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On the Accuracy of the Generalized Gamma Approximation to Generalized Negative Binomial Random Sums

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Abstract: We investigate the proximity in terms of zeta-structured metrics of generalized negative binomial random sums to generalized gamma distribution with the corresponding parameters, extending thus the zeta-structured estimates of the rate of convergence in the Rényi theorem. In particular, we derive upper bounds for the Kantorovich and the Kolmogorov metrics in the law of large numbers for negative binomial random sums of i.i.d. random variables with nonzero first moments and finite second moments. Our method is based on the representation of the generalized negative binomial distribution with the shape and exponent power parameters no greater than one as a mixed geometric law and the infinite divisibility of the negative binomial distribution.

Keywords: Rényi theorem; law of large numbers; Kantorovich distance; Kolmogorov metric; zeta-structured metrics; geometric random sum; generalized negative binomial random sum; exponential distribution; generalized gamma distribution

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1. Introduction

Negative binomial and generalized negative binomial random sums naturally arise in various applications, such as meteorology, insurance, and financial mathematics. For example, in [1], it was demonstrated that the generalized negative binomial distribution has an excellent concordance with the empirical distribution of the duration of wet periods measured in days in Potsdam and Elista. Hence, the total precipitation volume per wet period forms a generalized negative binomial random sum. Various applications of geometric random sums, which are particular cases of the generalized negative binomial random sums can be found in [2]. A reasonable approach is to approximate the distributions of such random sums with specific distributions by applying limit theorems. Thus, it is of great practical importance to measure the accuracy of these approximations.

Recall that an r.v. $N_{r,p}$ has the negative binomial distribution with parameters $r > 0$ and $p \in (0,1)$, which is denoted by $N_{r,p} \sim \mathrm{NB}(r,p)$, if

$$\mathbf{P}(N_{r,p} = k) = \frac{\Gamma(k+r)}{k!\,\Gamma(r)} p^r (1-p)^k, \quad k = 0,1,2,\ldots, \tag{1}$$

where $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\,dx$ is the Euler gamma function.

In case of $r = 1$, it is reduced to the geometric distribution (starting from zero)—$N_{1,p} \sim \mathrm{Geom}_0(p)$—that is, $N_{1,p}$ has the sense of the number of fails before the first success in the scheme of Bernoulli trials with the success probability p. In the case of natural r, the negative binomial distribution can be represented as an r-fold convolution of the geometric distribution:

$$N_{r,p} \stackrel{d}{=} M_1 + \ldots + M_r, \tag{2}$$

where $M_1, \ldots, M_r \sim \mathrm{Geom}_0(p)$ are i.i.d. In this case, $N_{r,p}$ has the sense of the total number of fails before the r-th success in the Bernoulli scheme with the success probability p.

The negative binomial is known to be a mixed Poisson distribution with the mixing gamma distribution:

$$\mathbf{P}(N_{r,p} = k) = \int_0^\infty \frac{z^k}{k!} e^{-z} g\left(z; r, \frac{p}{1-p}\right) dz, \quad k = 0, 1, 2, \ldots, \tag{3}$$

where

$$g(z; r, \mu) = \frac{\mu^r z^{r-1}}{\Gamma(r)} e^{-\mu z} \mathbb{1}(z > 0), \tag{4}$$

is the density of the gamma distribution $G(r, \mu)$ with parameters $r > 0$ and $\mu > 0$.

A natural idea was to generalize the negative binomial distribution by using generalized gamma $\mathrm{GG}(r, \alpha, \mu)$ mixing with the density

$$g^*(z; r, \alpha, \mu) = \frac{|\alpha| \mu^r}{\Gamma(r)} z^{\alpha r - 1} e^{-\mu z^\alpha}, \quad z > 0, \tag{5}$$

instead of the ordinary one in Equation (3). The generalized gamma (GG) distribution has an additional parameter $\alpha \neq 0$, which plays a role of the exponent power of a gamma-r.v.: if $G_{r,\mu} \sim G(r, \mu)$, then $G_{r,\mu}^{1/\alpha} \sim \mathrm{GG}(r, \alpha, \mu)$. It was introduced in [3] as a special family of lifetime distributions containing both gamma and Weibull distributions. It also comprises inverse-gamma, half-normal, and some other distributions. A comprehensive list can be found in [4]. The properties of GG distributions are described in [3,5].

A generalization of Gleser's theorem (see [6], Theorem 3) states that the generalized gamma distribution $\mathrm{GG}(r, \alpha, \mu)$ with $r \in (0, 1)$, $\alpha \in (0, 1]$ and $\mu > 0$ is a mixture of exponential distributions:

$$g^*(z; r, \alpha, \mu) = \int_0^1 \frac{y}{1-y} e^{-\frac{y}{1-y} z} \cdot h(y; r, \alpha, \mu)\, dy, \quad z > 0, \tag{6}$$

where (see [4], Remark 3)

$$h(y; r, \alpha, \mu) = \frac{\mu^r}{\Gamma(r)\,\Gamma(1-r)} \cdot \frac{1}{(1-y)^2} \int_\mu^{+\infty} \frac{f\left(y(1-y)^{-1} z^{-1/\alpha}; \alpha, 1\right)}{(z-\mu)^r\, z^{1+2/\alpha}}\, dz, \quad y \in (0, 1), \tag{7}$$

is the density of a probability law concentrated on $(0, 1)$, and $f(x; \alpha, 1)$ is the density of the one-sided strictly stable distribution concentrated on the nonnegative half-line with the characteristic exponent α. The case of $\alpha = 1$ (which corresponds to the gamma distribution) is covered by the original Gleser's theorem with the simplified version of the mixing density:

$$h\left(y; r, 1, \frac{p}{1-p}\right) = \frac{p^r}{\Gamma(r)\,\Gamma(1-r)} \cdot \frac{(1-y)^{r-1}}{y(y-p)^r} \cdot \mathbb{1}\{p < y < 1\}. \tag{8}$$

A r.v. $N_{r,\alpha,\mu}$ is said to have the *generalized negative binomial* (GNB) distribution [4] with parameters $r > 0$, $\alpha \in \mathbb{R} \backslash \{0\}$ and $\mu > 0$, which is denoted by $N_{r,\alpha,\mu} \sim \mathrm{GNB}(r, \alpha, \mu)$, if it has a mixed Poisson distribution with the mixing GG distribution $\mathrm{GG}(r, \alpha, \mu)$:

$$\mathbf{P}(N_{r,\alpha,\mu} = k) = \int_0^\infty \frac{z^k}{k!} e^{-z} g^*(z; r, \alpha, \mu)\, dz, \quad k = 0, 1, 2, \ldots, \tag{9}$$

If $\alpha = 1$, then the GNB distribution reduces to the negative binomial one, namely, $\mathrm{GNB}(r, 1, \mu) = \mathrm{NB}\left(r, \frac{\mu}{\mu+1}\right)$, or $\mathrm{GNB}\left(r, 1, \frac{p}{1-p}\right) = \mathrm{NB}(r, p)$ for $p \in (0, 1)$.

Recently, it was discovered in [4], (Theorem 2) that the GNB distribution with $r \in (0,1)$, $\alpha \in (0,1]$ and $\mu > 0$ is a mixed geometric distribution,

$$\mathbf{P}(N_{r,\alpha,\mu} = k) = \int_0^1 y(1-y)^k \cdot h(y; r, \alpha, \mu)\, dy, \quad k = 0, 1, 2, \ldots, \tag{10}$$

with the same mixing density h as in Equation (6).

In [4] (Theorem 5), the class of limit distributions for GNB random sums was described; in particular, it was shown that the properly normalized GNB random sum of i.i.d. nonnegative random variables with finite expectations converges to the GG distribution. In the present paper, we will not only extend this result to alternating random summands with nonzero means, but will also construct some estimates of the rate of this convergence. Our reasoning will be essentially based on representation (10), which implies, as we will show in the proof of our Theorem 1 below, that the distribution of the generalized negative binomial random sum of i.i.d. r.v.s is also a mixed geometric random sum and hence allows us to extend some existing estimates in the Rényi theorem to GNB random sums.

Let $X_1, X_2, \ldots$ be i.i.d. r.v.s with $\mathbf{E}X_1 \neq 0$, $S_n := \sum_{k=1}^n X_k$, $n \in \mathbb{N}$, $S_0 := 0$, $N_p \sim \mathrm{Geom}_0(p)$ be independent of $\{X_1, X_2, \ldots\}$, $\mathcal{S}_p := \dfrac{S_{N_p}}{\mathbf{E}S_{N_p}} = \dfrac{pS_{N_p}}{(1-p)\mathbf{E}X_1}$, $\mathscr{E} \sim \mathrm{Exp}(1) := \mathrm{GG}(1,1,1)$. The celebrated Rényi theorem states that if $X_n \geq 0$ a.s. for all $n \in \mathbb{N}$, then $\mathscr{L}(\mathcal{S}_p) \to \mathrm{Exp}(1)$ as $p \to 0$. There exists an extensive literature where the rate of convergence in the Rényi theorem is studied. Let us mention the classic monograph [2], which is devoted specifically to geometric summation, and papers [7–13]. Monograph [14] also contains an estimate of the rate of convergence of the geometric distribution to the exponential law. The most common are estimates in the Kolmogorov (uniform) metric,

$$\rho(X,Y) \equiv \rho(\mathscr{L}(X), \mathscr{L}(Y)) := \sup_{x\in\mathbb{R}} |F(x) - G(x)| = \sup_{f\in\mathcal{F}_K} \left| \int_{\mathbb{R}} f\, dF - \int_{\mathbb{R}} f\, dG \right|, \tag{11}$$

where $F(x) = \mathbf{P}(X < x)$, $G(x) = \mathbf{P}(Y < x)$, $\mathcal{F}_K = \left\{ \mathbb{1}_{(-\infty,a)}(x) \mid a \in \mathbb{R} \right\}$, and in Zolotarev's ζ-metrics

$$\zeta_s(X,Y) \equiv \zeta_s(\mathscr{L}(X), \mathscr{L}(Y)) := \sup_{f\in\mathcal{F}_s^\infty} \left| \int_{\mathbb{R}} f\, dF - \int_{\mathbb{R}} f\, dG \right|, \quad s > 0, \tag{12}$$

where

$$\mathcal{F}_s^\infty := \left\{ f\colon \mathbb{R} \to \mathbb{R} \,\middle|\, f \text{ is bounded}, |f^{(m)}(x) - f^{(m)}(y)| \leq |x-y|^\alpha \quad \forall x, y \in \mathbb{R} \right\},$$

$s = m + \alpha$, $m \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,1]$, which are simple ζ-structured metrics, i.e., can be represented as

$$\zeta_{\mathcal{F}}(X,Y) \equiv \zeta_{\mathcal{F}}(\mathscr{L}(X), \mathscr{L}(Y)) := \sup_{f\in\mathcal{F}} \left| \mathbf{E}f(X) - \mathbf{E}f(Y) \right|, \tag{13}$$

with specific classes $\mathcal{F}$ of real-valued bounded Borel functions on $\mathbb{R}$, namely, $\rho = \zeta_{\mathcal{F}_K}$, $\zeta_s = \zeta_{\mathcal{F}_s^\infty}$.

In particular, Brown [7] proved that

$$\rho(\mathcal{S}_p, \mathscr{E}) \leq p\, \frac{\mathbf{E}X_1^2}{(\mathbf{E}X_1)^2} \max\left\{ 1, \frac{1}{2(1-p)} \right\}, \text{ if } X_n \geq 0 \text{ a.s.} \tag{14}$$

Kalashnikov [2] provides the following bounds for the shifted geometric random sums with $\mathbf{E}X_1 = 1$:

$$\zeta_s\Big(pS_{N_p+1},\mathscr{E}\Big) \le p^{s-1}\,\zeta_s(X_1,\mathscr{E}), \tag{15}$$

$$\zeta_1\Big(pS_{N_p+1},\mathscr{E}\Big) \le p\,\zeta_1(X_1,\mathscr{E}) + 2(1-p)p^{s-1}\,\zeta_s(X_1,\mathscr{E}), \tag{16}$$

for all $s \in (1,2]$.

The present authors obtained [13] the following estimate for the Kantorovich ζ_1 distance, extending the one in [11] to the alternating random summands:

$$\zeta_1(S_p,\mathscr{E}) \le \frac{p}{1-p}\cdot\frac{\mathbf{E}X_1^2}{(\mathbf{E}X_1)^2}. \tag{17}$$

The aim of the present paper is to extend the ζ-structured estimates of the rate of convergence of geometric random sums to generalized negative binomial random sums for $r \in (0,1)$, in particular, to extend bounds (14) and (17) to negative binomial random sums for arbitrary $r > 0$.

2. Main Results

Theorem 1. *Let $X_1, X_2, \ldots$ be a sequence of arbitrary r.v.s, $S_n := \sum_{i=1}^n X_i$ for $n \in \mathbb{N}$, $S_0 := 0$. Let $r \in (0,1), \alpha \in (0,1]$, $\mu > 0$, $N_{r,\alpha,\mu} \sim \mathrm{GNB}(r,\alpha,\mu)$ be independent of $\{X_n\}$, $G^*_{r,\alpha,\mu} \sim \mathrm{GG}(r,\alpha,\mu)$ and $\zeta_{\mathcal{F}}$ be a ζ-structured metric w.r.t. some class $\mathcal{F}$ or real bounded Borel functions. Then*

$$\zeta_{\mathcal{F}}\Big(S_{N_{r,\alpha,\mu}}, G^*_{r,\alpha,\mu}\Big) \le \int_0^1 \zeta_{\mathcal{F}}\Big(S_{N_y}, \frac{1-y}{y}\mathscr{E}\Big)\cdot h(y;r,\alpha,\mu)\,dy, \tag{18}$$

where $N_y \sim \mathrm{Geom}_0(y)$ is independent of $\{X_n\}$, $\mathscr{E} \sim \mathrm{Exp}(1)$ and $h(y)$ is given in Equation (7). In particular, if $\{X_n\}$ are i.i.d. square integrable r.v.'s with $\mathbf{E}X_n = 1$; then, for all $r \in (0,1)$, $\mu > 0$, and $\alpha \in (0,1]$, we have

$$\zeta_1\Big(S_{N_{r,\alpha,\mu}}, G^*_{r,\alpha,\mu}\Big) \le \mathbf{E}X_1^2, \tag{19}$$

and if, in addition, all $X_n \ge 0$ a.s., then also

$$\rho\Big(S_{N_{r,\alpha,\mu}}, G^*_{r,\alpha,\mu}\Big) \le \int_0^1 \min\Big\{1 \wedge \frac{y\,\mathbf{E}X_1^2}{1-y}\Big\} h(y;r,\alpha,\mu)\,dy, \tag{20}$$

$$\rho\Big(S_{N_{r,p}}, G_{r,p/(1-p)}\Big) \le \frac{p^r(2\mathbf{E}X_1^2 + r^{-1} + 1)}{\Gamma(r)\,\Gamma(2-r)}, \quad p \in (0,1). \tag{21}$$

Proof. To begin with, we prove that the distribution of the random sum $S_{N_{r,\alpha,\mu}}$ is a mixture of geometric random sums. Indeed, according to the rule of total probability, representation (10) and Tonelli's theorem, for any Borel set $B \subset \mathbb{R}$, we have

$$\begin{aligned}
\mathbf{P}\big(S_{N_{r,\alpha,\mu}} \in B\big) &= \sum_{n=0}^{\infty} \mathbf{P}\big(N_{r,\alpha,\mu} = n\big)\mathbf{P}(S_n \in B) \\
&= \sum_{n=0}^{\infty}\left[\int_0^1 y(1-y)^n \cdot h(y;r,\alpha,\mu)\,dy\right]\mathbf{P}(S_n \in B) \\
&= \int_0^1 \left[\sum_{n=0}^{\infty} y(1-y)^n\,\mathbf{P}(S_n \in B)\right] h(y;r,\alpha,\mu)\,dy \\
&= \int_0^1 \mathbf{P}(S_{N_y} \in B)\cdot h(y;r,\alpha,\mu)\,dy.
\end{aligned}$$

Hence, for any $f \in \mathcal{F}$,

$$\mathbf{E}f\big(S_{N_{r,\alpha,\mu}}\big) = \int_0^1 \mathbf{E}f\big(S_{N_y}\big)\cdot h(y;r,\alpha,\mu)\,dy.$$

Similarly, using Equation (6), we obtain

$$\mathbf{E}f(G^*_{r,\alpha,\mu}) = \int_0^\infty f(z)\, g^*(z;r,\alpha,\mu)\,dz = \int_0^1 \mathbf{E}f\left(\frac{1-y}{y}\mathscr{E}\right)\cdot h(y;r,\alpha,\mu)\,dy.$$

Therefore, due to the triangle inequality,

$$\zeta_{\mathcal{F}}\left(S_{N_{r,\alpha,\mu}}, G^*_{r,\alpha,\mu}\right) = \sup_{f\in\mathcal{F}}\left|\mathbf{E}f(S_{N_{r,\alpha,\mu}}) - \mathbf{E}f(G^*_{r,\alpha,\mu})\right| \le \int_0^1 \zeta_{\mathcal{F}}\left(S_{N_y}, \frac{1-y}{y}\mathscr{E}\right)\cdot h(y;r,\alpha,\mu)\,dy,$$

which coincides with (18).

If $\{X_n\}$ are i.i.d. square integrable r.v.s with $\mathbf{E}X_n = 1$, then from (17) and the homogeneity of Kantorovich metric, we have

$$\zeta_1\left(S_{N_y}, \frac{1-y}{y}\mathscr{E}\right) = \frac{1-y}{y}\zeta_1(S_y, \mathscr{E}) \le \mathbf{E}X_1^2 =: \alpha_2.$$

Substituting this into (18) and observing that $h(y;r,\alpha,\mu)$ integrates to one as a probability density with respect to y, we arrive at (19).

Let us prove (20) and (21). In case of $X_n \ge 0$ a.s., due to (14), we have

$$\rho\left(S_{N_y}, \frac{1-y}{y}\mathscr{E}\right) = \rho(S_y, \mathscr{E}) \le \frac{\alpha_2 y}{1-y}\max\{1-y, 0.5\}, \quad y\in(0,1),$$

on the one hand. On the other hand, the Kolmogorov metric is always bounded by one, which allows us to write

$$\rho\left(S_{N_y}, \frac{1-y}{y}\mathscr{E}\right) \le \min\left\{\frac{\alpha_2 y}{1-y}\max\{1-y,0.5\},\, 1\right\} \le \frac{\alpha_2 y}{1-y}\wedge 1 \le \begin{cases} \frac{\alpha_2 y}{1-y}, & y\le 0.5,\\ 1, & y > 0.5.\end{cases}$$

Substituting this into (18), we obtain

$$\rho\left(S_{N_{r,\alpha,\mu}}, G^*_{r,\alpha,\mu}\right) \le \int_0^1 \left(\frac{\alpha_2 y}{1-y}\wedge 1\right) h(y;r,\alpha,\mu)\,dy.$$

In particular with $\alpha = 1$ and $\mu = p/(1-p)$, we have

$$\rho\left(S_{N_{r,p}}, G_{r,\mu}\right) \le \frac{p^r}{\Gamma(r)\,\Gamma(1-r)}\int_p^1 \left(\frac{\alpha_2 y}{1-y}\wedge 1\right)\frac{(1-y)^{r-1}}{y(y-p)^r}\,dy \le \frac{p^r(\alpha_2 I_1(p) + I_2(p))}{\Gamma(r)\,\Gamma(1-r)},$$

where

$$I_1(p) = \int_{p\wedge 0.5}^{0.5}\frac{(1-y)^{r-2}}{(y-p)^r}\,dy, \quad I_2(p) = \int_{0.5\vee p}^1 \frac{(1-y)^{r-1}}{y(y-p)^r}\,dy, \quad p\in(0,1).$$

Changing the variable $y = z/(1+z)$ we obtain

$$I_1(p) = \int_{p/(1-p)\wedge 1}^1 \frac{dz}{(z(1-p)-p)^r} = \frac{(1-2p)_+^{1-r}}{(1-r)(1-p)} \le \frac{2}{1-r},$$

$$\begin{aligned} I_2(p) &= \int_{1\vee\frac{p}{1-p}}^\infty \frac{dz}{z(z(1-p)-p)^r} = \int_{(1-2p)_+}^\infty \frac{dx}{(x+p)x^r} \\ &\le \frac{1}{p+(1-2p)_+}\int_{(1-2p)_+}^1 \frac{dx}{x^r} + \int_1^\infty \frac{dx}{x^{r+1}} = \frac{1-(1-2p)_+^{1-r}}{(p+(1-2p)_+)(1-r)} + \frac{1}{r}\end{aligned}$$

for all $p \in (0,1)$. Observe that

$$A(p) := \frac{1-(1-2p)_+^{1-r}}{p+(1-2p)_+} = \begin{cases} \dfrac{1-(1-2p)^{1-r}}{1-p} \le 2, & p \le 0.5, \\ p^{-1} \le 2, & p > 0.5, \end{cases}$$

so that $A(p) \le 2$ and $I_2(p) \le 2(1-r)^{-1} + r^{-1}$ uniformly with respect to $p \in (0,1)$. Hence, we finally obtain

$$\rho\Big(S_{N_{r,p}}, G_{r,\mu}\Big) \le \frac{p^r}{\Gamma(r)\,\Gamma(1-r)}\left(\frac{2\alpha_2}{1-r} + \frac{2}{1-r} + \frac{1}{r}\right) = \frac{p^r(2\alpha_2 + r^{-1} + 1)}{\Gamma(r)\,\Gamma(2-r)}. \quad \square$$

Recall that the metric $\zeta_{\mathcal{F}}$ is called *regular* if

$$\zeta_{\mathcal{F}}(X+Z, Y+Z) \le \zeta_{\mathcal{F}}(X,Y)$$

for Z independent of X and Y. In particular, $\zeta_{\mathcal{F}}$ is regular if $\mathcal{F}$ is closed w.r.t. shifts, i.e., $f \in \mathcal{F}$ yields $f(\,\cdot\,+c) \in \mathcal{F}$ for any $c \in \mathbb{R}$.

Theorem 2. *Let $X_1, X_2, \dots$ be a sequence of i.i.d. r.v.s, $S_n := \sum_{i=1}^n X_i$ for $n \in \mathbb{N}$, $S_0 := 0$. Let $r > 0$, $p \in (0,1)$, $N_{r,p} \sim \mathrm{NB}(r,p)$ be independent of $\{X_n\}$, $N_p := N_{1,p}$, $\mu = \frac{p}{1-p}$, $G_{r,\mu} \sim \mathrm{G}(r,\mu)$, $\mathscr{E} \stackrel{d}{=} G_{1,1} \sim \mathrm{Exp}(1)$ and $\zeta_{\mathcal{F}}$ be a regular ζ-structured metric. Then*

$$\zeta_{\mathcal{F}}\Big(S_{N_{r,p}}, G_{r,\mu}\Big) \le \lfloor r \rfloor\, \zeta_{\mathcal{F}}\left(S_{N_p}, \frac{1-p}{p}\mathscr{E}\right) + \int_0^1 \zeta_{\mathcal{F}}\left(S_{N_y}, \frac{1-y}{y}\mathscr{E}\right) \cdot h(y;\{r\},1,\mu)\,dy, \quad (22)$$

where the second term in the r.-h.s. vanishes for $r \in \mathbb{N}$, $h\Big(y; r, 1, \frac{p}{1-p}\Big)$ is given in (8), $\lfloor r \rfloor$ is the largest integer that does not exceed r, and $\{r\} = r - \lfloor r \rfloor$ is the fractional part of r. In particular, if $X_1, X_2, \dots$ are square integrable with $\mathbf{E}X_n = 1$, then

$$\zeta_1\left(\frac{p\,S_{N_{r,p}}}{r(1-p)}, G_{r,r}\right) \le \frac{\lceil r \rceil}{r} \cdot \frac{p\,\mathbf{E}X_1^2}{1-p}, \quad (23)$$

where $\lceil r \rceil$ is the least integer no less than r, and if, in addition, $X_n \ge 0$ a.s. for all $n \in \mathbb{N}$, then also

$$\rho\Big(S_{N_{r,p}}, G_{r,\mu}\Big) = \rho\left(\frac{p\,S_{N_{r,p}}}{r(1-p)}, G_{r,r}\right) \le \lfloor r \rfloor\, p \max\left\{1, \frac{1}{2(1-p)}\right\} \mathbf{E}X_1^2 + \frac{p^{\{r\}}(2\,\mathbf{E}X_1^2 + \{r\}^{-1} + 1)}{\Gamma(\{r\})\,\Gamma(2-\{r\})}, \quad (24)$$

where the second term vanishes in case of $r \in \mathbb{N}$.

Proof. First, consider the case of $r \in \mathbb{N}$. Due to Equation (2), the distribution of $S_{N_{r,p}}$ coincides with the r-fold convolution of the distribution of geometric random sums; i.e., $S_{N_{r,p}} \stackrel{d}{=} \sum\limits_{k=1}^{r} S_{N_p^{(k)}}$, where $\big\{S_{N_p^{(k)}}\big\}$ are independent copies of the geometric random sum S_{N_p}, $N_p \sim \mathrm{Geom}_0(p)$, N_p being independent of $\{X_n\}$. Due to the reproducibility of the gamma distribution w.r.t. the shape parameter r, $G_{r,\mu} \stackrel{d}{=} \sum_{k=1}^r \mathscr{E}_\mu^{(k)}$, where $\big\{\mathscr{E}_\mu^{(k)}\big\}$ are independent (and independent of everything else) copies of $\mathscr{E}_\mu \stackrel{d}{=} \mathscr{E}/\mu \sim \mathrm{Exp}(\mu)$. Next, by applying the triangle inequality and the property of regularity of $\zeta_{\mathcal{F}}$ for r times, we obtain

$$\begin{aligned}\zeta_{\mathcal{F}}(S_{N_{r,p}}, G_{r,\mu}) &= \zeta_{\mathcal{F}}\left(\sum_{k=1}^{r} S_{N_p^{(k)}}, \sum_{k=1}^{r} \mathscr{E}_\mu^{(k)}\right) \\ &\le \zeta_{\mathcal{F}}\left(\sum_{k=1}^{r} S_{N_p^{(k)}}, \sum_{k=1}^{r-1} S_{N_p^{(k)}} + \mathscr{E}_\mu^{(r)}\right) + \zeta_{\mathcal{F}}\left(\sum_{k=1}^{r-1} S_{N_p^{(k)}} + \mathscr{E}_\mu^{(r)}, \sum_{k=1}^{r} \mathscr{E}_\mu^{(k)}\right) \\ &\le \zeta_{\mathcal{F}}\left(S_{N_p^{(r)}}, \mathscr{E}_\mu^{(r)}\right) + \zeta_{\mathcal{F}}\left(\sum_{k=1}^{r-1} S_{N_p^{(k)}}, \sum_{k=1}^{r-1} \mathscr{E}_\mu^{(k)}\right) \le \ldots \le r\,\zeta_{\mathcal{F}}(S_{N_p}, \mathscr{E}_\mu).\end{aligned}$$

If $r \notin \mathbb{N}$, then, due to the reproducibility of the negative binomial distribution w.r.t. the shape parameter r, we have $N_{r,p} \stackrel{d}{=} N_{\lfloor r\rfloor,p} + N_{\{r\},p}$ with independent r.v.s. on the r.-h.s. Using the same reasoning as in the above chain of inequalities, we get

$$\begin{aligned}\zeta_{\mathcal{F}}\left(S_{N_{r,p}}, G_{r,\mu}\right) &= \zeta_{\mathcal{F}}\left(S_{N_{\lfloor r\rfloor,p}} + S_{N_{\{r\},p}}, G_{\lfloor r\rfloor,\mu} + G_{\{r\},\mu}\right) \\ &\le \zeta_{\mathcal{F}}\left(S_{N_{\lfloor r\rfloor,p}}, G_{\lfloor r\rfloor,\mu}\right) + \zeta_{\mathcal{F}}\left(S_{N_{\{r\},p}}, G_{\{r\},\mu}\right) \\ &\le \lfloor r\rfloor \zeta_{\mathcal{F}}(S_{N_p}, \mathscr{E}_\mu) + \zeta_{\mathcal{F}}\left(S_{N_{\{r\},p}}, G_{\{r\},\mu}\right).\end{aligned}$$

Applying (18) to bound $\zeta_{\mathcal{F}}\left(S_{N_{\{r\},p}}, G_{\{r\},\mu}\right)$ from above, we obtain (22).

Furthermore, if $X_1, X_2, \ldots$ are square-integrable with $\mathbf{E}X_n = 1$, then by using upper bounds (17) for $\zeta_1(S_{N_p}, \mathscr{E}_\mu)$ and (19) for $\zeta_1(S_{N_{\{r\},p}}, G_{\{r\},\mu})$, together with the homogeneity of the Kantorovich metric, we obtain

$$\begin{aligned}\zeta_1(S_{N_{r,p}}, G_{r,\mu}) &\le \lfloor r\rfloor \zeta_1(S_{N_p}, \mathscr{E}_\mu) + \zeta_1(S_{N_{\{r\},p}}, G_{\{r\},\mu})\mathbb{1}(r \notin \mathbb{N}) \\ &\le \lfloor r\rfloor \mathbf{E}X_1^2 + \mathbf{E}X_1^2 \cdot \mathbb{1}(r \notin \mathbb{N}) = \lceil r\rceil \cdot \mathbf{E}X_1^2,\end{aligned}$$

and hence, using the homogeneity again, we have

$$\zeta_1\left(\frac{p\,S_{N_{r,p}}}{r(1-p)}, G_{r,r}\right) = \frac{p}{r(1-p)}\zeta_1\left(S_{N_{r,p}}, G_{r,\mu}\right) \le \frac{\lceil r\rceil}{r} \cdot \frac{p\,\mathbf{E}X_1^2}{1-p}.$$

Similarly, for the Kolmogorov metric using (14) and (21) we obtain

$$\begin{aligned}\rho\left(\frac{p\,S_{N_{r,p}}}{r(1-p)}, G_{r,r}\right) &= \rho\left(S_{N_{r,p}}, G_{r,\mu}\right) \le \lfloor r\rfloor \rho\left(S_{N_p}, \mathscr{E}_\mu\right) + \rho\left(S_{N_{\{r\},p}}, G_{\{r\},\mu}\right)\mathbb{1}(r \notin \mathbb{N}) \\ &\le \lfloor r\rfloor\, p \max\left\{1, \frac{1}{2(1-p)}\right\} \mathbf{E}X_1^2 + \frac{p^{\{r\}}(2\,\mathbf{E}X_1^2 + \{r\}^{-1} + 1)}{\Gamma(\{r\})\,\Gamma(2-\{r\})}\mathbb{1}(r \notin \mathbb{N}). \quad \square\end{aligned}$$

3. Conclusions

In the present paper, we derived the proximity in terms of zeta-structured metrics of generalized negative binomial random sums to generalized gamma distribution by using that of the geometric random sums to the exponential distribution. As a special case, we gave the corresponding upper bounds for the Kantorovich and the Kolmogorov metrics in the law of large numbers for negative binomial random sums of i.i.d. random variables with nonzero first moments and finite second moments.

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Abbreviations

The following abbreviations are used in this manuscript:

r.v.	random variable
i.i.d.	independent identically distributed
d.f.	distribution function
a.s.	almost sure
a.c.	absolute continuity, absolutely continuous
w.r.t.	with respect to
r.-h.s.	right-hand side

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