpha\right)(1-\gamma)^{2} \tau^{-\left(2-\beta_{f}(1-\gamma)-\gamma-\alpha\right)}: 0<\beta_{f}<1\right\} \tag{43}
\end{equation*}
$$

If we set

$$
\mu=2-\beta_{f}(1-\gamma)-\gamma-\alpha
$$

then (43) becomes

$$
\begin{equation*}
q_{0}>\max \{c(\mu): 1-\alpha<\mu<2-\alpha-\gamma\} \tag{44}
\end{equation*}
$$

where $c(\mu)$ is defined by (4). Hence, condition (5) is not only sufficient, but also necessary for the existence of an $\mathcal{N}_{2}$-type solution and so (43) is sharp for (3) to have property $A$.

For example, set $\alpha=\gamma=0$ and $\tau=0.35$. By virtue of (44), we conclude that (5) has property $A$, if

$$
q_{0}>2.1327
$$

which is depicted in Figure 1—see the orange line. We can also observe from Figure 1 (see the green line) that if

$$
q_{0}<\max \{c(\mu): \mu<0\} \simeq 2.944,
$$

then (5) has a couple of Kneser solutions tending to zero asymptotically.


Figure 1. Graph of $c(\mu)$ for $\alpha=\gamma=0$ and $\tau=0.35$.
The remaining open problem stated below in Remark 2 is to prove a general criterion for the nonexistence of Kneser solutions of (1), which would reduce to

$$
q_{0}>\max \{c(\mu): \mu<0\}
$$

when applied to the Euler equation (5).

Next, we consider the situation when $r_{1}$ and $r_{2}$ are not of same type.
Example 2. Consider the third-order delay differential equation

$$
\begin{equation*}
\left(e^{-t} y^{\prime \prime}(t)\right)^{\prime}+q(t) y(\tau t)=0, \quad \tau \in(0,1), \quad t>1 \tag{45}
\end{equation*}
$$

It is easy to verify that

$$
r_{1}(t)=1, \quad r_{2}=e^{-t}, \quad R_{1}(t) \sim t, \quad R_{2}(t) \sim e^{t}, \quad R_{12} \sim e^{t} .
$$

Then,

$$
\lambda_{*}=\liminf _{t \rightarrow \infty} \frac{R_{12}(t)}{R_{12}(\tau t)}=\liminf _{t \rightarrow \infty} e^{(1-\tau) t}=\infty
$$

and

$$
\begin{equation*}
\beta_{*}=\liminf _{t \rightarrow \infty} R_{2}(t) R_{12}(\tau(t)) q(t) r_{2}(t)=\liminf _{t \rightarrow \infty} q(t) e^{\tau t}>0 \tag{46}
\end{equation*}
$$

Clearly, a positive $\beta_{*}$ implies that the integral (37) is divergent, i.e.,

$$
\int_{1}^{\infty} e^{s} \int_{s}^{\infty} q(u) \mathrm{d} u \mathrm{~d} s=\int_{1}^{\infty} q(s)\left(e^{s}-e\right) \mathrm{d} s=\infty .
$$

Hence, if (46) holds, all assumptions of Theorem 1 are satisfied and Equation (45) has property $A$.

Finally, we illustrate the case with non-proportional delay argument.
Example 3. Consider the third-order delay differential equation

$$
\begin{equation*}
y^{\prime \prime \prime}(t)+\frac{1}{t^{2}} y(t-2 \ln t)=0, \quad t>t-2 \ln t>1 \tag{47}
\end{equation*}
$$

It is easy to verify that

$$
\begin{aligned}
r_{1}(t) & =r_{2}(t)=1, \quad R_{1}(t) \sim t, \quad R_{2}(t) \sim t, \quad R_{12} \sim \frac{t^{2}}{2} \\
\lambda_{*} & =\liminf _{t \rightarrow \infty} \frac{R_{12}(t)}{R_{12}(\tau(t))}=\liminf _{t \rightarrow \infty} \frac{t^{2}}{(t-2 \ln t)^{2}}=1
\end{aligned}
$$

and

$$
\beta_{*}=\liminf _{t \rightarrow \infty} R_{2}(t) R_{12}(\tau(t)) q(t) r_{2}(t)=\infty .
$$

Hence, all assumptions of Theorem 6 are satisfied and Equation (47) has property $A$, that is, any nonoscillatory solution tends to zero asymptotically. One such solution is $y(t)=e^{-t}$.

Remark 1. In the paper, we suggested new oscillation criteria for property $A$ of a class of general third-order delay differential equations by employing a novel iterative technique. In a particular case when the functions $r_{i}$ are of the same type, a single condition guarantees property $A$ of (1), see Theorems 3 and 4. We stress that our criteria remove a restrictive condition that $\tau(t)$ is a nondecreasing function, they are also applicable in the ordinary case $\tau(t)=t$ and, most importantly, they are sharp when applied to general third-order delay Euler-type differential equations, see Example 1.

Remark 2. It is well-known, see, e.g. [3], that the delay argument can cause the oscillation of all solutions of (1). However, the problem of obtaining conditions for the nonexistence of Kneser solutions of (1) which would be sharp for the Euler equation (3) is nontrivial and we leave this question open for future research. How to extend the sharp results of the paper to the class of neutral third-order differential equations also remains open at the moment.

Author Contributions: Conceptualization, I.J.; methodology, I.J.; investigation, I.J., G.E.C., S.R.G. and J.D.; writing—original draft preparation, I.J.; writing—review and editing, I.J., G.E.C., S.R.G. and J.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research has been supported by the Slovak Research and Development Agency under contracts No. APVV-19-0590 and No. APVV-18-0373.

Institutional Review Board Statement: Not applicable.
Informed Consent Statement: Not applicable.
Data Availability Statement: Not applicable.
Acknowledgments: The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.
Conflicts of Interest: The authors declare no conflict of interest.

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