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An Equivalent Linear Programming Form of General Linear Fractional Programming: A Duality Approach

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Abstract: Linear fractional programming has been an important planning tool for the past four decades. The main contribution of this study is to show, under some assumptions, for a linear programming problem, that there are two different dual problems (one linear programming and one linear fractional functional programming) that are equivalent. In other words, we formulate a linear programming problem that is equivalent to the general linear fractional functional programming problem. These equivalent models have some interesting properties which help us to prove the related duality theorems in an easy manner. A traditional data envelopment analysis (DEA) model is taken, as an instance, to illustrate the applicability of the proposed approach.

Keywords: linear fractional programming; linear programming; duality; data envelopment analysis (DEA)

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1. Introduction

Fractional programming has been attracting the attention of a fair number of researchers all over the world (see [1]). The author of [2] presented the last bibliography with 520 entries, mainly from the period 1997–2019, which emphasizes the amount of effort that has been made in the field. This bibliography is an extension of eight other bibliographies previously published by him. Some researchers have been studying the role of duality in fractional programming. The authors of [3] dealt with programming with linear fractional functionals and utilized a transformation of variables to obtain an equivalent linear programming form of general linear fractional programming. The author of [4] studied the duality for a special class of linear fractional functionals programming problem where its dual is a linear programming problem. The authors of [5] showed that the dual of a linear fractional program, under some assumptions, is itself a fractional linear program. The author of [6] investigated the primal–dual relation in linear fractional programming when the constraints are in an equation form. It was concluded that some primal–dual relations in such problems may not hold. The author of [7] extended a dual linear fractional program of a given linear fractional program which resulted in necessary and sufficient conditions for the optimality of a given feasible solution. The authors of [8] extended the duality in linear fractional programming and developed a dual linear programming model for a general maximization linear fractional functionals programming problem. They proved duality theorems in linear fractional functionals programming. The authors of [9] and [10] extended fuzzy duality in linear fractional programming problems under uncertainty. Recently, the author of [11] presented a linear fractional programming problem and its dual problem under a fuzzy environment using hyperbolic membership functions.

In this paper, we first show that the dual linear programming model of the dual model proposed by [8] is the same linear programming model that was formulated by [3] as an equivalent model of linear fractional functionals programming problems. Next, we utilize some interesting properties obtained in this study and also the duality properties to prove (in an easy manner) that the linear programming and linear fractional functional programming models are equivalent. There is an *involutory property of duality* in linear programming which states that the dual of the dual is the original (primal) model (see [12]). We show that in linear fractional functional programming the dual of the dual is also a model that is equivalent to the original model. We use a numerical example to show the applicability of the proposed linear programming model.

As a special case of the general maximization linear fractional functional programming problem, we consider the well-known data envelopment analysis (DEA) approach. DEA, originated by [13], is an operations research method to evaluate the efficiency score of a set of similar decision-making units (DMUs) in which each DMU uses multiple inputs to produce multiple outputs. DEA assigns a weight (multiplier) to each input and output and to evaluate the efficiency score of a specific DMU, finds the maximum ratio of the weighted sum of outputs to the weighted sum of inputs subject to the condition that the same ratio for all DMUs must be less than or equal to unity.

2. Related Work

Consider the following general linear fractional programming:

$$\begin{array}{c} \text{maximize } f(\boldsymbol{x}) = \frac{\boldsymbol{cx}+\alpha}{\boldsymbol{dx}+\beta} \\ \text{subject to } \ \boldsymbol{x} \in X_F \\ \text{where } X_F = \{\boldsymbol{x} : \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}_n\} \end{array} \tag{1}$$

Here, $\mathbf{A} = [a_{ij}]_{m\times n}$ is the constraint (technology) matrix, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the decision variable vector, $\boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ is the right-hand side column vector, $\boldsymbol{c} = (c_1, \ldots, c_n)$ and $\boldsymbol{d} = (d_1, \ldots, d_n)$ are row vectors, α and β are arbitrarily scalar constants, and $\boldsymbol{0}_n$ is the origin in $\mathbb{R}^n$. It is assumed that $\boldsymbol{dx} + \beta > 0$ for all $\boldsymbol{x} \in X_F$, the objective function is continuously differentiable, and that the feasible region X_F is regular (nonempty and bounded).

The authors of [8] formulated the following linear programming and proved that it is the dual of general linear fractional programming (1):

$$\begin{array}{c} \text{minimize } g(\boldsymbol{y}, z) = z \\ \text{subject to } \ (\boldsymbol{y}, z) \in X_D \\ \text{where } X_D = \left\{(\boldsymbol{y}, z) : \mathbf{A}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{d}^{\mathrm{T}}z \geq \boldsymbol{c}^{\mathrm{T}}, -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} + \beta z = \alpha, \boldsymbol{y} \geq \mathbf{0}_m,\ z \text{ free}\right\} \end{array} \tag{2}$$

Here, the symbol T denotes transposition, $\boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ is a decision variable vector, and $z \in \mathbb{R}$ is a decision variable. The authors of [8] stated and proved the well-known weak duality and complementary slackness theorems for models (1) and (2).

3. Dual of the Dual in Linear Fractional Programming

This section shows that in contrast to linear programming where the dual linear program is itself a linear program, the dual linear fractional programming problem is not necessarily a linear fractional programming problem. The linear programming of model (2) is the dual of the general linear fractional programming (1) (see [8]). We first formulate the dual linear programming of model (2) to obtain the following linear programming problem and then show that the following model is equivalent (not necessarily equal) to the general linear fractional programming (1):

$$\begin{aligned} &\text{maximize } h(\boldsymbol{x},\lambda) = \boldsymbol{cx} + \alpha\lambda \\ &\text{subject to } (\boldsymbol{x},\lambda) \in X_L \\ &\text{where } X_L = \{(\boldsymbol{x},\lambda) : \boldsymbol{dx} + \beta\lambda = 1,\ \boldsymbol{Ax} - \lambda\boldsymbol{b} \le \boldsymbol{0}_m,\ \boldsymbol{x} \ge \boldsymbol{0}_n, \quad \lambda \text{ free}\} \end{aligned} \tag{3}$$

As a result, there are two various models which are the dual of the dual model (2), i.e., linear fractional programming (1) and linear programming (3) problems. In other words, we can conclude that a linear programming model and a fractional programming model might have a common dual problem. We utilize the dual model proposed by [8] to prove that models (1) and (3) are equivalent. Toward this end, we first prove $(\boldsymbol{x}, 0) \notin X_L$ and then state some interesting propositions which show the relationship between the feasible regions and objective functions of the linear fractional programming problem (1) and linear programming (3) problems.

Lemma 1. *For all* $(\boldsymbol{x},\lambda) \in X_L$ *we have* $\lambda \neq 0$.

Proof. Suppose that $(\mathbf{x}, 0) \in \mathrm{X_L}$, then $\mathrm{D} = \{\mathbf{x} : \mathbf{dx} = 1, \mathbf{Ax} \le \mathbf{0}_\mathrm{m},\ \mathbf{x} \ge \mathbf{0}_\mathrm{n}\}$ is a nonempty set. Subsequently, $\mathbf{x} \in \mathrm{D}$ is a recession direction for $\mathrm{X_F}$ (for more details about recession direction, we refer the reader to [12]) which contradicts the given assumption that $\mathrm{X_F}$ is bounded. □

Referencing Lemma 1 helps us to validate the following corollaries:

Corollary 1. $(\mathbf{x},\lambda) \in \mathbb{R}^{\mathrm{n}+1}$ *is a feasible solution of* X_L *if and only if* $\frac{1}{\lambda}\mathbf{x} \in \mathbb{R}^n$ *is a feasible solution of* X_F.

Corollary 2. $\boldsymbol{x} \in \mathbb{R}^n$ *is a feasible solution for* X_F *if and only if* $\left(\frac{\boldsymbol{x}}{\boldsymbol{dx}+\beta}, \frac{1}{\boldsymbol{dx}+\beta}\right) \in \mathbb{R}^{n+1}$ *is a feasible solution of* X_L.

Corollary 3. *For a given* $\boldsymbol{x} \in \mathbb{R}^n$ *in* X_F, $f(\boldsymbol{x}) = h\left(\frac{\boldsymbol{x}}{\boldsymbol{dx}+\beta}, \frac{1}{\boldsymbol{dx}+\beta}\right)$.

Corollary 4. *For a given* $(\boldsymbol{x},\lambda) \in \mathbb{R}^{n+1}$ *in* X_L, $h\,(\boldsymbol{x},\lambda) = f\left(\frac{\boldsymbol{x}}{\lambda}\right)$.

According to Lemma 1 and the given assumptions for model (1), all denominators in Corollaries 1–4 are nonzero. These corollaries help us to prove the following theorem.

Theorem 1. *The linear programming model* (3) *and the general linear fractional programming* (1) *are equivalent.*

Proof. Let $\boldsymbol{x}^*$ be an optimal solution for the general linear fractional programming (1). We show $\left(\frac{\boldsymbol{x}^*}{\boldsymbol{dx}^*+\beta}, \frac{1}{\boldsymbol{dx}^*+\beta}\right)$ is an optimal solution for the linear programming model (3). From the optimality conditions for dual linear programming, there is an optimal solution for model (2), say $(\boldsymbol{y}^*, z^*)$, in which $f(\boldsymbol{x}^*) = g(\boldsymbol{y}^*, z)$. From Corollary 2, $\left(\frac{\boldsymbol{x}^*}{\boldsymbol{dx}^*+\beta}, \frac{1}{\boldsymbol{dx}^*+\beta}\right) \in X_L$ and clearly $h\left(\frac{\boldsymbol{x}^*}{\boldsymbol{dx}^*+\beta}, \frac{1}{\boldsymbol{dx}^*+\beta}\right) = g(\boldsymbol{y}^*, z)$. Hence, $\left(\frac{\boldsymbol{x}^*}{\boldsymbol{dx}^*+\beta}, \frac{1}{\boldsymbol{dx}^*+\beta}\right)$ is an optimal solution for model (3) and $h\left(\frac{\boldsymbol{x}^*}{\boldsymbol{dx}^*+\beta}, \frac{1}{\boldsymbol{dx}^*+\beta}\right) = f(\boldsymbol{x}^*)$.

Conversely, let $(\boldsymbol{x}^*, \lambda^*)$ be the optimal solution of the linear programming model (3). We prove that $\frac{1}{\lambda^*}\boldsymbol{x}^*$ is an optimal solution for the linear fractional programming (1). From Corollary 1, $\frac{1}{\lambda^*}\boldsymbol{x}^* \in X_F$, which leads to

$$f\left(\frac{1}{\lambda^*}\boldsymbol{x}^*\right) = \frac{(\boldsymbol{c}(\boldsymbol{x}^*/\lambda^*) + \alpha)}{(\boldsymbol{d}(\boldsymbol{x}^*/\lambda^*) + \beta)} = \frac{\boldsymbol{cx}^* + \alpha\lambda^*}{\boldsymbol{dx}^* + \beta\lambda^*} = \boldsymbol{cx}^* + \alpha\lambda^* = h(\boldsymbol{x}^*, \lambda^*)$$

On the contrary, suppose $\boldsymbol{w}^*$ is an optimal solution for model (1) with $f(\boldsymbol{w}^*) > f\left(\frac{1}{\lambda^*}\boldsymbol{x}^*\right)$. According to Corollary 2, the vector $\left(\frac{\boldsymbol{w}^*}{\boldsymbol{d}\boldsymbol{w}^*+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{w}^*+\beta}\right) \in X_L$ and

$$h\left(\frac{\boldsymbol{w}^*}{\boldsymbol{d}\boldsymbol{w}^*+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{w}^*+\beta}\right) = f(\boldsymbol{w}^*) > f\left(\frac{1}{\lambda^*}\boldsymbol{x}^*\right) = h(\boldsymbol{x}^*, \lambda^*)$$

which shows that there is a feasible solution with a higher objective value than the optimal solution, which is impossible. □

The following corollaries can be directly obtained from Theorem 1:

Corollary 5. *If $\boldsymbol{x}^*$ is an optimal solution of the general fractional programming model* (1), *then $\left(\frac{1}{\lambda^*}\boldsymbol{x}^*\right)$ is an optimal solution of the equivalent linear programming model* (3).

Corollary 6. *If $(\boldsymbol{x}^*, \lambda^*)$ is an optimal solution of the linear programming model* (3), *then $\left(\frac{1}{\lambda^*}\boldsymbol{x}^*\right)$ is an optimal solution of the equivalent linear fractional programming model* (1).

Theorem 2 (Weak Duality Theorem). *For all $\boldsymbol{x}$ in X_F and for all $(\boldsymbol{y}, z)$ in X_D, we have $f(\boldsymbol{x}) \leq g(\boldsymbol{y}, z)$.*

Proof. Given $\boldsymbol{x}_0 \in X_F$ and $(\boldsymbol{y}_0, z_0) \in X_D$ and considering Corollaries 2 and 3, we have $\left(\frac{\boldsymbol{x}_0}{\boldsymbol{d}\boldsymbol{x}_0+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{x}_0+\beta}\right) \in X_L$ and $f(\boldsymbol{x}_0) = h\left(\frac{\boldsymbol{x}_0}{\boldsymbol{d}\boldsymbol{x}_0+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{x}_0+\beta}\right)$. Referencing the weak duality theorem for linear programming models (2) and (3) reveals that $h(\boldsymbol{x}_0, \lambda) = f(\boldsymbol{x}_0) \leq g(\boldsymbol{y}_0, z_0)$ which completes the proof. □

Theorem 3. *If $\boldsymbol{x}_0 \in X_F$ and $(\boldsymbol{y}_0, z_0) \in X_D$ with $f(\boldsymbol{x}_0) = g(\boldsymbol{y}_0, z_0)$, then $\boldsymbol{x}_0$ and $(\boldsymbol{y}_0, z_0)$ are the optimal solutions to their respective problems.*

Proof. Considering Corollaries 2 and 3, we have $\left(\frac{\boldsymbol{x}_0}{\boldsymbol{d}\boldsymbol{x}_0+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{x}_0+\beta}\right) \in X_L$ and $f(\boldsymbol{x}_0) = h\left(\frac{\boldsymbol{x}_0}{\boldsymbol{d}\boldsymbol{x}_0+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{x}_0+\beta}\right)$ which result in $h\left(\frac{\boldsymbol{x}_0}{\boldsymbol{d}\boldsymbol{x}_0+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{x}_0+\beta}\right) = g(\boldsymbol{y}_0, z_0)$. From the optimality conditions for linear programming problem models (2) and (3), $\left(\frac{\boldsymbol{x}_0}{\boldsymbol{d}\boldsymbol{x}_0+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{x}_0+\beta}\right)$ and $(\boldsymbol{y}_0, z_0)$ are the optimal solutions for models (3) and (2), respectively. From Corollary 5, $\boldsymbol{x}_0$ is an optimal solution for model (1) which completes the proof. □

Corollary 7. *Let $\boldsymbol{x}^*$ and $(\boldsymbol{y}^*, z^*)$ be the optimal solutions of models (1) and (2), respectively. $\left(\overline{\boldsymbol{x}}^*, \overline{\lambda}^*\right) = \left(\frac{\boldsymbol{x}^*}{\boldsymbol{d}\boldsymbol{x}^*+\beta}, \frac{1}{\boldsymbol{d}\boldsymbol{x}^*+\beta}\right)$ is an optimal solution of model* (3) *and $f(x^*) = g(\boldsymbol{y}^*, z^*) = h\left(\overline{\boldsymbol{x}}^*, \overline{\lambda}^*\right)$.*

Analogously, the strong direct duality and complementary slackness theorems can be proved.

4. Numerical Example

The authors of [8] employed the following fractional linear programming model, as an illustrative example, including two decision variables:

$$\begin{array}{rl} \text{maximize } f(x_1, x_2) = & \frac{3x_1+5x_2}{x_1+x_2+2} \\ \text{subject to } & x_1 + x_2 \leq 6 \\ & 3x_1 + 8x_2 \leq 24 \\ & x_1, x_2 \geq 0 \end{array} \tag{4}$$

The authors verified that the following linear programming model is the dual of the fractional linear programming (4):

$$\begin{aligned} &\text{minimize } g(y_1, y_2, z) = z \\ &\text{subject to } \; y_1 + \; 3y_2 + z \geq 3 \\ &\quad y_1 + \; 8y_2 + z \geq 5 \\ &\quad 6y_1 + 24y_2 - 2z = 0 \\ &\quad y_1, y_2 \geq 0, \; z \text{ free} \end{aligned} \tag{5}$$

By applying Theorem 1, we obtain the following linear programming problem that is equivalent to the fractional linear programming model (4):

$$\begin{aligned} &\text{maximize } h(x_1, x_2) = 3x_1 + 5x_2 \\ &\text{subject to } \; x_1 + \; x_2 + 2\lambda = 1 \\ &\quad x_1 + \; x_2 - 6\lambda \leq 0 \\ &\quad 3x_1 + 8x_2 - 24\lambda \leq 0 \\ &\quad x_1, x_2 \geq 0, \; \lambda \text{ free} \end{aligned} \tag{6}$$

It is easy to obtain the optimal solution $\left(x_1^*, x_2^*, \lambda^*\right) = \left(0, \frac{3}{5}, \frac{1}{5}\right)$ for the linear programming problem (6) and therefore applying Corollary 6 $\left(\lambda^* x_1^*, \lambda^* x_2^*\right) = (0, 3)$ is an optimal solution for the fractional linear programming problem (4) that is consistent with the results obtained in [8]. Note that the optimal objective value of model (6), $h^* = h\left(0, \frac{3}{5}\right) = 3$, is equal to those for model (4), $f^* = f(0, 3) = \frac{15}{5} = 3$, as we expected.

5. A Special Case (DEA Method)

In this section, we consider the fractional functional programming problem in DEA as a special case of the general linear fractional functional programming problem (1) where $\alpha = \beta = 0$:

$$\begin{aligned} &\text{maximize } f(\boldsymbol{x}) = \frac{\boldsymbol{cx}}{\boldsymbol{dx}} \\ &\text{subject to } \; \boldsymbol{x} \in X_F \\ &\text{where } X_F = \{\boldsymbol{x} : \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}_n\} \end{aligned} \tag{7}$$

According to the proposed approach in this study, the following linear programming model is equivalent to the above linear fractional model:

$$\begin{aligned} &\text{maximize } f(\boldsymbol{x}) = \boldsymbol{cx} \\ &\text{subject to } \; \boldsymbol{x} \in X_L \\ &\text{where } X_L = \{\boldsymbol{x} : \boldsymbol{dx} = 1, \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}_n\} \end{aligned} \tag{8}$$

As a result, to obtain an equivalent linear programming form of the linear fractional model (7), we easily set the denominator of the model equal to 1, include it in the feasible region, and maximize the numerator, as is done in the equivalent linear programming model (8).

It should be highlighted here that the following model is the dual of both models (7) and (8):

$$\begin{aligned} &\text{minimize } g(\boldsymbol{y}, z) = z \\ &\text{subject to } \; (\boldsymbol{y}, z) \in X_D \\ &\text{where } X_D = \left\{(\boldsymbol{y}, z) : \mathbf{A}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{d}^{\mathrm{T}} z \geq \boldsymbol{c}^{\mathrm{T}}, -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} = 0, \boldsymbol{y} \geq \mathbf{0}_m, \; z \text{ free}\right\} \end{aligned} \tag{9}$$

Assume there are n DMUs ($\text{DMU}_j : j = 1, \ldots, n$) of which DMU_j consumes m inputs $\boldsymbol{x}_j = \left(x_{1j}, \ldots, x_{mj}\right)$ to produce s outputs $\boldsymbol{y}_j = \left(y_{1j}, \ldots, y_{sj}\right)$. The authors of [13] formulated the following fractional functional programming to evaluate the efficiency score of

$\mathrm{DMU}_{o\in\{1,\ldots,n\}}$, the unit under consideration, which is known as the Charnes, Cooper, and Rhodes (CCR) model under the constant returns-to-scale (CRS) assumption:

$$\begin{aligned} &\text{maximize } e_F(\boldsymbol{u},\boldsymbol{v}) = \frac{(\boldsymbol{y}_0,\mathbf{0}_m^T)(\boldsymbol{u},\boldsymbol{v})}{(\mathbf{0}_s^T,\boldsymbol{x}_0)(\boldsymbol{u},\boldsymbol{v})} \\ &\text{subject to } (\boldsymbol{u},\boldsymbol{v}) \in X_F \\ &\text{where } X_F = \left\{ (\boldsymbol{u},\boldsymbol{v}) : \begin{bmatrix} \boldsymbol{y}_1 & -\boldsymbol{x}_1 \\ \vdots & \vdots \\ \boldsymbol{y}_n & -\boldsymbol{x}_n \end{bmatrix} (\boldsymbol{u},\boldsymbol{v}) \leq \mathbf{0}_n, \boldsymbol{u} \geq \mathbf{0}_s, \boldsymbol{v} \geq \mathbf{0}_m \right\} \end{aligned} \tag{10}$$

Here, $\begin{bmatrix} \boldsymbol{y}_1 & -\boldsymbol{x}_1 \\ \vdots & \vdots \\ \boldsymbol{y}_n & -\boldsymbol{x}_n \end{bmatrix}_{n\times(s+m)}$ is the constraint matrix, $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_s \end{bmatrix}$ are (decision variables) weights (multipliers) for inputs and outputs, respectively. In fact, since the weights in model (10) are derived from the data in favor of the unit under evaluation instead of being fixed in advance for all the units, the model is called the multiplier model (for more details, see [14]).

According to [8], the following linear programming model is the dual of fractional functional programming (10):

$$\begin{aligned} &\text{minimize } e_D(\theta) = \theta \\ &\text{subject to } (\boldsymbol{\lambda},\theta) \in X_D \\ &\text{where } X_D = \left\{ (\boldsymbol{\lambda},\theta) : \begin{bmatrix} \boldsymbol{y}_1^T & \cdots & \boldsymbol{y}_n^T \\ -\boldsymbol{x}_1^T & \cdots & -\boldsymbol{x}_n^T \end{bmatrix} \boldsymbol{\lambda} + \begin{pmatrix} \mathbf{0}_s \\ \boldsymbol{x}_o^T \end{pmatrix} \theta \geq \begin{pmatrix} \boldsymbol{y}_o^T \\ \mathbf{0}_m \end{pmatrix}, \boldsymbol{\lambda} \geq \mathbf{0}_n, \theta \text{ free} \right\} \end{aligned} \tag{11}$$

Here, $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is the intensity vector and, as a matter of fact, the nonnegative linear combination vector of DMUs $\lambda_1 \begin{pmatrix} \boldsymbol{y}_1^T \\ -\boldsymbol{x}_1^T \end{pmatrix} + \lambda_2 \begin{pmatrix} \boldsymbol{y}_2^T \\ -\boldsymbol{x}_2^T \end{pmatrix} + \ldots + \lambda_n \begin{pmatrix} \boldsymbol{y}_n^T \\ -\boldsymbol{x}_n^T \end{pmatrix}$ is compared with $\begin{pmatrix} \boldsymbol{y}_o^T \\ -\theta\boldsymbol{x}_o^T \end{pmatrix}$. If the optimal θ, denoted by θ^*, is less than one, then the obtained linear combination vector outperforms DMU_o.

Since the feasible region of the linear programming (11) envelops all data by the efficient frontier, it is called the *envelopment* model (for more details, see [15]).

Considering the proposed approach in this paper, the following linear programming model which is the dual of model (11) is equivalent to the fractional functional programming (10):

$$\begin{aligned} &\text{maximize } e_L(\boldsymbol{\mu},\boldsymbol{\eta}) = (\boldsymbol{y}_0,\mathbf{0}_m^T)(\boldsymbol{\mu},\boldsymbol{\eta}) \\ &\text{subject to } (\boldsymbol{\mu},\boldsymbol{\eta}) \in X_L \\ &\text{where } X_L = \left\{ (\boldsymbol{\mu},\boldsymbol{\eta}) : (\mathbf{0}_s^T,\boldsymbol{x}_0)(\boldsymbol{\mu},\boldsymbol{\eta}) = 1, \begin{bmatrix} \boldsymbol{y}_1 & -\boldsymbol{x}_1 \\ \vdots & \vdots \\ \boldsymbol{y}_n & -\boldsymbol{x}_n \end{bmatrix} (\boldsymbol{\mu},\boldsymbol{\eta}) \leq \mathbf{0}_n, \boldsymbol{\mu} \geq \mathbf{0}_s, \boldsymbol{\eta} \geq \mathbf{0}_m \right\} \end{aligned} \tag{12}$$

Here, $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_m \end{bmatrix}$ and $\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_s \end{bmatrix}$ are the multipliers for inputs and outputs, respectively.

The above linear programming problem is called the *multiplier form* of the CCR model. It should be mentioned here that to obtain the equivalent linear programming form of the linear fractional programming (10), we just set the denominator of the model equal to 1 and then consider it as a new constraint.

Let $(\boldsymbol{u}_F^*, \boldsymbol{v}_F^*)$, $(\boldsymbol{\lambda}^*, \theta^*)$, and $(\boldsymbol{u}_L^*, \boldsymbol{v}_L^*)$ be the optimal solutions for models (10)–(12), respectively. As inspection makes clear, $e_D(\boldsymbol{\lambda}^*, \theta^*) = e_L(\boldsymbol{u}_L^*, \boldsymbol{v}_L^*)$ and $(\boldsymbol{0}_s^T, \boldsymbol{x}_0)(\boldsymbol{u}_F^*, \boldsymbol{v}_F^*) \times e_F(\boldsymbol{u}_F^*, \boldsymbol{v}_F^*) = e_L(\boldsymbol{u}_L^*, \boldsymbol{v}_L^*)$ (for more details, see [16]).

As a result, there is an identical dual problem for both fractional functional programming (10) and linear programming (12) which points out that these models are equivalent.

Numerical Case Study (Healthcare Systems)

Table 1 exhibits data of twelve hospitals with two inputs (Doctors and Nurses) and two outputs (Inpatients and Outpatients), as adopted from [17]:

Table 1. Data for 12 hospitals.

Hospitals	Inputs		Outputs	
	Doctors	Nurses	Inpatients	Outpatients
H_1	25	148	33	189
H_2	46	234	65	175
H_3	32	193	70	142
H_4	39	207	63	234
H_5	23	150	87	161
H_6	36	287	91	225
H_7	50	203	84	258
H_8	16	196	68	294
H_9	21	176	33	320
H_{10}	31	159	75	295
H_{11}	45	225	73	277
H_{12}	43	207	69	293

The following fractional CCR model evaluates the performance of the first hospital:

$$\begin{gathered}
\text{maximize } e_F(u_1, u_2, v_1, v_2) = \frac{33u_1+189u_2}{25v_1+148v_2} \\
\text{subject to} \\
\frac{33u_1+189u_2}{25v_1+148v_2} \le 1\ (H_1),\ \frac{65u_1+175u_2}{46v_1+234v_2} \le 1(H_2),\ \frac{70u_1+142u_2}{32v_1+193v_2} \le 1\ (H_3) \\
\frac{63u_1+234u_2}{39v_1+207v_2} \le 1\ (H_4),\ \frac{87u_1+161u_2}{23v_1+150v_2} \le 1(H_5),\ \frac{91u_1+225u_2}{36v_1+287v_2} \le 1\ (H_6) \\
\frac{84u_1+258u_2}{50v_1+203v_2} \le 1\ (H_7),\ \frac{68u_1+294u_2}{16v_1+196v_2} \le 1(H_8),\ \frac{33u_1+320u_2}{21v_1+176v_2} \le 1\ (H_9) \\
\frac{75u_1+295u_2}{31v_1+159v_2} \le 1\ (H_{10}),\ \frac{73u_1+277u_2}{45v_1+225v_2} \le 1(H_{11}),\ \frac{69u_1+293u_2}{43v_1+207v_2} \le 1(H_{12}) \\
u_1, u_2, v_1, v_2 \ge 0
\end{gathered} \tag{13}$$

The optimal solution and the optimal objective value of the fractional CCR model are $(u_1^*, u_2^*, v_1^*, v_2^*) = (265.29,\ 950.15,\ 0,\ 539.99)$ and $e_F^*(u_1^*, u_2^*, v_1^*, v_2^*) = 0.693$, respectively, which points out that the first hospital is inefficient and its efficiency score is 0.693.

According to [8], the following linear programming model is the dual of fractional functional programming (13):

$$\begin{gathered}
\text{minimize } e_D(\theta) = \theta \\
\text{subject to} \\
33\lambda_1 + 65\lambda_2 + 70\lambda_3 + 63\lambda_4 + 87\lambda_5 + 91\lambda_6 + 84\lambda_7 + 68\lambda_8 + 33\lambda_9 + 75\lambda_{10} + 73\lambda_{11} + 69\lambda_{12} \ge 33 \\
189\lambda_1 + 175\lambda_2 + 142\lambda_3 + 234\lambda_4 + 161\lambda_5 + 225\lambda_6 + 258\lambda_7 + 294\lambda_8 + 320\lambda_9 + 295\lambda_{10} + 277\lambda_{11} + 293\lambda_{12} \ge 189 \\
25\lambda_1 + 46\lambda_2 + 32\lambda_3 + 39\lambda_4 + 23\lambda_5 + 36\lambda_6 + 50\lambda_7 + 16\lambda_8 + 21\lambda_9 + 31\lambda_{10} + 45\lambda_{11} + 43\lambda_{12} \le \theta 25 \\
148\lambda_1 + 234\lambda_2 + 193\lambda_3 + 207\lambda_4 + 150\lambda_5 + 287\lambda_6 + 203\lambda_7 + 196\lambda_8 + 176\lambda_9 + 159\lambda_{10} + 225\lambda_{11} + 207\lambda_{12} \le \theta 148 \\
\lambda_j \ge 0\ j = 1, \ldots, 12,\ \theta \text{ free}
\end{gathered} \tag{14}$$

At optimality, $\theta^* = 0.693$ & $\lambda_j^* = \begin{cases} 0.200 & j = 9 \\ 0.423 & j = 10 \\ 0 & \text{otherwise} \end{cases}$. As a result, the efficiency score of H_1 is 0.693 and this hospital is not efficient which is consistent with the results obtained by the fractional CCR model (13). Moreover, the reference set for H_1 is $\{H_9, H_{10}\}$ and $\lambda_9^* = 0.200$ and $\lambda_{10}^* = 0.423$ show the proportions contributed by H_9 and H_{10} to the point used to evaluate H_1 and hence the hospital under evaluation is technically inefficient.

The following model is also another alternative dual linear programming model of the model (14) which is not identical to the fractional CCR model (13):

$$\begin{array}{c} \text{maximize } e_L(\mu_1, \mu_2) = 33\mu_1 + 189\mu_2 \\ \text{subject to} \\ 25\eta_1 + 148\eta_2 = 1 \\ 33\mu_1 + 189\mu_2 - (25\eta_1 + 148\eta_2) \le 0\ (H_1),\ 65\mu_1 + 175\mu_2 - (46\eta_1 + 234\eta_2) \le 0\ (H_2) \\ 70\mu_1 + 142\mu_2 - (32\eta_1 + 193\eta_2) \le 0\ (H_3), 63\mu_1 + 234\mu_2 - (39\eta_1 + 207\eta_2) \le 0\ (H_4) \\ 87\mu_1 + 161\mu_2 - (23\eta_1 + 150\eta_2) \le 0\ (H_5), 91\mu_1 + 225\mu_2 - (36\eta_1 + 287\eta_2) \le 0\ (H_6) \\ 84\mu_1 + 258\mu_2 - (50\eta_1 + 203\eta_2) \le 0\ (H_7),\ 68\mu_1 + 294\mu_2 - (16\eta_1 + 196\eta_2) \le 0\ (H_8) \\ 33\mu_1 + 320\mu_2 - (21\eta_1 + 176\eta_2) \le 0\ (H_9), 75\mu_1 + 295\mu_2 - (31\eta_1 + 159\eta_2) \le 0\ (H_{10}) \\ 73\mu_1 + 277\mu_2 - (45\eta_1 + 225\eta_2) \le 0 (H_{11}), 69\mu_1 + 293\mu_2 - (43\eta_1 + 207\eta) \le 0\ (H_{12}) \\ \mu_1, \mu_2, \eta_1, \eta_2 \ge 0 \end{array} \tag{15}$$

This example illustrates that the dual of the dual model (14) is not necessarily the original model. As a matter of fact, these models are equivalent. In other words, the linear model (15) is equivalent to the fractional linear programming problem (13) (see Theorem 1) and both are the dual model. It should be highlighted here that the optimal solution for the linear model (15) is $\left(\eta_1^*, \eta_2^*, \mu_1^*, \mu_2^*\right) = \left(1.80 \times 10^{-3}, 6.45 \times 10^{-3},\ 0,\ 3.67 \times 10^{-3}\right)$ with $e_L^* = e_L\left(\mu_1^*, \mu_2^*\right) = 0.693$. In summary, the optimal objective values of models (13–15) are identical, i.e., $e_F\left(u_1^*, u_2^*, v_1^*, v_2^*\right) = e_D(\theta^*) = e_L\left(\mu_1^*, \mu_2^*\right) = 0.693$.

The main aim of this paper is to point out that, in contrast to linear programming, in fractional linear programming, the dual of the "dual" is not necessarily the original model. As a matter of fact, the dual of the dual is a linear programming problem that is equivalent to the fractional linear programming problem. Toward this end, we employed the approach of [8] to formulating a linear programming problem from a general fractional linear programming problem. Although both problems have an identical dual problem, they are equivalent, not equal. In other words, we demonstrated that for a linear programming problem, under some assumptions, two different dual problems exist that are indeed equivalent. We provided a numerical example along with one of the most important DEA models ($\alpha = \beta = 0$), to illustrate the provided theorems. We applied the method of [8] to directly find the dual of the linear fractional DEA model and then utilized the suggested approach in this study to directly obtain its equivalent linear DEA model. Extending our approach for $\alpha > 0$ ($\beta > 0$) leads to another DEA model with an input (output) orientation form under the variable returns-to-scale assumption (for a deeper discussion about different orientations and returns-to-scale assumptions, we refer the reader to [18]). An interesting further research direction is investigating duality in linear fractional programming with multiple objective functions. Finally, our approach can be applied to a wide range of practical applications, including energy [19], the banking industry [20], manufacturing systems [21], finance [22], and sport [23], among others.

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