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Article

On Effectively Indiscernible Projective Sets and the Leibniz-Mycielski Axiom

Ali Enayat [1,†], Vladimir Kanovei [2,*,†] and Vassily Lyubetsky [2,*,†]

[1] Department of Philosophy, Linguistics, and Theory of Science, University of Gothenburg, 405 30 Gothenburg, Sweden; ali.enayat@gu.se
[2] Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute), 127051 Moscow, Russia
* Correspondence: kanovei@iitp.ru (V.K.); lyubetsk@iitp.ru (V.L.).
† These authors contributed equally to this work.

Abstract: Examples of effectively indiscernible projective sets of real numbers in various models of set theory are presented. We prove that it is true, in Miller and Laver generic extensions of the constructible universe, that there exists a lightface $\varPi^1_2$ equivalence relation on the set of all nonconstructible reals, having exactly two equivalence classes, neither one of which is ordinal definable, and therefore the classes are OD-indiscernible. A similar but somewhat weaker result is obtained for Silver extensions. The other main result is that for any n, starting with 2, the existence of a pair of countable disjoint OD-indiscernible sets, whose associated equivalence relation belongs to lightface $\varPi^1_n$, does not imply the existence of such a pair with the associated relation in Σ^1_n or in a lower class.

Keywords: indiscernible sets; Leibniz-Mycielski axiom; projective hierarchy; generic models; ordinal definability; Miller forcing; Laver forcing; Silver forcing

MSC: 03E35, 03E15

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1. Introduction

Questions related to the definability of mathematical objects have often been in the focus of discussions of the foundations of mathematics. In particular, an early discussion between Hadamard, Borel, Baire, and Lebesgue, published in [1], emphasized a notable divergence of their positions regarding pure existence proofs in mathematics, effectiveness, the axiom of choice, definability, and other foundational issues.

As far as the definability issues are concerned, the principal ideas were first elaborated in precise mathematical terms by Tarski in his seminal papers [2–4], and others (See a comprehensive review by Addison [5]). Yet another view on definability was developed by Tarski in [6]. As logical notions are invariant under one-to-one and onto transformations of the universe of discourse, definable sets turn out to be invariant under automorphisms. This concept of invariance has found applications in various fields of mathematics. A notable recent instance is developed in the book [7] by Alexandru and Ciobanu, which studies the theory of finitely supported sets; such sets are equipped with actions of the group of all permutations of some basic elements called atoms satisfying a finite support requirement. In this theory basic choice principles fail and some paradoxes such as Banach–Tarski are eliminated.

As evidenced by [1], the initial discussion of definability aspects in the early years of twentieth century was largely inspired by the introduction of the axiom of choice **AC**. (We leave aside issues related to the Richard paradox, which was resolved by the Gödel–Tarski truth undefinability theorem in classical mathematics. See [4,8] on the ensuing 'definability of definable' problem by Tarski and its recent solution.) The axiom of choice **AC** states that something exists even if it cannot be effectively defined or constructed. Fraenkel and Mostowski introduced in 1930s the permutation models to prove the independence of **AC** and some other axioms in set theory with atoms. In 1938–1940, Gödel [9] proved that **AC** is consistent with the axioms of **NBG** class theory and **ZFC** set theory. Cohen [10] proved in 1963 the independence of **AC** from the standard axioms of **ZF** set theory, using a version of his *forcing method* derived from the Fraenkel–Mostowski permutation method, see [11–13]. Forcing tools are also connected with invariant sets described in [7], with permutation models, with Ramsey theory [14], etc. Forcing techniques are well described in [15,16].

2. The Leibniz-Mycielski Axiom

This paper is devoted to the investigation of another question in connection with the problem of definability in mathematical foundations that is intimately related to Leibniz's well-known principle of the *identity of indiscernibles* ([17], p. 304). Leibniz's principle states that no two distinct substances exactly resemble each other, thus the principle can be construed as prescribing a logical relationship between objects and properties: *any two distinct objects must differ in at least one property*.

Leibniz's principle suggests the following model-theoretic definition introduced in [18]: a structure M in a first order language $\mathcal{L}$ is *Leibnizian* if M contains no pair of distinct indiscernibles, i.e.,

(1) there are no distinct elements $a \neq b$ in M such that, for every formula $\varphi(x)$ of $\mathcal{L}$ with one free variable x, the following holds: $M \models \big(\varphi(a) \iff \varphi(b)\big)$.

For example, the field $\mathbb{R}$ of real numbers is Leibnizian (since distinct real numbers have distinct Dedekind cuts), but the field $\mathbb{C}$ of complex numbers is not. Indeed the complex numbers i and $-i$ are indiscernible, simply because the conjugation map sending $a + bi$ to $a - bi$ is easily seen to be an automorphism of the field $\mathbb{C}$ of complex numbers. Generally, first order properties of an object in a structure M are preserved by automorphisms of M, so any structure with a nontrivial automorphism (such as $\mathbb{C}$ with conjugation) is not Leibnizian. Similarly, the ordered group of integers is not Leibnizian since f(x) = $-$x is a group automorphism. On the other hand, the ordered set of natural numbers and the hereditarily finite sets are pointwise definable, and hence Leibnizian models.

Further, by cardinality considerations, if the language of M is countable, and M is a Leibnizian structure, then M is, at most, cardinality continuum.So any structure M for a countable language that has cardinality higher than continuum is not Leibnizian. This gives lots of examples of non-Leibnizian models, including rigid ones (those having no automorphisms except for the identity map) since $\langle A\,; <\rangle$ is rigid if $\langle A\,; <\rangle$ is wellordered, so any structure of size greater than continuum that carries a well-ordering as part of its language, is not Leibnizian.

Generally it is well-known that any first order theory which possesses an infinite model also possesses a model that contains distinct indiscernibles; this is an immediate consequence of the venerable Ehrenfeucht–Mostowski theorem ([19], Theorem 3.3.10). Therefore the property of being Leibnizian cannot be guaranteed by any set of sentences of first order logic. However, Mycielski [20] formulated an axiom in the usual first order language of set theory that captures the spirit of Leibniz's principle considered with respect to the whole set theoretic universe $\mathbf{V}$ (as opposed to the case of a particular model $M \in \mathbf{V}$ as in (1) above).

Here we may note that a straightforward reformulation of (1) in the case $M = \mathbf{V}$ as

(2) there are no distinct sets $a \neq b$ in the set universe $\mathbf{V}$, such that, for every $\in$-formula $\varphi(x)$ with one free variable x, we have: $\varphi(a) \iff \varphi(b)$

is mathematically incorrect because modern foundations of mathematics do not allow quantifiers over arbitrary formulas followed by a reference to the truth of these formulas in the set universe. Surprisingly, this problem can be circumvented by allowing arbitrary *ordinals* as parameters. This leads to the following reformulation of (2):

(3) there are no distinct sets $a \neq b$ in $\mathbf{V}$, such that, for every $\in$-formula $\varphi(x, y_1, \ldots, y_m)$ and any ordinals $\gamma_1, \ldots, \gamma_m$, we have: $\varphi(a, \gamma_1, \ldots, \gamma_m) \iff \varphi(b, \gamma_1, \ldots, \gamma_m)$.

At first glance, this does not appear any better than (2). Yet Mycielski [20] uses the methods of ordinal definability [21] to reformulate (3) as follows:

(4) there are no distinct sets $a \neq b$ in $\mathbf{V}$, such that, for every $\in$-formula $\varphi(x, y_1, \ldots, y_m)$ and any ordinals β and $\gamma_1, \ldots, \gamma_m < \beta$ with $a, b \in \mathbf{V}_\beta$, we have:

$$\mathbf{V}_\beta \models (\varphi(a, \gamma_1, \ldots, \gamma_m) \iff \varphi(b, \gamma_1, \ldots, \gamma_m)),$$

where $\mathbf{V}_\beta$ is the β-th level of the von Neumann hierarchy consisting of sets of ordinal-rank less than β.

We may note now that, unlike (3), the sentence (4) is mathematically expressible, since so is the *satisfiability relation* "$\mathbf{V}_\beta \models (\varphi(a, \gamma_1, \ldots, \gamma_m) \iff \varphi(b, \gamma_1, \ldots, \gamma_m))$" (with $a, b, \beta, \gamma_1, \ldots, \gamma_m$, and φ as well, as free variables) by the fact that Tarski's definition of truth in a structure can be implemented in **ZF** for any structure whose universe of discourse forms a set (as opposed to a proper class). On the other hand, sentence (4) is a perfect approximation of (3). Indeed it follows from the *Reflection Principle* (true in **ZF** by e.g., [15], Theorem 12.14), that for any formula $\varphi(\cdot, \cdot)$ it holds that for every ordinal β_0 there is a larger ordinal $\beta > \beta_0$ such that we have

$$\varphi(a, \gamma_1, \ldots, \gamma_m) \iff (\mathbf{V}_\beta \models \varphi(a, \gamma_1, \ldots, \gamma_m))$$

for all ordinals $\gamma_1, \ldots, \gamma_m < \beta$ and all $a \in \mathbf{V}_\beta$.

We extract from (4) a mathematically correct notion of indiscernibility:

(5) sets a, b are OD-*indiscernible* if for every $\in$-formula $\varphi(\cdot, \cdot)$ and any ordinals β and $\gamma_1, \ldots, \gamma_m < \beta$ with $a, b \in \mathbf{V}_\beta$, we have:

$$\mathbf{V}_\beta \models (\varphi(a, \gamma_1, \ldots, \gamma_m) \iff \varphi(b, \gamma_1, \ldots, \gamma_m)),$$

and subsequently the following formulation of the *Leibniz-Mycielski axiom*:

$\mathbf{LM}_{\mathrm{OD}}$: any two OD-indiscernible sets a, b are equal to each other,

which is obviously equivalent to (4). We recall that OD means ordinal-definable and i.e., ordinals are allowed as parameters in defining formulas. In other words, a set x is *ordinal definable*, briefly OD, if there is such a formula $\varphi(x)$ with ordinals as parameters that x is the only set satisfying $\varphi(x)$. See [21] or ([15], Chapter 13) on ordinal definability, where the mathematical correctness of this notion is established in sufficient detail, on the basis of a technical definition that utilizes the same idea as (5), that is, a set x is *ordinal definable* if there is an ordinal β, and a formula $\varphi(x)$ with ordinals smaller than β as parameters, such that that x is the only set in $\mathbf{V}_\beta$ satisfying $\mathbf{V}_\beta \models \varphi(x)$.

For the purpose of bookkeeping, we give the original formulation of **LM** proposed by Mycielski under the name A_2'; the **LM** terminology was suggested in [22].

LM: if $a \neq b$ then there is an ordinal β such that a, b belong to $\mathbf{V}_\beta$ and

$$\mathsf{Th}(\mathbf{V}_\beta, \in, a) \neq \mathsf{Th}(\mathbf{V}_\beta, \in, b),$$

where $\mathsf{Th}(\mathbf{V}_\beta, \in, a)$ is the *first order theory* of the structure $(\mathbf{V}_\beta, \in, a)$, and a is viewed as a distinguished constant. In other words, by contraposition, **LM** postulates that any two sets a, b, indiscernible in all sets of the form $\mathbf{V}_\beta$ to which they both belong, are equal to each other.

As shown in Enayat [22], **LM** is equivalent to the existence of a parameter-free definable global form of the Kinna-Wagner Selection Principle, and more specifically, to the existence of a parameter-free definable injection of the set universe $\mathbf{V}$ into the class of subsets of ordinals. The equivalence of **LM** and $\mathbf{LM}_{\mathrm{OD}}$ is proved in ([18] Lemma 2.1.1), see also ([23], Theorem 3.7). A number of consistency and independence results related to **LM** is established in [22]. In particular, assuming the consistency of **ZF** itself, the negation of **LM** is consistent with **ZFC**, and the negation of the axiom of choice **AC** is consistent with $\mathbf{ZF} + \mathbf{LM}$.

3. The Results

Papers [18,22–24] present a number of results on the status of the Leibniz-Mycielski axiom in different models of set theory. Their general meaning is that **LM**, or equivalently $\mathbf{LM}_{\mathrm{OD}}$, holds in the Gödel universe $\mathbf{L}$ of constructible sets and similar models that have a definable well-ordering of the set universe, but fails in some models containing generic pairs of reals $x, y \in \omega^\omega$, since their *real* $\mathbf{L}$*-degrees* $[x]_{\mathbf{L}} = \{z \in \omega^\omega : \mathbf{L}[x] = \mathbf{L}[z]\}$ and $[y]_{\mathbf{L}}$ are OD-indiscernible.

The problem of construction of models in which $\mathbf{LM}_{\mathrm{OD}}$ fails, but there is no such generic pairs, is also discussed in [22]. This problem has recently been solved in [25], where it is established that $\mathbf{LM}_{\mathrm{OD}}$ fails in generic extensions $\mathbf{L}[a]$ of $\mathbf{L}$ by means of a Sacks-generic real a (in fact an unpublished theorem of Solovay) or by an $\mathbb{E}_0$-large generic real a, but in both cases $\mathbf{L}[a]$ contains no generic pairs because of the *minimality* property of the Sacks and $\mathbb{E}_0$-large generic extensions (just two constructibility degrees: the trivial and the maximal). In this paper, we extend this result to the case of the Miller, Laver, and Silver forcing notions, known to have the same minimality property.

The following two theorems give a partial answer to Problem 11.1 in [25].

Theorem 1. *It is true in Miller and Laver extensions of* $\mathbf{L}$ *that* $\mathbf{LM}_{\mathrm{OD}}$ *fails, and, more specifically, there is a* Π^1_2*, hence* OD*, equivalence relation* Q *on* $\omega^\omega \smallsetminus \mathbf{L}$ *that has exactly two equivalence classes, say* M, N*, and neither of those is an* OD *set, hence the classes are* OD*-indiscernible.*

As usual, we make use of slanted lightface greek letters for *effective* projective classes (*Kleene classes*) Σ^1_n, Π^1_n, Δ^1_n. Effective projective hierarchy has been accepted as a universal tool of estimation of complexity of reals and sets of reals, and generally points and pointsets in recursively presented Polish spaces like $\mathbb{R}$, the Baire space ω^ω, or the Cantor space 2^ω, in the parameter-free case, see e.g., Moschovakis ([26], 3E). Thus if $\mathscr{Y}$ is a recursively presented Polish space and $n \geq 1$ then a set $X \subseteq \mathscr{Y}$ is Σ^1_n if there is a semirecursive (i.e., a recursive union of basic nbhds) set $P \subseteq \mathscr{Y} \times (\omega^\omega)^n$ such that

$$X = \{y \in \mathscr{Y} : \exists\, x_1 \in \omega^\omega\, \forall\, x_2 \in \omega^\omega \ldots \exists\,(\forall\,) x_n \in \omega^\omega\, (\langle y, x_1, x_2, \ldots, x_n \rangle \in P)\},$$

Π^1_n is obtained similarly but with $\forall$ as the leftmost quantifier, and $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$. We have $\Sigma^1_n \not\subseteq \Pi^1_n \not\subseteq \Sigma^1_n$ and $\Sigma^1_n \cup \Pi^1_n \subsetneqq \Delta^1_{n+1}$ for all n, so that the classes properly increase in any recursively presented Polish space. See also ([15], Section 25). or ([27], Section 1).

If real parameters are admitted then the extended "boldface" classes are denoted by $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, $\boldsymbol{\Delta}^1_n$, with bolface upright greek letters; they coincide with the classes $\mathbf{A}_n$, $\mathbf{CA}_n$, $\mathbf{B}_n$ of classical projective hierarchy, see Kechris [28].

We may note that Π^1_2 in Theorem 1 is the least possible complexity of such examples, see an explanation in [25], at the end of Section 1.

Theorem 2. *It is true in Silver extensions of* $\mathbf{L}$ *that* $\mathbf{LM}_{\mathrm{OD}}$ *fails, and, more specifically, there is an* OD *equivalence relation* Q *on the set* **Sil** *of all reals Silver-generic over* $\mathbf{L}$*, that has exactly two equivalence classes, and neither of those is an* OD *set, so the classes are* OD*-indiscernible.*

Definition 1. *Let a strong counterexample to* $\mathbf{LM}_{\mathrm{OD}}$ *be any* OD *(unordered) pair* $\{M, N\}$ *of disjoint non-*OD *sets* $M, N \subseteq \omega^\omega$*. If the associated equivalence relation*

$$x \mathrel{\mathsf{E}} y, \text{ iff } x, y \in M \text{ or } x, y \in N$$

on the set $M \cup N$ *belongs to a class* Π^1_n*, resp.,* Σ^1_n*, then we say that* $\{M, N\}$ *is a strong* Π^1_n*-counterexample, resp., a strong* Σ^1_n*-counterexample.*

In this terminology, Theorem 1 says that Miller and Laver extensions of $\mathbf{L}$ contain strong Π^1_2-counterexamples. Such a result can be viewed as the best possible in matters of definability, because strong Σ^1_2-counterexamples do not exist by the Σ^1_2 basis theorem. (The latter asserts that any non-empty Σ^1_2 set contains a Δ^1_2 element, ([26], 4E.5).) Descriptive classification of the counterexample given by Theorem 2 (and basically of the set **Sil** itself) is not known yet. Another model containing a strong Π^1_2-counterexample to $\mathbf{LM}_{\mathrm{OD}}$ is defined in [29] on the basis of the forcing product technique. Some other strong counterexamples are discussed in [23,30].

The next theorem is the other main result of this paper. It continues the research line of recent papers [31,32] aimed at constructing generic models in which some property of reals or pointsets is fulfilled at a given projective level.

Theorem 3. *Let* $\mathtt{n} \geq 3$*. There is a generic extension* $\mathbf{L}[a]$ *of* $\mathbf{L}$ *by a real* $a \in 2^\omega$*, in which:*

(i) *there exists a strong* $\Pi^1_{\mathtt{n}}$ *counterexample to* $\mathbf{LM}_{\mathrm{OD}}$*, which consists of two* **countable** *disjoint sets in* 2^ω*;*
(ii) *every countable* $\Sigma^1_{\mathtt{n}}$ *set in* 2^ω *contains only* OD *elements, and hence there is no strong* $\Sigma^1_{\mathtt{n}}$ *counterexample to* $\mathbf{LM}_{\mathrm{OD}}$ *which consists of two countable sets.*

The interest in countable sets in this theorem is motivated, in particular, by a well-known problem by S. D. Friedman which focuses on the elements of definable countable sets (See Problem 10 in ([33], p. 209) or Problem 8 in ([34], Section 9)).

The counterexamples defined in the proof of Theorem 3(i) differ from those given in the proofs of Theorems 1 and 2. The latter involve a transfinite construction of an increasing sequence of Borel countable equivalence relations in $\mathbf{L}$, somewhat different for the three generic extensions considered, whereas the former (those for (i) of Theorem 3) are defined by a natural partition of the $\mathbb{E}_0$-class $[a]_{\mathbb{E}_0}$ of a into two $\mathbb{E}_0^{\mathrm{even}}$-classes. and this depends on very special properties of the forcings involved in the proof of Theorem 3. See Example 1 in the next section on the equivalence relations $\mathbb{E}_0$ and $\mathbb{E}_0^{\mathrm{even}}$.

4. Preliminaries for the Proof of Theorems 1 and 2

The equivalence relation for the proof of Theorem 1 will be defined by means of a fairly complicated transfinite construction. The following are several key definitions involved in the construction.

Definition 2. *An equivalence relation* E *is countable, if every* E*-equivalence class is a finite or countable set. A dyadic pair of equivalence relations, or just a dyadic pair, is any pair* $\langle \mathsf{B}, \mathsf{E} \rangle$ *of countable Borel equivalence relations on the Baire space* ω^ω*, such that every* E*-class is the union of exactly two* B*-classes.*

A dyadic pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ *extends a dyadic pair* $\langle \mathsf{B}, \mathsf{E} \rangle$*, in symbol* $\langle \mathsf{B}, \mathsf{E} \rangle \preccurlyeq \langle \mathsf{B}', \mathsf{E}' \rangle$*, if* $\mathsf{B} \subseteq \mathsf{B}'$*,* $\mathsf{E} \subseteq \mathsf{E}'$*, and for every* $x, y \in \omega^\omega$*, if* $x \mathrel{\mathsf{E}} y$ *but* $x \not\mathrel{\mathsf{B}} y$ *then* $x \not\mathrel{\mathsf{B}'} y$*.*

Thus the extension of dyadic pairs comes down to merging equivalence classes, by necessity in countable groups (since only countable equivalence relations are considered), such that two B-subclasses of the same E-class do not merge to the same B'-class when extending a dyadic pair $\langle \mathsf{B}, \mathsf{E} \rangle$ to some $\langle \mathsf{B}', \mathsf{E}' \rangle$ since by definition if $x \mathrel{\mathsf{E}} y$ but $x \not\mathrel{\mathsf{B}} y$ (the same E-class but different B-class) then $x \not\mathrel{\mathsf{B}'} y$.

Example 1. *To define an important example of a dyadic pair, if* $x, y \in \omega^\omega$ *then put* $\Delta(x, y) = \sum_{k<\omega} |x(k) - y(k)|$*, so* $\Delta(x, y)$ *either an integer or* $\pm\infty$*.*

Define $x \mathrel{\mathbb{E}_0} y$ *if* $\Delta(x, y)$ *is finite; this is equivalent to the set* $\{k : x(k) \neq y(k)\}$ *being finite, of course. Define* $x \mathrel{\mathbb{E}_0^{\text{even}}} y$ *iff* $\Delta(x, y)$ *is a finite even number (of any sign). Then* $\mathbb{E}_0^{\text{even}}$*,* $\mathbb{E}_0$ *are countable Borel equivalence relations, and each* $\mathbb{E}_0$*-class contains exactly two* $\mathbb{E}_0^{\text{even}}$*-classes, that is,* $\langle \mathbb{E}_0^{\text{even}}, \mathbb{E}_0 \rangle$ *is a dyadic pair.*

Another simple example of a dyadic pair involves the following extension of $\mathbb{E}_0$*:* $x \mathrel{\mathbb{E}_0^*} y$ *if* $\Delta(x, y)$ *is either finite or cofinite. Then* $\langle \mathbb{E}_0, \mathbb{E}_0^* \rangle$ *is a dyadic pair.*

Definition 3. *Let* $X \subseteq \omega^\omega$ *and* $f : X \to \omega^\omega$ *be any map. A dyadic pair* $\langle \mathsf{B}, \mathsf{E} \rangle$*:*

– corrals f *if* $f(x) \in [x]_{\mathsf{E}}$ *for all* $x \in X$*;*

– negatively corrals f *if* $f(x) \in [x]_{\mathsf{E}} \smallsetminus [x]_{\mathsf{B}}$ *for all* $x \in X$*.*

It is easy to see that if a dyadic pair $\langle \mathsf{B}, \mathsf{E} \rangle$ corrals (including negatively) some f then any dyadic pair that extends it necessarily corrals f (resp., corrals negatively).

To prove Theorem 1, we'll define a $\preccurlyeq$-increasing sequence $\{\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle\}_{\alpha<\omega_1}$ of dyadic pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$, whose terms eventually (i.e., a certain index) corral any Borel map, and for many maps, the corralling will be negative. Then the union $\mathsf{B} = \bigcup_\alpha \mathsf{B}_\alpha$ will be the desired equivalence relation.

5. The Miller Case: Superperfect Sets

We consider non-empty trees in $\omega^{<\omega}$ with no endpoints.

Recall that a tree $T \subseteq \omega^{<\omega}$ is *superperfect*, if for any *string* (a finite sequence) $s \in T$ there is a string $t \in T$ such that $s \subset t$ and the set $\{j < \omega : t^\frown j \in T\}$ is infinite, that is, t is *an infinite branching node* of T. Here and in the following we use the subset symbol $\subset$ to denote the relation of *proper extension* of finite sequences (strings), and also the the relation of extension between a finite and an infinite sequence. Accordingly, a set $X \subseteq \omega^\omega$ is *superperfect*, if it is closed, non-compact, and has no (non-empty) compact portions. (A *portion* of a set $X \subseteq \omega^\omega$ is any set of the form $X{\restriction}_u = \{x \in X : u \subset x\}$, where $u \in \omega^{<\omega}$.) If a set $X \subseteq \omega^\omega$ is superperfect then it is not σ-compact, and its non-empty portions are obviously superperfect and not σ-compact as well.

Lemma 1. *A tree* $T \subseteq \omega^{<\omega}$ *is superperfect iff the corresponding set* $X = [T] = \{x \in \omega^\omega : \forall n\, (x \restriction n \in T)\}$ *is superperfect.*

Proof. If $s \in T$ but there is no infinite branching node $t \in T$ above s then the portion $X{\restriction}_s$ in X is compact. Conversely, if $X{\restriction}_s$ is compact then T has no infinite branching nodes above s. $\square$

Lemma 2. *Any two superperfect sets $X, Y \subseteq \omega^\omega$ are homeomorphic.*

Proof. By Lemma 1 we can choose superperfect trees $S, T \subseteq \omega^{<\omega}$ satisfying $X = [T]$ and $Y = [S]$. We first consider the case when $S = \omega^{<\omega}$, i.e., $Y = \omega^\omega$. Define a map $h_T : \omega^{<\omega} \to T$ as follows. Put $h_T(\Lambda) = \Lambda$ where Λ is the empty string. Assume that a string $h_T(u) = s \in T$ is defined. As T is superperfect, there is a number $m > \mathtt{lh}\, s$ (the length of s) such that the set $T_{sm} = \{t \in T : s \subset t \wedge \mathtt{lh}\, t = m\}$ is infinite. Let $m = m(u)$ be the least such, and let $T_{sm} = \{t_k : k < \omega\}$, without repetitions. Put $h_T(u^\frown k) = t_k$ for all k. This ends the inductive step of the construction of h_T.

It follows by construction that $u \subset v$ is equivalent to $h_T(u) \subset h_T(v)$. Thus if $a \in \omega^\omega$, then $H_T(a) = \bigcup_n h_T(a \restriction n) \in X = [T]$. The mapping $H_T : \omega^\omega \to X$ is continuous and 1-1. To prove that $\mathtt{ran}\, H_T = X$ let $b \in X$. If $b \notin \mathtt{ran}\, H_T$ then there is a largest string $s = h_T(u)$ such that $s \subset b$. Define $m = m(u)$ and $T_{sm} \subseteq T$ as above. As $s \subset b$, then there is a unique string $t \in T_{sm}$ with $t \subset b$. However $t = h_T(u^\frown k)$ for some k by construction. This contradicts the choice of s. We conclude that $\mathtt{ran}\, H_T = X$, and this completes the case $Y = \omega^\omega$.

In the general case, a required homeomorphism $H_{ST} = H_{YX} : Y \xrightarrow{\text{onto}} X$ can be defined to be equal to the superposition $H_{ST} = H_T \circ {H_S}^{-1}$. $\square$

Mappings of the form $H_{ST} = H_{YX} : Y \xrightarrow{\text{onto}} X$ as in the proof of Lemma 2 will be called *canonical homeomorphisms* of superperfect sets.

Lemma 3. *If $H : X \xrightarrow{\text{onto}} Y$ is a homeomorphism of superperfect sets $X, Y \subseteq \omega^\omega$, and $X' \subseteq X$ is superperfect, then its H-image $H[X'] = \{H(x) : x \in X'\}$ is superperfect as well.*

Proof. Make use of the fact that homeomorphisms preserve compactness. $\square$

Recall that Miller forcing consists of all superperfect trees in $\omega^{<\omega}$, or equivalently, all superperfect sets $X \subseteq \omega^\omega$, ordered by inclusion (smaller conditions are stronger).

Proposition 1 (See e.g., [35])**.** *Miller forcing adjoins a real $a \in \omega^\omega$ of minimal degree, preserves $\aleph_1$, and has continuous reading of names: if a real $a \in \omega^\omega$ is Miller-generic over $\mathbf{L}$ and $b \in \mathbf{L}[a] \cap \omega^\omega$, then $b = f(a)$ for some continuous map $f : \omega^\omega \to \omega^\omega$ coded in $\mathbf{L}$.*

6. The Miller Case: Canonization

Canonization theorems are known in many areas of mathematics. They have the following typical form: each structure of a certain type contains a large substructure from some canonical list. The proof of Theorem 1 uses the next theorem of this type.

Theorem 4 (total canonization for Miller forcing)**.** *Let E be a Borel equivalence relation on a superperfect set $X \subseteq \omega^\omega$. There is a superperfect set $Y \subseteq X$ on which E coincides:*

- *either* (I) *with the total relation TOT that makes all elements equivalent;*
- *or* (II) *the equality, i.e., Y is a partial E-transversal.*

If in addition E is a countable relation (Definition 2)*, then* (I) *is impossible.*

Thus, Borel equivalence relations have two canonical types on superperfect sets in the Baire space ω^ω, namely, the total relation TOT and the equality.

Proof. According to 6.16 in [14], any superperfect set X either is covered by a countable number of E-equivalence classes and a countable number of compact sets, or there is a superperfect subset $Y \subseteq X$ of E-inequivalent elements. In the second case, we immediately have (II). So let's consider the first case. Since superperfect sets are not σ-compact, there is such an E-equivalence class C that $C \cap X$ is not covered by a σ-compact set. Then, by the Hurewicz theorem ([28], 7.10), there exists a superperfect set $Y \subseteq C \cap X$, which gives (I). $\square$

Corollary 1. *If $X \subseteq \omega^\omega$ is superperfect, and $f : X \to \omega^\omega$ is a Borel map, then there is a superperfect set $Y \subseteq X$ such that $f \restriction Y$ is a bijection or a constant.*

Proof. Apply Theorem 4 for the equivalence relation $x \mathrel{\mathsf{E}} y$ iff $f(x) = f(y)$ on X. $\square$

Corollary 2. *If $X \subseteq \omega^\omega$ is superperfect, and $A \subseteq X$ is a Borel set, then there is a superperfect set $Y \subseteq X$ such that either $Y \subseteq A$ or $Y \subseteq X \smallsetminus A$.*

Proof. Define a Borel equivalence relation E on X by: $x \mathrel{\mathsf{E}} y$ iff either $x, y \in A$ or $x, y \in X \smallsetminus A$. Apply Theorem 4. $\square$

7. The Miller Case: Corralling Borel Maps

Coming back to the notion of corralling, we now prove two key lemmas which aim to extend a given dyadic pair so that the extension corrals a given map.

Lemma 4. *Let $\langle \mathsf{B}, \mathsf{E} \rangle$ be a dyadic pair, $X \subseteq \omega^\omega$ a superperfect set, and $f : X \to \omega^\omega$ a Borel $1-1$ map. There exist a superperfect set $Y \subseteq X$ and a dyadic pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ which extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and corrals $f \restriction Y$.*

Proof. The sets $X' = \{x \in X : x \mathrel{\mathsf{E}} f(x)\}$ and $X'' = \{x \in X : x \not\mathrel{\mathsf{E}} f(x)\}$ are Borel, therefore Corollary 2 yields a superperfect set X_0 such that $X_0 \subseteq X'$ or $X_0 \subseteq X''$. If $X_0 \subseteq X'$ then $\langle \mathsf{B}, \mathsf{E} \rangle$ itself corrals $f \restriction X_0$. Therefore assume that $X_0 \subseteq X''$, i.e., $x \not\mathrel{\mathsf{E}} f(x)$ for all $x \in X_0$. Theorem 4 gives a superperfect set $X_1 \subseteq X_0$ such that the relations E, B coincide with the equality on X_1.

Define an auxiliary equivalence relation $\widehat{\mathsf{E}}$ on X_1 so that $x \mathrel{\widehat{\mathsf{E}}} y$ iff $f(x) \mathrel{\mathsf{E}} f(y)$, and define $\widehat{\mathsf{B}}$ similarly. Consider the $\subseteq$-least equivalence relation F on X, containing all pairs of the form $\langle x, y \rangle$, where $f(x) \mathrel{\mathsf{E}} y$. The relations $\widehat{\mathsf{E}}, \widehat{\mathsf{B}}, \mathsf{F}$ are countable Borel equivalence relations on X_1. (That F is Borel is implied by ([28], Lemma 18.12), because all quantifiers in the definition of F have countable domains, which in turn follows from the countability of E and bijectivity of f.) Theorem 4 implies that there is a superperfect set $Y \subseteq X_1$ such that the relations $\widehat{\mathsf{E}}, \widehat{\mathsf{B}}, \mathsf{F}$ coincide with the equality on Y, and so do E, B by the above. In particular, by the choice of X_0, if $x, y \in Y$ then $x \not\mathrel{\mathsf{E}} f(y)$.

Now define equivalence relations B', E' as follows.

If $x \in \omega^\omega$ and the E-class $[x]_{\mathsf{E}}$ does **not** intersect *the critical domain* $\Delta = Y \cup \{f(x) : x \in Y\}$, then put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}}$ and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}}$. However, in the domain Δ some classes will be merged. Namely, let $x \in Y$. Then the class $[x]_{\mathsf{E}}$ has to merge with the class $[f(x)]_{\mathsf{E}}$. Therefore we put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup [f(x)]_{\mathsf{E}}$ and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup [f(x)]_{\mathsf{B}}$, and define the other B'-class inside $[x]_{\mathsf{E}'}$ to be equal to $[x]_{\mathsf{E}'} \smallsetminus [x]_{\mathsf{B}'}$. Then E', B' are Borel countable equivalence relations, $\langle \mathsf{B}', \mathsf{E}' \rangle$ is a dyadic pair extending $\langle \mathsf{B}, \mathsf{E} \rangle$, and corralling $f \restriction Y$ as $f(x) \in [x]_{\mathsf{B}'}$ for all $x \in Y$ by construction. $\square$

Lemma 5. *Let $\langle \mathsf{B}, \mathsf{E} \rangle$ be a dyadic pair, and $X \subseteq \omega^\omega$ be a superperfect set. There exist superperfect sets $Y, W \subseteq X$ and a dyadic pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ which extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and negatively corrals the canonical homeomorphism $g = H_{YW}$.*

Proof. By Theorem 4 there is a superperfect set $X_0 \subseteq X$ such that E is the equality on X_0. Let $Y, W \subseteq X_0$ be *disjoint* superperfect subsets. Then $[Y]_{\mathsf{E}}$, $[W]_{\mathsf{E}}$ are disjoint too by the choice of X_0. Lemma 2 gives a canonical homeomorphism $g = H_{YW} : Y \xrightarrow{\text{onto}} W$. Define equivalence relations E', B' as follows.

If $x \in \omega^\omega$ and the E-class $[x]_{\mathsf{E}}$ does **not** intersect *the critical domain* X_0 then put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}}$ and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}}$. However, if $x \in Y$ then, to merge $[x]_{\mathsf{E}}$ with $[g(x)]_{\mathsf{E}}$, we define $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup [g(x)]_{\mathsf{E}}$. Further define the B'-class

$$[x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup ([g(x)]_{\mathsf{E}} \smallsetminus [g(x)]_{\mathsf{B}}),$$

and take $([x]_{\mathsf{E}} \smallsetminus [x]_{\mathsf{B}}) \cup [g(x)]_{\mathsf{B}}$ as the other B'-class inside $[x]_{\mathsf{E}'}$. Then E', B' are countable Borel equivalence relations, and $\langle \mathsf{B}', \mathsf{E}' \rangle$ is a dyadic pair that extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and negatively corrals g. $\square$

8. The Miller Case: Increasing Sequence of Dyadic Pairs

The next theorem asserts the existence of a transfinite increasing sequence of dyadic pairs with strong corralling and definability properties.

Theorem 5 (in L). *There is a $\preccurlyeq$-increasing sequence of dyadic pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$, $\alpha < \omega_1$, beginning with $\mathsf{E}_0 = \mathbb{E}_0$ and $\mathsf{B}_0 = \mathbb{E}_0^{\text{even}}$ as in Example 1 and satisfying the following:*

(i) *if $X, X_1 \subseteq 2^\omega$ are superperfect sets, $f : X \to X_1$ is Borel and $1-1$, then there is an ordinal $\alpha < \omega_1$ and a superperfect set $X' \subseteq X$ such that $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ corrals $f \restriction X'$;*

(ii) *if $X \subseteq \omega^{<\omega}$ is a superperfect set then there is an ordinal $\alpha < \omega_1$ and superperfect sets $Y, W \subseteq X$ such that $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ negatively corrals $g = H_{YW}$;*

(iii) *the sequence of pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ is Δ^1_2 in the codes, in the sense that there exist Δ^1_2 sequences of codes for Borel sets B_α and E_α.*

Proof of Theorem 5—Part 1. (Part 2 will be completed in Section 9.) We argue in L. We define a sequence required by transfinite induction based on Lemmas 4 and 5. This is accomplished as follows.

1°. Fix an enumeration $\{\widehat{X}_\alpha, \widehat{f}_\alpha\}_{\alpha<\omega_1}$ of all pairs $\langle X, f \rangle$, where $f : \omega^\omega \to \omega^\omega$ is continuous, $X \subseteq \omega^\omega$ is a superperfect set, and $f \restriction X$ is $1-1$.

The beginning of induction. Take $\mathsf{E}_0 = \mathbb{E}_0$ and $\mathsf{B}_0 = \mathbb{E}_0^{\text{even}}$ as in Example 1.

Successor step. Assume that $\alpha < \omega_1$ and $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle = \langle \mathsf{B}, \mathsf{E} \rangle$ is already defined. By Lemma 4 there exist a superperfect set $X' \subseteq \widehat{X}_\alpha$ and a dyadic pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ extending $\langle \mathsf{B}, \mathsf{E} \rangle$ and corralling $\widehat{f}_\alpha \restriction X'$. By Lemma 5 there exist superperfect sets $Y, W \subseteq X'$ and a dyadic pair $\langle \mathsf{B}'', \mathsf{E}'' \rangle$ extending $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ and corralling the canonical homeomorphism $g = H_{YW}$ negatively. Let $Y_\alpha = Y$, $W_\alpha = W$, $\langle \mathsf{B}_{\alpha+1}, \mathsf{E}_{\alpha+1} \rangle = \langle \mathsf{B}'', \mathsf{E}'' \rangle$. The relations between the αth and $(\alpha+1)$th steps are formulated as follows in terms of 1°:

2°. (1) $Y, W \subseteq \widehat{X}_\alpha$ are superperfect sets,

(2) $\langle \mathsf{B}'', \mathsf{E}'' \rangle$ is a dyadic pair extending $\langle \mathsf{B}, \mathsf{E} \rangle$,

(3) $\langle \mathsf{B}'', \mathsf{E}'' \rangle$ corrals $\widehat{f}_\alpha \restriction Y$,

(4) $\langle \mathsf{B}'', \mathsf{E}'' \rangle$ corrals $g = H_{YW}$ negatively.

Limit step. If $\lambda < \omega_1$ is limit then put $\mathsf{E}_\lambda = \bigcup_{\alpha<\lambda} \mathsf{E}_\alpha$ and $\mathsf{B}_\lambda = \bigcup_{\alpha<\lambda} \mathsf{B}_\alpha$.

Remark 1. *Whatever way we choose Y, W, $\langle \mathsf{B}'', \mathsf{E}'' \rangle$ in accordance with 2° at all successor construction steps, the resulting sequence meets the requirements of (i) and (ii). To also satisfy (iii), we'll make the construction more precise in the next Section.*

This ends Part 1 of the proof of Theorem 5. $\square$

9. The Sequence of Dyadic Pairs: Definability

Proof of Theorem 5—Part 2. In continuation of the proof of Theorem 5, we are going to recall some definitions and results concerning the encoding of ordinals and Borel sets and effective descriptive set theory. **We continue to argue in L in the course of the proof.**

3°. If $x \in \omega^\omega$ and $Y \subseteq \omega^\omega$ is a countable $\Sigma^1_1(x)$ set (i.e., a Σ^1_1-definable set with x as a parameter), then $Y \subseteq \Delta^1_1(x)$. See e.g., ([27], 2.10.5).

4°. If $\Phi(x, y, \dots)$ is a Π^1_1 formula then $\exists\, y \in \Delta^1_1(x)\, \Phi(x, y, \dots)$ is transformable to Π^1_1 form. See e.g., ([26], 4d.3) or ([27], 2.8.6).

5°. Fix a recursive enumeration $\mathbb{Q} = \{r_k : k < \omega\}$ of the rationals. If $w \in \omega^\omega$ then let $Q_w = \{r_k : w(k) = 0\}$. Put $\mathbf{WO} = \{x \in \omega^\omega : Q_x \text{ is wellordered}\}$ (Π^1_1 set of *codes of countable ordinals*). If $w \in \mathbf{WO}$ then $|w| < \omega_1$ is the order type of Q_w.

6°. There is a Π^1_1 set $\mathscr{E} \subseteq \omega^\omega$ of *codes* for Borel sets in $\omega^\omega \times \omega^\omega$. If $\varepsilon \in \mathscr{E}$ then a Borel set $\boldsymbol{E}_\varepsilon \subseteq \omega^\omega \times \omega^\omega$ coded by ε is defined, and there exist ternary Σ^1_1 relations $\boldsymbol{R}$, $\boldsymbol{R}'$ on ω^ω such that if $\varepsilon \in \mathscr{E}$ and $x, y \in \omega^\omega$ then $\langle x, y\rangle \in \boldsymbol{E}_\varepsilon \iff \boldsymbol{R}(\varepsilon, x, y) \iff \neg\,\boldsymbol{R}'(\varepsilon, x, y)$. See e.g., ([36], Section 1D).

7°. $\mathcal{T} \subseteq \mathscr{P}(\omega^{<\omega})$ is the set of all trees $T \subseteq \omega^{<\omega}$. If $T \in \mathcal{T}$ then $[T] = \{x \in \omega^\omega : \forall\, n\, (x \restriction n \in T)\}$ is the corresponding closed set of all *paths* trough T, and $\overline{[T]} = \omega^\omega \smallsetminus \overline{T}$ is its open complement.

8°. A *code of continuous function* $\omega^\omega \to \omega^\omega$ will be any "matrix" $\tau = \{T^n_k\}_{k,n<\omega}$ of trees $T^n_k \in \mathcal{T}$, satisfying the following: if $n < \omega$ then the open sets $\overline{[T^n_k]}$, $k < \omega$, are pairwise disjoint and their union is ω^ω. Let $\mathscr{F}$ be the set of all such codes (a Π^1_1 set in $\mathcal{T}^{\omega\times\omega}$). If $\tau = \{T^n_k\}_{k,n<\omega} \in \mathscr{F}$ then a continuous $\mathbb{f}_\tau : \omega^\omega \to \omega^\omega$ is defined by $f(x)(n) = k$ iff $x \in \overline{[T^n_k]}$.

Lemma 6. *The following sets and relations belong to* Π^1_1*:*

(i) $\mathscr{E}^{\mathtt{cnt}} = \{\varepsilon \in \mathscr{E} : \boldsymbol{E}_\varepsilon \text{ is a countable equivalence relation on } \omega^\omega\}$;

(ii) *the set* $\{\langle \beta, \varepsilon\rangle \in \mathscr{E} \times \mathscr{E} : \boldsymbol{E}_\beta \subseteq \boldsymbol{E}_\varepsilon\}$;

(iii) *the set* $\mathscr{E}^{\mathrm{DP}} = \{\langle \beta, \varepsilon\rangle : \beta, \varepsilon \in \mathscr{E}^{\mathtt{cnt}} \wedge \langle \boldsymbol{E}_\beta, \boldsymbol{E}_\varepsilon\rangle \text{ is a dyadic pair}\}$ *and the relation of extension of coded dyadic pairs as in Definition 2*;

(iv) $\mathbf{SPT} = \{T \in \mathcal{T} : [T] \text{ is a superperfect tree in } \omega^{<\omega}\}$;

(v) *the set* $\{\langle T, \tau\rangle : T \in \mathbf{SPT} \wedge \tau \in \mathscr{F} \wedge \mathbb{f}_\tau \restriction [T] \text{ is a bijection}\}$;

(vi) *the set*

$$\{\langle \beta, \varepsilon, \tau, T\rangle : T \in \mathbf{SPT} \wedge \tau \in \mathscr{F} \wedge \langle \beta, \varepsilon\rangle \in \mathscr{E}^{\mathrm{DP}} \wedge \langle \boldsymbol{E}_\beta, \boldsymbol{E}_\varepsilon\rangle \text{ corrals } \mathbb{f}_\tau \restriction [T]\},$$

and the same for negative corralling.

Proof. (i) Let $\varepsilon \in \mathscr{E}$. Then $\boldsymbol{E}_\varepsilon$ is an equivalence relation if and only if:

$$\forall\, x\, \boldsymbol{R}'(\varepsilon, x, x) \wedge \forall\, x, y\, \big(\boldsymbol{R}(\varepsilon, x, y) \implies \neg\,\boldsymbol{R}'(\varepsilon, y, x)\big) \wedge$$
$$\wedge\, \forall\, x, y, z \big(\boldsymbol{R}(\varepsilon, x, y) \wedge \boldsymbol{R}(\varepsilon, y, z) \implies \neg\,\boldsymbol{R}'(\varepsilon, x, z)\big),$$

which is Π^1_1. Further by 3° $\boldsymbol{E}_\varepsilon$ is a **countable** equivalence relation iff

$$\forall\, x, y\, (\boldsymbol{R}(\varepsilon, x, y) \implies y \in \Delta^1_1(\varepsilon, x)),$$

or equivalently, $\forall\, x, y\, (\boldsymbol{R}(\varepsilon, x, y) \implies \exists\, y' \in \Delta^1_1\,(\varepsilon, x)\, (y' = y))$. This is Π^1_1 by 4°.

(iii) If $\beta, \varepsilon \in \mathscr{E}^{\mathtt{cnt}}$ and $\boldsymbol{E}_\beta \subseteq \boldsymbol{E}_\varepsilon$ then $\langle \boldsymbol{E}_\beta, \boldsymbol{E}_\varepsilon\rangle$ is a dyadic pair iff first, within any triple of $\boldsymbol{E}_\varepsilon$-equivalent reals there is a pair of $\boldsymbol{E}_\beta$-equivalent ones, and second, it holds $\forall\, x\, \exists\, y, y' \in \Delta^1_1(\varepsilon, x)\, (y\; \boldsymbol{E}_\varepsilon\; y' \wedge \neg\, y\; \boldsymbol{E}_\beta\; y')$. This belongs to Π^1_1 by 4°.

(ii), (iv), (v), (vi) is verified by similar arguments.

This ends the proof of Lemma 6. □

Definition 4 (in $\mathbf{L}$)**.** *Let $\mathscr{X}$ be the set of all pairs $\langle T, \tau\rangle \in \mathbf{SPT} \times \mathscr{F}$ such that $\mathbb{f}_\tau \restriction [T]$ is a bijection. If $\alpha < \omega_1$ then let $\langle T_\alpha, \tau_\alpha\rangle$ be the αth element of $\mathscr{X}$ in the sense of Gödel's wellordering $<_{\mathbf{L}}$. We put $\widehat{X}_\alpha = [T_\alpha]$, $\widehat{f}_\alpha = \mathbb{f}_{\tau_\alpha}$. The sequence $\{\widehat{X}_\alpha, \widehat{f}_\alpha\}_{\alpha<\omega_1}$ then satisfies 1° of Section 8.*

Now let $\Phi(\langle\alpha, \beta, \varepsilon\rangle, \langle\beta', \varepsilon', T\rangle)$ be the formula:

(∗) $\alpha < \omega_1$; the pairs $\langle\beta, \varepsilon\rangle$, $\langle\beta', \varepsilon'\rangle$ belong to $\mathscr{E}^{\mathrm{DP}}$; $T \in \mathbf{SPT}$; and the pairs $\langle \mathsf{B}, \mathsf{E}\rangle = \langle \boldsymbol{E}_\beta, \boldsymbol{E}_\varepsilon\rangle$, $\langle \mathsf{B}'', \mathsf{E}''\rangle = \langle \boldsymbol{E}_{\beta'}, \boldsymbol{E}_{\varepsilon'}\rangle$ and the set $Y = [T]$ satisfy condition 2° of Section 8 with $\widehat{X}_\alpha$ and $\widehat{f}_\alpha$ as in Definition 4.

If $\alpha < \omega_1$ and $\langle\beta, \varepsilon\rangle \in \mathscr{E}^{\mathrm{DP}}$ then there exists a triple $\langle\beta', \varepsilon', T\rangle$ satisfying the relation $\Phi(\langle\alpha, \beta, \varepsilon\rangle, \langle\beta', \varepsilon', T\rangle)$. (The successor step in Section 8.) Let

$$\pi_\alpha(\beta, \varepsilon) = \langle\beta'_\alpha(\beta, \varepsilon), \varepsilon'_\alpha(\beta, \varepsilon), T_\alpha(\beta, \varepsilon)\rangle$$

denote the $<_{\mathbf{L}}$-least of such triples. This allows us to define a sequence $\{\langle\beta_\alpha, \varepsilon_\alpha\rangle\}_{\alpha<\omega_1}$ of pairs $\langle\beta_\alpha, \varepsilon_\alpha\rangle \in \mathscr{E}^{\mathrm{DP}}$ by transfinite induction as follows.

As $\beta_0, \varepsilon_0 \in \mathscr{E}^{\mathtt{cnt}}$ we take any pair of computable codes for equivalence relations $\mathbb{E}_0^{\mathrm{even}}$ and $\mathbb{E}_0$, so that $\langle\beta_0, \varepsilon_0\rangle \in \mathscr{E}^{\mathrm{DP}}$. On successor steps, if $\langle\beta_\alpha, \varepsilon_\alpha\rangle \in \mathscr{E}^{\mathrm{DP}}$ is defined then let $\beta_{\alpha+1} = \beta'_\alpha(\beta, \varepsilon)$ and $\varepsilon_{\alpha+1} = \varepsilon'_\alpha(\beta, \varepsilon)$ via (∗). On limit steps $\lambda < \omega_1$, if $\langle\beta_\alpha, \varepsilon_\alpha\rangle \in \mathscr{E}^{\mathrm{DP}}$ is defined for all $\alpha < \lambda$ then let $\langle\beta_\lambda, \varepsilon_\lambda\rangle$ be the $<_{\mathbf{L}}$-pair of codes in $\mathscr{E}^{\mathtt{cnt}}$ satisfying $\boldsymbol{E}_{\beta_\lambda} = \bigcup_{\alpha<\lambda} \boldsymbol{E}_{\beta_\alpha}$ and $\boldsymbol{E}_{\varepsilon_\lambda} = \bigcup_{\alpha<\lambda} \boldsymbol{E}_{\varepsilon_\alpha}$.

It follows from Remark 1 that the sequence of pairs $\langle \boldsymbol{E}_{\beta_\alpha}, \boldsymbol{E}_{\varepsilon_\alpha}\rangle$, $\alpha < \omega_1$, defined this way, satisfies (i), (ii) of Theorem 5. Let's check that (iii) is satisfied as well. This is the content of the next lemma.

Lemma 7 (= (iii) of Theorem 5)**.** *The set $\{\langle w, \beta_{|w|}, \varepsilon_{|w|}\rangle : w \in \mathbf{WO}\}$ is Δ^1_2.*

Proof. We still argue in $\mathbf{L}$. We observe that $\varPi^1_1$-formulas, involved in the definitions of sets and relations considered by Lemma 6, are absolute for transitive models of $\mathbf{ZFC}^-$ ($\mathbf{ZFC}$ without the Power Set axiom and with AC in the form of the wellorderability principle) by the Mostowski absoluteness theorem [15, Theorem 25.4]. Gödel's wellordering $<_{\mathbf{L}}$ is absolute as well for models of $\mathbf{ZFC}^- + (\mathbf{V} = \mathbf{L})$. This implies the absoluteness, in the same sense, of the mappings $\alpha, \beta, \varepsilon \longmapsto \pi_\alpha(\beta, \varepsilon)$ and $\alpha \longmapsto \langle\beta_\alpha, \varepsilon_\alpha\rangle$. Let $\Psi(\mathfrak{M}, \alpha, \varepsilon, \beta)$ be the formula:

$\mathfrak{M}$ is a countable transitive model of $\mathbf{ZFC}^- + (\mathbf{V} = \mathbf{L})$, $\alpha \in \mathbf{Ord}$, and $\alpha, \varepsilon, \beta \in \mathfrak{M}$.

Then

$$\begin{aligned}\langle\beta, \varepsilon\rangle = \langle\beta_\alpha, \varepsilon_\alpha\rangle \quad &\Longleftrightarrow \quad \exists\, \mathfrak{M}\, (\Psi(\mathfrak{M}, \alpha, \varepsilon, \beta) \wedge \mathfrak{M} \models \langle\beta, \varepsilon\rangle = \langle\beta_\alpha, \varepsilon_\alpha\rangle) \\ &\Longleftrightarrow \quad \forall\, \mathfrak{M}\, (\Psi(\mathfrak{M}, \alpha, \varepsilon, \beta) \Longrightarrow \mathfrak{M} \models \langle\beta, \varepsilon\rangle = \langle\beta_\alpha, \varepsilon_\alpha\rangle)\end{aligned}$$

because of the absoluteness mentioned. Here the quantifiers over countable transitive models can be eliminated in terms of a standard coding of such models by reals, see e.g., ([36], Section 2B) The related set of codes is $\varPi^1_1$. (We have to express the wellorderability of the inner ordinals.) Hence the class of the given set is Σ^1_2 by the first equivalence, and $\varPi^1_2$ for the second one. This completes Lemma 7. □

The proof of Theorem 5 is accomplished. □

10. The Miller Case: Last Stage

Proof of Theorem 1, the Miller case. To prove Theorem 1 for Miller extensions, we fix, in $\mathbf{L}$, a $\preccurlyeq$-increasing sequence of dyadic pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$, $\alpha < \omega_1$, which satisfies (i), (ii), (iii) of Theorem 5.

We argue in a Miller extension $\mathbf{L}[a_0]$**,** where $a_0 \in \omega^\omega$ is a Miller-generic real over $\mathbf{L}$. Define $\mathsf{B} = \bigcup_{\alpha<\omega_1} \mathsf{B}_\alpha$; thus $x \mathrel{\mathsf{B}} y$ iff $x \mathrel{\mathsf{B}_\alpha} y$ for some $\alpha < \omega_1$. (The Borel sets B_α, E_α are formally defined in $\mathbf{L}$, but we identify them with their *extensions*—Borel sets in $\mathbf{L}[a_0]$ with the same codes.) Define $\mathsf{E}=\bigcup_{\alpha<\omega_1}\mathsf{E}_\alpha$ similarly. Consider the domain $U=\omega^\omega\smallsetminus\mathbf{L}$ of all new reals in $\mathbf{L}[a_0]$. Then $a_0\in U$ and all reals in U have the same $\mathbf{L}$-degree because Miller reals are minimal, see Proposition 1.

Lemma 8. *It is true in* $\mathbf{L}[a_0]$ *that*:

(i) E, B *are equivalence relations and* B *is a sub-relation of* E;

(ii) B *is* Σ^1_2;

(iii) *all reals* $x,y\in U$ *are* E*-equivalent*;

(iv) *there exist exactly two* B*-classes of reals* $x\in U$*—let them be* M,N;

(v) *the sets* M,N *are not* OD.

Proof. (i) To see that E is an equivalence relation, let $a,b,c\in U$ and we have $a\ \mathsf{E}\ b$ and $a\ \mathsf{E}\ c$. Then by construction $a\ \mathsf{E}_\alpha\ b$ and $a\ \mathsf{E}_\alpha\ c$ hold for some $\alpha<\omega_1$. However to be an equivalence relation is absolute by Shoenfield [15, Theorem 25.20]. We conclude that $b\ \mathsf{B}_\alpha\ c$, as required.

(ii) follows from Theorem 5(iii).

(iii) Let $b\in U$; we will prove that $a_0\ \mathsf{E}\ b$. Proposition 1 gives a continuous function $f:\omega^\omega\to\omega^\omega$ coded in $\mathbf{L}$ and such that $b=f(a_0)$. We know by Corollary 1 that any superperfect $X\subseteq\omega^\omega$ contains a superperfect subset $Y\subseteq X$, on which f is 1–1 or a constant. Therefore by the genericity there is a superperfect set $Y\subseteq\omega^\omega$ coded in $\mathbf{L}$ and such that $a_0\in Y$ and $f\restriction Y$ is 1–1 or a constant. If f is a constant, $f(x)=z_0\in\omega^\omega$ for all $x\in Y$, then $f(a_0)=b=z_0\in\mathbf{L}$, which contradicts the choice of $b\notin\mathbf{L}$. Thus $f\restriction Y$ is a bijection. By the genericity of a_0 and Theorem 5(i) there is a superperfect set $Z\subseteq\omega^\omega$ coded in $\mathbf{L}$, such that $a_0\in Z$ and E_α corrals $f\restriction Z$ for some α. In particular, $\langle a_0,b\rangle\in\mathsf{E}_\alpha$, hence $a_0\ \mathsf{E}\ b$, as required.

(iv) Let $a,b\in U$; prove that the three reals $a_0,a,b\in U$ cannot be pairwise B-inequivalent. We have $a_0\ \mathsf{E}\ a\ \mathsf{E}\ b$ by (iii), hence there is an ordinal $\alpha<\omega_1$ such that $a_0\ \mathsf{E}_\alpha\ a\ \mathsf{E}_\alpha\ b$. However "to have exactly two B_α-classes in every E_α-class" is absolute by Shoenfield. Therefore a_0,a,b cannot be B_α-inequivalent, as required.

(v) Suppose to the contrary that M,N are OD. Let $a_0\in M$. (The case $a_0\in N$ is similar.) Then M is forced over $\mathbf{L}$, i.e.,, there is a superperfect set $Z\subseteq\omega^\omega$ such that (*) $a_0\in Z$, and all Miller reals over $\mathbf{L}$ which belong to Z in $\mathbf{L}[a_0]$ are pairwise B-equivalent. By Theorem 5(ii), there are $\alpha<\omega_1$ and superperfect sets $Y,W\subseteq Z$ coded in $\mathbf{L}$, such that $a_0\in Y$ and E_α corrals $g=H_{YW}$ negatively. Then $c=g(a_0)$ satisfies $a_0\ \mathsf{E}_\alpha\ c$ but $\neg\,(a_0\ \mathsf{B}_\alpha\ c)$, and hence $b\ \not\mathsf{B}\ c$. However, $a_0,c\in Z$, and c is Miller-generic along with a_0 by Lemma 3. This contradicts (*). □

Thus it holds in the Miller generic model $\mathbf{L}[a_0]$ that B is a Σ^1_2 equivalence relation on ω^ω, the nonconstructible domain $U=\omega^\omega\smallsetminus\mathbf{L}$ (a Π^1_2 set) is equal to the union of two (non-empty) B-classes, and these classes are non-OD. Now to prove Theorem 1 it remains to check that $\mathsf{Q}=\mathsf{B}\restriction U$ is a Π^1_2 relation. If $x\in\omega^\omega$ then let $x^-\in\omega^\omega$ be defined by $x^-(0)=x(0)+1$ and $x^-(k)=x(k)$ for $k\geq 1$. Then $x\ \mathbb{E}_0\ x^-$ but $x\ \not\mathsf{B}_0\ x^-$, since B_0 is $\mathbb{E}_0^{\text{even}}$. This implies $x\ \not\mathsf{B}_\alpha\ x^-$ for all α, hence $x\ \not\mathsf{B}\ x^-$. Thus if $x,y\in U$ then $x\ \mathsf{Q}\ y$ is equivalent to $x\ \not\mathsf{B}\ y^-$. This implies the required result. □

11. The Laver Case in Theorem 1

Proof of Theorem 1, the Laver case. The Laver case in Theorem 1 does not differ from the Miller case because Laver forcing admits the same total canonization theorem for countable Borel equivalence relations (even for those classifiable by countable structures) as Theorem 4 provides for Miller forcing— see ([14], Theorem 6.53). We shall not elaborate on this case. □

12. The Silver Case: Canonization and Corralling

Proof of Theorem 2, the Silver case. The Silver case in Theorem 2 requires a bit different organization of the arguments. Recall that by ([14], Section 8.2) a *Silver cube* is any set $X \subseteq 2^\omega$ of the form $X = [p] = \{x \in 2^\omega : p \subset x\}$, where $p : \mathtt{dom}\, p \to 2 = \{0,1\}$ and $\mathtt{dom}\, p \subseteq \omega$ is a coinfinite set.

Silver forcing consists of all Silver cubes $X \subseteq 2^\omega$, ordered by inclusion.

The basic canonization reference is the following theorem ([14], Theorem 8.6).

Theorem 6. *If* E *is an equivalence relation on a Silver cube* $X \subseteq 2^\omega$ *classifiable by countable structures* (this includes the case of countable Borel equivalences), *then there is a Silver subcube* $Y \subseteq X$ *on which either* E *equals to the total equivalence relation* TOT *or* $\mathsf{E} \subseteq \mathbb{E}_0$. □

The "or" clause here is admittedly weaker than the one in Theorem 4. This is why we have to significantly change the flow of arguments.

Corollary 3 (of Theorem 6)**.** *If* $X \subseteq 2^\omega$ *is a Silver cube and* E *a Borel countable equivalence relation on* 2^ω *then there is a Silver cube* $Y \subseteq X$ *such that* $\mathsf{E} \subseteq \mathbb{E}_0$ *on* Y.

If in addition $\mathbb{E}_0 \subseteq \mathsf{E}$ *on* X *then* E *will be equal to* $\mathbb{E}_0$ *on such an* Y.

Corollary 4 (similar to Corollary 2)**.** *If* $X \subseteq \omega^\omega$ *is a Silver cube, and* $A \subseteq X$ *is a Borel set, then there is a Silver cube* $Y \subseteq X$ *such that either* $Y \subseteq A$ *or* $Y \subseteq X \smallsetminus A$.

Definition 5. *Let* $X \subseteq 2^\omega$. *A map* $f : X \to 2^\omega$ *is* X*-regular if for any Silver cube* $Y \subseteq X$ *and any countable Borel equivalence relation* E *with* $\mathbb{E}_0 \subseteq \mathsf{E}$ *there is a Silver cube* $Z \subseteq Y$ *such that* E *is equal to* $\mathbb{E}_0$ *on the set* $f[Z] = \{f(z) : z \in Z\}$.

Now we prove two canonization lemmas for Silver cubes, similar but not really identic to the results in Section 7.

Lemma 9. *Let* $X \subseteq 2^\omega$ *be a Silver cube,* $f : X \to 2^\omega$ *be Borel* 1–1 *and* X*-regular map, and* $\langle \mathsf{B}, \mathsf{E} \rangle$ *be a dyadic pair. Then there exists a Silver cube* $Y \subseteq X$ *and a dyadic pair* $\langle \mathsf{B}', \mathsf{E}' \rangle$ *which extends* $\langle \mathsf{B}, \mathsf{E} \rangle$ *and corrals* $f \restriction Y$.

Proof. As in the proof of Lemma 4, we w.l.o.g. assume that $x \not\mathrel{\mathsf{E}} f(x)$ for all $x \in X$. Define countable Borel equivalence relations $\widehat{\mathsf{E}}$, $\widehat{\mathsf{B}}$, F on X as in the proof of Lemma 4. By Corollary 3 and the regularity of f there is a Silver cube $Y \subseteq X$ such that the relations $\widehat{\mathsf{B}}$, $\widehat{\mathsf{E}}$, F are subrelations of $\mathbb{E}_0$ on Y and $\widehat{\mathsf{B}}$, $\widehat{\mathsf{E}}$ coinside on Y.

Now define equivalence relations B', E' as follows.

If $z \in 2^\omega$ and the E-class $[z]_{\mathsf{E}}$ does **not** intersect the set $\Delta = Y \cup \{f(x) : x \in Y\}$, then put $[z]_{\mathsf{E}'} = [z]_{\mathsf{E}}$ and $[z]_{\mathsf{B}'} = [z]_{\mathsf{B}}$. Next suppose that $x \in Y$. Then the class $[x]_{\mathsf{E}}$ has to merge with $[f(x)]_{\mathsf{E}}$. Therefore we put

$$[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup \bigcup_{x' \in Y,\, x' \mathbb{E}_0 x} [f(x')]_{\mathsf{E}} \quad \text{and} \quad [x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup \bigcup_{x' \in Y,\, x' \mathbb{E}_0 x} [f(x')]_{\mathsf{B}}\,, \tag{4}$$

and naturally define the other B'-class inside $[x]_{\mathsf{E}'}$ to be equal to $[x]_{\mathsf{E}'} \smallsetminus [x]_{\mathsf{B}'}$.

To see that E' is an equivalence relation, it suffices to prove that if $x, y \in Y$ then either $x \mathbin{\mathbb{E}_0} y$— and then obviously $[x]_{\mathsf{E}'} = [y]_{\mathsf{E}'}$, or else $[x]_{\mathsf{E}'} \cap [y]_{\mathsf{E}'} = \varnothing$. Assume that $a \in [x]_{\mathsf{E}'} \cap [y]_{\mathsf{E}'}$. By (4), there exist $x', y' \in Y$ such that $x' \mathbin{\mathbb{E}_0} x$, $y' \mathbin{\mathbb{E}_0} y$, and

$$a \in \big([x']_{\mathsf{E}} \cup [f(x')]_{\mathsf{E}}\big) \cap \big([y']_{\mathsf{E}} \cup [f(y')]_{\mathsf{E}}\big)\,.$$

- If now $a \in [x']_{\mathsf{E}} \cap [y']_{\mathsf{E}}$ then $x' \mathbin{\mathsf{E}} y'$, hence $x' \mathbin{\mathbb{E}_0} y'$ and $x \mathbin{\mathbb{E}_0} y$.
- If $a \in [x']_{\mathsf{E}} \cap [f(y')]_{\mathsf{E}}$ then $x' \mathbin{\mathsf{E}} f(y')$, hence $x' \mathbin{\mathsf{F}} y'$ and $x' \mathbin{\mathbb{E}_0} y'$, $x \mathbin{\mathbb{E}_0} y$.

- If $a \in [f(x')]_{\mathsf{E}} \cap [f(y')]_{\mathsf{E}}$ then $f(x') \mathrel{\mathsf{E}} f(y')$, hence $x' \mathrel{\widehat{\mathsf{E}}} y'$ and $x' \mathrel{\mathbb{E}_0} y'$, $x \mathrel{\mathbb{E}_0} y$.

  Therefore E' is an equivalence relation.

  That B' is an equivalence relation is verified by the same arguments.

  That B', E' are *countable* Borel equivalence relations is easy.

  That $\mathsf{B} \subseteq \mathsf{B}' \subseteq \mathsf{E}'$ and $\mathsf{E} \subseteq \mathsf{E}'$ hold by construction, as well as $x \mathrel{\mathsf{E}'} f(x)$ for $x \in Y$.

  It remains to check that if $a \mathrel{\mathsf{B}'} b$ and $a \mathrel{\mathsf{E}} b$ then $a \mathrel{\mathsf{B}} b$. By (4), there exist $x', x'' \in Y$

such that $a \in [x']_{\mathsf{B}} \cup [f(x')]_{\mathsf{B}}$ and $b \in [x'']_{\mathsf{B}} \cup [f(x'')]_{\mathsf{B}}$. We also know that $a \mathrel{\mathsf{E}} b$.

- If now $a \in [x']_{\mathsf{B}}$ and $b \in [x'']_{\mathsf{B}}$ then immediately $x' \mathrel{\mathsf{E}} x''$, hence $x' \mathrel{\mathsf{B}} x''$ as we have $\mathsf{E} = \mathsf{B}$ on Y, and we conclude that $a \mathrel{\mathsf{B}} b$.
- If $a \in [x']_{\mathsf{B}}$ but $b \in [f(x'')]_{\mathsf{B}}$, then $x' \mathrel{\mathsf{E}} f(x'')$, $x' \mathrel{\mathsf{F}} x''$, and further $x' \mathrel{\mathsf{E}} x''$ (as $x', x'' \in Y$), so finally $x'' \mathrel{\mathsf{E}} f(x'')$, which cannot be by the w.l.o.g. assumption at the beginning of the proof.
- Finally let $a \in [f(x')]_{\mathsf{B}}$ and $b \in [f(x'')]_{\mathsf{B}}$, so that $f(x') \mathrel{\mathsf{E}} f(x'')$. However, E coincides with B on $f[Y]$. It follows that $f(x') \mathrel{\mathsf{B}} f(x'')$, and hence once again $a \mathrel{\mathsf{B}} b$.

This ends the proof of Lemma 9. $\square$

Let $X = [p]$ and $Y = [q]$ be Silver cubes defined as above, p, q being partial functions $\omega \to 2$ with coinfinite domains. To define a *canonical homeomorphism* $h = S_{pq} = S_{XY} : X \xrightarrow{\text{onto}} Y$, let $\omega \smallsetminus \mathtt{dom}\, p = \{k_j^p : j < \omega\}$ and $\omega \smallsetminus \mathtt{dom}\, q = \{k_j^q : j < \omega\}$ in the order of increase. Now let $x \in X$. Define $y = S_{XY}(x) \in Y$ so that $y(k) = q(k)$ in case $k \in \mathtt{dom}\, q$, and $y(k_j^q) = x(k_j^p)$ for all $j < \omega$.

Lemma 10. *Let $\langle \mathsf{B}, \mathsf{E} \rangle$ be a dyadic pair with $\mathbb{E}_0 \subseteq \mathsf{B}$, and $X \subseteq 2^\omega$ be a Silver cube. There exist Silver cubes $Y, W \subseteq X$ and a dyadic pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ which extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and negatively corrals the canonical homeomorphism $g = S_{YW}$.*

Proof. By Theorem 6 there is a Silver cube $X_0 \subseteq X$ on which B and E are equal to $\mathbb{E}_0$. One easily defines disjoint Silver cubes $Y, W \subseteq X_0$ such that $[Y]_{\mathsf{E}}$, $[W]_{\mathsf{E}}$ are disjoint too. Define equivalence relations E', B' as follows (similar to the proof of Lemma 5).

If $x \in 2^\omega$ and the E-class $[x]_{\mathsf{E}}$ does **not** intersect *the critical domain* X_0 then put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}}$ and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}}$. However, if $x \in Y$ then, to merge $[x]_{\mathsf{E}}$ with $[g(x)]_{\mathsf{E}}$, we define $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup [g(x)]_{\mathsf{E}}$. Further define the B'-class

$$[x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup ([g(x)]_{\mathsf{E}} \smallsetminus [g(x)]_{\mathsf{B}}),$$

and take $([x]_{\mathsf{E}} \smallsetminus [x]_{\mathsf{B}}) \cup [g(x)]_{\mathsf{B}}$ as the other B'-class inside $[x]_{\mathsf{E}'}$. Then E', B' are countable Borel equivalence relations, and $\langle \mathsf{B}', \mathsf{E}' \rangle$ is a dyadic pair that extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and negatively corrals g. $\square$

13. The Silver Case: Last Stage

Arguing in L, Theorem 5 takes the following form for Silver cubes:

Theorem 7 (in L). *There is a $\preccurlyeq$-increasing sequence of dyadic pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$, $\alpha < \omega_1$, beginning with $\mathsf{B}_0 = \mathbb{E}_0$ and $\mathsf{E}_0 = \mathbb{E}_0^*$ as in Example 1, and satisfying the following:*

(I) *if $X \subseteq \omega^\omega$ is a Silver cube and $f : X \to \omega^\omega$ is Borel, 1–1, and X-regular, then there is an ordinal $\alpha < \omega_1$ and a Silver cube $X' \subseteq X$ such that $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ corrals $f \restriction X'$;*

(II) *if $X \subseteq \omega^\omega$ is a Silver cube then there is an ordinal $\alpha < \omega_1$ and Silver cubes $Y, W \subseteq X$ such that $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ negatively corrals $g = S_{YW}$;*

(III) *the sequence of pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ is Δ^1_4 in the codes, in the sense that there exist Δ^1_4 sequences of codes for Borel sets B_α and E_α.*

Proof. The proof is based on the results of Section 12, and is pretty analogous to the proof of Theorem 5, so we skip it altogether. The only notable moment is the class Δ^1_4 in (III) instead of Δ^1_2—this is because that the notion of regularity as in Definition 5 is Π^1_3 in the codes. □

Proof of Theorem 2. Now to prove Theorem 2, we fix, in $\mathbf{L}$, a $\preccurlyeq$-increasing sequence of dyadic pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$, $\alpha < \omega_1$, which satisfies (I), (II), (III) of Theorem 7.

We argue in a Silver extension $\mathbf{L}[a_0]$, where $a_0 \in \omega^\omega$ is a Silver-generic real over $\mathbf{L}$. Define $\mathsf{B} = \bigcup_{\alpha<\omega_1} \mathsf{B}_\alpha$; thus $x \mathrel{\mathsf{B}} y$ iff $x \mathrel{\mathsf{B}_\alpha} y$ for some $\alpha < \omega_1$. Define $\mathsf{E} = \bigcup_{\alpha<\omega_1} \mathsf{E}_\alpha$ similarly. Consider the domain $\mathbf{Sil}$ of all reals in $\mathbf{L}[a_0] \cap 2^\omega$ Silver-generic over $\mathbf{L}$. Then $a_0 \in \mathbf{Sil}$ and all reals in $\mathbf{Sil}$ have the same $\mathbf{L}$-degree because Silver reals are minimal. Then $\mathbf{Sil}$ is OD, but we don't know whether it is a projective set in any Σ^1_n.

Lemma 11. *It is true in $\mathbf{L}[a_0]$ that:*

(i) E, B *are equivalence relations and* B *is a sub-relation of* E;
(ii) *the relation* B *is* Σ^1_4;
(iii) *all reals* $x, y \in \mathbf{Sil}$ *are* E*-equivalent*;
(iv) *there exist exactly two* B*-classes of reals* $x \in \mathbf{Sil}$*—let them be* M, N;
(v) *the sets* M, N *are not* OD.

Proof. Similar to Lemma 8, but with one extra issue in the proof of (iii).

(i) similar to (i) of Lemma 8.

(ii) follows from Theorem 7(III).

(iii) Let $b \in \mathbf{Sil}$; prove that $a_0 \mathrel{\mathsf{E}} b$. Silver forcing has continuous reading of names, and hence there is a continuous function $f : 2^\omega \to 2^\omega$ coded in $\mathbf{L}$ and satisfying $b = f(a_0)$. Arguing as in the proof of Lemma 8(iii), we check that there is a Silver cube $X \subseteq 2^\omega$ coded in $\mathbf{L}$ and such that $a_0 \in X$, $f \restriction X$ is 1–1, and X Silver-forces that $f(a_0)$ is Silver-generic too, so that (*) if $a \in X$ is Silver-generic then so is $f(a)$.

We assert that f is X-regular (in $\mathbf{L}$). Indeed let E be a countable Borel equivalence relation coded in $\mathbf{L}$ and such that $\mathbb{E}_0 \subseteq \mathsf{E}$, and let $Y \subseteq X$ be a Silver cube coded in $\mathbf{L}$. Consider any Silver-generic $a \in Y$. Then $c = f(a)$ is Silver-generic as well by (*). Therefore by Corollary 3 there is a Silver cube $W \subseteq 2^\omega$, coded in $\mathbf{L}$, containing b, and such that $\mathsf{E} = \mathbb{E}_0$ on W. There is a Silver cube $Z \subseteq Y$ coded in $\mathbf{L}$, which Silver-forces that $f(a) \in W$, in the sense that any Silver-generic real $a \in Z$ satisfies $f(a) \in W$. As f is continuous, this easily implies $f[Z] \subseteq W$, and hence $\mathsf{E} = \mathbb{E}_0$ on $f[Z]$, as required.

We conclude that indeed f is Y-regular. Now by the genericity of a_0 and Theorem 7(I) there is a Silver cube $Z \subseteq 2^\omega$ coded in $\mathbf{L}$, such that $a_0 \in Z$ and E_α corrals $f \restriction Z$ for some α. In particular, $\langle a_0, b \rangle \in \mathsf{E}_\alpha$, hence $a_0 \mathrel{\mathsf{E}} b$, as required.

(iv) similar to (iv) of Lemma 8.

(v) Suppose to the contrary that M, N are OD. Let $a_0 \in M$. Then M is forced over $\mathbf{L}$, i.e., there is a Silver cube $Z \subseteq 2^\omega$ such that (†) $a_0 \in Z$ and all Silver reals $c \in Z$ in $\mathbf{L}[a_0]$ are pairwise B-equivalent. By Theorem 7(II), there are $\alpha < \omega_1$ and Silver cubes $Y, W \subseteq Z$ coded in $\mathbf{L}$, such that $a_0 \in Y$ and E_α corrals $g = S_{YW}$ negatively. Then $c = g(a_0)$ satisfies $a_0 \mathrel{\mathsf{E}_\alpha} c$ but $\neg\,(a_0 \mathrel{\mathsf{B}_\alpha} c)$, and hence $b \not\mathrel{\mathsf{B}} c$. However, $a_0, c \in Z$, and c is Silver-generic along with a_0. However, this contradicts (†). □

Thus it holds in the Miller generic model $\mathbf{L}[a_0]$ that B is a Σ^1_4 equivalence relation on 2^ω, the domain $\mathbf{Sil}$ (an OD set) is equal to the union of two (non-empty) B-classes, and these classes are non-OD. □

14. Theorem 3: Indiscernible Countable Sets of Reals

To establish Theorem 3 we make use of the forcing notion $\mathbb{P} = \mathbb{P}_{\mathbb{n}} \in \mathbf{L}$ defined in [31] for a given $\mathbb{n} \geq 2$. (We distinguish $\mathbb{n}$ by the blackboard font to specify that its value is fixed during the course of the proof of Theorem 3.) It satisfies the following conditions.

1*. $\mathbb{P} \in \mathbf{L}$ and $\mathbb{P}$ consists of Silver cubes in 2^ω.

2*. If $s \in 2^{<\omega}$, $X \in \mathbb{P}$ and $X{\restriction}_s = \{a \in X : s \subset a\} \neq \varnothing$ then $X{\restriction}_s$ belongs to $\mathbb{P}$—therefore *$\mathbb{P}$ adjoins a generic real $a \in 2^\omega$, $a \notin \mathbf{L}$.*

Below, if $s \in 2^{<\omega}$ and $x \in 2^\omega$ then $s \cdot x \in 2^\omega$ is defined so that $(s \cdot x)(k) = s(k) +_2 x(k)$ (where $+_2$ is the addition mod 2).

3*. The forcing notion $\mathbb{P}$ is $\mathbb{E}_0$*-invariant*, in the sense that if $X \in \mathbb{P}$ and $s \in 2^{<\omega}$ then the Silver cube $s \cdot X = \{s \cdot a : a \in X\}$ belongs to $\mathbb{P}$. It follows that *if a real $a \in 2^\omega$ is $\mathbb{P}$-generic over $\mathbf{L}$, then any real $b \in [a]_{\mathbb{E}_0}$ is $\mathbb{P}$-generic over $\mathbf{L}$, too.* In other words, $\mathbb{P}$ adjoins *a whole $\mathbb{E}_0$-class $[a]_{\mathbb{E}_0}$ of $\mathbb{P}$-generic reals.*

Note: in this Section we assume that $[a]_{\mathbb{E}_0} = \{b \in 2^\omega : a \; \mathbb{E}_0 \; b\}$ in the domain 2^ω.

4*. Conversely, *if reals $a \in 2^\omega$ and $b \in 2^\omega \cap \mathbf{L}[a]$ are $\mathbb{P}$-generic over $\mathbf{L}$ then $b \in [a]_{\mathbb{E}_0}$.*

5*. The property of "being a $\mathbb{P}$-generic real in 2^ω over $\mathbf{L}$" is $\Pi^1_{\mathbb{n}}$ in any generic extension of $\mathbf{L}$. (Recall that $\mathbb{n} \geq 2$ is fixed.)

6*. If a real $a \in 2^\omega$ is $\mathbb{P}$-generic over $\mathbf{L}$, then it is true in $\mathbf{L}[a]$ that

(1) (by 3*, 4*, 5*) *$[a]_{\mathbb{E}_0}$ is a $\Pi^1_{\mathbb{n}}$-set without* OD *elements,* but

(2) *every countable $\Sigma^1_{\mathbb{n}}$ set $X \subseteq \omega^\omega$ consists of* OD *elements.*

Earlier results in this direction include a model in [37] containing a Π^1_2 $\mathbb{E}_0$-class in 2^ω without OD elements, which is equivalent to the case $\mathbb{n} = 2$ in 6*. This involves an invariant (in the sense of 3*) "Silver" modification $\mathbb{P} = \mathbb{P}_2$ of a forcing notion, say $\mathbb{J}$, introduced by Jensen in [38] for the construction of a model with a nonconstructible Π^1_2 real singleton. See also 28A in [15] about this forcing. Here the invariance means that, similarly to 3* above, instead of a single generic real a, as in [38], $\mathbb{P}_2$ adjoins the entire $\mathbb{E}_0$-equivalence class $[a]_{\mathbb{E}_0}$ that consists of $\mathbb{P}_2$-generic reals. Another method of forcing a countable non-empty Π^1_2 set of non-OD reals was developed in [39]. Following Enayat's idea in [22], the method utilizes the finite-support product $\mathbb{J}^\omega$ of Jensen's forcing $\mathbb{J}$.

See ([31], Introduction) for a more detailed survey of the problem of existence of a countable non-empty OD set of reals containing no OD elements.

Proof of Theorem 3. Let $\mathbb{P} \in \mathbf{L}$ be a forcing notion satisfying 1*–6*. Consider a real $a_0 \in 2^\omega$ $\mathbb{P}$-generic over $\mathbf{L}$. It is true in the extension $\mathbf{L}[a_0]$ that the $\mathbb{E}_0$-class $[a_0]_{\mathbb{E}_0}$ is a $\Pi^1_{\mathbb{n}}$ set without OD elements by 6*(1). Define $b_0 \in 2^\omega$ by $b_0(0) = 1 - a_0(0)$ and $b_0(k) = a_0(k)$ for all $k \geq 1$. Then $[a_0]_{\mathbb{E}_0} = [a_0]_{\mathbb{E}_0^{\rm even}} \cup [b_0]_{\mathbb{E}_0^{\rm even}}$, a partition of the $\mathbb{E}_0$-class $[a_0]_{\mathbb{E}_0}$ into two $\mathbb{E}_0^{\rm even}$-classes. (See Example 1.)

We claim that the pair of the sets $M = [a_0]_{\mathbb{E}_0^{\rm even}}$ and $N = [b_0]_{\mathbb{E}_0^{\rm even}}$ is a strong $\Pi^1_{\mathbb{n}}$ counterexample to $\mathbf{LM}_{\rm OD}$ in $\mathbf{L}[a_0]$ in the sense of Definition 1.

Indeed the set $M \cup N = [a_0]_{\mathbb{E}_0}$ is $\Pi^1_{\mathbb{n}}$ whereas $\mathbb{E}_0^{\rm even}$ is an arithmetically definable relation. It follows that the associated equivalence E on $M \cup N$ (with M, N as the only equivalence classes) is $\Pi^1_{\mathbb{n}}$ as well. It remains to check that the set $M = [a_0]_{\mathbb{E}_0^{\rm even}}$ is not OD in $\mathbf{L}[a_0]$. Suppose to the contrary that $[a_0]_{\mathbb{E}_0^{\rm even}} = \{x \in 2^\omega : \varphi(x)\}$, where $\varphi(x)$ is a formula with ordinals as parameters. This is forced by some $X \in \mathbb{P}$ with $a_0 \in X$, so that if $a \in X$ is $\mathbb{P}$-generic over $\mathbf{L}$ then $[a]_{\mathbb{E}_0^{\rm even}} = \{x \in 2^\omega : \varphi(x)\}$ in $\mathbf{L}[a]$.

By definition, there is a partial function $p : \mathtt{dom}\, p \to 2$, $p \in \mathbf{L}$, with coinfinite domain $\mathtt{dom}\, p \subseteq \omega$, such that $X = [p] = \{x \in 2^\omega : p \subset x\}$. Let $m = \mathtt{min}\,(\omega \smallsetminus \mathtt{dom}\, p)$ and $s = 0^m{}^\frown 1$, so that $s \in 2^{<\omega}$ is a string of m zeros followed by a single 1; $\mathtt{dom}\, s = m + 1$. Then $s \cdot X = X$, and hence the real $c_0 = s \cdot a_0$ belongs to X along with a_0 itself and is generic by 3*. It follows that $[c_0]_{\mathbb{E}_0^{\rm even}} = \{x \in 2^\omega : \varphi(x)\}$ in $\mathbf{L}[c_0] = \mathbf{L}[a_0]$ by the choice of T. We conclude that $[a_0]_{\mathbb{E}_0^{\rm even}} = [c_0]_{\mathbb{E}_0^{\rm even}}$. However, on the other hand, $a_0 \; \mathbb{E}_0^{\rm even} \; c_0$ obviously fails, because the set $a_0 \,\Delta\, c_0 = \{m\}$ contains exactly one (an odd number) element. The contradiction completes the proof of (i) of Theorem 3.

To check (ii) apply 6*(2). $\square$

15. Conclusions and Discussion

In this study, different forcing and descriptive set theoretic tools were employed to construct of strong counterexamples to the Leibniz–Mycielski axiom $\mathbf{LM}_{\mathrm{OD}}$ in non-product generic extensions of $\mathbf{L}$ by a single generic real. The first main result (Theorem 1) shows that the Solovay strong Π^1_2 counterexample exists in non-product extensions $\mathbf{L}[a]$ of the constructible universe $\mathbf{L}$ by Miller-generic and Laver-generic reals a. Theorem 2 provides a slightly weaker result for Silver-generic reals. All together, these results significantly contribute to the project, initiated in the recent paper [25], of constructing strong counterexamples to $\mathbf{LM}_{\mathrm{OD}}$ (in the sense of Definition 1) in non-product generic models.

The other main result (Theorem 3) deals with countable strong counterexamples to $\mathbf{LM}_{\mathrm{OD}}$ in various projective classes. A model of **ZFC** is defined, in which, for a given $\mathfrak{n} \geq 2$, there exists a strong $\Pi^1_{\mathfrak{n}}$ countable counterexample to $\mathbf{LM}_{\mathrm{OD}}$ whereas there is no any strong countable $\Sigma^1_{\mathfrak{n}}$ counterexample, and hence no such counterexample in lower classes. This significant result extends the research line of resent papers [31,32,40] aimed at theorems that assert that the strength of various important statements of descriptive set theory (like the existence of strong counterexamples to $\mathbf{LM}_{\mathrm{OD}}$) properly depends on the associated projective class.

As for possible continuation of this research line, it can be connected with different coding systems like [41,42], different generic models like e.g., [43,44], and different problems, of course. Of the latter, let us reiterate the problem formulated in [25]:

Problem 1. *Extend the results of Theorem 1 or at least Theorem 2 to generic extensions of* $\mathbf{L}$*, the constructible universe, by a single Cohen-generic or Solovay-random real. Other popular forcing notions that adjoin a single generic real are also of interest here.*

It would be no less interesting to find a forcing of this type for which Theorem 2 definitely fails.

So far the result is known for the Sacks and $\mathbb{E}_0$-large generic reals from [25], and for Miller, Laver, Silver generic reals just from this paper.

To explain the main difficulty, consider the case of Solovay-random forcing, which consists of all (constructible) trees $T \subseteq 2^{<\omega}$ such that the according set $[T] \subseteq 2^\omega$ has a positive probability measure. Let us come back to Lemma 4. We have to re-prove it for the case of Solovay-random forcing, that is, with "superperfect" replaced by "closed subset of 2^ω of a positive probability measure" (twice). In fact it would be sufficient to consider the case when $f : X \to 2^\omega$ is a map 1–1, continuous and (measure 0)-preserving both ways. Then the domain of the counterexample in the Solovay-random extension would be equal to the set **Rand** of all random reals $b \in \mathbf{L} \cap 2^\omega$ satisfyng $\mathbf{L}[b] = \mathbf{L}[a]$ (Compare to **Sil** in Section 13).

The larger component E' of the extending and corralling dyadic pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ required could be equal to the $\subseteq$-smallest equivalence relation E' which includes both E and the graph of f; its countability would follow by standard descriptive set theoretic technique. However, an appropriate extension B' of B causes problems. We used f itself for that purpose in the proofs of Lemma 4 (and Lemma 9 in the Silver case, too) — but that was possible only after shrinking X via Theorem 4 (canonization). Unfortunately no appropriate canonization results are known for sets of positive measure. If f is not canonized then one immediately encounters problems while attempting to make use of f in the definition of B'. For instance how can one define the extended equivalence class $[x]_{\mathsf{B}'} = [y]_{\mathsf{B}'}$ if reals $x, y \in X$ are B-equivalent but the images $f(x)$, $f(y)$ are E-equivalent but not B-equivalent?

Thus Problem 1 remains open for the time being.

Finally, the following problem aims at separating countable and uncountable definable counterexamples to $\mathbf{LM}_{\mathrm{OD}}$.

Problem 2. *Prove that countable* OD *counterexamples to* $\mathbf{LM}_{\mathrm{OD}}$ *do not exist in Sacks, Miller, Laver, Silver extensions of* $\mathbf{L}$ *by a single generic real.*

The following two problems were suggested by an anonymous referee. They are related to set theory with atoms, hence, most likely, they may need methods quite different from those used in this article.

Problem 3. *Are there some independence results regarding* **LM** *in the Zermelo–Fraenkel set theory with atoms?*

Recall that **ZFA** is obtained from **ZF** by weakening the axiom of extensionality to allow *atoms* (also known as urelements). Atoms are distinct elements with no assumed internal structure (they contain no elements, thus extensionality fails in ZFA). It seems that **LM** may need more effort to be formulable, as a first order axiom, in the context of **ZFA** and the theory of finitely supported structures (fss) described in [7] (with roots in permutation models of **ZFA**). Indeed, the passage from (3) to (4) in Section 2 is based on the Reflection Principle, and that needs Foundation, absent in **ZFA**.

Problem 4. *Is* **LM** *consistent/inconsistent in the theory of finitely supported structures (fss)?*

Note that the Kinna–Wagner selection principle **KW** fails in this theory of fss [7], whereas it is known that the global form of **KW** is equivalent to **LM** in **ZF** [22]. However, the results in **ZF** are not necessarily valid when translated into an atomic set theory. For instance, the claim, that Kurepa's maximal antichain principle implies axiom of choice, is true in **ZF** but fails in **ZFA**, see [45] or ([13], Section 9.1).

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