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Solving a System of Nonlinear Integral Equations via Common Fixed Point Theorems on Bicomplex Partial Metric Space

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Abstract: In this paper, we introduce the notion of bicomplex partial metric space and prove some common fixed point theorems. The presented results generalize and expand some of the literature's well-known results. An example and application on bicomplex partial metric space is given.

Keywords: integral equations; bicomplex partial metric space; common fixed point

MSC: 47H9; 47H10; 30G35; 46N99; 54H25

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1. Introduction

Serge [1] made a pioneering attempt in the development of special algebra. He conceptualized commutative generalization of complex numbers, bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebra. Subsequently, in the 1930s, researchers contributed in this area [2–4]. The next fifty years failed to witness any advancement in this field. Later, Price [5] developed the bicomplex algebra and function theory. Recent works in this subject [6] find some significant applications in different fields of mathematical sciences as well as other "branches of science and technology (see, for instance [7–9] and reference therein)". An impressive body of work has been developed by a number of researchers. Among them, an important work on elementary functions of bicomplex numbers has been done by Luna-Elizaarrarás, Shapiro, Struppa and Vajiac [10]. Choi, Datta, Biswa, and Islam [11] proved some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. Jebril [12] proved some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. In 2017, Dhivya and Marudai [13] introduced the concept of complex partial metric space and suggested a plan to expand the results and proved the following common fixed point theorems under the rational expression contraction condition.

Theorem 1. *Let $(\mathfrak{W}, \preceq)$ be a partially ordered set and suppose that there exists a complex partial metric ϱ_{cb} in $\mathfrak{W}$ such that $(\mathfrak{W}, \varrho_{cb})$ is a complete complex partial metric space. Let $\Gamma, \Lambda : \mathfrak{W} \to \mathfrak{W}$ be a pair of weakly increasing mapping and suppose that, for every comparable $\sigma, \psi \in \mathfrak{W}$, we have either*

$$\varrho_{cb}(\Gamma\sigma, \Lambda\psi) \preceq a\frac{\varrho_{cb}(\sigma, \Gamma\sigma)\varrho_{cb}(\psi, \Lambda\psi)}{\varrho_{cb}(\sigma, \psi)} + b\varrho_{cb}(\sigma, \psi),$$

for $\varrho_{cb}(\sigma, \psi) \neq 0$ *with* $a \geq 0, b \geq 0, a + b < 1$, *or*

$$\varrho_{cb}(\Gamma\sigma, \Lambda\psi) = 0 \text{ if } \varrho_{cb}(\sigma, \psi) = 0.$$

If Γ *or* Λ *is continuous; then,* Γ *and* Λ *have a common fixed point* $\propto \in \mathfrak{W}$ *and* $\varrho_{cb}(\propto, \propto) = 0$.

In 2019, Gunaseelan and Mishra [14] proved coupled fixed point theorems on complex partial metric space using different types of contractive conditions. In 2021, Gunaseelan, Arul Joseph, Yongji, and Zhaohui [15] proved common fixed point theorems on complex partial metric space. In 2021, Beg, Kumar Datta, and Pal [16] proved fixed point theorems on bicomplex valued metric spaces. Usually, in a metric space, self distance is zero (i.e., $\varrho_{cb}(\sigma, \psi) = 0$), but, in partial metric space, the self distance need not be equal to zero. In this paper, inspired by Theorem 1, here we prove some common fixed point theorems on bicomplex partial metric space with an application.

2. Preliminaries

Throughout this paper, we denote the set of real, complex, and bicomplex numbers respectively as $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$. Segre [1] defined the bicomplex number as follows:

$$\xi = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 i_1 i_2,$$

where $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4 \in \mathbb{C}_0$, and independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we denote that the set of bicomplex numbers $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\xi : \xi = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 i_1 i_2, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_2 = \{\xi : \xi = \mathfrak{z}_1 + i_2 \mathfrak{z}_2, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}_1\},$$

where $\mathfrak{z}_1 = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 \in \mathbb{C}_1$ and $\mathfrak{z}_2 = \mathfrak{a}_3 + \mathfrak{a}_4 i_1 \in \mathbb{C}_1$. If $\xi = \mathfrak{z}_1 + i_2 \mathfrak{z}_2$ and $\eta = \mathfrak{w}_1 + i_2 \mathfrak{w}_2$ be any two bicomplex numbers, then the sum is $\xi \pm \eta = (\mathfrak{z}_1 + i_2 \mathfrak{z}_2) \pm (\mathfrak{w}_1 + i_2 \mathfrak{w}_2) = \mathfrak{z}_1 \pm \mathfrak{w}_1 + i_2(\mathfrak{z}_2 \pm \mathfrak{w}_2)$ and the product is $\xi.\eta = (\mathfrak{z}_1 + i_2 \mathfrak{z}_2)(\mathfrak{w}_1 + i_2 \mathfrak{w}_2) = (\mathfrak{z}_1 \mathfrak{w}_1 - \mathfrak{z}_2 \mathfrak{w}_2) + i_2(\mathfrak{z}_1 \mathfrak{w}_2 + \mathfrak{z}_2 \mathfrak{w}_1)$.

Definition 1. *Ref. [5] Let* ξ *and* η *be elements in* $\mathbb{C}_2$. *If* $\xi^2 = \xi$, *then* ξ *is called an idempotent element. If* $\xi \neq 0$, $\eta \neq 0$, *and* $\xi\eta = 0$, *then* ξ *and* η *are called divisors of zero.*

There are four idempotent elements in $\mathbb{C}_2$, they are $0, 1, \mathfrak{e}_1 = \frac{1+i_1 i_2}{2}, \mathfrak{e}_2 = \frac{1-i_1 i_2}{2}$ out of which $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are nontrivial such that $\mathfrak{e}_1 + \mathfrak{e}_2 = 1$ and $\mathfrak{e}_1 \mathfrak{e}_2 = 0$. Every bicomplex number $\mathfrak{z}_1 + i_2 \mathfrak{z}_2$ can be uniquely expressed as the combination of $\mathfrak{e}_1$ and $\mathfrak{e}_2$, namely

$$\xi = \mathfrak{z}_1 + i_2 \mathfrak{z}_2 = (\mathfrak{z}_1 - i_1 \mathfrak{z}_2)\mathfrak{e}_1 + (\mathfrak{z}_1 + i_1 \mathfrak{z}_2)\mathfrak{e}_2.$$

This representation of ξ is known as the idempotent representation of bicomplex number and the complex coefficients $\xi_1 = (\mathfrak{z}_1 - i_1 \mathfrak{z}_2)$ and $\xi_2 = (\mathfrak{z}_1 + i_1 \mathfrak{z}_2)$ are known as idempotent components of the bicomplex number ξ.

An element $\xi = \mathfrak{z}_1 + i_2 \mathfrak{z}_2 \in \mathbb{C}_2$ is said to be invertible if there exists another element η in $\mathbb{C}_2$ such that $\xi\eta = 1$ and η is said to be inverse (multiplicative) of ξ. Consequently, ξ is said to be the inverse (multiplicative) of η. An element which has an inverse in $\mathbb{C}_2$ is said to be the non-singular element of $\mathbb{C}_2$ and an element which does not have an inverse in $\mathbb{C}_2$ is said to be the singular element of $\mathbb{C}_2$.

An element $\xi = \mathfrak{z}_1 + i_2\mathfrak{z}_2 \in \mathbb{C}_2$ is non-singular if and only if $|\mathfrak{z}_1^2 + \mathfrak{z}_2^2| \neq 0$ and singular if and only if $|\mathfrak{z}_1^2 + \mathfrak{z}_2^2| = 0$. The inverse of ξ is defined as

$$\xi^{-1} = \eta = \frac{\mathfrak{z} - i_2\mathfrak{z}_2}{\mathfrak{z}_1^2 + \mathfrak{z}_2^2}.$$

Zero is the only element in $\mathbb{C}_0$ which does not have multiplicative inverse and in $\mathbb{C}_1$, $0 = 0 + i0$ is the only element which does not have a multiplicative inverse. We denote the set of singular elements of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\mathfrak{O}_0$ and $\mathfrak{O}_1$, respectively. However, there is more than one element in $\mathbb{C}_2$, which does not have multiplicative inverse, and we denote this set by $\mathfrak{O}_2$ and clearly $\mathfrak{O}_0 = \mathfrak{O}_1 \subset \mathfrak{O}_2$.

A bicomplex number $\xi = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} \mathfrak{a}_1 & \mathfrak{a}_2 \\ \mathfrak{a}_3 & \mathfrak{a}_4 \end{pmatrix}$$

is degenerated. In that case, ξ^{-1} exists, and it is also degenerated.

The norm $||.||$ of $\mathbb{C}_2$ is a positive real valued function and $||.|| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} ||\xi|| = ||\mathfrak{z}_1 + i_2 z_2|| &= \{|\mathfrak{z}|^2 + |\mathfrak{z}|^2\}^{\frac{1}{2}} \\ &= \left[\frac{|(\mathfrak{z}_1 - i_1\mathfrak{z}_2)|^2 + |(\mathfrak{z}_1 + i_1\mathfrak{z}_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= (\mathfrak{a}_1^2 + \mathfrak{a}_2^2 + \mathfrak{a}_1^2 + \mathfrak{a}_3^2 + \mathfrak{a}_4^2)^{\frac{1}{2}}, \end{aligned}$$

where $\xi = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 i_1 i_2 = \mathfrak{z}_1 + i_2\mathfrak{z}_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to defined norm is a normed linear space; in addition, $\mathbb{C}_2$ is complete; therefore, $\mathbb{C}_2$ is the Banach space. If $\xi, \eta \in \mathbb{C}_2$, then $||\xi\eta|| \leq \sqrt{2}||\xi||||\eta||$ holds instead of $||\xi\eta|| \leq ||\xi||||\eta||$. Therefore, $\mathbb{C}_2$ is not the Banach algebra. The partial order relation $\preceq_{i_2}$ on $\mathbb{C}_2$ is defined as: Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\xi = \mathfrak{z}_1 + i_2\mathfrak{z}_2$, $\eta = \mathfrak{w}_1 + i_2\mathfrak{w}_2 \in \mathbb{C}_2$ then $\xi \preceq_{i_2} \eta$ if and only if $\mathfrak{z}_1 \preceq \mathfrak{w}_1$ and $\mathfrak{z}_2 \preceq \mathfrak{w}_2$, i.e., $\xi \preceq_{i_2} \eta$ if one of the following conditions is satisfied:

(a) $\mathfrak{z}_1 = \mathfrak{w}_1, \mathfrak{z}_2 = \mathfrak{w}_2$,
(b) $\mathfrak{z}_1 \prec \mathfrak{w}_1, \mathfrak{z}_2 = \mathfrak{w}_2$,
(c) $\mathfrak{z}_1 = \mathfrak{w}_1, \mathfrak{z}_2 \prec \mathfrak{w}_2$, and
(d) $\mathfrak{z}_1 \prec \mathfrak{w}_1, \mathfrak{z}_2 \prec \mathfrak{w}_2$,

In particular, we can write $\xi \precnsim_{i_2} \eta$ if $\xi \preceq_{i_2} \eta$ and $\xi \neq \eta$ i.e., one of (b), (c), and (d) is satisfied, and we will write $\xi \prec_{i_2} \eta$ if only (d) is satisfied.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$, we can verify the following:

(1) $\xi \preceq_{i_2} \eta \Rightarrow ||\xi|| \leq ||\eta||$,
(2) $||\xi + \eta|| \leq ||\xi|| + ||\eta||$,
(3) $||\mathfrak{a}\xi|| = \mathfrak{a}||\xi||$, where $\mathfrak{a}$ is a non-negative real number,
(4) $||\xi\eta|| \leq \sqrt{2}||\xi||||\eta||$ and the equality holds only when at least one of ξ and η is degenerated,
(5) $||\xi^{-1}|| = ||\xi||^{-1}$ if ξ is a degenerated bicomplex number with $0 \prec \xi$,
(6) $||\frac{\xi}{\eta}|| = \frac{||\xi||}{||\eta||}$, if η is a degenerated bicomplex number.

Now, let us recall some basic concepts and notations, which will be used in the sequel.

Definition 2. *A bicomplex partial metric on a non-void set $\mathfrak{W}$ is a function $\varrho_{bcb} : \mathfrak{W} \times \mathfrak{W} \to \mathbb{C}_2^+$ such that, for all $\sigma, \psi, \vartheta \in \mathfrak{W}$:*

(i) $0 \preceq_{i_2} \varrho_{bcb}(\sigma, \sigma) \preceq_{i_2} \varrho_{bcb}(\sigma, \psi)$, *(small self-distances)*
(ii) $\varrho_{bcb}(\sigma, \psi) = \varrho_{bcb}(\psi, \sigma)$ *(symmetry)*

(iii) $\varrho_{bcb}(\sigma,\sigma) = \varrho_{bcb}(\sigma,\psi) = \varrho_{bcb}(\psi,\psi)$ *if and only if* $\sigma = \psi$ *(equality)*
(iv) $\varrho_{bcb}(\sigma,\psi) \preceq_{i_2} \varrho_{bcb}(\sigma,\vartheta) + \varrho_{bcb}(\vartheta,\psi) - \varrho_{bcb}(\vartheta,\vartheta)$ *(triangularity).*

A bicomplex partial metric space is a pair $(\mathfrak{W}, \varrho_{bcb})$ *such that* $\mathfrak{W}$ *is a non-void set, and* ϱ_{bcb} *is the bicomplex partial metric on* $\mathfrak{W}$*.*

Example 1. *Let* $\mathfrak{W} = \{1,3,4,7\}$ *be a set endowed with the classical bicomplex partial metric* $\varrho_{bcb}(\sigma,\psi) = (1+i_2)\max\{\sigma,\psi\}, \ \forall\,\sigma,\psi \in \mathfrak{W}$,

$\varrho_{cb}(\sigma,\psi)$	1	3	4	7
1	$(1+i_2)$	$(1+i_2)3$	$(1+i_2)4$	$(1+i_2)7$
3	$(1+i_2)3$	$(1+i_2)3$	$(1+i_2)4$	$(1+i_2)7$
4	$(1+i_2)4$	$(1+i_2)4$	$(1+i_2)4$	$(1+i_2)7$
7	$(1+i_2)7$	$(1+i_2)7$	$(1+i_2)7$	$(1+i_2)7$

Then, (i)*,* (ii)*, and* (iii) *of Definition 2 are obvious for the function* ϱ_{bcb}*. Let* $\sigma = 1$*,* $\psi = 3$*,* $\vartheta = 4 \in \mathfrak{W}$ *be arbitrary.*

Now,

$$\begin{aligned}\varrho_{bcb}(1,3) = (1+i_2)3 &\preceq_{i_2} 4(1+i_2) + 4(1+i_2) - 4(1+i_2)\\ &= \varrho_{bcb}(1,4) + \varrho_{bcb}(4,3) - \varrho_{bcb}(4,4).\end{aligned}$$

Therefore, $\varrho_{bcb}(\sigma,\psi) \preceq_{i_2} \varrho_{bcb}(\sigma,\vartheta) + \varrho_{bcb}(\vartheta,\psi) - \varrho_{bcb}(\vartheta,\vartheta)$*. Hence,* $(\mathfrak{W}, \varrho_{bcb})$ *is a bicomplex partial metric space.*

For the bicomplex partial metric space ϱ_{bcb} on $\mathfrak{W}$, the function $\mathfrak{d}_{\varrho_{bcb}} : \mathfrak{W} \times \mathfrak{W} \to \mathbb{C}_2^+$ given by $\mathfrak{d}_{\varrho_{bcb}} = 2\varrho_{bcb}(\sigma,\psi) - \varrho_{bcb}(\sigma,\sigma) - \varrho_{bcb}(\psi,\psi)$ is a usual metric on $\mathfrak{W}$. Each bicomplex partial metric ϱ_{bcb} on $\mathfrak{W}$ generates a topology $\tau_{\varrho_{bcb}}$ on $\mathfrak{W}$ with the base family of open ϱ_{bcb}-balls $\{\mathfrak{B}_{\varrho_{bcb}}(\sigma,\epsilon) : \sigma \in \mathfrak{W}, \epsilon \succ_{i_2} 0\}$, where $\mathfrak{B}_{\varrho_{bcb}}(\sigma,\epsilon) = \{\psi \in \mathfrak{W} : \varrho_{bcb}(\sigma,\psi) \prec_{i_2} \varrho_{bcb}(\sigma,\sigma) + \epsilon\}$ for all $\sigma \in \mathfrak{W}$ and $0 \prec_{i_2} \epsilon \in \mathbb{C}_2^+$.

A bicomplex valued metric space is a bicomplex partial metric space. However, a bicomplex partial metric space need not be a bicomplex valued metric space. The above Example 1 illustrates such a bicomplex partial metric space.

Note that self distance need not be zero, for example $\varrho_{bcb}(1,1) = 1 + i_2 \neq 0$. Now, the metric induced by ϱ_{bcb} is as follows: $\mathfrak{d}_{\varrho_{bcb}} = 2\varrho_{bcb}(\sigma,\psi) - \varrho_{bcb}(\sigma,\sigma) - \varrho_{bcb}(\psi,\psi)$; without loss of generality, suppose $\sigma \geq \psi$ then $\mathfrak{d}_{\varrho_{bcb}} = 2\{\max\{\sigma,\psi\} + i_2\max\{\sigma,\psi\}\} - \{\sigma + i_2\sigma\} - \{\psi + i_2\psi\}$. Therefore, $\mathfrak{d}_{\varrho_{bcb}}(\sigma,\psi) = |\sigma - \psi| + i_2|\sigma - \psi|$.

Theorem 2. *Let* $(\mathfrak{W}, \varrho_{bcb})$ *be a bicomplex partial metric space, then* $(\mathfrak{W}, \varrho_{bcb})$ *is* $\mathcal{T}_0$*.*

Proof. Supposing $\sigma, \psi \in \mathfrak{W}$ and $\sigma \neq \psi$, from condition (i) and (iii) in Definition 2, we get

$$\varrho_{bcb}(\sigma,\sigma) \prec_{i_2} \varrho_{bcb}(\sigma,\psi)$$
$$\text{OR}$$
$$\varrho_{bcb}(\psi,\psi) \prec_{i_2} \varrho_{bcb}(\sigma,\psi).$$

Suppose that $\varrho_{bcb}(\sigma,\sigma) \prec_{i_2} \varrho_{bcb}(\sigma,\psi)$. Then, we have $0 \prec_{i_2} \varrho_{bcb}(\sigma,\psi) - \varrho_{bcb}(\sigma,\sigma)$. Now, let $\mathfrak{r} \in \mathbb{C}_2^+$ such that $0 \prec_{i_2} \mathfrak{r} \prec_{i_2} \varrho_{bcb}(\sigma,\psi) - \varrho_{bcb}(\sigma,\sigma)$. Therefore, $\sigma \in \mathfrak{B}_{\varrho_{bcb}}(\sigma,\mathfrak{r})$ and $\psi \notin \mathfrak{B}_{\varrho_{bcb}}(\sigma,\mathfrak{r})$. Hence, $(\mathfrak{W}, \varrho_{bcb})$ is $\mathcal{T}_0$. □

Definition 3. *Let* $(\mathfrak{W}, \varrho_{bcb})$ *be a bicomplex partial metric space. A sequence* $\{\sigma_\alpha\}$ *in* $\mathfrak{W}$ *is said to be a convergent and converges to* $\sigma \in \mathfrak{W}$ *if, for every* $0 \prec_{i_2} \epsilon \in \mathbb{C}_2^+$*, there exists* $\mathfrak{N} \in \mathbb{N}$ *such that* $\sigma_\alpha \in \mathfrak{B}_{\varrho_{bcb}}(\sigma,\epsilon)$ *for all* $\alpha \geq \mathfrak{N}$*, and it is denoted by* $\lim\limits_{\alpha\to\infty} \sigma_\alpha = \sigma$*.*

Lemma 1. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a bicomplex partial metric space. A sequence $\{\sigma_\alpha\} \in \mathfrak{W}$ converges to $\sigma \in \mathfrak{W}$ iff $\varrho_{bcb}(\sigma, \sigma) = \lim_{\alpha \to \infty} \varrho_{bcb}(\sigma, \sigma_\alpha)$.*

Proof. Assume that $\{\sigma_n\}$ converges to σ. Let $\epsilon > 0$ be any real number. Suppose

$$\mathfrak{r} = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}.$$

Then, $0 \prec_{i_2} \mathfrak{r} \in \mathbb{C}_2^+$ and, for this $\mathfrak{r}$, there is a natural number $\mathfrak{N} \in \mathbb{N}$ such that $\sigma_\alpha \in \mathfrak{B}_{\varrho_{bcb}}(\sigma, \mathfrak{r})$ for all $\alpha \geq \mathfrak{N}$ i.e., $\varrho_{bcb}(\sigma_\alpha, \sigma) \prec_{i_2} \mathfrak{r} + \varrho_{bcb}(\sigma, \sigma)$. Therefore,

$$||\varrho_{bcb}(\sigma_\alpha, \sigma) - \varrho_{bcb}(\sigma, \sigma)|| < \epsilon \text{ for all } \alpha \geq \mathfrak{N}.$$

Therefore, $\varrho_{bcb}(\sigma_\alpha, \sigma) \to \varrho_{bcb}(\sigma, \sigma)$ as $\alpha \to \infty$.

Conversely, assume that $\varrho_{bcb}(\sigma_\alpha, \sigma) \to \varrho_{bcb}(\sigma, \sigma)$ as $\alpha \to \infty$. Then, for each $0 \prec_{i_2} \mathfrak{r} \in \mathbb{C}_2^+$, there exists a real $\epsilon > 0$ such that, for all $\xi \in \mathbb{C}_2^+$,

$$||\xi|| < \epsilon \Rightarrow \xi \prec_{i_2} \mathfrak{r}.$$

Then, for this $\epsilon > 0$, there exists $\mathfrak{N} \in \mathbb{N}$ such that

$$||\varrho_{bcb}(\sigma_\alpha, \sigma) - \varrho_{bcb}(\sigma, \sigma)|| < \epsilon \text{ for all } \alpha \geq \mathfrak{N}.$$

Therefore,

$$\varrho_{bcb}(\sigma_\alpha, \sigma) \prec_{i_2} \mathfrak{r} + \varrho_{bcb}(\sigma, \sigma) \text{ for all } \alpha \geq \mathfrak{N}.$$

Hence, $\{\sigma_\alpha\}$ converges to a point σ. □

Definition 4. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a bicomplex partial metric space. A sequence $\{\sigma_\alpha\}$ in $\mathfrak{W}$ is said to be a Cauchy sequence in $(\mathfrak{W}, \varrho_{bcb})$ if, for any $\epsilon > 0$, there exist $\mathfrak{a} \in \mathbb{C}_2^+$ and $\mathfrak{N} \in \mathbb{N}$ such that $||\varrho_{bcb}(\sigma_\beta, \sigma_\alpha) - \mathfrak{a}|| < \epsilon$ for all $\alpha, \beta \in \mathbb{N}$ and $\alpha, \beta \geq \mathfrak{N}$.*

Definition 5. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a bicomplex partial metric space. Let $\{\sigma_\alpha\}$ be any sequence in $\mathfrak{W}$. Then,*

(i) If every Cauchy sequence in $\mathfrak{W}$ is convergent in $\mathfrak{W}$, then $(\mathfrak{W}, \varrho_{bcb})$ is said to be a complete bicomplex partial metric space.

(ii) A mapping $\Gamma : \mathfrak{W} \to \mathfrak{W}$ is said to be continuous at $\sigma_0 \in \mathfrak{W}$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\Gamma(\mathfrak{B}_{\varrho_{bcb}}(\sigma_0, \delta)) \subset \mathfrak{B}_{\varrho_{bcb}}(\Gamma(\sigma_0, \epsilon))$.

Lemma 2. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a bicomplex partial metric space and $\{\sigma_\alpha\}$ be a sequence in $\mathfrak{W}$. Then, $\{\sigma_\alpha\}$ is a Cauchy sequence in $\mathfrak{W}$ iff $\lim_{\alpha \to \infty} \varrho_{bcb}(\sigma_\alpha, \sigma_\beta) = \varrho_{bcb}(\sigma, \sigma)$.*

Proof. Assume that $\{\sigma_\alpha\}$ is a Cauchy sequence in $\mathfrak{W}$. Let $\epsilon > 0$ be any real number. Suppose

$$\mathfrak{r} = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}.$$

Then, $0 \prec_{i_2} \mathfrak{r} \in \mathbb{C}_2^+$ and, for this $\mathfrak{r}$, there is a natural number $\mathfrak{N} \in \mathbb{N}$ such that $\sigma_\alpha \in \mathfrak{B}_{\varrho_{bcb}}(\sigma_\beta, \mathfrak{r})$ for all $\alpha, \beta \geq \mathfrak{N}$ i.e., $\varrho_{bcb}(\sigma_\alpha, \sigma_\beta) \prec_{i_2} \mathfrak{r} + \varrho_{bcb}(\sigma, \sigma)$. Therefore,

$$||\varrho_{bcb}(\sigma_\alpha, \sigma_\beta) - \varrho_{bcb}(\sigma, \sigma)|| < \epsilon \text{ for all } \alpha, \beta \geq \mathfrak{N}.$$

Therefore, $\varrho_{bcb}(\sigma_\alpha, \sigma_\beta) \to \varrho_{bcb}(\sigma, \sigma)$ as $\alpha, \beta \to \infty$.

Conversely, assume that $\varrho_{bcb}(\sigma_\alpha,\sigma_\beta) \to \varrho_{bcb}(\sigma,\sigma)$ as $\alpha,\beta\to\infty$. Then, for each $0 \prec_{i_2} \mathfrak{r} \in \mathbb{C}_2^+$, there exists a real $\epsilon > 0$ such that, for all $\xi \in \mathbb{C}_2^+$,

$$||\xi|| < \epsilon \Rightarrow \xi \prec_{i_2} \mathfrak{r}.$$

Then, for this $\epsilon > 0$, there exists $\mathfrak{N} \in \mathbb{N}$ such that

$$||\varrho_{bcb}(\sigma_\alpha,\sigma_\beta) - \varrho_{bcb}(\sigma,\sigma)|| < \epsilon \text{ for all } \alpha,\beta \geq \mathfrak{N}.$$

Therefore,

$$\varrho_{bcb}(\sigma_\alpha,\sigma_\beta) \prec_{i_2} \mathfrak{r} + \varrho_{bcb}(\sigma,\sigma) \text{ for all } \alpha,\beta \geq \mathfrak{N}.$$

Hence, $\{\sigma_\alpha\}$ is a Cauchy sequence. □

Definition 6. *Let Γ and Λ be self mappings of non-void set $\mathfrak{W}$. A point $\sigma \in \mathfrak{W}$ is called a common fixed point of Γ and Λ if $\sigma = \Gamma\sigma = \Lambda\sigma$.*

3. Main Results

Theorem 3. *Let $(\mathfrak{W},\varrho_{bcb})$ be a complete bicomplex partial metric space and $\Gamma,\Lambda\colon \mathfrak{W}\to\mathfrak{W}$ be two continuous mappings such that*

$$\begin{aligned}\varrho_{bcb}(\Gamma\sigma,\Lambda\psi) \preceq_{i_2} \curlywedge \max\{&\varrho_{bcb}(\sigma,\psi),\varrho_{bcb}(\sigma,\Gamma\sigma),\varrho_{bcb}(\psi,\Lambda\psi),\\ &\frac{1}{2}(\varrho_{bcb}(\sigma,\Lambda\psi)+\varrho_{bcb}(\psi,\Gamma\sigma))\},\end{aligned} \tag{1}$$

for all $\sigma,\psi \in \mathfrak{W}$, where $0 \leq \curlywedge < 1$. Then, the pair (Γ,Λ) has a unique common fixed point and $\varrho_{bcb}(\sigma^*,\sigma^*) = 0$.

Proof. Let σ_0 be arbitrary point in $\mathfrak{W}$ and define a sequence $\{\sigma_\alpha\}$ as follows:

$$\sigma_{2\alpha+1} = \Gamma\sigma_{2\alpha} \quad \text{and} \quad \sigma_{2\alpha+2} = \Lambda\sigma_{2\alpha+1}, \alpha = 0,1,2,\ldots. \tag{2}$$

Then, by (1) and (2), we obtain

$$\begin{aligned}\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}) &= \varrho_{bcb}(\Gamma\sigma_{2\alpha},\Lambda\sigma_{2\alpha+1})\\ &\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\varrho_{bcb}(\sigma_{2\alpha},\Gamma\sigma_{2\alpha}),\varrho_{bcb}(\sigma_{2\alpha+1},\Lambda\sigma_{2\alpha+1}),\\ &\quad \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha},\Lambda\sigma_{2\alpha+1})+\varrho_{bcb}(\sigma_{2\alpha+1},\Gamma\sigma_{2\alpha}))\}\\ &\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}),\\ &\quad \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+2})+\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+1}))\}\\ &\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}),\\ &\quad \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})+\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2})-\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+1})\\ &\quad +\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+1}))\}\\ &= \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}),\\ &\quad \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})+\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}))\}.\end{aligned}$$

Case I: If $\max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}),$
$\frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})+\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}))\} = \varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2})$, then we have

$$\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}) \preceq_{i_2} \curlywedge\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}).$$

This implies $\curlywedge \geq 1$, which is a contradiction.

Case II: If $\max\{\varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}), \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}), \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}) + \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}))\}$ $= \varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1})$, then we have

$$\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}) \preceq_{i_2} \curlywedge \varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}). \tag{3}$$

From the next step, we have

$$\begin{aligned}\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \curlywedge \max\{&\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}), \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}),\\ &\frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}) + \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}))\}.\end{aligned}$$

The following three cases arise, and we have

Case IIa:

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \curlywedge \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}),$$

which implies $\curlywedge \geq 1$ and is a contradiction.

Case IIb:

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \curlywedge \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}). \tag{4}$$

From (3) and (4), $\forall \alpha = 0, 1, 2, \ldots$, we get

$$\varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) \preceq_{i_2} \curlywedge \varrho_{bcb}(\sigma_{\alpha}, \sigma_{\alpha+1}) \preceq_{i_2} \ldots \preceq_{i_2} \curlywedge^{\alpha+1} \varrho_{bcb}(\sigma_0, \sigma_1).$$

For $\beta, \alpha \in \mathbb{N}$, with $\beta > \alpha$, we have

$$\begin{aligned}\varrho_{bcb}(\sigma_\alpha, \sigma_\beta) &\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_\beta) - \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+1})\\ &\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_\beta)\\ &\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2}, \sigma_\beta)\\ &\quad - \varrho_{bcb}(\sigma_{\alpha+2}, \sigma_{\alpha+2})\\ &\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2}, \sigma_\beta)\\ &\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2}, \sigma_{\alpha+3})\\ &\quad + \ldots + \varrho_{bcb}(\sigma_{\beta-2}, \sigma_{\beta-1}) + \varrho_{bcb}(\sigma_{\beta-1}, \sigma_\beta).\end{aligned}$$

Moreover, by using (4), we get

$$\begin{aligned}\varrho_{bcb}(\sigma_\alpha, \sigma_\beta) &\preceq_{i_2} \curlywedge^\alpha \varrho_{bcb}(\sigma_0, \sigma_1) + \curlywedge^{\alpha+1} \varrho_{bcb}(\sigma_0, \sigma_1) + \curlywedge^{\alpha+2} \varrho_{bcb}(\sigma_0, \sigma_1)\\ &\quad + \ldots + \curlywedge^{\beta-2} \varrho_{bcb}(\sigma_0, \sigma_1) + \curlywedge^{\beta-1} \varrho_{bcb}(\sigma_0, \sigma_1)\\ &= \sum_{i=1}^{\beta-\alpha} \curlywedge^{i+\alpha-1} \varrho_{bcb}(\sigma_0, \sigma_1).\end{aligned}$$

Therefore,

$$\begin{aligned}||\varrho_{bcb}(\sigma_\alpha, \sigma_\beta)|| &\leq \sum_{i=1}^{\beta-\alpha} \curlywedge^{i+\alpha-1} ||\varrho_{bcb}(\sigma_0, \sigma_1)|| = \sum_{t=\alpha}^{\beta-1} \curlywedge^t ||\varrho_{bcb}(\sigma_0, \sigma_1)||\\ &\leq \sum_{i=\alpha}^{\infty} ||\varrho_{bcb}(\sigma_0, \sigma_1)||\\ &= \frac{\curlywedge^\alpha}{1-\curlywedge} ||\varrho_{bcb}(\sigma_0, \sigma_1)||.\end{aligned}$$

Then, we have

$$||\varrho_{bcb}(\sigma_\alpha, \sigma_\beta)|| \leq \frac{\curlywedge^\alpha}{1-\curlywedge}||\varrho_{bcb}(\sigma_0, \sigma_1)|| \to 0 \quad as \quad \alpha \to \infty.$$

Hence, $\{\sigma_\alpha\}$ is a Cauchy sequence in $\mathfrak{W}$.

Case IIc:

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \curlywedge\frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}) + \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3})).$$

This implies that

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \frac{\curlywedge}{(2-\curlywedge)}\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}). \tag{5}$$

Since $a := \dfrac{\curlywedge}{2-\curlywedge} < 1$, we get $\varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) \preceq_{i_2} a\varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1})$. Using Case IIb, we get that $\{\sigma_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{W}$.

Case III:

If $\max\{\varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}), \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}), \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}) + \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}))\} = \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}) + \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}))$. Then, we have

$$\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}) \preceq_{i_2} \frac{\curlywedge}{2}(\varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}) + \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2})).$$

Hence,

$$\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}) \preceq_{i_2} \frac{\curlywedge}{2-\curlywedge}\varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}). \tag{6}$$

For the next step, we have

$$\begin{aligned}\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \curlywedge\max\{&\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}), \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}),\\ &\frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}) + \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}))\}.\end{aligned}$$

Then, we have the following three cases:

Case IIIa:

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \curlywedge\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}),$$

which implies $\curlywedge \geq 1$, which is a contradiction.

Case IIIb:

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \curlywedge\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}). \tag{7}$$

Then, by (6) and (7), we get $\varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) \preceq_{i_2} \gamma\varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1})$, where $\gamma = \max\left\{\curlywedge, \dfrac{\curlywedge}{2-\curlywedge}\right\} < 1$. Hence, $\{\sigma_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{W}$.

Case IIIc:

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}) + \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3})).$$

Hence, we obtain

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \frac{\curlywedge}{(2-\curlywedge)}\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}). \tag{8}$$

Using (6) and (8) yield

$$\varrho_{bcb}(\sigma_{\alpha+1},\sigma_{\alpha+2}) \preceq_{i_2} \wr\varrho_{bcb}(\sigma_\alpha,\sigma_{\alpha+1}), \tag{9}$$

where $0 \leq \wr = \dfrac{人}{2-人} < 1$.

Then, $\forall\alpha = 0,1,2,\ldots$, we get

$$\varrho_{bcb}(\sigma_{\alpha+1},\sigma_{\alpha+2}) \preceq_{i_2} \wr\varrho_{bcb}(\sigma_\alpha,\sigma_{\alpha+1}) \preceq_{i_2} \cdots \preceq_{i_2} \wr^{\alpha+1}\varrho_{bcb}(\sigma_0,\sigma_1).$$

For $\beta,\alpha \in \mathbb{N}$, with $\beta > \alpha$, we have

$$\begin{aligned}
\varrho_{bcb}(\sigma_\alpha,\sigma_\beta) &\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha,\sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1},\sigma_\beta) - \varrho_{bcb}(\sigma_{\alpha+1},\sigma_{\alpha+1})\\
&\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha,\sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1},\sigma_\beta)\\
&\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha,\sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1},\sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2},\sigma_\beta)\\
&\quad - \varrho_{bcb}(\sigma_{\alpha+2},\sigma_{\alpha+2})\\
&\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha,\sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1},\sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2},\sigma_\beta)\\
&\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha,\sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1},\sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2},\sigma_{\alpha+3})\\
&\quad + \ldots + \varrho_{bcb}(\sigma_{\beta-2},\sigma_{\beta-1})) + \varrho_{bcb}(\sigma_{\beta-1},\sigma_\beta).
\end{aligned}$$

Using (9), we get

$$\begin{aligned}
\varrho_{bcb}(\sigma_\alpha,\sigma_\beta) &\preceq_{i_2} \wr^\alpha\varrho_{bcb}(\sigma_0,\sigma_1) + \wr^{\alpha+1}\varrho_{bcb}(\sigma_0,\sigma_1) + \wr^{\alpha+2}\varrho_{bcb}(\sigma_0,\sigma_1)\\
&\quad + \ldots + \wr^{\beta-2}\varrho_{bcb}(\sigma_0,\sigma_1) + \wr^{\beta-1}\varrho_{bcb}(\sigma_0,\sigma_1)\\
&= \sum_{i=1}^{\beta-\alpha} \wr^{i+\alpha-1}\varrho_{bcb}(\sigma_0,\sigma_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
||\varrho_{bcb}(\sigma_\alpha,\sigma_\beta)|| \leq \sum_{i=1}^{\beta-\alpha} \wr^{i+\alpha-1}||\varrho_{bcb}(\sigma_0,\sigma_1)|| &= \sum_{t=\alpha}^{\beta-1} \wr^t||\varrho_{bcb}(\sigma_0,\sigma_1)||\\
&\leq \sum_{i=\alpha}^{\infty} \wr^t||\varrho_{bcb}(\sigma_0,\sigma_1)||\\
&= \frac{\wr^\alpha}{1-\wr}||\varrho_{bcb}(\sigma_0,\sigma_1)||.
\end{aligned}$$

Hence, we have

$$||\varrho_{bcb}(\sigma_\alpha,\sigma_\beta)|| \leq \frac{\wr^\alpha}{1-\wr}||\varrho_{bcb}(\sigma_0,\sigma_1)|| \to 0 \quad as \quad \alpha \to \infty.$$

Hence, $\{\sigma_\alpha\}$ is a Cauchy sequence in $\mathfrak{W}$. In all the above discussed cases, we get that the sequence $\{\sigma_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence. Since $\mathfrak{W}$ is complete, there exists $\sigma^* \in \mathfrak{W}$ such that $\sigma_\alpha \to \sigma^*$ as $\alpha \to \infty$ and

$$\varrho_{bcb}(\sigma^*,\sigma^*) = \lim_{\alpha\to\infty} \varrho_{bcb}(\sigma^*,\sigma_\alpha) = \lim_{\alpha\to\infty} \varrho_{bcb}(\sigma_\alpha,\sigma_\alpha) = 0.$$

By the continuity of Γ, it follows that $\sigma_{2\alpha+1} = \Gamma\sigma_{2\alpha} \to \Gamma\sigma^*$ as $\alpha \to \infty$.

$$\text{i.e., } \varrho_{bcb}(\Gamma\sigma^*,\Gamma\sigma^*) = \lim_{\alpha\to\infty} \varrho_{bcb}(\Gamma\sigma^*,\Gamma\sigma_{2\alpha}) = \lim_{\alpha\to\infty} \varrho_{bcb}(\Gamma\sigma_{2\alpha},\Gamma\sigma_{2\alpha}).$$

However,

$$\varrho_{bcb}(\Gamma\sigma^*,\Gamma\sigma^*) = \lim_{\alpha\to\infty}\varrho_{bcb}(\Gamma\sigma_{2\alpha},\Gamma\sigma_{2\alpha}) = \lim_{\alpha\to\infty}\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+1}) = 0.$$

Next, we have to prove that σ^* is a fixed point of Γ.

$$\varrho_{bcb}(\Gamma\sigma^*,\sigma^*) \preceq_{i_2} \varrho_{bcb}(\Gamma\sigma^*,\Gamma\sigma_{2\alpha}) + \varrho_{bcb}(\Gamma\sigma_{2\alpha},\sigma^*) - \varrho_{bcb}(\Gamma\sigma_{2\alpha},\Gamma\sigma_{2\alpha}).$$

As $\alpha \to \infty$, we obtain $||\varrho_{bcb}(\Gamma\sigma^*,\sigma^*)|| \leq 0$. Thus, $\wp_{bcb}(\Gamma\sigma^*,\sigma^*) = 0$. Hence, $\varrho_{bcb}(\sigma^*,\sigma^*) = \varrho_{bcb}(\sigma^*,\Gamma\sigma^*) = \varrho_{bcb}(\Gamma\sigma^*,\Gamma\sigma^*) = 0$ and $\Gamma\sigma^* = \sigma^*$. In the same way, we have $\sigma^* \in \mathfrak{W}$ such that $\sigma_\alpha \to \sigma^*$ as $\alpha \to \infty$ and

$$\varrho_{bcb}(\sigma^*,\sigma^*) = \lim_{\alpha\to\infty}\varrho_{bcb}(\sigma^*,\sigma_\alpha) = \lim_{\alpha\to\infty}\varrho_{bcb}(\sigma_\alpha,\sigma_\alpha) = 0.$$

By the continuity of Γ, it follows $\sigma_{2\alpha+2} = \Lambda\sigma_{2\alpha+1} \to \Lambda\sigma^*$ as $\alpha \to \infty$.

$$\text{i.e., } \varrho_{bcb}(\Lambda\sigma^*,\Lambda\sigma^*) = \lim_{\alpha\to\infty}\varrho_{bcb}(\Lambda\sigma^*,\Lambda\sigma_{2\alpha+1}) = \lim_{\alpha\to\infty}\varrho_{bcb}(\Lambda\sigma_{2\alpha+1},\Lambda\sigma_{2\alpha+1}).$$

However,

$$\varrho_{bcb}(\Lambda\sigma^*,\Lambda\sigma^*) = \lim_{\alpha\to\infty}\varrho_{bcb}(\Lambda\sigma_{2\alpha+1},\Lambda\sigma_{2\alpha+1}) = \lim_{\alpha\to\infty}\varrho_{bcb}(\sigma_{2\alpha+2},\sigma_{2\alpha+2}) = 0.$$

Next, we have to prove that σ^* is a fixed point of Λ.

$$\varrho_{bcb}(\Lambda\sigma^*,\sigma^*) \preceq_{i_2} \varrho_{bcb}(\Lambda\sigma^*,\Lambda\sigma_{2\alpha+1}) + \varrho_{bcb}(\Lambda\sigma_{2\alpha+1},\sigma^*) - \varrho_{bcb}(\Lambda\sigma_{2\alpha+1},\Gamma\sigma_{2\alpha+1}).$$

As $\alpha \to \infty$, we obtain $||\varrho_{bcb}(\Lambda\sigma^*,\sigma^*)|| \leq 0$. Thus, $\wp_{bcb}(\Lambda\sigma^*,\sigma^*) = 0$. Hence, $\varrho_{bcb}(\sigma^*,\sigma^*) = \varrho_{bcb}(\sigma^*,\Lambda\sigma^*) = \varrho_{bcb}(\Lambda\sigma^*,\Lambda\sigma^*) = 0$ and $\Lambda\sigma^* = \sigma^*$. Therefore, σ^* is a common fixed point of the pair (Γ,Λ).

To prove uniqueness, let us consider $\psi^* \in \mathfrak{W}$ as another common fixed point for the pair (Γ,Λ). Then,

$$\begin{aligned}
\varrho_{bcb}(\sigma^*,\psi^*) &= \varrho_{bcb}(\Gamma\sigma^*,\Lambda\psi^*) \\
&\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma^*,\psi^*),\varrho_{bcb}(\sigma^*,\Gamma\sigma^*),\varrho_{bcb}(\psi^*,\Lambda\psi^*), \\
&\quad \frac{1}{2}(\varrho_{bcb}(\sigma^*,\Lambda\psi^*) + \varrho_{bcb}(\psi^*,\Gamma\sigma^*))\} \\
&\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma^*,\psi^*),\varrho_{bcb}(\sigma^*,\sigma^*),\varrho_{bcb}(\psi^*,\psi^*), \\
&\quad \frac{1}{2}(\varrho_{bcb}(\sigma^*,\psi^*) + \varrho_{bcb}(\psi^*,\sigma^*))\} \\
&\preceq_{i_2} \curlywedge\varrho_{bcb}(\sigma^*,\psi^*).
\end{aligned}$$

This implies that $\sigma^* = \psi^*$. □

In the absence of the continuity condition for the mappings Γ and Λ, we get the the following theorem.

Theorem 4. *Let $(\mathfrak{W},\varrho_{bcb})$ be a complete bicomplex partial metric space and $\Gamma,\Lambda\colon \mathfrak{W} \to \mathfrak{W}$ be two mappings such that*

$$\begin{aligned}
\varrho_{bcb}(\Gamma\sigma,\Lambda\psi) \preceq_{i_2} &\curlywedge \max\{\varrho_{bcb}(\sigma,\psi),\varrho_{bcb}(\sigma,\Gamma\sigma),\varrho_{bcb}(\psi,\Lambda\psi), \\
&\frac{1}{2}(\varrho_{bcb}(\sigma,\Lambda\psi) + \varrho_{bcb}(\psi,\Gamma\sigma))\},
\end{aligned} \tag{10}$$

for all $\sigma,\psi \in \mathfrak{W}$, where $0 \leq \curlywedge < 1$. Then, the pair (Γ,Λ) has a unique common fixed point and $\varrho_{bcb}(\sigma^*,\sigma^*) = 0$.

Proof. Following from Theorem 3, we get that the sequence $\{\sigma_\alpha\}$ is a Cauchy sequence. Since $\mathfrak{W}$ is complete, there exists $\sigma^* \in \mathfrak{W}$ such that $\sigma_\alpha \to \sigma^*$ as $\alpha \to \infty$.

Since Γ and Λ are not continuous, we have $\varrho_{bcb}(\sigma^*, \Gamma\sigma^*) = \vartheta > 0$.

Then, we estimate

$$\begin{aligned}
\vartheta &= \varrho_{bcb}(\sigma^*, \Gamma\sigma^*) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \varrho_{bcb}(\sigma_{2\alpha+2}, \Gamma\sigma^*) - \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+2}) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \varrho_{bcb}(\sigma_{2\alpha+2}, \Gamma\sigma^*) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \varrho_{bcb}(\Lambda\sigma_{2\alpha+1}, \Gamma\sigma^*) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma^*), \varrho_{bcb}(\sigma_{2\alpha+1}, \Lambda\sigma_{2\alpha+1}), \varrho_{bcb}(\sigma^*, \Gamma\sigma^*), \\
&\quad \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha+1}, \Gamma\sigma^*) + \varrho_{bcb}(\sigma^*, \Lambda\sigma_{2\alpha+1}))\} \\
&\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma^*), \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}), \varrho_{bcb}(\sigma^*, \Gamma\sigma^*), \\
&\quad \frac{1}{2}(\varrho_{bcb}(\sigma_{2\alpha+1}, \Gamma\sigma^*) + \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}))\} \\
&\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \curlywedge \varrho_{bcb}(\sigma^*, \Gamma\sigma^*) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \curlywedge \vartheta.
\end{aligned}$$

This yields

$$||\vartheta|| \leq ||\varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2})|| + \curlywedge ||\vartheta||.$$

Hence, $\curlywedge \geq 1$, which is a contradiction. Then, $\sigma^* = \Gamma\sigma^*$. In the same way, we obtain $\sigma^* = \Lambda\sigma^*$. Hence, σ^* is a common fixed point for the pair (Γ, Λ) and $\varrho_{bcb}(\sigma^*, \sigma^*) = \varrho_{bcb}(\sigma^*, \Lambda\sigma^*) = \varrho_{bcb}(\Lambda\sigma^*, \Lambda\sigma^*) = 0$. For uniqueness of the common fixed point, σ^* follows from Theorem 3. $\square$

For $\Gamma = \Lambda$, we get the following fixed points results on bicomplex partial metric space.

Theorem 5. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a complete bicomplex partial metric space and $\Gamma\colon \mathfrak{W} \to \mathfrak{W}$ be a continuous mapping such that*

$$\begin{aligned}
\varrho_{bcb}(\Gamma\sigma, \Gamma\psi) \preceq_{i_2} & \curlywedge \max\{\varrho_{bcb}(\sigma, \psi), \varrho_{bcb}(\sigma, \Gamma\sigma), \varrho_{bcb}(\psi, \Gamma\psi), \\
& \frac{1}{2}(\varrho_{bcb}(\sigma, \Gamma\psi) + \varrho_{bcb}(\psi, \Gamma\sigma))\},
\end{aligned} \tag{11}$$

for all $\sigma, \psi \in \mathfrak{W}$, where $0 \leq \curlywedge < 1$. Then, the pair Γ has a unique fixed point and $\varrho_{bcb}(\sigma^*, \sigma^*) = 0$.

Remark 1. *Similarly, we get a fixed point result in the absence of continuity condition for the mapping Γ.*

Corollary 1. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a complete bicomplex partial metric space and $\Lambda\colon \mathfrak{W} \to \mathfrak{W}$ be a continuous mapping such that*

$$\begin{aligned}
\varrho_{bcb}(\Lambda^\alpha\sigma, \Lambda^\alpha\psi) \preceq_{i_2} & \curlywedge \max\{\varrho_{bcb}(\sigma, \psi), \varrho_{bcb}(\sigma, \Lambda^\alpha\sigma), \varrho_{bcb}(\psi, \Lambda^\alpha\psi), \\
& \frac{1}{2}(\varrho_{bcb}(\sigma, \Lambda^\alpha\psi) + \varrho_{bcb}(\psi, \Lambda^\alpha\sigma))\},
\end{aligned}$$

for all $\sigma, \psi \in \mathfrak{W}$, where $0 \leq \curlywedge < 1, \alpha \in \mathbb{N}$. Then, Λ has a unique fixed point and $\varrho_{bcb}(\sigma^*, \sigma^*) = 0$.

Proof. By Theorem 3, we get $\sigma^* \in \mathfrak{W}$ such that $\Lambda^\alpha\sigma^* = \sigma^*$ and $\varrho_{bcb}(\sigma^*,\sigma^*) = 0$. Then, we get

$$\begin{aligned}
\varrho_{bcb}(\Lambda\sigma^*,\sigma^*) &= \varrho_{bcb}(\Lambda\Psi^\alpha\sigma^*,\Lambda^n\sigma^*) = \varrho_{bcb}(\Lambda^\alpha\Lambda\sigma^*,\Lambda^\alpha\sigma^*)\\
&\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\Lambda\sigma^*,\sigma^*),\varrho_{bcb}(\Lambda\sigma^*,\Lambda^\alpha\Lambda\sigma^*),\varrho_{bcb}(\sigma^*,\Lambda^\alpha\sigma^*),\\
&\quad \frac{1}{2}(\varrho_{bcb}(\Lambda\sigma^*,\Lambda^\alpha\sigma^*)+\varrho_{bcb}(\sigma^*,\Lambda^\alpha\Lambda\sigma^*))\}\\
&\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\Lambda\sigma^*,\sigma^*),\varrho_{bcb}(\Lambda\sigma^*,\Lambda\sigma^*),\varrho_{bcb}(\sigma^*,\sigma^*),\\
&\quad \frac{1}{2}(\varrho_{bcb}(\Lambda\sigma^*,\sigma^*)+\varrho_{bcb}(\sigma^*,\Lambda\sigma^*))\}\\
&= \curlywedge\varrho_{bcb}(\Lambda\sigma^*,\sigma^*).
\end{aligned}$$

Hence, $\Lambda^\alpha\sigma^* = \Lambda\sigma^* = \sigma^*$. Then, Λ has a unique fixed point. □

Remark 2. *From the above Corollary 1, similarly, we get a fixed point result in the absence of continuity condition for the mapping Λ.*

Next, we will present a new generalization of a common fixed point theorem on bicomplex partial metric space.

Theorem 6. *Let $(\mathfrak{W},\varrho_{bcb})$ be a complete bicomplex partial metric space with non singular $1+\varrho_{bcb}(\sigma,\psi)$ and $||1+\varrho_{bcb}(\sigma,\psi)|| \neq 0$ and $\Gamma,\Lambda\colon \mathfrak{W}\to\mathfrak{W}$ be two continuous mappings such that*

$$\begin{aligned}
\varrho_{bcb}(\Gamma\sigma,\Lambda\psi) \preceq_{i_2} \curlywedge \max\bigg\{&\varrho_{bcb}(\sigma,\psi),\frac{\varrho_{bcb}(\sigma,\Gamma\sigma)\varrho_{bcb}(\psi,\Lambda\psi)}{1+\varrho_{bcb}(\sigma,\psi)},\\
&\frac{\varrho_{bcb}(\sigma,\Gamma\sigma)\varrho_{bcb}(\Gamma\sigma,\Lambda\psi)}{1+\varrho_{bcb}(\sigma,\psi)}\bigg\},
\end{aligned} \tag{12}$$

for all $\sigma,\psi\in\mathfrak{W}$, where $0 \leq \curlywedge < 1$. Then, the pair (Γ,Λ) has a unique common fixed point and $\varrho_{bcb}(\sigma^*,\sigma^*) = 0$.

Proof. Let σ_0 be arbitrary point in $\mathfrak{W}$ and define a sequence $\{\sigma_\alpha\}$ as follows:

$$\sigma_{2\alpha+1} = \Gamma\sigma_{2\alpha} \quad and \quad \sigma_{2\alpha+2} = \Lambda\sigma_{2\alpha+1}, \alpha = 0,1,2,\dots. \tag{13}$$

Then, by (12) and (13), we obtain

$$\begin{aligned}
\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}) &= \varrho_{bcb}(\Gamma\sigma_{2\alpha},\Lambda\sigma_{2\alpha+1})\\
&\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\\
&\quad \frac{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})\varrho_{bcb}(\Lambda\sigma_{2\alpha+1},\Gamma\sigma_{2\alpha})}{1+\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})},\\
&\quad \frac{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})\varrho_{bcb}(\Gamma\sigma_{2\alpha},\Lambda\sigma_{2\alpha+1})}{1+\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})}\}\\
&\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\\
&\quad \frac{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2})}{1+\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})},\\
&\quad \frac{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2})}{1+\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1})}\}\\
&\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2})\}.
\end{aligned}$$

If $\max\{\varrho_{bcb}(\sigma_{2\alpha},\sigma_{2\alpha+1}),\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2})\} = \varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2})$, then

$$\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}) \preceq_{i_2} \curlywedge\varrho_{bcb}(\sigma_{2\alpha+1},\sigma_{2\alpha+2}).$$

This shows that $\curlywedge \geq 1$, which is a contradiction. Therefore,

$$\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}) \preceq_{i_2} \curlywedge \varrho_{bcb}(\sigma_{2\alpha}, \sigma_{2\alpha+1}). \tag{14}$$

Similarly, we obtain

$$\varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+3}) \preceq_{i_2} \curlywedge \varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2}). \tag{15}$$

From (14) and (15), $\forall \alpha = 0, 1, 2, \ldots$, we get

$$\varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) \preceq_{i_2} \curlywedge \varrho_{bcb}(\sigma_{\alpha}, \sigma_{\alpha+1}) \preceq_{i_2} \cdots \preceq_{i_2} \curlywedge^{\alpha+1} \varrho_{bcb}(\sigma_0, \sigma_1). \tag{16}$$

For $\beta, \alpha \in \mathbb{N}$, with $\beta > \alpha$, we have

$$\begin{aligned}
\varrho_{bcb}(\sigma_\alpha, \sigma_\beta) &\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_\beta) - \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+1}) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_\beta) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2}, \sigma_\beta) \\
&\quad - \varrho_{bcb}(\sigma_{\alpha+2}, \sigma_{\alpha+2}) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2}, \sigma_\beta) \\
&\preceq_{i_2} \varrho_{bcb}(\sigma_\alpha, \sigma_{\alpha+1}) + \varrho_{bcb}(\sigma_{\alpha+1}, \sigma_{\alpha+2}) + \varrho_{bcb}(\sigma_{\alpha+2}, \sigma_{\alpha+3}) \\
&\quad + \ldots + \varrho_{bcb}(\sigma_{\beta-2}, \sigma_{\beta-1}) + s^{\beta-\alpha} \varrho_{bcb}(\sigma_{\beta-1}, \sigma_\beta).
\end{aligned}$$

By using (16), we get

$$\begin{aligned}
\varrho_{bcb}(\sigma_\alpha, \sigma_\beta) &\preceq_{i_2} \curlywedge^{\alpha} \varrho_{bcb}(\sigma_0, \sigma_1) + \curlywedge^{\alpha+1} \varrho_{bcb}(\sigma_0, \sigma_1) + \curlywedge^{\alpha+2} \varrho_{bcb}(\sigma_0, \sigma_1) \\
&\quad + \ldots + \curlywedge^{\beta-2} \varrho_{bcb}(\sigma_0, \sigma_1) + \curlywedge^{\beta-1} \varrho_{bcb}(\sigma_0, \sigma_1) \\
&= \sum_{i=1}^{\beta-\alpha} \curlywedge^{i+\alpha-1} \varrho_{bcb}(\sigma_0, \sigma_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
||\varrho_{bcb}(\sigma_\alpha, \sigma_\beta)|| &\leq \sum_{i=1}^{\beta-\alpha} \curlywedge^{i+\alpha-1} ||\varrho_{bcb}(\sigma_0, \sigma_1)|| = \sum_{i=1}^{\beta-\alpha} \curlywedge^{t} ||\varrho_{bcb}(\sigma_0, \sigma_1)|| \\
&\leq \sum_{i=\alpha}^{\infty} \curlywedge^{t} ||\varrho_{bcb}(\sigma_0, \sigma_1)|| \\
&= \frac{\curlywedge^{\alpha}}{1 - \curlywedge} ||\varrho_{bcb}(\sigma_0, \sigma_1)||.
\end{aligned}$$

Hence, we have

$$||\varrho_{bcb}(\sigma_\alpha, \sigma_\beta)|| \leq \frac{\curlywedge^{\alpha}}{1 - \curlywedge} ||\varrho_{bcb}(\sigma_0, \sigma_1)|| \to 0 \quad as \quad \alpha \to \infty.$$

Hence, $\{\sigma_\alpha\}$ is a Cauchy sequence in $\mathfrak{W}$. Since $\mathfrak{W}$ is complete, then there exists $\sigma^* \in \mathfrak{W}$ such that $\sigma_\alpha \to \sigma^*$ as $\alpha \to \infty$ and

$$\varrho_{bcb}(\sigma^*, \sigma^*) = \lim_{\alpha \to \infty} \varrho_{bcb}(\sigma^*, \sigma_\alpha) = \lim_{\alpha \to \infty} \varrho_{bcb}(\sigma_\alpha, \sigma_\alpha) = 0.$$

Λ being continuous yields

$$\sigma^* = \lim_{\alpha \to \infty} \sigma_{2\alpha+2} = \lim_{\alpha \to \infty} \Lambda \sigma_{2\alpha+1} = \Lambda \lim_{\alpha \to \infty} \sigma_{2\alpha+1} = \Lambda \sigma^*.$$

Similarly, by the continuity of Γ, we get $\sigma^* = \Gamma\sigma^*$. Then, the pair (Γ, Λ) has a common fixed point. To prove uniqueness, let us consider that $\psi^* \in \mathfrak{W}$ is another common fixed point for the pair (Γ, Λ). Then,

$$\begin{aligned}\varrho_{bcb}(\sigma^*, \psi^*) &= \varrho_{bcb}(\Gamma\sigma^*, \Lambda\psi^*) \\ &\preceq_{i_2} \curlywedge \max\{\varrho_{bcb}(\sigma^*, \psi^*), \frac{\varrho_{bcb}(\sigma^*, \Gamma\sigma^*)\varrho_{bcb}(\psi^*, \Lambda\psi^*)}{1+\varrho_{bcb}(\sigma^*, \psi^*)}, \\ &\frac{\varrho_{bcb}(\sigma^*, \Gamma\sigma^*)\varrho_{bcb}(\Lambda\psi^*, \Gamma\sigma^*)}{1+\varrho_{bcb}(\sigma^*, \psi^*)}\} \\ &\preceq_{i_2} \curlywedge \varrho_{bcb}(\sigma^*, \psi^*).\end{aligned}$$

This implies that $\sigma^* = \psi^*$. □

In the absence of the continuity condition for the mapping Γ and Λ in Theorem 6, we obtain the following result.

Theorem 7. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a complete bicomplex partial metric space with non singular $1 + \varrho_{bcb}(\sigma, \psi)$ and $||1 + \varrho_{bcb}(\sigma, \psi)|| \neq 0$ and $\Gamma, \Lambda\colon \mathfrak{W} \to \mathfrak{W}$ be two mappings such that*

$$\begin{aligned}\varrho_{bcb}(\Gamma\sigma, \Lambda\psi) \preceq_{i_2} \curlywedge \max\Bigg\{&\varrho_{bcb}(\sigma, \psi), \frac{\varrho_{bcb}(\sigma, \Gamma\sigma)\varrho_{bcb}(\psi, \Lambda\psi)}{1+\varrho_{bcb}(\sigma, \psi)}, \\ &\frac{\varrho_{bcb}(\sigma, \Gamma\sigma)\varrho_{bcb}(\Gamma\sigma, \Lambda\psi)}{1+\varrho_{bcb}(\sigma, \psi)}\Bigg\},\end{aligned} \tag{17}$$

for all $\sigma, \psi \in \mathfrak{W}$, where $0 \leq \curlywedge < 1$. Then, the pair (Γ, Λ) has a unique common fixed point and $\varrho_{bcb}(\sigma^*, \sigma^*) = 0$.

Proof. Following from Theorem 6, we get that the sequence $\{\sigma_\alpha\}$ is a Cauchy sequence. Since $\mathfrak{W}$ is complete, then there exists $\sigma^* \in \mathfrak{W}$ such that $\sigma_\alpha \to \sigma^*$ as $\alpha \to \infty$ and

$$\varrho_{bcb}(\sigma^*, \sigma^*) = \lim_{\alpha\to\infty} \varrho_{bcb}(\sigma^*, \sigma_\alpha) = \lim_{\alpha\to\infty} \varrho_{bcb}(\sigma_\alpha, \sigma_\alpha) = 0.$$

Since Γ and Λ are not continuous, we have $\varrho_{bcb}(\sigma^*, \Gamma\sigma^*) = \vartheta > 0$.
Then, we estimate

$$\begin{aligned}\vartheta &= \varrho_{bcb}(\sigma^*, \Gamma\sigma^*) \\ &\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \varrho_{bcb}(\sigma_{2\alpha+2}, \Gamma\sigma^*) - \varrho_{bcb}(\sigma_{2\alpha+2}, \sigma_{2\alpha+2}) \\ &\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \varrho_{bcb}(\Gamma\sigma^*, \sigma_{2\alpha+2}) \\ &\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \varrho_{bcb}(\Gamma\sigma^*, \Lambda\sigma_{2\alpha+1}) \\ &\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \curlywedge \max\Bigg\{\varrho_{bcb}(\sigma^*, \sigma_{2\alpha+1}), \frac{\varrho_{bcb}(\sigma^*, \Gamma\sigma^*)\varrho_{bcb}(\sigma_{2\alpha+1}, \Lambda\sigma_{2\alpha+1})}{1+\varrho_{bcb}(\sigma^*, \sigma_{2\alpha+1})}, \\ &\frac{\varrho_{bcb}(\sigma^*, \Gamma\sigma_*)\varrho_{bcb}(\Gamma\sigma^*, \Lambda\sigma_{2\alpha+1})}{1+\varrho_{bcb}(\sigma^*, \sigma_{2\alpha+1})}\Bigg\} \\ &\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \curlywedge \max\Bigg\{\varrho_{bcb}(\sigma^*, \sigma_{2\alpha+1}), \frac{\varrho_{bcb}(\sigma^*, \Gamma\sigma^*)\varrho_{bcb}(\sigma_{2\alpha+1}, \sigma_{2\alpha+2})}{1+\varrho_{bcb}(\sigma^*, \sigma_{2\alpha+1})}\}, \\ &\frac{\varrho_{bcb}(\sigma^*, \Gamma\sigma_*)\varrho_{bcb}(\Gamma\sigma^*, \sigma_{2\alpha+2})}{1+\varrho_{bcb}(\sigma^*, \sigma_{2\alpha+1})}\Bigg\} \\ &\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \curlywedge \varrho_{bcb}(\sigma^*, \Gamma\sigma^*)^2 \\ &\preceq_{i_2} \varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2}) + \curlywedge \vartheta^2.\end{aligned}$$

This yields

$$||\vartheta|| \leq ||\varrho_{bcb}(\sigma^*, \sigma_{2\alpha+2})|| + \curlywedge||\vartheta||^2.$$

Hence, $\curlywedge \geq 1$, which is a contradiction. Then, $\sigma^* = \Gamma\sigma^*$. In the same way, we obtain $\sigma^* = \Lambda\sigma^*$. Hence, σ^* is a common fixed point for the pair (Γ, Λ). The uniqueness of the common fixed point σ^* follows from Theorem 6. $\square$

For $\Gamma = \Lambda$, we get the following fixed points results on bicomplex partial metric space.

Theorem 8. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a complete bicomplex partial metric space with non singular $1 + \varrho_{bcb}(\sigma, \psi)$ and $||1 + \varrho_{bcb}(\sigma, \psi)|| \neq 0$ and $\Gamma\colon \mathfrak{W} \to \mathfrak{W}$ be a continuous mapping such that*

$$\varrho_{bcb}(\Gamma\sigma, \Gamma\psi) \preceq_{i_2} \curlywedge \max\left\{\varrho_{bcb}(\sigma, \psi), \frac{\varrho_{bcb}(\sigma, \Gamma\sigma)\varrho_{bcb}(\psi, \Gamma\psi)}{1 + \varrho_{bcb}(\sigma, \psi)}, \frac{\varrho_{bcb}(\sigma, \Gamma\sigma)\varrho_{bcb}(\Gamma\sigma, \Gamma\psi)}{1 + \varrho_{bcb}(\sigma, \psi)}\right\},$$

for all $\sigma, \psi \in \mathfrak{W}$, where $0 \leq \curlywedge < 1$. Then, Γ has a unique fixed point and $\varrho_{bcb}(\sigma^*, \sigma^*) = 0$.

Remark 3. *Similarly, in the absence of the continuity condition, we can get a fixed point result on Γ.*

Corollary 2. *Let $(\mathfrak{W}, \varrho_{bcb})$ be a complete bicomplex partial metric space with non singular $1 + \varrho_{bcb}(\sigma, \psi)$ and $||1 + \varrho_{bcb}(\sigma, \psi)|| \neq 0$ and $\Gamma\colon \mathfrak{W} \to \mathfrak{W}$ be a continuous mapping such that*

$$\varrho_{bcb}(\Gamma^\alpha\sigma, \Gamma^\alpha\psi) \preceq_{i_2} \curlywedge \max\left\{\varrho_{bcb}(\sigma, \psi), \frac{\varrho_{bcb}(\sigma, \Gamma^\alpha\sigma)\varrho_{bcb}(\psi, \Gamma^\alpha\psi)}{1 + \varrho_{bcb}(\sigma, \psi)}, \frac{\varrho_{bcb}(\sigma, \Gamma^\alpha\sigma)\varrho_{bcb}(\Gamma^\alpha\sigma, \Gamma\psi)}{1 + \varrho_{bcb}(\sigma, \psi)}\right\},$$

for all $\sigma, \psi \in \mathfrak{W}$, where $0 \leq \curlywedge < 1$. Then, Γ has a unique fixed point and $\varrho_{bcb}(\sigma^*, \sigma^*) = 0$.

Proof. By Theorem 6, we get $\sigma^* \in \mathfrak{W}$ such that $\Gamma^\alpha\sigma^* = \sigma^*$ and $\varrho_{bcb}(\sigma^*, \sigma^*) = 0$. Then, we get

$$\begin{aligned}
\varrho_{bcb}(\Gamma\sigma^*, \sigma^*) &= \varrho_{bcb}(\Gamma\Gamma^\alpha\sigma^*, \Gamma^\alpha\sigma^*) = \varrho_{bcb}(\Gamma^\alpha\Gamma\sigma^*, \Gamma^\alpha\sigma^*) \\
&\preceq_{i_2} \curlywedge \max\left\{\varrho_{bcb}(\Gamma\sigma^*, \sigma^*), \frac{\varrho_{bcb}(\Gamma\sigma^*, \Gamma^\alpha\Gamma\sigma^*)\varrho_{bcb}(\sigma^*, \Gamma^\alpha\sigma^*)}{1 + \varrho_{bcb}(\Gamma\sigma^*, \sigma^*)}, \frac{\varrho_{bcb}(\Gamma\sigma^*, \Gamma^\alpha\Gamma\sigma^*)\varrho_{bcb}(\Gamma^\alpha\Gamma\sigma^*, \Gamma^\alpha\sigma^*)}{1 + \varrho_{bcb}(\Gamma\sigma^*, \sigma^*)}\right\} \\
&\preceq_{i_2} \curlywedge \max\left\{\varrho_{bcb}(\Gamma\sigma^*, \sigma^*), \frac{\varrho_{bcb}(\Gamma\sigma^*, \Gamma\Gamma^\alpha\sigma^*)\varrho_{bcb}(\sigma^*, \Gamma^\alpha\sigma^*)}{1 + \varrho_{bcb}(\Gamma\sigma^*, \sigma^*)}, \frac{\varrho_{bcb}(\Gamma\sigma^*, \Gamma\Gamma^\alpha\sigma^*)\varrho_{bcb}(\Gamma\Gamma^\alpha\sigma^*, \Gamma^\alpha\sigma^*)}{1 + \varrho_{bcb}(\Gamma\sigma^*, \sigma^*)}\right\} \\
&= \curlywedge\varrho_{bcb}(\Gamma\sigma^*, \sigma^*).
\end{aligned}$$

Hence, $\Gamma^\alpha\sigma^* = \Gamma\sigma^* = \sigma^*$. Then, Γ has a unique fixed point. $\square$

Remark 4. *From the above Corollary 2, similarly, we get a fixed point result in the absence of continuity condition for the mapping Γ.*

Example 2. *Let $\mathfrak{W} = \{1, 2, 3, 4\}$ be endowed with the order $\sigma \preceq_{i_2} \psi$ if and only if $\sigma \leq \psi$. Then, $\preceq_{i_2}$ is a partial order in $\mathfrak{W}$. Define the bicomplex partial metric space $\varrho_{bcb} : \mathfrak{W} \times \mathfrak{W} \to \mathbb{C}_2^+$ as follows:*

(σ, ψ)	$\varrho_{bcb}(\sigma, \psi)$
(1,1), (2,2)	0
(1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (3,3)	$e^{i_2 x}$
(1,4), (4,1), (2,4), (4,2), (3,4), (4,3), (4,4)	$3e^{i_2 x}$

Obviously, $(\mathfrak{W}, \varrho_{bcb})$ is a complete bicomplex partial metric space for $x \in [0, \frac{\pi}{2}]$. Define $\Gamma, \Lambda : \mathfrak{W} \to \mathfrak{W}$ by $\Gamma\sigma = 1$,

$$\Lambda(\sigma) = \begin{cases} 1 & \text{if } \sigma \in \{1,2,3\} \\ 2 & \text{if } \sigma = 4. \end{cases}$$

Clearly, Γ and Λ are continuous functions. Now, for $\curlywedge = \frac{1}{3}$, we consider the following cases:

(A) *If $\sigma = 1$ and $\psi \in G - \{4\}$, then $\Gamma(\sigma) = \Lambda(\psi) = 1$ and the conditions of Theorem 3 are satisfied.*

(B) *If $\sigma = 1$, $\psi = 4$, then $\Gamma\sigma = 1$, $\Lambda\psi = 2$,*

$$\begin{aligned} \varrho_{bcb}(\Gamma\sigma, \Lambda\psi) = e^{i_2 x} &\precsim_{i_2} 3 \curlywedge e^{i_2 x} \\ &= \curlywedge \max\{3e^{i_2 x}, 0, 3e^{i_2 x}, \\ &\quad \frac{1}{2}(e^{i_2 x} + 3e^{i_2 x})\} \\ &= \curlywedge \max\{\varrho_{bcb}(\sigma, \psi), \varrho_{bcb}(\sigma, \Gamma\sigma), \varrho_{bcb}(\psi, \Lambda\psi), \\ &\quad \frac{1}{2}(\varrho_{bcb}(\sigma, \Lambda\psi) + \varrho_{bcb}(\psi, \Gamma\sigma))\}, \end{aligned}$$

(C) *If $\sigma = 2$, $\psi = 4$, then $\Gamma\sigma = 1$, $\Lambda\psi = 2$,*

$$\begin{aligned} \varrho_{bcb}(\Gamma\sigma, \Lambda\psi) = e^{i_2 x} &\precsim_{i_2} 3 \curlywedge e^{i_2 x} \\ &= \curlywedge \max\{3e^{i_2 x}, e^{i_2 x}, 3e^{i_2 x}, \\ &\quad \frac{1}{2}(0 + 3e^{i_2 x})\} \\ &= \curlywedge \max\{\varrho_{bcb}(\sigma, \psi), \varrho_{bcb}(\sigma, \Gamma\sigma), \varrho_{bcb}(\psi, \Lambda\psi), \\ &\quad \frac{1}{2}(\varrho_{bcb}(\sigma, \Lambda\psi) + \varrho_{bcb}(\psi, \Gamma\sigma))\}, \end{aligned}$$

(D) *If $\sigma = 3$, $\psi = 4$, then $\Gamma\sigma = 1$, $\Lambda\psi = 2$,*

$$\begin{aligned} \varrho_{bcb}(\Gamma\sigma, \Lambda\psi) = e^{i_2 x} &\precsim_{i_2} 3 \curlywedge e^{i_2 x} \\ &= \curlywedge \max\{3e^{i_2 x}, e^{i_2 x}, 3e^{i_2 x}, \\ &\quad \frac{1}{2}(e^{i_2 x} + 3e^{i_2 x})\} \\ &= \curlywedge \max\{\varrho_{bcb}(\sigma, \psi), \varrho_{bcb}(\sigma, \Gamma\sigma), \varrho_{bcb}(\psi, \Lambda\psi), \\ &\quad \frac{1}{2}(\varrho_{bcb}(\sigma, \Lambda\psi) + \varrho_{bcb}(\psi, \Gamma\sigma))\}, \end{aligned}$$

(E) *If $\sigma = 4$, $\psi = 4$, then $\Gamma\sigma = 2$, $\Lambda\psi = 2$,*

$$\begin{aligned}\varrho_{bcb}(\Gamma\sigma, \Lambda\psi) = e^{i_2 x} &\preceq_{i_2} 3 \curlywedge e^{i_2 x} \\ &= \curlywedge \max\{3e^{i_2 x}, 3e^{i_2 x}, 3e^{i_2 x}, \\ &\quad \frac{1}{2}(3e^{i_2 x} + 3e^{i_2 x})\} \\ &= \curlywedge \max\{\varrho_{bcb}(\sigma, \psi), \varrho_{bcb}(\sigma, \Gamma\sigma), \varrho_{bcb}(\psi, \Lambda\psi), \\ &\quad \frac{1}{2}(\varrho_{bcb}(\sigma, \Lambda\psi) + \varrho_{bcb}(\psi, \Gamma\sigma))\},\end{aligned}$$

Moreover, for $\curlywedge = \frac{1}{3}$, with $\curlywedge < 1$, the conditions of Theorem 3 are satisfied. Therefore, 1 is the unique common fixed point of Γ and Λ.

4. Application

Let $\mathfrak{W} = C[\leftthreetimes_1, \leftthreetimes_2]$ be a set of all real continuous functions on $[\leftthreetimes_1, \leftthreetimes_2]$ equipped with metric $\varrho_{bcb}(\sigma, \psi) = (1 + i_2)(\max_{\sqcup \in [\leftthreetimes_1, \leftthreetimes_2]} |\sigma(\sqcup) - \psi(\sqcup)| + 2)$ for all $\sigma, \psi \in C[\leftthreetimes_1, \leftthreetimes_2]$, where $|.|$ is the usual real modulus. Then, $(\mathfrak{W}, \varrho_{bcb})$ is a complete bicomplex partial metric space. Now, we consider the system of nonlinear Fredholm integral equation

$$\sigma(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1} \int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}_1(\sqcup, \mathfrak{s}, \sigma(s))ds$$

and

$$\sigma(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1} \int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}_2(\sqcup, \mathfrak{s}, \sigma(s))ds,$$

where $\sqcup, \mathfrak{s} \in [\leftthreetimes_1, \leftthreetimes_2]$. Assume that $\mathfrak{K}_1, \mathfrak{K}_2 : [\leftthreetimes_1, \leftthreetimes_2] \times [\leftthreetimes_1, \leftthreetimes_2] \times \mathfrak{W} \to \mathbb{R}$ and $\mathfrak{v} : [\leftthreetimes_1, \leftthreetimes_2] \to \mathbb{R}$ are continuous, where $\mathfrak{v}(\sqcup)$ is a given function in $\mathfrak{W}$.

Theorem 9. *Suppose that $(\mathfrak{W}, d)$ is a bicomplex partial metric space equipped with metric $\varrho_{bcb}(\sigma, \psi) = (1 + i_2)(\max_{\sqcup \in [\leftthreetimes_1, \leftthreetimes_2]} |\sigma(\sqcup) - \psi(\sqcup)| + 2)$ for all $\sigma, \psi \in \mathfrak{W}$ and $\Gamma, \Lambda : \mathfrak{W} \to \mathfrak{W}$ be a continuous operator on $\mathfrak{W}$ defined by*

$$\Gamma\sigma(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1} \int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}_1(\sqcup, \mathfrak{s}, \sigma(s))ds \tag{18}$$

and

$$\Lambda\sigma(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1} \int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}_2(\sqcup, \mathfrak{s}, \sigma(s))ds. \tag{19}$$

If there exists $\wr > 0$ such that, for all $\sigma, \psi \in \mathfrak{W}$ with $\sigma \neq \psi$ and $\mathfrak{s}, \sqcup \in [\leftthreetimes_1, \leftthreetimes_2]$ satisfying the following inequality:

$$\begin{aligned}|\mathfrak{K}_1(\sqcup, \mathfrak{s}, \sigma(s)) - \mathfrak{K}_2(\sqcup, \mathfrak{s}, \psi(\mathfrak{s}))| \leq \wr \max\{&|\sigma(s) - \psi(\mathfrak{s})|, |\sigma(s) - \Gamma\sigma(\mathfrak{s})|, \\ &|\psi(s) - \Lambda\psi(\mathfrak{s})|, \\ &\frac{1}{2}(|\sigma(s) - \Lambda\psi(s)| + |\psi(s) - \Gamma\sigma(s)|)\},\end{aligned} \tag{20}$$

then the integral operators defined by (18) and (19) have a common unique solution.

Proof. Consider,

$$
\begin{aligned}
(1+i_2)(|\Gamma\sigma(\sqcup)-\Lambda\psi(\sqcup)|+2) &= \frac{(1+i_2)}{|\curlywedge_2-\curlywedge_1|}\left(\left|\int_{\curlywedge_1}^{\curlywedge_2}\mathfrak{K}_1(\sqcup,\mathfrak{s},\sigma(s))ds\right.\right.\\
&\quad\left.\left.-\int_{\curlywedge_1}^{\curlywedge_2}\mathfrak{K}_2(\sqcup,\mathfrak{s},\psi(\mathfrak{s}))ds\right|+2\right)\\
&\leq \frac{(1+i_2)}{|\curlywedge_2-\curlywedge_1|}\left(\int_{\curlywedge_1}^{\curlywedge_2}|\mathfrak{K}_1(\sqcup,\mathfrak{s},\sigma(s))\right.\\
&\quad\left.-\mathfrak{K}_2(\sqcup,\mathfrak{s},\psi(\mathfrak{s}))|ds+2\right)\\
&\leq \frac{(1+i_2)\wr}{|\curlywedge_2-\curlywedge_1|}\left(\int_{\curlywedge_1}^{\curlywedge_2}\max\{|\sigma(s)-\psi(\mathfrak{s})|,\right.\\
&\quad|\sigma(s)-\Gamma\sigma(\mathfrak{s})|,|\psi(s)-\Lambda\psi(\mathfrak{s})|,\\
&\quad\left.\frac{1}{2}(|\sigma(s)-\Lambda\psi(s)|+|\psi(s)-\Gamma\sigma(s)|)\}+2\right)\\
&\leq \frac{\wr}{|\curlywedge_2-\curlywedge_1|}\int_{\curlywedge_1}^{\curlywedge_2}\max\{(1+i_2)|\sigma(s)-\psi(\mathfrak{s})|+2,\\
&\quad(1+i_2)|\sigma(s)-\Gamma\sigma(\mathfrak{s})|+2,\\
&\quad(1+i_2)|\psi(s)-\Lambda\psi(\mathfrak{s})|+2,\\
&\quad\frac{1}{2}((1+i_2)|\sigma(s)-\Lambda\psi(s)|+2\\
&\quad+(1+i_2)|\psi(s)-\Gamma\sigma(s)|+2)\}.
\end{aligned}
$$

Taking the maximum on both sides for all $\sqcup\in[\curlywedge_1,\curlywedge_2]$, we obtain

$$
\begin{aligned}
\varrho_{bcb}(\Gamma\sigma,\Gamma\psi)) &= (1+i_2)(\max_{\sqcup\in[\curlywedge_1,\curlywedge_2]}|\Gamma\sigma(\sqcup)-\Lambda\psi(\sqcup)|+2)\\
&\leq \frac{\wr}{|\curlywedge_2-\curlywedge_1|}\max_{\sqcup\in[\curlywedge_1,\curlywedge_2]}\int_{\curlywedge_1}^{\curlywedge_2}\max\{(1+i_2)|\sigma(s)-\psi(\mathfrak{s})|+2,\\
&\quad(1+i_2)|\sigma(s)-\Gamma\sigma(\mathfrak{s})|+2,(1+i_2)|\psi(s)-\Lambda\psi(\mathfrak{s})|+2,\\
&\quad\frac{1}{2}((1+i_2)|\sigma(s)-\Lambda\psi(s)|+2+(1+i_2)|\psi(s)-\Gamma\sigma(s)|+2)\}\\
&\leq \frac{\wr}{|\curlywedge_2-\curlywedge_1|}\max[\max_{\nabla\in[\curlywedge_1,\curlywedge_2]}\{(1+i_2)|\sigma(\nabla)-\psi(\nabla)|+2,\\
&\quad(1+i_2)|\sigma(\nabla)-\Gamma\psi(\nabla)|+2,(1+i_2)|\psi(\nabla)-\Gamma\psi(\nabla)|+2,\\
&\quad\frac{1}{2}((1+i_2)|\sigma(\nabla)-\Lambda\psi(\nabla)|+2\\
&\quad+(1+i_2)|\psi(\nabla)-\Gamma\sigma(\nabla)|+2)\}]\int_{\curlywedge_1}^{\curlywedge_2}ds\\
&\\
&= \curlywedge\max\{\varrho_{bcb}(\sigma,\psi),\varrho_{bcb}(\sigma,\Gamma\sigma),\varrho_{bcb}(\psi,\Lambda\psi),\\
&\quad\frac{1}{2}(\varrho_{bcb}(\sigma,\Lambda\psi)+\varrho_{bcb}(\psi,\Gamma\sigma))\}.
\end{aligned}
$$

Hence, all the conditions of Theorem 3 are satisfied and so the integral operators Γ and Λ defined by (18) and (19) have a common unique solution. □

5. Conclusions

In this paper, we proved some common fixed point theorems on bicomplex partial metric space. In addition, we find a common unique solution of a system of nonlinear

Fredholm integral equations, and we support our theoretical results by an example that we explain.

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