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The Cohomological Genus and Symplectic Genus for 4-Manifolds of Rational or Ruled Types

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Abstract: An important problem in low dimensional topology is to understand the properties of embedded or immersed surfaces in 4-dimensional manifolds. In this article, we estimate the lower genus bound of closed, connected, smoothly embedded, oriented surfaces in a smooth, closed, connected, oriented 4-manifold with the cohomology algebra of a rational or ruled surface. Our genus bound depends only on the cohomology algebra rather than on the geometric structure of the 4-manifold. It provides evidence for the genus minimizing property of rational and ruled surfaces.

Keywords: minimal genus; symplectic genus; rational surface; ruled surface

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1. Introduction

Let X be a smooth, closed, connected, oriented 4-manifold and C a closed, connected, smoothly embedded, oriented surface in X . We are interested in the topology of C or, specifically, the genus $g(C)$ of C .

If X is a complex surface and C is a complex curve, the well known adjunction formula in algebraic geometry (for example, see Sections 1.4 and 2.1 in [1]) shows that

$$g(C) = \frac{[C] \cdot [C] + K([C]) + 2}{2}.$$

Here, $K \in H^2(X; \mathbb{Z})$ is the canonical class for the complex structure, $[C] \in H_2(X; \mathbb{Z})$ is the homology class of C and $[C] \cdot [C]$ denotes the value of symmetric bilinear form

$$Q_X : H_2(X; \mathbb{Z}) / \text{Tor} \times H_2(X; \mathbb{Z}) / \text{Tor} \rightarrow \mathbb{Z}.$$

If oriented, smoothly embedded surfaces C_1, C_2 intersect transversely, then $[C_1] \cdot [C_2] = Q_X([C_1], [C_2])$ is the signed count of $C_1 \cap C_2$. Thus Q_X is called the intersection form of X . By Poincaré duality, we can identify homology group $H_2(X; \mathbb{Z})$ with cohomology group $H^2(X; \mathbb{Z})$ and Q_X is equivalent to the cup product in cohomology ring

$$Q_X : H^2(X; \mathbb{Z}) / \text{Tor} \times H^2(X; \mathbb{Z}) / \text{Tor} \rightarrow \mathbb{Z},$$

which is a unimodular symmetric bilinear form, that is, $\det(Q_X) = \pm 1$. Let $(b^+(X), b^-(X))$ be the signature of Q_X .

The adjunction formula shows that the genus of a complex curve C is completely determined by its homology class $[C]$, the symmetric bilinear form Q_X and the canonical class K (or the complex structure). This observation suggests that the genus of embedded surfaces should be explored from the viewpoint of homology theory. To fix a homology class $A \in H^2(X; \mathbb{Z})$, we want to know the minimal genus of a connected surface representing A , or the minimal genus function

$$mg_X(A) = \min\{g(C) | C : \text{connected, smoothly embedded surface with } [C] = PD(A)\}.$$

For rich history on this function, see the excellent surveys of Lawson [2,3].

Suppose X is the complex projective plane $\mathbb{CP}^2$. If C, C' are smooth surfaces in X such that $[C] = [C']$ and C are complex curves, the Thom conjecture, which was proved in [4], says that

$$g(C') \geq g(C) = \frac{[C] \cdot [C] + K([C]) + 2}{2}.$$

It shows that complex curves in $\mathbb{CP}^2$ have a minimal genus within their homology classes.

If X has a symplectic structure ω , we can also use a compatible, almost complex, structure to define the symplectic canonical class K_ω . The symplectic Thom conjecture, proved in [5,6], states that a symplectic surface has minimal genus within its homology class, and, under some conditions for (X, ω) , each surface C satisfies the adjunction inequality:

$$g(C) \geq \frac{[C] \cdot [C] + |K_\omega([C])| + 2}{2}.$$

In both cases, the lower bound of genus depends on the intersection form and the complex or symplectic structure. Moreover, the computation is descended to linear spaces and can be achieved by elementary linear algebra.

For example, the product of Riemann surfaces $X = S^2 \times T^2$ has homology group $H_2(X; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. $H_2(X; \mathbb{Z})$ has a basis $B = [pt \times T^2]$, $F = [S^2 \times pt]$ with $B \cdot B = F \cdot F = 0$, $B \cdot F = 1$ and $K = -2B$. Assume C is a complex torus on X with $[C] = aB + bF$, then it satisfies

$$g(C) = \frac{(aB + bF) \cdot (aB + bF) + (-2B) \cdot (aB + bF) + 2}{2} = ab - b + 1 = 1.$$

So $a = 1$ or $b = 0$ and $[C] = B + bF$ or aB . It is easy to construct an embedded torus in the class $B + bF$.

Later, an adjunction type lower bound h for mg_X was introduced in [7] by Strle, and was extended to a more general situation in [8]. See also the beautiful reformulation in [9]. Unlike the well known adjunction bounds in [4–6], this bound is purely in terms of the cohomology algebra of X under the assumption $b^+(X) = 1$. More precisely, let $\Lambda_X = H^*(X; \mathbb{Z})/Tor$ be the cohomology algebra of X modulo torsion. Then h is completely determined by Λ_X when $b^+(X) = 1$, and is called the cohomological genus function of X . This allows us to transform the study of mg_X , which depends on the geometric structure of X , to the computation of h , which is based on the algebraic structure of Λ_X and is much easier to deal with than mg_X .

It is shown in [8] that the cohomological genus function h provides a sharp bound for mg_X when $2\tilde{\chi}(X) + 3\sigma(X) \geq 0$. Here, $\tilde{\chi}(X)$ is the modified Euler number of X and $\sigma(X) = b^+(X) - b^-(X)$ is the signature of X (see Section 2.1 for a precise definition). The sharp bound is realized when $X = Y \# (S^1 \times S^3)$ with Y being a rational or ruled surface.

In this note, we will deal with the complementary case. We speculate that rational surfaces or ruled surfaces with $2\tilde{\chi} + 3\sigma < 0$ have the same property.

Conjecture 1. *Let X be a smooth 4-manifold with the cohomology algebra of a rational surface or ruled surface Z with $2\tilde{\chi} + 3\sigma < 0$. Then there exists a cohomology algebra isomorphism $\rho : \Lambda_X \rightarrow \Lambda_Z$ such that, for any class $A \in \Lambda_X^2$ with $A \cdot A \geq 0$, the minimal genus of A is bounded below by the minimal genus of $\rho(A)$, $mg_X(A) \geq mg_Z(\rho(A))$.*

We are able to partially verify the conjecture.

Theorem 1. *Let X be a smooth 4-manifold with the cohomology algebra of a rational surface or ruled surface Z with $2\tilde{\chi} + 3\sigma < 0$ and $\rho : \Lambda_X \rightarrow \Lambda_Z$ is any cohomology algebra isomorphism. Let*

$c_0 = \rho^{-1}(-K_Z) \in \Lambda_X^2$, where K_Z is the canonical class of Z regarded as a complex surface. For all classes $A \in \Lambda_X^2$ with $A \cdot A > 0$ and $c_0 \cdot A \geq 0$, we have $mg_X(A) \geq mg_X(\rho(A))$. Specifically,

$$mg_X(A) \geq \frac{A \cdot A - c_0 \cdot A + 2}{2}.$$

Various terminologies will be reviewed in Section 2. One important ingredient of the proof of Theorem 1 is the symplectic genus introduced in [10].

In this note, we will also make some explicit estimates of $h(A)$ when $c_0 \cdot A < 0$. In this case, our estimates are not optimal, but are easy to compute and effective.

The organization of this note is as follows: in Section 2, we review the notions of cohomology algebra, cohomological genus and adjunction classes from [8], and their general properties, which are needed throughout the article. In Section 3, we prove Theorem 1 and explicitly compute h for a rational or ruled type cohomology algebra with $2\tilde{\chi} + 3\sigma < 0$ when A satisfies $A \cdot A > 0$ and $c_0 \cdot A \geq 0$. In Section 4, we estimate h in the remaining case.

2. Cohomology Algebra of $b^+ = 1$ Type and Cohomological Genus

2.1. Cohomology Algebra and Quadratic Form

Let $\Lambda = \bigoplus_{i=0}^4 \Lambda^i$ be a finitely generated graded commutative algebra over $\mathbb{Z}$ with each summand Λ^i a free abelian group, and a group isomorphism $p : \Lambda^4 \cong \mathbb{Z}$. (Λ, p) , or simply Λ , is called a cohomology algebra if, with respect to p , the products $\Lambda^i \times \Lambda^{4-i} \rightarrow \Lambda^4 \cong \mathbb{Z}$ are duality pairings, in the sense that

$$\Lambda^i \rightarrow \text{Hom}_{\mathbb{Z}}(\Lambda^{4-i}, \mathbb{Z})$$

are isomorphisms of groups. Let $T : \Lambda^1 \times \Lambda^1 \rightarrow \Lambda^2$ be the skew-symmetric pairing and $\Gamma : \Lambda^2 \times \Lambda^2 \rightarrow \Lambda^4 \cong \mathbb{Z}$ the symmetric form.

Let $b_i(\Lambda)$ be the rank of Λ^i , $\tilde{b}_1(\Lambda)$ the rank of T , $\chi(\Lambda) = \sum (-1)^i b_i(\Lambda)$ the Euler number, and $\tilde{\chi}(\Lambda) = 2 + b_2(\Lambda) - 2\tilde{b}_1(\Lambda)$ the modified Euler number. Denote the signature type of Γ by $(b^+(\Lambda), b^-(\Lambda))$, and let $\sigma(\Lambda) = b^+(\Lambda) - b^-(\Lambda)$ denote the signature. Λ is called an algebra of $b^+ = k$ type if $b^+(\Lambda) = k$. A class $\xi \in \Lambda^2$ is called characteristic if $\Gamma(\xi, \eta) \equiv \Gamma(\eta, \eta) \pmod{2}$ for any $\eta \in \Lambda^2$.

For X a smooth, closed, connected, oriented 4-manifold, let $\Lambda_X = H^*(X; \mathbb{Z})/\text{Tor}$. Then Λ_X is a cohomology algebra. We define the cohomological invariants of X as $*(X) = (\Lambda_X, *, b_i, \tilde{b}_1, \chi, \tilde{\chi}, \sigma)$.

Example 1. The following algebra of $b^+ = 0$ type Λ_S is modeled on $S^1 \times S^3$: $\Lambda_S^2 = 0$, $\Lambda_S^1 = \Lambda_S^3 = \mathbb{Z}$ with trivial T pairing.

Given two cohomology algebras Λ_a and Λ_b , their direct sum $\Lambda_a \oplus \Lambda_b$ is defined as the cohomology algebra Λ with $\Lambda^i = \Lambda_a^i \oplus \Lambda_b^i$ for $1 \leq i \leq 3$, and the bilinear pairings being the direct sum. It is clear that $b^+(\Lambda_a \oplus \Lambda_b) = b^+(\Lambda_a) + b^+(\Lambda_b)$.

Λ is called Lefschetz if T is non-degenerate, or equivalently, $\tilde{b}_1(\Lambda) = b_1(\Lambda)$. Λ' is called the Lefschetz reduction of Λ if Λ' is Lefschetz and $\Lambda \cong \Lambda' \oplus l\Lambda_S$ for some l . It is easy to see that Λ' is well defined up to isomorphism, so we will denote any Lefschetz reduction of Λ by Λ_{red} . A simple but useful fact is that $\tilde{b}_1(\Lambda) = \tilde{b}_1(\Lambda_{red})$ and $\tilde{\chi}(\Lambda) = \tilde{\chi}(\Lambda_{red})$.

Suppose now Λ is a cohomology algebra of $b^+ = 1$ type. A special feature is that the image of T is either 0 or 1 dimensional, see [8], Paragraph 2.1.1. Since T is skew symmetric, $\tilde{b}_1(\Lambda)$ is always an even number.

Notice that the symmetric bilinear pairing Γ is unimodular. We will often abbreviate $\Gamma(x, y)$ as $x \cdot y$. It induces a unimodular quadratic form $Q : \Lambda^2 \rightarrow \mathbb{Z}$ as $Q(x) = \Gamma(x, x)$. $Q(x)$ is called the square of x . Γ is of even type if $Q(x)$ is even for any vector $x \in \Lambda^2$. Otherwise, Γ is called the odd type. Γ is called *definite*, or *indefinite* if $\min\{b^+, b^-\} = 0$ or ≥ 1 , respectively.

It is well known that indefinite unimodular symmetric forms are classified by their rank, signature and type (e.g., [11]). When $b^+ = 1$, the list of unimodular symmetric forms is

$$\langle 1 \rangle \oplus n\langle -1 \rangle, \quad U \oplus q(-E), \quad \text{where } n, q \in \mathbb{N} \cup \{0\}.$$

Here, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E are the hyperbolic lattice and the (positive definite) E_8 lattice, respectively.

Remark 1. It is conjectured in [12] that $q \leq 1$ when $b^+ = 1$. If this is true, then for any Lefschetz type algebra Λ with $b^+ = 1$ and $2\tilde{\chi} + 3\sigma < 0$, there is a rational or ruled surface Z with $\Lambda_Z = \Lambda$.

2.2. Cohomological Genus Function h

A class $c \in \Lambda^2$ is called an adjunction class if it is characteristic and either of the following conditions is satisfied,

- (I) $c \cdot c > \sigma(\Lambda)$,
- (II) $c \cdot c \geq 2\tilde{\chi}(\Lambda) + 3\sigma(\Lambda)$ and c pairs non-trivially with $\text{Im } T$ when T is non-trivial.

Definition 1. Let $A \in \Lambda^2$. For any class c of adjunction type, introduce the c -genus of A ,

$$h_c(A) = \begin{cases} \frac{A \cdot A - |c \cdot A| + 2}{2} & \text{if } A \neq 0 \\ 0 & \text{if } A = 0. \end{cases} \quad (1)$$

Let $h_\Lambda(A) = \max h_c(A)$, where the maximum is taken among all adjunction classes of Λ . h_Λ is called the cohomological genus of Λ . When there is no confusion, we just use $h(A)$ for $h_\Lambda(A)$.

For X , a smooth, closed, connected, oriented 4-manifold with $b^+(X) = 1$ and any cohomology class $A \in \Lambda_X^2$ with $A \cdot A \geq 0$, there is the inequality (Theorem 1.4 in [8])

$$mg_X(A) \geq h(A).$$

This inequality was proved in [7] when $b^+ = 1$ and $b_1 = 0$, via L^2 moduli spaces of manifolds with cylindrical ends. The proof in [8] uses the wall crossing formula for Seiberg–Witten invariants.

2.3. $\text{Aut}(\Lambda)$ and Adjunction Classes

Let Λ be a cohomology algebra of $b^+ = 1$ type and $\text{Aut}(\Lambda)$ the automorphism group of Λ . Let $\mathcal{C}_\Lambda$ be the set of adjunction classes of Λ . We discuss the properties of this set and their consequences for the function h .

To compute or estimate $h(A)$, we want to minimize $|c \cdot A|$ among all $c \in \mathcal{C}_\Lambda$.

If T is trivial, then there are only type I adjunction classes. A well-known result says that characteristic classes satisfy $c \cdot c \equiv \sigma(\Lambda) \pmod{8}$. So $c \cdot c > \sigma(\Lambda)$ is equivalent to $c \cdot c \geq \sigma(\Lambda) + 8$. By direct computation, we also have $2\tilde{\chi}(\Lambda) + 3\sigma(\Lambda) = \sigma(\Lambda) + 8 - 4\tilde{b}_1(\Lambda)$.

Lemma 1. When T is non-trivial, that is, $\tilde{b}_1(\Lambda) \geq 2$, a type I adjunction class is also of type II if and only if it pairs non-trivially with $\text{Im } T$.

Since an adjunction class c is characteristic, $h_c(A)$ is always an integer. It also satisfies $h_c(A) = h_c(-A)$, hence $h(A) = h(-A)$. We remark that the minimal genus function mg_X also has this symmetry.

Lemma 2 ([8]). $\mathcal{C}_\Lambda$ is preserved by $\text{Aut}(\Lambda)$. Consequently, h is invariant under $\text{Aut}(\Lambda)$. Under the natural identification between Λ^2 and Λ_{red}^2 , $\mathcal{C}_\Lambda = \mathcal{C}_{\Lambda_{\text{red}}}$. Consequently, $h_\Lambda = h_{\Lambda_{\text{red}}}$.

By this invariance property, to determine the function h , it suffices to pick an element, called reduced element, in each orbit of $Aut(\Lambda)$ and to calculate the value of h . When $2\tilde{\chi} + 3\sigma \geq 0$, this was carried out in [7,8] completely.

3. Calculation of h in Terms of the Symplectic Genus

3.1. Cohomology Algebras of Rational or Ruled Type with $2\tilde{\chi} + 3\sigma < 0$

We say that an algebra Λ is of rational or ruled type if $\Lambda \cong \Lambda_Z$ for some rational or ruled manifold Z . Notice that such Λ is Lefschetz and of $b^+ = 1$ type, thus $2\tilde{\chi}(\Lambda) + 3\sigma(\Lambda) = \sigma(\Lambda) + 8 - 4\tilde{b}_1(\Lambda)$. Cohomology algebras of rational or ruled type with $2\tilde{\chi} + 3\sigma < 0$ can be listed as

1. T trivial and $\Gamma = \langle 1 \rangle \oplus n\langle -1 \rangle, n \geq 10$,
2. $\tilde{b}_1(\Lambda) = 2g \geq 4$ and $\Gamma = U$,
3. $\tilde{b}_1(\Lambda) = 2g \geq 4$ and $\Gamma = \langle 1 \rangle \oplus \langle -1 \rangle$,
4. $\tilde{b}_1(\Lambda) = 2g \geq 2$ and $\Gamma = U \oplus n\langle -1 \rangle, n \geq 1$.

3.1.1. Standard basis and reduced classes

We now introduce the notion of standard bases of Λ^2 in the four cases. The standard bases are equivalent under $Aut(\Lambda)$. Notice that, whenever T is trivial, we have $Aut(\Lambda) = Aut(\Gamma)$.

We also introduce reduced classes under a choice of standard basis. As mentioned in the paragraph after Lemma 2, we only have to consider the c -genus for reduced classes in the computation of h .

Case (1): Let $n = b^- \geq 10$ and $\mathcal{B} = \{H, E_1, \dots, E_n\}$ be a pairwise orthogonal basis for Λ^2 such that $H \cdot H = 1 = -E_i \cdot E_i$.

Reduced classes ([13,14]): $aH - \sum_{i=1}^n b_i E_i$ with $b_1 \geq b_2 \geq \dots \geq b_n \geq 0$ and $a \geq b_1 + b_2 + b_3$.

For the remaining cases, assume $\text{Im } T$ is generated by F . In particular, $F \cdot F = 0$. Note that F is preserved by $Aut(\Lambda)$ up to sign.

Case (2): Choose a basis $\mathcal{B} = \{F, B\}$ of Λ^2 with $B \cdot B = 0$ and $F \cdot B = 1$.

The automorphism group is $\mathbb{Z}_2$ generated by $-Id$.

Reduced classes: $aB + bF$ with $a > 0$, or $a = 0, b \geq 0$.

Case (3): Choose a basis $\mathcal{B} = \{F, B\}$ of Λ^2 with $B \cdot B = 1, F \cdot B = 1$.

The automorphism group is $\mathbb{Z}_2$ generated by $-Id$.

Reduced classes: $aB + bF$ with $a > 0$, or $a = 0, b \geq 0$.

Case (4): Choose a basis $\mathcal{B} = \{F, B, E_i\}$ of Λ^2 with $B \cdot B = 0, F \cdot B = 1, E_i \cdot E_i = -1, E_i \cdot E_j = E_i \cdot F = E_i \cdot B = 0, 1 \leq i \neq j \leq n$.

Reduced classes (Definition 3.3 in [10]): $aB + bF - \sum_{i=1}^n e_i E_i$ with $a > 0, a \geq e_1 \geq e_2 \geq \dots \geq e_n \geq 0$ or $bF, b \geq 0$.

3.1.2. Geometric Standard Basis for Rational or Ruled Surfaces

For a rational surface Z , a standard basis is said to be geometric if each class is represented by an embedded sphere. For a ruled surface Z over a genus g surface, a standard basis is said to be geometric if F and E_i are represented by embedded spheres, B is represented by an embedded surface of genus g .

Geometric standard bases always exist and are equivalent under $D(Z)$. Here, $D(Z)$ is the geometric automorphism group of Z , which is the image of $\text{Diff}^+(Z) \rightarrow Aut(\Lambda(Z))$.

Lemma 3. Any class of non-negative square in the four cases is equivalent to a reduced class under $Aut(\Lambda)$.

Proof. Lemma 3.4 (1) in [10] says that, for a rational or ruled surface, any class of non-negative square in the four cases is equivalent to a reduced class under $D(Z)$. Since $D(Z)$ is a subgroup of $Aut(\Lambda)$, any class of non-negative square is equivalent to a reduced class under $Aut(\Lambda)$. $\square$

3.1.3. The Adjunction Classes

We describe the adjunction classes in terms of a standard basis.

Lemma 4. *Let c be an adjunction class. Then, c is of type I for a rational surface type, and of type II for an irrational ruled surface type.*

Proof. For a rational surface type, there are only type I adjunction classes since T is trivial.

By Lemma 1, we just need to check that c is non-trivial on F . Let $c = \alpha B + \beta F$ for Cases (2) and (3), and $c = \alpha B + \beta F + \sum_{i=1}^n \gamma_i E_i$ with γ_i odd for Case (4).

If c is trivial on F , then $\alpha = 0$. So c cannot be of type I since $c \cdot c = 0 = \sigma$ for Cases (2) and (3), and $c \cdot c = -\sum_{i=1}^n \gamma_i^2 \leq -n = \sigma$ for Case (4). $\square$

Generally, it is not easy to evaluate $h(A)$ since there are always infinitely many adjunction classes. We will use the notion of symplectic genus in [10] to evaluate $h(A)$ when $c_0 \cdot A \geq 0$.

3.1.4. The Adjunction Class c_0

Proposition 1. *Suppose a basis $\mathcal{B}$ is given in each case. There is a unique $c_0 \in \Lambda^2$ such that*

- (i) c_0 is characteristic and $c_0 \cdot E_i = 1$.
- (ii) $c_0 \cdot c_0 = 2\tilde{\chi} + 3\sigma$,
- (iii) c_0 is reduced,
- (iv) $|c_0 \cdot F| = 2$ when T is non-trivial.

Explicitly, with respect to the standard basis $\mathcal{B}$, c_0 is given in each case as follows.

- Case (1): $c_0 = 3H - \sum_{i=1}^n E_i$
- Case (2): $c_0 = 2B + (2 - 2g)F$,
- Case (3): $c_0 = 2B + (1 - 2g)F$,
- Case (4): $c_0 = 2B + (2 - 2g)F - \sum_{i=1}^n E_i$.

Since standard bases are equivalent under $Aut(\Lambda)$, the associated c_0 are also equivalent under $Aut(\Lambda)$.

Proof. The existence of c_0 is given by the explicit expression.

To establish the uniqueness, note that the coefficient of each E_i is 1 in Cases (1) and (4) by (i), and the coefficient of B is 2 in Cases (2)–(4) by (iii) and (iv). Finally, by (ii) and (iii), the coefficient of H is 3 in Case (1), and by (ii), the coefficient of F is uniquely determined in Cases (2)–(4). $\square$

We write down $h_{c_0}(A)$ explicitly, when $c_0 \cdot A \geq 0$.

- Case (1): $\binom{a-1}{2} - \sum_{i=1}^n \binom{b_i}{2}$
- Case (2): $(a-1)(b-1) + ga$
- Case (3): $(a-1)(\frac{a}{2} + b - 1) + ga$
- Case (4): $(a-1)(b-1) + ga - \frac{1}{2} \sum_{i=1}^n e_i(e_i - 1)$.

3.2. Symplectic Genus of a Rational or Ruled Surface

To define the symplectic genus, first recall the notion of the K -symplectic cone. A class $K \in H^2(X; \mathbb{Z})$ is called a symplectic canonical class if it is the canonical class of some orientation-compatible symplectic structure on X . Let $\mathcal{K}$ be the set of symplectic canonical classes. For any $K \in \mathcal{K}$, we introduce the K -symplectic cone:

$$\mathcal{C}_K = \{[\omega] | \omega \text{ is an orientation-compatible symplectic form, } K_\omega = K\}.$$

For $e \in \Lambda_X^2$, consider the subset $\mathcal{K}_e$ of $\mathcal{K}$:

$$\mathcal{K}_e = \{K \in \mathcal{K} \mid \text{there is a class } \tau \in \mathcal{C}_K \text{ such that } \tau \cdot e > 0\}.$$

For $K \in \mathcal{K}_e$, define the K -symplectic genus $\eta_K(e)$ to be $\frac{1}{2}(K \cdot e + e \cdot e) + 1$. Finally, the symplectic genus of e is defined as

$$\eta(e) = \max_{K \in \mathcal{K}_e} \eta_K(e).$$

By Lemma 3.2 in [10], $\eta(e)$ has the following basic properties.

- $\eta(e)$ is no bigger than the minimal genus of e , and they are both equal to the K -symplectic genus $\eta_K(e)$ if e is represented by a connected ω -symplectic surface for some symplectic form ω with $K_\omega = K$. Here K_ω is the symplectic canonical class of ω
- $\eta(e) = \eta(-e)$, and $D(X)$ -equivalent classes have the same η .

For a rational or ruled surface, the symplectic genus η has additional properties. Let Z be a non-minimal rational or irrational ruled surface with a geometric standard basis. Let c_0 be the associated adjunction class. In [10], c_0 is denoted as $-K_0$.

- (Lemma 3.4 (5) in [10]) If e is a reduced class with non-negative square, then $-c_0 \in \mathcal{K}(e)$ and consequently $\eta(e)$ is given by $\eta_{-c_0}(e)$.
- (Lemma 3.6 in [10]) The symplectic genus of any class with positive square or a primitive class with square 0 is non-negative.
- (Proposition 3.5 in [10]) Suppose e is a reduced class. If $e \cdot e \geq \eta(e) - 1$, then $e \cdot e \geq 0$ and e is represented by a symplectic surface. Moreover, if e is either a class of positive square or a primitive class with square 0, e is represented by a connected symplectic surface, and therefore its minimal genus is given by its symplectic genus.

We are now ready to prove Theorem 1.

Proposition 2. Let X be a 4-manifold with the cohomology ring of a rational surface or ruled surface Z with $2\tilde{\chi} + 3\sigma < 0$. Given a standard basis of Λ_X^2 and a geometric standard basis of Λ_Z^2 , we can identify Λ_X^2 with Λ_Z^2 and let c_0 be the associated adjunction class. For a reduced class A with $A \cdot A > 0$, or $A \cdot A = 0$ and A is primitive, and $c_0 \cdot A \geq 0$, we have

$$mg_X(A) \geq h(A) = h_{c_0}(A) = mg_Z(A).$$

Proof. If $c_0 \cdot A \geq 0$, then $|c_0 \cdot A| = c_0 \cdot A$. Hence $h_{c_0}(A) = \eta_{-c_0}(A)$.

Since $\eta(A) = \eta_{-c_0}(A)$ by Lemma 3.4 (5) in [10], $A \cdot A \geq \eta(A) - 1$ is the same as

$$A \cdot A \geq \eta_{-c_0}(A) - 1 = \frac{1}{2}(A \cdot A - c_0 \cdot A),$$

which is the same as $A \cdot A \geq -c_0 \cdot A$. This is certainly satisfied if $A \cdot A \geq 0$ and $c_0 \cdot A \geq 0$.

By Proposition 3.5 in [10], $\eta(A) = mg_Z(A)$. Therefore we have

$$mg_X(A) \geq h(A) = h_{c_0}(A) = mg_Z(A),$$

if A is a reduced class with $A \cdot A > 0$, or $A \cdot A = 0$ and primitive, and $c_0 \cdot A \geq 0$. $\square$

Proof of Theorem 1. Since a reduced class A with sufficiently big H coefficient in Case (1), or positive B coefficient and sufficiently big F coefficient in Cases (2)–(4) satisfies $A \cdot A > 0$ and $c_0 \cdot A \geq 0$, it follows from Proposition 2 that there are infinitely many classes as claimed. $\square$

Remark 2. It follows from Theorem 1 and Lemma 2 and Lemma 4.1 [8] that if $\tilde{Z}$ is a rational surface or a ruled surface with $2\tilde{\chi} + 3\sigma < 0$ and $Z = \tilde{Z} \# l(S^1 \times S^3)$, then for any 4-manifold X with the cohomology ring of Z and an identification $\rho : \Lambda_X \rightarrow \Lambda_Z$, there are infinitely many homology classes $A \in \Lambda_X^2$ with $A \cdot A > 0$ and $mg_X(A) \geq mg_Z(\rho(A))$.

If the conjecture on the even intersection form in [12] mentioned in Remark 1 holds, then any 4-manifold X with $b^+ = 1$ and $2\tilde{\chi} + 3\sigma < 0$ has the cohomology ring of a rational or ruled surface.

Example 2. Let $Z = \mathbb{CP}^2 \# n \overline{\mathbb{CP}^2}$, $n \geq 10$. For any $b > 0$, $n \geq 10$, $A = bH - \sum_{i=1}^n bE_i$ satisfies $A \cdot A > 0$, $c_0 \cdot A \geq 0$.

By standard topological surgery, we can construct an embedded smooth surface C in Z such that $[C] = PD(A)$ and

$$g(C) = \frac{b^2 n(n-1)}{2} - bn + 1 = h_{c_0}(A)$$

is the minimal genus.

4. Estimates of h When $c_0 \cdot A < 0$

In this section, let X be a 4-manifold with the cohomology ring of a rational surface or ruled surface Z with $2\chi + 3\sigma < 0$. Given a standard basis of Λ_X^2 let c_0 be the associated canonical class. Let A be a reduced class with $A \cdot A \geq 0$, and $c_0 \cdot A < 0$. We will estimate $h(A)$.

4.1. Rational Surface Type (Case (1))

Let $\mathcal{B} = \{H, E_1, \dots, E_n\}$ be a standard basis of Λ^2 such that $H^2 = 1 = -E_i^2$. Let $A = aH - \sum_{i=1}^n b_i E_i$ be a reduced class with $A \cdot A \geq 0$. So $b_1 \geq b_2 \geq \dots \geq b_n \geq 0$ and $a \geq b_1 + b_2 + b_3$. For convenience, we assume that $b_n > 0$. Otherwise, it reduces to a smaller n case.

Recall that $c_0 = 3H - \sum_{i=1}^n E_i$, so $c_0 \cdot A < 0$ means that $3a - \sum_{i=1}^n b_i < 0$. It is also clear that $3a - \sum_{i=1}^9 b_i \geq 0$.

Let $c = kH - \sum_{i=1}^n e_i E_i$ be an adjunction class. Then, k and e_i are odd and $c \cdot c = k^2 - \sum e_i^2 \geq \sigma + 8 = 9 - n$.

Lemma 5. Suppose $n \geq 10$, $A = aH - \sum_{i=1}^n b_i E_i$ is a reduced class with $A \cdot A \geq 0$, and $3a - \sum_{i=1}^n b_i < 0$. We can find an adjunction class c_1 such that $|c_1 \cdot A| \leq b_{10}$. Consequently, we have the uniform and simple bound,

$$h(A) \geq h_{c_1}(A) = \frac{1}{2}(A \cdot A - |c_1 \cdot A| + 2) \geq \frac{1}{2}A \cdot A - \frac{b_{10}}{2} + 1.$$

Proof. Let $c_1 = 3H + \sum_{i=1}^n \epsilon_i E_i$ where $\epsilon_i = \pm 1$ are defined inductively as follows. Let $S_0 = 3a$, $S_p = 3a + \sum_{i=1}^p \epsilon_i b_i$, $1 \leq p \leq n$ be the partial sums of $c_1 \cdot A = 3a + \sum_{i=1}^n \epsilon_i b_i$. Then,

$$\epsilon_{p+1} = -\text{sgn}(S_p) = \begin{cases} -1 & \text{if } S_p \geq 0, \\ 1 & \text{if } S_p < 0, \end{cases} \quad 0 \leq p \leq n-1. \quad (2)$$

The sign rule is to make the partial sums S_p oscillate around 0 with minimal amplitudes. The sign rule Equation (2) determines a subsequence of indices

$$10 \leq l_1 < l_2 < \dots < l_m \leq n, \quad (3)$$

where the sequence of partial sums $\{S_p\}$ change monotonicity.

Let $l_0 = 0$. When $l_j + 1 \leq p \leq l_{j+1}$, $0 \leq j \leq m-1$, and j is even, we have $\epsilon_p = -1$, and the subsequence S_p strictly decreases until it becomes negative. So $-b_{l_{j+1}} \leq S_{l_{j+1}} < 0$, when j is even.

When $l_j + 1 \leq p \leq l_{j+1}$, $0 \leq j \leq m-1$ and j are odd, we have $\epsilon_p = 1$, and the subsequence S_p strictly increases until it becomes non-negative. So $0 \leq S_{l_{j+1}} < b_{l_{j+1}}$ when j is odd.

If $l_m = n$, then we have $|S_n| \leq b_n$. If $l_m < n$, then the remaining string has two cases according to the sign of S_{l_m} . In all cases, we can verify that $|c_1 \cdot A| = |S_n| \leq b_{l_m}$.

Since $l_m \geq 10$, we have a uniform bound $|c_1 \cdot A| \leq b_{10}$. The genus bound follows immediately. $\square$

Example 3. Let $a = 16, b_1 = 6, b_2 = \dots = b_9 = 5 > b_{10} = b_{11} = 3$. Then $c_1 = 3H - \sum_{i=1}^{10} E_i + E_{11}$. We have $A \cdot A = 2 \geq 0$, $c_0 \cdot A = -4 < 0$, and $|c_1 \cdot A| = 2 < |c_0 \cdot A| = 4$.

Similarly, for the sequence of primitive classes $A = (3s + 1)H - (s + 1)b_1 - \sum_{i=2}^9 sE_i - 3E_{10} - 3E_{11}$, $s \geq 5$, $c_1 = 3H - \sum_{i=1}^{10} E_i + E_{11}$. Since $A \cdot A = 4s - 18 \geq 0$ and $c_1 \cdot A = 2$ we have $h(A) \geq h_{c_1}(A) = 2s - 9$. Note that $h(A)$ could be arbitrarily large for this sequence of primitive classes.

Remark 3. For primitive classes A with $A \cdot A > 0$ and $A_d = dA$, $h(A_d)$ is estimated in Theorem 11.1 and Corollary 11.2 in [7]. In particular, if A is characteristic, $h(A_d) \geq \binom{d}{2} A \cdot A$. It is remarked that the leading term for $h(A_d)$ is, by a factor of 2, better than the bounds for the divisible classes A_d in [15,16].

4.2. Minimal Irrational Ruled Surface Type (Cases (2) and (3))

Let c be an adjunction class. By Lemma 1, c is of type II.

Proposition 3. Let Λ be a cohomology algebra of irrational ruled surface type and $\{B, F\}$ a standard basis of Λ^2 . Let $A = aB + bF$ be a reduced class with $A \cdot A \geq 0$ and $c_0 \cdot A < 0$. There is a uniform estimate for $h(A)$:

- $h(A) \geq (a - 1)b + 1$ when $B \cdot B = 0$.
- $h(A) \geq (a - 1)(\frac{a}{2} + b) + 1$ when $B \cdot B = 1$.

Proof. In the case $B \cdot B = 0$, $A \cdot A = 2ab \geq 0$ and $c_0 \cdot A = (2 - 2g)a + 2b < 0$ imply $(g - 1)a > b \geq 0$. An adjunction class has the form $c = 2pB + 2qF$, and satisfies

$$c \cdot c = 8pq \geq 2\tilde{\chi} + 3\sigma = 8 - 8g \quad \text{and} \quad c \cdot F = 2p \neq 0.$$

We take adjunction class $c_1 = 2B$. Then $c_1 \cdot A = 2b \geq 0$, and

$$h(A) \geq h_{c_1}(A) = \frac{1}{2}(2ab - 2b + 2) = (a - 1)b + 1.$$

In the case $B \cdot B = 1$, $A \cdot A = a(a + 2b) \geq 0$ and $c_0 \cdot A = (3 - 2g)a + 2b < 0$ imply $a > 0, a + 2b \geq 0$. An adjunction class has the form $c = 2pB + (2q - 1)F$, and satisfies

$$c \cdot c = 4p(p + 2q - 1) \geq 2\tilde{\chi} + 3\sigma = 8 - 8g \quad \text{and} \quad c \cdot F = 2p \neq 0.$$

We take adjunction class $c_2 = 2B - F$, then $c_2 \cdot A = a + 2b \geq 0$, and

$$h(A) \geq h_{c_2}(A) = \frac{1}{2}(a^2 + 2ab - (a + 2b) + 2) = (a - 1)(\frac{a}{2} + b) + 1.$$

□

Remark 4. This bound is uniform and very simple but not optimal in general. Consider the even intersection form case $B \cdot B = 0$. Suppose $A = aB + bF$ with $A \cdot A = 2ab \leq 2g - 2$.

1. Then, $c = 2aB - 2bF$ is an adjunction class and $c \cdot A = 0$. Therefore, we have $h(A) = h_c(A) = A \cdot A + 1 = ab + 1$.
2. For the divisible class $A' = tA$, we also have $h(A') = h_c(A') = A' \cdot A' + 1$.

Remark 5. The set of adjunction classes $\mathcal{C}_\Lambda$ is bigger when $b_1(\Lambda)$ increases. This is true since $\chi(\Lambda)$ becomes smaller as $b_1(\Lambda)$ increases. Consequently, we sometimes obtain stronger estimates for the same class $A = aB + bF$ when $b_1(\Lambda)$ increases, although the uniform estimate in Proposition 3 is independent of $b_1(\Lambda)$. For example, if $B \cdot B = 0$ and $g = 2$, for $A = B + 2F$, we can choose $c = 2B - 2F$ and get $h_c(A) = 2$. When $B \cdot B = 0$ and $g > 2$, we can choose $c = 2B - 4F$ and get $h(A) = h_c(A) = 3$.

4.3. Non-Minimal Irrational Ruled Surface Type (Case (4))

Choose a standard basis $\mathcal{B} = \{F, B, E_1, \dots, E_n\}$ for Λ^2 with $B \cdot B = 0 = F \cdot F$, $F \cdot B = 1$, $E_i \cdot E_i = -1$, $E_i \cdot E_j = E_i \cdot F = E_i \cdot B = 0$. Recall that $g \geq 1$.

Let $A = aB + bF - \sum_{i=1}^n e_i E_i$ be a reduced class with $A \cdot A = 2ab - \sum_{i=1}^n e_i^2 \geq 0$. So $a \geq e_1 \geq \dots \geq e_n \geq 0$, $b \geq 0$. We assume $e_1 \geq 1$, otherwise it reduces to the $n = 0$ case.

Recall that $c_0 = 2B + (2 - 2g)F - \sum_{i=1}^n E_i$. So $c_0 \cdot A < 0$ implies $2b + (2 - 2g)a - \sum_{i=1}^n e_i < 0$.

Lemma 6. Let $A = aB + bF - \sum_{i=1}^n e_i E_i$ be a reduced class with $A \cdot A \geq 0$, $e_1 \geq 1$, and satisfy $2b + (2 - 2g)a - \sum_{i=1}^n e_i < 0$. Then we have

$$h(A) \geq ab + 1 - \frac{1}{2} \left(\sum_{i=1}^n e_i^2 + \max\{2b - \sum_{i=1}^n e_i, e_1\} \right).$$

Proof. An adjunction class has the form $c = 2pB + 2qF + \sum_{i=1}^n r_i E_i$, where $p, q \in \mathbb{Z}$, r_i are odd integers. c is of type II, so

$$c \cdot c = 8pq - \sum_{i=1}^n r_i^2 \geq 2\chi + 3\sigma = 8 - 8g - n, \text{ and } c \cdot F = 2p \neq 0.$$

We will choose an adjunction class $c_1 = 2B + \sum_{i=1}^n r_i E_i$ with $r_i = \pm 1$. So,

$$c_1 \cdot A = 2b + \sum_{i=1}^n r_i e_i.$$

If $2b - \sum_{i=1}^n e_i \geq 0$, then we take all $r_i = -1$, and get $c_1 \cdot A = 2b - \sum_{i=1}^n e_i$.

If $2b - \sum_{i=1}^n e_i < 0$, let $S_0 = 2b$, $S_j = 2b + \sum_{i=1}^j r_i e_i$, $1 \leq j \leq n$, be the partial sums of $c_1 \cdot A$. Applying the oscillating sequence method and sign rule Equation (2) in Section 4.1, we can deduce that $|c_1 \cdot A| = |S_n| \leq e_1$.

Thus, we obtain an estimate $|c_1 \cdot A| \leq \max\{2b - \sum_{i=1}^n e_i, e_1\}$, and have the desired genus bound. $\square$

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