mathematics

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On the Dimension of a New Class of Derivation Lie Algebras Associated to Singularities

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Abstract: Let $(V,0) = \{(z_1,\cdots,z_n) \in \mathbb{C}^n : f(z_1,\cdots,z_n) = 0\}$ be an isolated hypersurface singularity with $mult(f) = m$. Let $J_k(f)$ be the ideal generated by all k-th order partial derivatives of f. For $1 \leq k \leq m-1$, the new object $\mathcal{L}_k(V)$ is defined to be the Lie algebra of derivations of the new k-th local algebra $M_k(V)$, where $M_k(V) := \mathcal{O}_n/((f) + J_1(f) + \cdots + J_k(f))$. Its dimension is denoted as $\delta_k(V)$. This number $\delta_k(V)$ is a new numerical analytic invariant. In this article we compute $\mathcal{L}_4(V)$ for fewnomial isolated singularities (binomial, trinomial) and obtain the formulas of $\delta_4(V)$. We also verify a sharp upper estimate conjecture for the $\delta_4(V)$ for large class of singularities. Furthermore, we verify another inequality conjecture: $\delta_{(k+1)}(V) < \delta_k(V)$, $k = 3$ for low-dimensional fewnomial singularities.

Keywords: isolated hypersurface singularity; Lie algebra; local algebra

MSC: 14B05; 32S05

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Citation: Hussain, N.; Yau, S.S.-T.; Zuo, H. On the Dimension of a New Class of Derivation Lie Algebras Associated to Singularities. *Mathematics* **2021**, *9*, 1650. https://doi.org/10.3390/math9141650

Academic Editor: Jan L. Cieśliński

Received: 26 May 2021
Accepted: 9 July 2021
Published: 13 July 2021

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1. Introduction

Let G be a semi-simple Lie group and $\mathcal{G}$ be its Lie algebra. Suppose G acts on $\mathcal{G}$ by the adjoint action. Let $\mathcal{G}/G$ be the variety corresponding to the G-invariant polynomials on $\mathcal{G}$. The quotient morphism $\gamma : \mathcal{G} \to \mathcal{G}/G$ was intensively studied by Kostant ([1,2]). Let $\mathcal{H} \subset \mathcal{G}$ be a Cartan subalgebra of $\mathcal{G}$ and W be the corresponding Weyl group.

(i) The space $\mathcal{G}/G$ may be identified with the set of semi-simple G classes in $\mathcal{G}$ such that γ maps an element $x \in \mathcal{G}$ to the class of its semi-simple part x_s. Thus $\gamma^{-1}(0) = N(\mathcal{G})$ is the nilpotent variety. An element $x \in N(\mathcal{G})$ is termed "regular" (resp., "subregular") if its centralizer has a minimal dimension (resp., minimal dimension + 2).

(ii) By a theorem of Chevalley, the space $\mathcal{G}/G$ is isomorphic to $\mathcal{H}/W$, an affine space of dimension $r = \text{rank}(\mathcal{G})$. The isomorphism is given by the map of a semi-simple class to its intersection with $\mathcal{H}$ (a W orbit).

Brieskorn [3] obtained the following beautiful theorem, which was conjectured by Grothendieck [4], which establishes connections between the simple singularities and the simple Lie algebras.

Theorem 1 ([3]). *Let $\mathcal{G}$ be a simple Lie algebra over $\mathbb{C}$ of type A_r, D_r, E_r. Then*

(i) The intersection of the variety $N(\mathcal{G})$ of the nilpotent elements of $\mathcal{G}$ with a transverse slice S to the subregular orbit, which has codimension 2 in $N(\mathcal{G})$, is a surface $S \cap N(\mathcal{G})$ with an isolated rational double point of the type corresponding to the algebra $\mathcal{G}$.

(ii) The restriction of the quotient $\gamma : \mathcal{G} \to \mathcal{H}/W$ to the slice S is a realization of a semi-universal deformation of the singularity in $S \cap N(\mathcal{G})$.

The details of this Brieskorn's theory can be found in Slodowy's papers ([5,6]).

Finite dimensional Lie algebras are a semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Simple Lie algebras and semi-simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras, since Brieskorn gave a beautiful connection between simple Lie algebras and simple singularities. So it is extremely important and natural to establish connections between singularities and solvable (nilpotent) Lie algebras.

We use $\mathcal{O}_n$ to denote the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$. $\mathcal{O}_n$ has a unique maximal ideal $\mathfrak{m}$, which is generated by germs of holomorphic functions which vanish at the origin. For any isolated hypersurface singularity $(V,0) \subset (\mathbb{C}^n,0)$ where $V = \{f = 0\}$ Yau considers the Lie algebra of derivations of moduli algebra [7] $A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, i.e., $L(V) := \mathrm{Der}(A(V), A(V))$. The finite dimensional Lie algbra $L(V)$ is solvable ([8,9]). $L(V)$ is called the Yau algebra of V in [10,11] in order to distinguish from Lie algebras of other types of singularities ([12,13]). Yau algebra plays an important role in singularity theory([14,15]). In recent years, Yau, Zuo, Hussain and their collaborators ([16–19]) have constructed many new natural connections between the set of isolated hypersurface singularities and the set of finite dimensional solvable (nilpotent) Lie algebras. They introduced three different ways to associate Lie algebras to isolated hypersurface singularities. These constructions are useful to study the solvable (nilpotent) Lie algebras from the geometric point of view ([16]). Yau, Zuo, and their collaborators have been systematically studying various derivation Lie algebras of isolated hypersurface singularities (see, e.g., [16–31]).

In this paper, we are interested in the new series of derivation Lie algebras which are firstly introduced in Let $(V,0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^n,0) \to (\mathbb{C},0)$. The multiplicity $mult(f)$ of the singularity $(V,0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0.

Definition 1. *Let $(V,0) = \{(x_1,\cdots,x_n) \in \mathbb{C}^n : f(x_1,\cdots,x_n) = 0\}$ be an isolated hypersurface singularity with $mult(f) = m$. Let $J_k(f)$ be the ideal generated by all the k-th order partial derivative of f, i.e., $J_k(f) =< \frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}} \mid 1 \le i_1,\cdots,i_k \le n >$. For $1 \le k \le m$, we define the new k-th local algebra, $M_k(V) := \mathcal{O}_n/(f + J_1(f) + \cdots + J_k(f))$. In particular, $M_m(V) = 0$, $M_1(V) = A(V)$, and $M_2(V) = H_1(V)$.*

Remark 1. *If f defines a weighted homogeneous isolated singularity at the origin, then $f \in J_1(f) \subset J_2(f) \subset \cdots \subset J_k(f)$, thus $M_k(V) = \mathcal{O}_n/(f + J_1(f) + \cdots + J_k(f)) = \mathcal{O}_n/(J_k(f))$.*

The k-th local algebra $M_k(V)$ is a contact invariant of $(V,0)$, i.e., it depends only on the isomorphism class of $(V,0)$. The dimension of $M_k(V)$ is denoted by $d_k(V)$. It is a new numerical analytic invariant of an isolated hypersurface singularity.

Theorem 2. *Suppose $(V,0) = \{(x_1,\cdots,x_n) \in \mathbb{C}^n : f(x_1,\cdots,x_n) = 0\}$ and $(W,0) = \{(x_1,\cdots,x_n) \in \mathbb{C}^n : g(x_1,\cdots,x_n) = 0\}$ are isolated hypersurface singularities. If $(V,0)$ is biholomorphically equivalent to $(W,0)$, then $M_k(V)$ is isomorphic to $M_k(W)$ as a $\mathbb{C}$-algebra for all $1 \le k \le m$, where $m = mult(f) = mult(g)$.*

Based on Theorem 2, it is natural to introduce the new series of k-th derivation Lie algebras $\mathcal{L}_k(V)$ which are defined to be the Lie algebra of derivations of the k-th local algebra $M_k(V)$, i.e., $\mathcal{L}_k(V) = Der(M_k(V), M_k(V))$. Its dimension is denoted as $\delta_k(V)$. This number $\delta_k(V)$ is also a new numerical analytic invariant. In particular, $\mathcal{L}_1(V) = L(V)$. Therefore, $\mathcal{L}_k(V)$ is a generalization of Yau algebra.

An interesting general question is that can we find some topological invariants to bound an analytic invariant of singularities. In particular, we ask the following question: can we bound sharply the analytic invariant $\delta_k(V)$ by only using the weight types for

weighted homogeneous isolated hypersurface singularities? We propose the following sharp upper estimate conjecture.

Remark 2. *In dimension one and two, the weighted types are topological invariants for weighted homogeneous isolated hypersurface singularities* [32,33].

Conjecture 1 ([34]). *For each* $0 \leq k \leq n$, *assume that* $\delta_k(\{x_1^{a_1} + \cdots + x_n^{a_n} = 0\}) = h_k(a_1, \cdots, a_n)$. *Let* $(V, 0) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \cdots, x_n) = 0\}, (n \geq 2)$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2, \cdots, x_n)$ *of weight type* $(w_1, w_2, \cdots, w_n; 1)$. *Then,* $\delta_k(V) \leq h_k(1/w_1, \cdots, 1/w_n)$.

It is also interesting to compare dimensions between $\mathcal{L}_k(V)$.

Conjecture 2 ([34]). *With notations above, let* $(V, 0)$ *be an isolated hypersurface singularity which is defined by* $f \in \mathcal{O}_n$, $n \geq 2$. *Then*

$$\delta_{(k+1)}(V) < \delta_k(V), k \geq 1.$$

The Conjecture 1 holds true for following cases:

(1) Binomial singularities (see Definition 5) when $k = 1$ [31];
(2) Trinomial singularities (see Definition 5) when $k = 1$ [23];
(3) Binomial and trinomial singularities when $k = 2$ [19];
(4) Binomial and trinomial singularities when $k = 3$ [34].

Conjecture 2 holds true for binomial and trinomial singularities when $k = 1, 2$ [34].

The purpose of this article is to verify Conjecture 1 (Conjecture 2, resp.) for binomial and trinomial singularities when $k = 4$ ($k = 3$, resp.). We obtain the following main results.

Theorem 3. *Let* $(V(f), 0) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n : x_1^{a_1} + \cdots + x_n^{a_n} = 0\}$, *where* a_i *are fixed natural numbers,* $(n \geq 2;\ a_i \geq 6,\ 1 \leq i \leq n)$. *Then*

$$\delta_4(V(f)) = h_4(a_1, \cdots, a_n) = \sum_{j=1}^{n} \frac{a_j - 5}{a_j - 4} \prod_{i=1}^{n} (a_i - 4).$$

Theorem 4. *Let* $(V, 0)$ *be a binomial singularity (see Corollary 1) defined by the weighted homogeneous polynomial* $f(x_1, x_2)$ *with weight type* $(w_1, w_2; 1)$ *and* $mult(f) \geq 6$. *Then*

$$\delta_4(V) \leq h_4(\frac{1}{w_1}, \frac{1}{w_2}) = \sum_{j=1}^{2} \frac{\frac{1}{w_j} - 5}{\frac{1}{w_j} - 4} \prod_{i=1}^{2} (\frac{1}{w_i} - 4).$$

Theorem 5. *Let* $(V, 0)$ *be a trinomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2, x_3)$ *with weight type* $(w_1, w_2, w_3; 1)$ *(see Proposition 2) and* $mult(f) \geq 6$. *Then*

$$\delta_4(V) \leq h_4(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}) = \sum_{j=1}^{3} \frac{\frac{1}{w_j} - 5}{\frac{1}{w_j} - 4} \prod_{i=1}^{3} (\frac{1}{w_i} - 4).$$

Theorem 6. *Let* $(V, 0)$ *be a binomial singularity (see Corollary 1) defined by the weighted homogeneous polynomial* $f(x_1, x_2)$ *with weight type* $(w_1, w_2; 1)$ *and* $mult(f) \geq 6$. *Then*

$$\delta_4(V) < \delta_3(V).$$

Theorem 7. *Let $(V,0)$ be a trinomial singularity which is defined by the weighted homogeneous polynomial $f(x_1,x_2,x_3)$ (see Proposition 2) with weight type $(w_1,w_2,w_3;1)$ and $mult(f) \geq 6$. Then*

$$\delta_4(V) < \delta_3(V).$$

2. Derivation Lie Algebras of Isolated Singularities

In this section we shall briefly provide the basic definitions and important results which will be used to compute the derivation Lie algebras of isolated hypersurface singularities.

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $Der(A,A)$ (or $Der(A)$) with the bracket defined by the commutator of linear endomorphisms.

Theorem 8 ([35]). *For finite dimensional commutative associative algebras with units A, B, and $S = A \otimes B$ are a tensor product, then*

$$DerS \cong (DerA) \otimes B + A \otimes (DerB). \tag{1}$$

We shall use this formula in the following.

Definition 2. *Let J be an ideal in an analytic algebra S (i.e., $\mathcal{O}_n/I$). Then $Der_J S \subseteq Der_{\mathbb{C}} S$ is Lie subalgebra of all $\sigma \in Der_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.*

The following well-known results are used to compute the derivations.

Theorem 9 ([31]). *Let J be an ideal in $R = \mathbb{C}\{x_1, \cdots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

Definition 3. *Let $(V,0)$ be an isolated hypersurface singularity. The new series of k-th derivation Lie algebras $\mathcal{L}_k(V)$ (or $\mathcal{L}_k((V,0))$) which are defined to be the Lie algebra of derivations of the k-th local algebra $M_k(V)$, i.e., $\mathcal{L}_k(V) = Der(M_k(V), M_k(V))$. Its dimension is denoted as $\delta_k(V)$ (or $\delta_k((V,0))$). This number $\delta_k(V)$ is also a new numerical analytic invariant*

Definition 4. *A polynomial $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ is weighted homogeneous if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with a non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the weighted homogeneous degree (w-degree) of f with respect to weights w_j and is denoted* $\deg f$. *The collection $(w;d) = (w_1, \cdots, w_n; d)$ is called the weight type of f.*

Definition 5 ([36]). *An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by an n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. The 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.*

Proposition 1 ([31]). *Let f be a weighted homogeneous fewnomial isolated singularity with $mult(f) \geq 3$. Then f is analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \geq 2$.

Proposition 1 has the following immediate corollary.

Corollary 1 ([31]). *Each binomial isolated singularity is analytically equivalent to one from the three series: (A)* $x_1^{a_1} + x_2^{a_2}$, *(B)* $x_1^{a_1}x_2 + x_2^{a_2}$, *(C)* $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

Wolfgang and Atsushi [37] gave the following classification of fewnomial singularities in case of three variables.

Proposition 2 ([37]). *Let* $f(x_1, x_2, x_3)$ *be a weighted homogeneous fewnomial isolated singularity with* $mult(f) \geq 3$. *Then* f *is analytically equivalent to following five types:*
Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
Type 2. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$,
Type 3. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$,
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_1$,
Type 5. $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$.

3. Proof of Theorems

We need to prove several propositions first in order to prove the main theorems,

Proposition 3. *Let* $(V(f), 0)$ *be a weighted homogeneous fewnomial isolated singularity which is defined by* $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$, *where* a_i *are fixed natural numbers,* $(a_i \geq 6, i = 1, 2, \cdots, n)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}; 1)$. *Then*

$$\delta_4(V(f)) = \sum_{j=1}^{n} \frac{a_j - 5}{a_j - 4} \prod_{i=1}^{n} (a_i - 4).$$

Proof. The generalized moduli algebra $M_4(V)$ has dimension $\prod_{i=1}^{n}(a_i - 4)$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n}, 0 \leq i_1 \leq a_1 - 5, 0 \leq i_2 \leq a_2 - 5, \cdots, 0 \leq i_n \leq a_n - 5\},$$

with following relations:

$$x_1^{a_1-4} = 0, x_2^{a_2-4} = 0, x_3^{a_3-4} = 0, \cdots, x_n^{a_n-4} = 0. \tag{2}$$

In order to compute a derivation D of $M_4(V)$ it suffices to indicate its values on the generators $x_1, x_2, \cdots, x_n$ which can be written in terms of the monomial basis. Without loss of generality, we write

$$Dx_j = \sum_{i_1=0}^{a_1-5} \sum_{i_2=0}^{a_2-5} \cdots \sum_{i_n=0}^{a_n-5} c^j_{i_1,i_2,\cdots,i_n} x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n},\ j = 1, 2, \cdots, n.$$

It follows from relations (2) that one easily finds the necessary and sufficient conditions defining a derivation of $M_4(V)$ as follows:

$$\begin{aligned}
c^1_{0,i_2,i_3,,\cdots,i_n} &= 0; 0 \leq i_2 \leq a_2 - 5, 0 \leq i_3 \leq a_3 - 5, \cdots, 0 \leq i_n \leq a_n - 5;\\
c^2_{i_1,0,i_3,,\cdots,i_n} &= 0; 0 \leq i_1 \leq a_1 - 5, 0 \leq i_3 \leq a_3 - 5, \cdots, 0 \leq i_n \leq a_n - 5;\\
c^3_{i_1,i_2,0,\cdots,i_n} &= 0; 0 \leq i_1 \leq a_1 - 5, 0 \leq i_2 \leq a_2 - 5, \cdots, 0 \leq i_n \leq a_n - 5;\\
&\vdots\\
c^n_{i_1,i_2,i_3,\cdots,i_{n-1},0} &= 0; 0 \leq i_1 \leq a_1 - 5, 0 \leq i_2 \leq a_2 - 5, \cdots, 0 \leq i_{n-1} \leq a_{n-1} - 5.
\end{aligned}$$

Therefore, we obtain the following bases of the Lie algebra in question:

$$x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_1,\ 1\le i_1\le a_1-5, 0\le i_2\le a_2-5, 0\le i_3\le a_3-5,\cdots, 0\le i_n\le a_n-5;$$
$$x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_2,\ 0\le i_1\le a_1-5, 1\le i_2\le a_2-5, 0\le i_3\le a_3-5,\cdots, 0\le i_n\le a_n-5;$$
$$x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_3,\ 0\le i_1\le a_1-5, 0\le i_2\le a_2-5, 1\le i_3\le a_3-5, 0\le i_4\le a_4-5,$$
$$0\le i_5\le a_5-5, 0\le i_6\le a_6-5,\cdots, 0\le i_n\le a_n-5;$$
$$\vdots$$
$$x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_n,\ 0\le i_1\le a_1-5, 0\le i_2\le a_2-5, 0\le i_3\le a_3-5,\cdots, 1\le i_n\le a_n-5.$$

Therefore, we have the following formula

$$\delta_4(V)=\sum_{j=1}^{n}\frac{a_j-5}{a_j-4}\prod_{i=1}^{n}(a_i-4).$$

□

Remark 3. *Let $(V,0)$ be a binomial isolated singularity of type A which is defined by $f=x_1^{a_1}+x_2^{a_2}$ ($a_1\ge 6,\ a_2\ge 6$) with weight type $(\frac{1}{a_1},\frac{1}{a_2};1)$. Then it follows from Proposition 3 that*

$$\delta_4(V)=2a_1a_2-9(a_1+a_2)+40.$$

Proposition 4. *Let $(V,0)$ be a binomial singularity of type B defined by $f=x_1^{a_1}x_2+x_2^{a_2}$ ($a_1\ge 5,\ a_2\ge 6$) with weight type $(\frac{a_2-1}{a_1a_2},\frac{1}{a_2};1)$. Then,*

$$\delta_4(V)=2a_1a_2-9(a_1+a_2)+43.$$

Furthermore, assuming that $mult(f)\ge 6$, we have

$$2a_1a_2-9(a_1+a_2)+43\le\frac{2a_1a_2^2}{a_2-1}-9(\frac{a_1a_2}{a_2-1}+a_2)+40.$$

Proof. The generalized moduli algebra

$$M_4(V)=\mathbb{C}\{x_1,x_2\}/(f_{x_1x_1x_1x_1},f_{x_2x_2x_2x_2},f_{x_1x_1x_1x_2},f_{x_1x_1x_2x_2},f_{x_1x_2x_2x_2})$$

has dimension $a_1a_2-4(a_1+a_2)+17$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}, 0\le i_1\le a_1-5; 0\le i_2\le a_2-5; x_1^{a_1-4}\}. \tag{3}$$

In order to compute a derivation D of $M_4(V)$ it suffices to indicate its values on the generators x_1,x_2 which can be written in terms of the basis (3). Without loss of generality, we write

$$Dx_j=\sum_{i_1=0}^{a_1-5}\sum_{i_2=0}^{a_2-5}c_{i_1,i_2}^{j}x_1^{i_1}x_2^{i_2}+c_{a_1-4,0}^{j}x_1^{a_1-4},\ j=1,2.$$

We obtain the following description of the Lie algebra in question. The following derivations form bases of $\mathrm{Der}M_4(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1\le i_1\le a_1-5, 0\le i_2\le a_2-5; x_1^{i_1}x_2^{i_2}\partial_2, 0\le i_1\le a_1-5, 1\le i_2\le a_2-5;$$
$$x_2^{a_2-5}\partial_1; x_1^{a_1-4}\partial_1; x_1^{a_1-4}\partial_2.$$

Therefore, we have the following formula

$$\delta_4(V) = 2a_1a_2 - 9(a_1 + a_2) + 43.$$

It follows from Proposition 3 we have

$$h_4(a_1, a_2) = 2a_1a_2 - 9(a_1 + a_2) + 40.$$

After putting the weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$ of binomial isolated singularity of type B, we have

$$h_4(\frac{1}{w_1}, \frac{1}{w_2}) = \frac{2a_1a_2^2}{a_2 - 1} - 9(\frac{a_1a_2}{a_2 - 1} + a_2) + 40.$$

Finally we need to show that

$$2a_1a_2 - 9(a_1 + a_2) + 43 \leq \frac{2a_1a_2^2}{a_2 - 1} - 9(\frac{a_1a_2}{a_2 - 1} + a_2) + 40. \tag{4}$$

After solving (4) we have $a_1(a_2 - 8) + a_2(a_1 - 4) + 4 \geq 0$. □

Proposition 5. *Let $(V, 0)$ be a binomial singularity of type C defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 5, a_2 \geq 5)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$.*

$$\delta_4(V) = \begin{cases} 2a_1a_2 - 9(a_1 + a_2) + 46; & a_1 \geq 6, a_2 \geq 6 \\ a_2 - 1; & a_1 = 5, a_2 \geq 5. \end{cases}$$

Furthermore, assuming that $mult(f) \geq 7$, we have

$$2a_1a_2 - 9(a_1 + a_2) + 46 \leq \frac{2(a_1a_2 - 1)^2}{(a_1 - 1)(a_2 - 1)} - 9(a_1a_2 - 1)(\frac{a_1 + a_2 - 2}{(a_1 - 1)(a_2 - 1)}) + 40.$$

Proof. The generalized moduli algebra

$$M_4(V) = \mathbb{C}\{x_1, x_2\}/(f_{x_1x_1x_1x_1}, f_{x_2x_2x_2x_2}, f_{x_1x_1x_1x_2}, f_{x_1x_1x_2x_2}, f_{x_1x_2x_2x_2})$$

has dimension $a_1a_2 - 4(a_1 + a_2) + 18$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 5; 0 \leq i_2 \leq a_2 - 5; x_1^{a_1-4}; x_2^{a_2-4}\}. \tag{5}$$

In order to compute a derivation D of $M_4(V)$, it suffices to indicate its values on the generators x_1, x_2 which can be written in terms of the basis (5). Without loss of generality, we write

$$Dx_j = \sum_{i_1=0}^{a_1-5}\sum_{i_2=0}^{a_2-5} c_{i_1,i_2}^j x_1^{i_1}x_2^{i_2} + c_{a_1-4,0}^j x_1^{a_1-4} + c_{0,a_2-4}^j x_2^{a_2-4},\ j = 1, 2.$$

We obtain the following description of the Lie algebra in question. The following derivations form bases of Der$M_4(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1 - 5, 0 \leq i_2 \leq a_2 - 5; x_1^{i_1}x_2^{i_2}\partial_2, 0 \leq i_1 \leq a_1 - 5, 1 \leq i_2 \leq a_2 - 5;$$

$$x_2^{a_2-5}\partial_1; x_2^{a_2-4}\partial_1; x_1^{a_1-4}\partial_1; x_2^{a_2-4}\partial_2; x_1^{a_1-5}\partial_2; x_1^{a_1-4}\partial_2.$$

Therefore, we have the following formula

$$\delta_4(V) = 2a_1a_2 - 9(a_1 + a_2) + 46.$$

In the case of $a_1 = 5, a_2 \geq 5$, we have following bases of Lie algebra:

$$x_2^{i_2}\partial_2, 1 \leq i_2 \leq a_2 - 4; x_2^{a_2-4}\partial_1; x_1\partial_1; x_1\partial_2.$$

It follows from Proposition 3 and binomial singularity of type C that we have

$$h_4(\frac{1}{w_1}, \frac{1}{w_2}) = \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 9(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1}) + 40.$$

Finally, we need to show that

$$2a_1a_2 - 9(a_1+a_2) + 46 \leq \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 9(a_1a_2-1)(\frac{a_1+a_2-2}{(a_1-1)(a_2-1)}) + 40. \quad (6)$$

After solving (6), we have
$a_1a_2^2[(a_2-3)(a_1-3) - a_1(a_2-6)] + a_2^3 + 4a_1^2a_2 + 10a_2^2(a_1-4) + 6a_1a_2(a_1-4) + 3a_1^2(a_2-4) + a_1a_2(a_1-4) + 15a_1 + 2(a_2-4) \geq 0$.
Similarly, we can prove Conjecture 1 for $a_1 = 5, a_2 \geq 5$. □

Remark 4. *Let $(V,0)$ be a trinomial singularity of type 1 (see Proposition 2) defined by $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ ($a_1 \geq 6, a_2 \geq 6, a_3 \geq 6$) with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}; 1)$. Then it follows from Proposition 3 that*

$$\delta_4(V) = 3a_1a_2a_3 + 56(a_1+a_2+a_3) - 13(a_1a_2 + a_1a_3 + a_2a_3) - 240.$$

Proposition 6. *Let $(V,0)$ be a trinomial singularity of type 2 defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ ($a_1 \geq 5, a_2 \geq 5, a_3 \geq 6$) with weight type $(\frac{1-a_3+a_2a_3}{a_1a_2a_3}, \frac{a_3-1}{a_2a_3}, \frac{1}{a_3}; 1)$. Then*

$$\delta_4(V) = \begin{cases} 3a_1a_2a_3 - 13(a_1a_2 + a_1a_3 + a_2a_3) + 60(a_1+a_3) & \\ +56a_2 - 275; & a_1 \geq 5, a_2 \geq 6, a_3 \geq 6 \\ 2a_1a_3 - 5a_1 - 7a_3 + 15; & a_1 \geq 5, a_2 = 5, a_3 \geq 6. \end{cases}$$

Furthermore, assuming that $a_1 \geq 5, a_2 \geq 6, a_3 \geq 6$, we have

$$3a_1a_2a_3 - 13(a_1a_2 + a_1a_3 + a_2a_3) + 60(a_1+a_3) + 56a_2 - 275 \leq \frac{3a_1a_2^2a_3^3}{(1-a_3+a_2a_3)(a_3-1)}$$
$$- 13(\frac{a_1a_2^2a_3^2}{(1-a_3+a_2a_3)(a_3-1)} + \frac{a_1a_2a_3^2}{1-a_3+a_2a_3} + \frac{a_2a_3^2}{a_3-1}) + 56(\frac{a_1a_2a_3}{1-a_3+a_2a_3}$$
$$+ \frac{a_2a_3}{a_3-1} + a_3) - 240.$$

Proof. The moduli algebra $M_4(V)$ has dimension $(a_1a_2a_3 - 4(a_1a_2 + a_1a_3 + a_2a_3) + 17(a_1 + a_3) + 16a_2 - 72)$ and has a monomial basis of the form:

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0 \leq i_1 \leq a_1 - 5; 0 \leq i_2 \leq a_2 - 5; 0 \leq i_3 \leq a_3 - 5; x_1^{a_1-4}x_3^{i_3}, 0 \leq i_3 \leq a_3 - 5;$$
$$x_1^{i_1}x_3^{a_3-4}, 0 \leq i_1 \leq a_1 - 5\}.$$

In order to compute a derivation D of $M_4(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the bases. Thus we can write

$$Dx_j = \sum_{i_1=0}^{a_1-5}\sum_{i_2=0}^{a_2-5}\sum_{i_3=0}^{a_3-5} c_{i_1,i_2,i_3}^j x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_1=0}^{a_1-5} c_{i_1,0,a_3-4}^j x_1^{i_1}x_3^{a_3-4} + \sum_{i_3=0}^{a_3-5} c_{a_1-4,0,i_3}^j x_3^{i_3}x_1^{a_1-4},\ j = 1,2,3.$$

Using the above derivations we obtain the following description of the Lie algebras in question. The derivations represented by the following vector fields form bases of $\mathrm{Der}M_4(V)$:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1\le i_1\le a_1-5, 0\le i_2\le a_2-5, 0\le i_3\le a_3-5; x_1^{a_1-4}x_3^{i_3}\partial_1,\ 0\le i_3\le a_3-5,$$
$$x_2^{a_2-5}x_3^{i_3}\partial_1,\ 1\le i_3\le a_3-5; x_1^{i_1}x_2^{a_2-4}\partial_1,\ 0\le i_1\le a_1-5,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 0\le i_1\le a_1-5, 1\le i_2\le a_2-5, 0\le i_3\le a_3-5; x_1^{a_1-4}x_3^{i_3}\partial_2,\ 0\le i_3\le a_3-5,$$
$$x_1^{i_1}x_2^{a_2-4}\partial_2,\ 0\le i_1\le a_1-5; x_1^{i_1}x_3^{a_3-5}\partial_2,\ 1\le i_1\le a_1-5,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0\le i_1\le a_1-5, 0\le i_2\le a_2-5, 1\le i_3\le a_3-5, x_1^{i_1}x_2^{a_2-4}\partial_3,\ 0\le i_1\le a_1-5,$$
$$x_1^{a_1-4}x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-5.$$

Therefore, we have

$$\delta_4(V)=3a_1a_2a_3-13(a_1a_2+a_1a_3+a_2a_3)+60(a_1+a_3)+56a_2-275.$$

In the case of $a_1\ge 5, a_2=5, a_3\ge 6$, we obtain the following basis:

$$x_1^{i_1}x_3^{i_3}\partial_1,\ 1\le i_1\le a_1-4, 0\le i_3\le a_3-5; x_1^{i_1}x_2\partial_1,\ 0\le i_1\le a_1-5,$$
$$x_1^{i_1}x_2\partial_2,\ 0\le i_1\le a_1-5; x_1^{i_1}x_3^{a_3-5}\partial_2,\ 1\le i_1\le a_1-4,$$
$$x_1^{i_1}x_3^{i_3}\partial_3,\ 0\le i_1\le a_1-4, 1\le i_3\le a_3-5; x_1^{i_1}x_2\partial_3,\ 0\le i_1\le a_1-5.$$

We have

$$\delta_4(V)=2a_1a_3-5a_1-7a_3+15.$$

Next, we need to show that when $a_1\ge 5, a_2\ge 6, a_3\ge 6$, then

$$3a_1a_2a_3-13(a_1a_2+a_1a_3+a_2a_3)+60(a_1+a_3)+56a_2-275\le\frac{3a_1a_2^2a_3^3}{(1-a_3+a_2a_3)(a_3-1)}$$
$$-13(\frac{a_1a_2^2a_3^2}{(1-a_3+a_2a_3)(a_3-1)}+\frac{a_1a_2a_3^2}{1-a_3+a_2a_3}+\frac{a_2a_3^2}{a_3-1})+56(\frac{a_1a_2a_3}{1-a_3+a_2a_3}$$
$$+\frac{a_2a_3}{a_3-1}+a_3)-240.$$

After simplification we obtain
$(a_1-3)^3(a_2-5)a_3+(a_2-4)a_1a_3((a_3-3)(a_1-5)+(a_2-3)(a_3-3))+a_2(3a_3-4)(a_1-3)+a_2(a_1-2)+6\ge 0$.
Similarly, we can prove Conjecture 1 for $a_1\ge 5, a_2=5, a_3\ge 6$. □

Proposition 7. *Let $(V,0)$ be a trinomial singularity of type 3 defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}x_1$ ($a_1\ge 5, a_2\ge 5, a_3\ge 5$) with weight type*

$$(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3},\frac{1-a_1+a_1a_3}{1+a_1a_2a_3},\frac{1-a_2+a_1a_2}{1+a_1a_2a_3};1).$$

Then

$$\delta_4(V)=\begin{cases}3a_1a_2a_3+60(a_1+a_2+a_3)-13(a_1a_2+a_1a_3+a_2a_3)\\ -291; & a_1\ge 6, a_2\ge 6, a_3\ge 6\\ 2a_2a_3-7a_2-5a_3+19; & a_1=5, a_2\ge 6, a_3\ge 5\\ 2a_1a_3-5a_1-7a_3+19; & a_1\ge 5, a_2=5, a_3\ge 5\\ 2a_1a_2-7a_1-5a_2+19; & a_1\ge 6, a_2\ge 6, a_3=5\end{cases}$$

Furthermore, assuming that $a_1\ge 6, a_2\ge 6, a_3\ge 6$, we have
$3a_1a_2a_3+60(a_1+a_2+a_3)-13(a_1a_2+a_1a_3+a_2a_3)-291\le\frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)}$

$+56(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3}+\frac{1+a_1a_2a_3}{1-a_1+a_1a_3}+\frac{1+a_1a_2a_3}{1-a_2+a_1a_2})-13(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)}+\frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)}$
$+\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)})-240.$

Proof. The moduli algebra $M_4(V)$ has dimension $(a_1a_2a_3-4(a_1a_2+a_1a_3+a_2a_3)+17(a_1+a_2+a_3)-76)$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0\le i_1\le a_1-5; 0\le i_2\le a_2-5; 0\le i_3\le a_3-5; x_1^{a_1-4}x_3^{i_3}, 0\le i_3\le a_3-5;$$
$$x_2^{i_2}x_3^{a_3-4}, 0\le i_2\le a_2-5; x_1^{i_1}x_2^{a_2-4}, 0\le i_1\le a_1-5\}.$$

In order to compute a derivation D of $M_3(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the bases. Thus we can write

$$Dx_j=\sum_{i_1=0}^{a_1-5}\sum_{i_2=0}^{a_2-5}\sum_{i_3=0}^{a_3-5}c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}+\sum_{i_1=0}^{a_1-5}c^j_{i_1,a_2-4,0}x_1^{i_1}x_2^{a_2-4}+\sum_{i_3=0}^{a_3-5}c^j_{a_1-4,0,i_3}x_1^{a_1-4}x_3^{i_3}$$
$$+\sum_{i_2=0}^{a_2-5}c^j_{0,i_2,a_3-4}x_2^{i_2}x_3^{a_3-4},\ j=1,2,3.$$

We obtain the following bases of the Lie algebra in question:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1\le i_1\le a_1-5, 0\le i_2\le a_2-5, 0\le i_3\le a_3-5; x_2^{i_2}x_3^{a_3-4}\partial_1,\ 0\le i_2\le a_2-6,$$
$$x_2^{a_2-5}x_3^{i_3}\partial_1,\ 1\le i_3\le a_3-4; x_1^{i_1}x_2^{a_2-4}\partial_1,\ 0\le i_1\le a_1-5; x_1^{a_1-4}x_3^{i_3}\partial_1,\ 0\le i_3\le a_3-5,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 0\le i_1\le a_1-5, 1\le i_2\le a_2-5, 0\le i_3\le a_3-5; x_1^{a_1-4}x_3^{i_3}\partial_2,\ 0\le i_3\le a_3-5,$$
$$x_1^{i_1}x_2^{a_2-4}\partial_2,\ 0\le i_1\le a_1-5; x_1^{i_1}x_3^{a_3-5}\partial_2,\ 1\le i_1\le a_1-5; x_2^{i_2}x_3^{a_3-4}\partial_2,\ 0\le i_2\le a_2-5,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0\le i_1\le a_1-5, 0\le i_2\le a_2-5, 1\le i_3\le a_3-5; x_1^{i_1}x_2^{a_2-4}\partial_3,\ 0\le i_1\le a_1-5,$$
$$x_1^{a_1-5}x_2^{i_2}\partial_3,\ 1\le i_2\le a_2-5; x_2^{i_2}x_3^{a_3-4}\partial_3,\ 0\le i_2\le a_2-5; x_1^{a_1-3}x_3^{i_3}\partial_3,\ 0\le i_3\le a_3-5.$$

Therefore, we have

$$\delta_4(V)=3a_1a_2a_3+60(a_1+a_2+a_3)-13(a_1a_2+a_1a_3+a_2a_3)-291.$$

In the case of $a_1=5, a_2\ge 6, a_3\ge 5$, we obtain the following basis:

$$x_2^{a_2-5}x_3^{i_3}\partial_1,\ 1\le i_3\le a_3-4; x_1x_3^{i_3}\partial_1,\ 0\le i_3\le a_3-5; x_2^{a_2-4}\partial_1; x_3^{a_3-4}\partial_2,$$
$$x_1x_3^{i_3}\partial_2,\ 0\le i_3\le a_3-5; x_2^{a_2-4}\partial_2; x_2^{i_2}x_3^{i_3}\partial_2,\ 1\le i_2\le a_2-5, 0\le i_3\le a_3-4,$$
$$x_2^{i_2}x_3^{i_3}\partial_3,\ 0\le i_2\le a_2-5, 1\le i_3\le a_3-4, x_1x_3^{i_3}\partial_3,\ 0\le i_3\le a_3-5; x_2^{a_2-4}\partial_3.$$

Therefore, we have

$$\delta_4(V)=2a_2a_3-7a_2-5a_3+19.$$

Similarly, we can obtain the basis of Lie algebra for $a_1\ge 5, a_2=5, a_3\ge 5$ and $a_1\ge 6, a_2\ge 6, a_3=5$.

Furthermore, we need to show that if when $a_1\ge 6, a_2\ge 6, a_3\ge 6$, then

$3a_1a_2a_3+60(a_1+a_2+a_3)-13(a_1a_2+a_1a_3+a_2a_3)-291\le\frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)}$
$+56(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3}+\frac{1+a_1a_2a_3}{1-a_1+a_1a_3}+\frac{1+a_1a_2a_3}{1-a_2+a_1a_2})-13(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)}+\frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)}$
$+\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)})-240.$

After simplification we obtain

$4(a_1a_2+a_2a_3+a_1a_3)+a_1(a_2-5)+a_2(a_3-5)+a_3(a_1-5)+4a_1^2[a_2(a_3-5)+a_3(a_2-5)]+3a_2^2[a_1(a_3-4)+a_3(a_1-5)]+5a_3^2[a_1(a_2-5)+a_2(a_1-4)]+2(a_1^2+a_2^2+a_3^2)+3(a_1^3a_2+$

$a_2^3a_3+a_3^3a_1)+2a_1^2a_2^2a_3^2+5(a_1a_2^2a_3+a_1a_2a_3^2)+2a_1^2a_2a_3+a_1a_2a_3[2a_1-9]+a_1^3a_2a_3^2(a_3-5)$ $(a_2-5)+a_1^2a_3^2(a_3-5)(a_1a_2-5)+a_1^2a_2a_3^2(a_3+a_2-6)+3a_1a_2a_3^3(a_1-5)+a_1^2a_2^3a_3(a_3-5)$ $(a_1-4)+a_1^2a_2^2(a_1-5)(a_2a_3-4)+a_1^3a_2a_3(a_2-5)+a_1^2a_2^2a_3(a_1-4+(a_3-5))+a_1a_2^2a_3^3(a_2-$ $5)(a_1-4)+a_2^2a_3^2(a_2-5)(a_1a_3-5)+10\geq 0$.

Similarly, we can prove Conjecture 1 for $a_1\geq 5, a_2=5, a_3\geq 5; a_1\geq 6, a_2\geq 6, a_3=5$ and $a_1=5, a_2\geq 6, a_3\geq 5$. □

Proposition 8. *Let $(V,0)$ be a trinomial singularity of type 4 which is defined by $f=x_1^{a_1}+x_2^{a_2}+x_3^{a_3}x_2$ ($a_1\geq 6, a_2\geq 6, a_3\geq 5$) with weight type $(\frac{1}{a_1},\frac{1}{a_2},\frac{a_2-1}{a_2a_3};1)$. Then*

$$\delta_4(V)=3a_1a_2a_3+60a_1+56(a_2+a_3)-13(a_1a_2+a_1a_3+a_2a_3)-257.$$

Furthermore, assuming that $mult(f)\geq 6$, we have
$3a_1a_2a_3+60a_1+56(a_2+a_3)-13(a_1a_2+a_1a_3+a_2a_3)-257\leq \frac{3a_2^2a_1a_3}{a_2-1}+56(a_1+a_2+\frac{a_2a_3}{a_2-1})$ $-13(a_1a_2+\frac{a_1a_2a_3}{a_2-1}+\frac{a_2^2a_3}{a_2-1})-240$.

Proof. It is easy to see that the moduli algebra $M_4(V)$ has dimension $(a_1a_2a_3-4(a_1a_2+a_1a_3+a_2a_3)+16(a_2+a_3)+17a_1-68)$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0\leq i_1\leq a_1-5; 0\leq i_2\leq a_2-5; 0\leq i_3\leq a_3-5; x_1^{i_1}x_3^{a_3-4}, 0\leq i_2\leq a_2-5\}.$$

In order to compute a derivation D of $M_4(V)$ it suffices to indicate its values on the generators x_1,x_2,x_3 which can be written in terms of bases. Thus we can write

$$Dx_j=\sum_{i_1=0}^{a_1-5}\sum_{i_2=0}^{a_2-5}\sum_{i_3=0}^{a_3-5}c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}+\sum_{i_1=0}^{a_1-5}c^j_{i_1,0,a_3-4}x_1^{i_1}x_3^{a_3-4},\ j=1,2,3.$$

We obtain the following bases of the Lie algebra in question:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1\leq i_1\leq a_1-5, 0\leq i_2\leq a_2-5, 0\leq i_3\leq a_3-5; x_1^{i_1}x_3^{a_3-4}\partial_1,\ 1\leq i_1\leq a_1-5,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1\leq i_1\leq a_1-5, 1\leq i_2\leq a_2-5, 0\leq i_3\leq a_3-5; x_1^{i_1}x_3^{a_3-4}\partial_2,\ 0\leq i_1\leq a_1-5,$$
$$x_2^{i_2}x_3^{i_3}\partial_2,\ 1\leq i_2\leq a_2-5, 0\leq i_3\leq a_3-5, x_1^{i_1}x_2^{a_2-5}\partial_3,\ 0\leq i_1\leq a_1-5$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0\leq i_1\leq a_1-5, 0\leq i_2\leq a_2-5, 1\leq i_3\leq a_3-5, x_1^{i_1}x_3^{a_3-4}\partial_3,\ 0\leq i_1\leq a_1-5.$$

Therefore, we have

$$\delta_4(V)=3a_1a_2a_3+60a_1+56(a_2+a_3)-13(a_1a_2+a_1a_3+a_2a_3)-257.$$

Furthermore, we need to show that if when $a_1\geq 6, a_2\geq 6, a_3\geq 5$, then

$3a_1a_2a_3+60a_1+56(a_2+a_3)-13(a_1a_2+a_1a_3+a_2a_3)-257\leq \frac{3a_2^2a_1a_3}{a_2-1}+56(a_1+a_2+\frac{a_2a_3}{a_2-1})$ $-13(a_1a_2+\frac{a_1a_2a_3}{a_2-1}+\frac{a_2^2a_3}{a_2-1})-240$.

After solving above inequality, we get

$$\frac{a_1a_3(2a_2-10)}{a_2-5}+a_2a_3+a_3(a_2-3)+\frac{6a_3}{a_2-4}+\frac{a_1[a_2(a_3-4)+5]}{a_2-4}\geq 0.$$

□

Proposition 9. *Let $(V,0)$ be a trinomial singularity of type 5 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ ($a_1 \geq 5, a_2 \geq 5, a_3 \geq 6$) with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1)$. Then*

$$\delta_4(V) = \begin{cases} 3a_1a_2a_3 + 56(a_1+a_2) + 64a_3 - 13(a_1a_2 + a_1a_3 + a_2a_3) & \\ -274; & a_1 \geq 6, a_2 \geq 6, a_3 \geq 6 \\ 2a_2a_3 - 9a_2 - 4a_3 + 20; & a_1 = 5, a_2 \geq 5, a_3 \geq 6 \end{cases}$$

Furthermore, assuming that $a_1 \geq 6, a_2 \geq 6, a_3 \geq 6$, we have

$3a_1a_2a_3 + 56(a_1+a_2) + 64a_3 - 13(a_1a_2 + a_1a_3 + a_2a_3) - 274 \leq \frac{3a_3(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + 56(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1} + a_3) - 13(\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + \frac{a_3(a_1a_2-1)}{a_1-1} + \frac{a_3(a_1a_2-1)}{a_2-1}) - 240.$

Proof. It is easy to see that the moduli algebra $M_4(V)$ has dimension $a_1a_2a_3 - 4(a_1a_2 + a_1a_3 + a_2a_3) + 16(a_1+a_2) + 18a_3 - 72$ and has a monomial basis of the form

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0 \leq i_1 \leq a_1-5; 0 \leq i_2 \leq a_2-5; 0 \leq i_3 \leq a_3-5; x_1^{a_1-4}x_3^{i_3}, 0 \leq i_3 \leq a_3-5; x_2^{a_2-4}x_3^{i_3}, 0 \leq i_3 \leq a_3-5\},$$

In order to compute a derivation D of $M_4(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of bases. Thus, we can write

$$Dx_j = \sum_{i_1=0}^{a_1-5}\sum_{i_2=0}^{a_2-5}\sum_{i_3=0}^{a_3-5} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_3=0}^{a_3-5} c^j_{a_1-4,0,i_3}x_1^{a_1-4}x_3^{i_3} + \sum_{i_3=0}^{a_3-5} c^j_{0,a_2-4,i_3}x_2^{a_2-4}x_3^{i_3},\ j = 1,2,3.$$

We obtain the following bases of the Lie algebra in question:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1-5, 0 \leq i_2 \leq a_2-5, 0 \leq i_3 \leq a_3-5; x_1^{a_1-4}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-5,$$
$$x_2^{a_2-4}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-5; x_2^{a_2-5}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-5,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 0 \leq i_1 \leq a_1-5, 1 \leq i_2 \leq a_2-5, 0 \leq i_3 \leq a_3-5; x_1^{a_1-4}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-5,$$
$$x_2^{a_2-4}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-5; x_1^{a_1-5}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-5,$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_1 \leq a_1-5, 0 \leq i_2 \leq a_2-5, 1 \leq i_3 \leq a_3-5; x_1^{a_1-4}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-5,$$
$$x_2^{a_2-4}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-5.$$

Therefore, we have

$$\delta_4(V) = 3a_1a_2a_3 + 56(a_1+a_2) + 64a_3 - 13(a_1a_2 + a_1a_3 + a_2a_3) - 274.$$

In the case of $a_1 = 5, a_2 \geq 5, a_3 \geq 6$, we obtain the following basis:

$$x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \leq i_2 \leq a_2-5, 0 \leq i_3 \leq a_3-5; x_2^{a_2-4}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-5,$$
$$x_1x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-5; x_2^{a_2-4}x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-5,$$
$$x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_2 \leq a_2-5, 1 \leq i_3 \leq a_3-5; x_1x_3^{i_3}\partial_2,\ 0 \leq i_3 \leq a_3-5,$$
$$x_1x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-5.$$

We have

$$\delta_4(V) = 2a_2a_3 - 9a_2 - 4a_3 + 20.$$

Next, we need to show that if when $a_1 \geq 6, a_2 \geq 6, a_3 \geq 6$, then

$3a_1a_2a_3 + 56(a_1+a_2) + 64a_3 - 13(a_1a_2 + a_1a_3 + a_2a_3) - 274 \leq \frac{3a_3(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + 56(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1} + a_3) - 13(\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + \frac{a_3(a_1a_2-1)}{a_1-1} + \frac{a_3(a_1a_2-1)}{a_2-1}) - 240.$

After simplification, we obtain

$$a_1(a_1-5)(a_2-4)(a_3+(a_1-3)a_2(a_2-5)a_3)+a_1^2(a_3-4)(a_2-3)+a_2^2a_1+4a_1(a_2-4)$$
$$+4a_2(a_1-4)+4a_3(a_1-3)+10a_1a_2+12a_1a_3+3a_2a_3+20a_2+a_1a_2(a_1-4)$$
$$+(a_1-3)a_2(a_2-4)(a_3-3)+(a_1-4)(a_3-5)+20\geq 0.$$

Similarly, we can prove that Conjecture 1 is also true for $a_1=5, a_2\geq 5, a_3\geq 6$, □

Proof of Theorem 3. It follows from Proposition 3 that Theorem 3 is true. □

Proof of Theorem 4. Since f is a binomial singularity, f is one of the following three types (see Corollary 1):
Type A. $x_1^{a_1}+x_2^{a_2}$,
Type B. $x_1^{a_1}x_2+x_2^{a_2}$,
Type C. $x_1^{a_1}x_2+x_2^{a_2}x_1$.
Theorem 4 is a corollary of Remark 3, Proposition 4, and Proposition 5. □

Proof of Theorem 5. Since f is a trinomial singularity, f is one of the following five types (see Proposition 2):
Type 1. $x_1^{a_1}+x_2^{a_2}+x_3^{a_3}$,
Type 2. $x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}$,
Type 3. $x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}x_1$,
Type 4. $x_1^{a_1}+x_2^{a_2}+x_3^{a_3}x_2$,
Type 5. $x_1^{a_1}x_2+x_2^{a_2}x_1+x_3^{a_3}$.
Theorem 5 is a corollary of Remark 4, Propositions 6–9. □

Proof of Theorem 6. It is easy to see, from Remark 3, Propositions 4 and 5, and Remark 3, Propositions 4 and 5 in [34], the inequality $\delta_4(V)<\delta_3(V)$ holds true. □

Proof of Theorem 7. It follows from Remark 4, Propositions 6–9, and Remark 4, and Propositions 6–9 in [34], that the inequality $\delta_4(V)<\delta_3(V)$ holds true. □

4. Conclusions

The $\delta_k(V)$ is a new analytic invariant of singularities. It is an interesting question to obtain the formula for computing $\delta_k(V)$. In this article we obtain the formulas of $\delta_4(V)$ for fewnomial isolated singularities (binomial, trinomial). We also verify a sharp upper estimate conjecture for the $\delta_4(V)$ for large class of singularities. Moreover, we verify another inequality conjecture: $\delta_{(k+1)}(V)<\delta_k(V)$, $k=3$ for low-dimensional fewnomial singularities. The present work may also shed some light on verifying the two inequality conjectures for general k.

Author Contributions: Conceptualization, N.H., S.S.-T.Y. and H.Z.; methodology, N.H., S.S.-T.Y. and H.Z.; software, N.H., S.S.-T.Y. and H.Z.; validation, N.H., S.S.-T.Y. and H.Z.; formal analysis, N.H., S.S.-T.Y. and H.Z.; investigation, N.H., S.S.-T.Y. and H.Z.; resources, N.H., S.S.-T.Y. and H.Z.; data curation, N.H., S.S.-T.Y. and H.Z; writing—original draft preparation, N.H., S.S.-T.Y. and H.Z.; writing—review and editing, N.H., S.S.-T.Y. and H.Z.; visualization, N.H., S.S.-T.Y. and H.Z.; supervision, N.H., S.S.-T.Y. and H.Z. All authors have read and agreed to the published version of the manuscript.

Funding: Both Yau and Zuo are supported by NSFC Grants 11961141005. Zuo is supported by NSFC Grant 11771231. Yau is supported by the Tsinghua University start-up fund and Tsinghua University Education Foundation fund (042202008). Hussain is supported by innovation team project of Humanities and Social Sciences in Colleges and Universities of Guangdong Province (No.: 2020wcxtd008).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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