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# An Intrinsic Value Approach to Valuation with Forward–Backward Loops in Dividend Paying Stocks

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**Abstract:** We formulate a claim valuation problem where the dynamics of the underlying asset process contain the claim value itself. The problem is motivated here by an equity valuation of a firm, with intermediary dividend payments that depend on both the underlying, that is, the assets of the company, and the equity value itself. Since the assets are reduced by the dividend payments, the entanglement of claim, claim value, and underlying is complete and numerically challenging because it forms a forward–backward stochastic system. We propose a numerical approach based on disentanglement of the forward–backward deterministic system for the intrinsic values, a parametric assumption of the claim value in its intrinsic value, and a simulation of the stochastic elements. We illustrate the method in a numerical example where the equity value is approximated efficiently, at least for the relevant ranges of the asset value.

**Keywords:** corporate finance; with-profit insurance; forward–backward stochastic differential equations; intrinsic value

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## 1. Introduction

We propose and demonstrate a simulation technique for claim valuation in a situation where this is fundamentally challenging, namely for non-simple contingent claims where both the claim payments and the underlying price process depend on the option value itself. A key motivating example is the equity valuation in a corporate finance framework for a dividend-paying company with a dividend strategy that depends on the equity value itself. A similarly challenging situation arises in with-profit insurance when dividends to policy holders depend on prospectively calculated liabilities themselves. The fundamental idea proposed here is to set up a deterministic as-if market of intrinsic values of both the price and the claim value process, and then assume a parametric relation between the value and the intrinsic value of the claim. From a practical point of view, the intrinsic value is particularly apt as a deterministic basis since it has a clear economic interpretation. We carry out all details and demonstrate the idea in a numerical example for the special case where the option value is approximated as a linear function of its intrinsic value.

Motivating examples, where the claim value appears in the dynamics of the underlying, can be found in both finance and insurance. In corporate finance, a main task is to calculate equity value as a claim on the assets in the presence of debt. This was first studied by [1], but the problem continues to attract interest, see for instance, [2,3]. If the company pays out dividends and these dividends depend on the equity value itself, we have a situation where the contingent claim consists of equity-dependent dividends and an ultimate sharing of assets with creditors. However, since the dividends are financed by the company's assets, even the underlying asset process is influenced by the equity value. Then, we have a triangular interdependence between the underlying, the claim on the underlying, and the value of the claim. The feedback effect from the value of the claim to

the underlying is non-standard, and is well-motivated by this corporate finance example, but is notoriously difficult to handle from a numerical point of view.

Another example in with-profit insurance is where a situation similar to the one in corporate finance arises for the valuation of liabilities. In with-profit insurance, the insurance company redistributes profits to the collective of policy holders in terms of dividends to policy holders but the extent and the timing are partly regulated by the financial authorities. In market valuation based accounting and solvency rules, the future redistribution strategy must be formalized in terms of so-called Future Management Actions that also include future investment decisions. It is natural to base the Future Management Actions, and thus the dividends to policy holders, on the future prospective liability value itself and, thereby, the involved forward–backward system appears again. The underlying assets of the company are reduced by dividends to policy holders and, thus, the assets, the dividends, and the liability value of policy holder dividends are completely entangled. The simpler case, where dividends depend on the assets (and other quantities that are easily calculated at every time point in a simulation), has only recently been formalized, for example, in [4,5]. The Future Management Actions in with-profit insurance naturally motivate our study but we stick to the corporate finance story when we present our method and numerical example below.

Feeding back the claim value into the underlying price dynamics creates a stochastic forward–backward differential system since the known side condition of the underlying (given the claim value) exists at the initial time point, whereas the known side condition of the claim value (given the underlying) exists at the terminal time point. For example, for standard diffusive financial markets, such a situation is solvable by PDE methods where the claim value is calculated for all asset values backward. In general markets (or if one for other reasons prefers, or is forced, to simulate), the feed-back feature is challenging. Our proposal is to disentangle the problem into essentially three parts. First, the forward–backward element of the problem is handled by an iteration in a deterministic world where both asset prices and claim prices are represented by their intrinsic values. Two iterative methods are proposed here. Second, based on assuming a parametric relation between the claim value and the intrinsic value of the claim, a standard forward Monte Carlo simulation is performed for a given parameter. Third, the simulation is performed a number of times for iterated determination of the parameter value consistent with the input value (current asset price) and the ultimate output value (current claim value) of the system.

Numerical techniques in general, and simulation techniques in particular, for the valuation of contingent claims are challenged by fundamental relations between the claims themselves, claim values and underlying state processes. A canonical example is the simulation for the valuation of American options where the decision about whether to exercise, and therefore the actual claim, depends on the value itself. Thus, in that case, there is a relation between claims and claim values but claim values are (usually) not fed back into the underlying stock price dynamics. For the American option case, least-squares Monte Carlo has become a dominant numerical technique, both theoretically and practically, since [6] introduced that idea to the domain.

The least-squares Monte Carlo is an example of a numerical method tailor-made for a specific version of a general forward–backward stochastic differential equation. Similar to that, other methods have also been proposed but they are typically constructed for certain cases with a special entanglement of the forward and backward equations. Compared to the American option valuation, we add here a layer of complexity and allow the underlying stock price dynamics to also depend on the value of the contingent claim. This completely entangles the forward and backward equations. It is outside the scope of this paper to clarify whether the least-squares Monte Carlo techniques and/or other specific approximation methods to other types of forward–backward stochastic differential equations can (be generalized to) cope with our case of value dependence in the underlying stock price. Even if they could, our approximation method based on an intrinsic value projection can

be justified as a practical alternative since the intrinsic value in finance and insurance is well-understood and has its clear intuitive meaning and merits.

The entanglement of claim and claim value is standard from American option theory and is acknowledged to be numerically challenging. Therefore, upper and lower bounds have been sought and found via a dual formulation of the optimal stopping problem by, for example, [7–9]. The upper and lower bounds obtained by [10] are particularly interesting in relation to our work as they also use the intrinsic value of the option as a state process in a recursive procedure. To calculate and simulate under the feedback effect into the asset price dynamics is new, to our knowledge, although both the corporate finance and the with-profit insurance applications and interpretations seem obvious.

After having discussed the relation to numerical techniques for American option valuation, we find it important to stress one final difference. Along with American option valuation comes the optimal stopping problem, and solving the value and the optimal stopping strategy are two sides of the same story. There is no optimization going on in our problem, which is a pure valuation problem. The computational difficulty arises purely from the specification of the dividend payment strategy and is, at least in this exposition, completely separated from any question about whether such a strategy is optimal in any sense. Therefore, our work also contains no discussion about optimal versus sub-optimal strategies such as, for example, that studied by [11]. It is, though, an interesting discussion—but also beyond our scope—to learn about which objective functions lead to equity-dependent dividends when considering dividends as an optimal control process.

The structure of the paper is as follows: in Section 2, we formalize and motivate the problem of the main application of equity valuation. In Section 3, we approximate the value of the claim that is presented by means of its intrinsic value and several iterative methods to calculate it. Section 4 presents the simulation part of the valuation and the second layer of iteration. A numerical study in Section 5 shows the quality of our method for the equity valuation.

## 2. The Problem

We consider a general financial market consisting of an (possibly stochastic) interest rate process $r = \big(r(t)\big)_{t\geq 0}$ and an asset $S = \big(S(t)\big)_{t\geq 0}$. We assume the financial market is free of arbitrage resulting in the existence of a (not necessarily unique) martingale measure $\mathbb{Q}$. We let $\big(\Omega, \mathcal{A}, \mathbb{Q}\big)$ be a complete probability space governing a 2-dimensional Brownian motion $W = \big(W_1, W_2\big)$, and denote by $\mathcal{F} = \big(\mathcal{F}_t\big)_{t\geq 0}$ the natural filtration of $W$. We assume that $W_1$ and $W_2$ are independent for simplicity, but the extension to correlated dynamics is straightforward.

In this model, the dynamics of the asset depend on the value process of an option derived from the asset itself. The option is a payment stream with continuous payments with a (possibly stochastic) payment rate $\phi$ and a lump sum payment $\Phi\big(S(T)\big)$ at time $T$. We denote the value of the future payments of the option at time $t \leq T$ by $V(t)$, and assume that $\phi$ is in the form $\phi\big(t, S(t), r(t), V(t)\big)$ for a deterministic function $\phi : \mathbb{R}^2_+ \times \mathbb{R}^2 \mapsto \mathbb{R}$. The dynamics of the asset are of the form

$$\mathrm{d}S(t) = g\big(t, S(t), r(t), V(t)\big)\mathrm{d}t + \sigma\big(t, S(t), r(t), V(t)\big)\mathrm{d}W_1(t), \tag{1}$$

for deterministic functions $g : \mathbb{R}^2_+ \times \mathbb{R}^2 \mapsto \mathbb{R}$ and $\sigma : \mathbb{R}^2_+ \times \mathbb{R}^2 \mapsto \mathbb{R}$. If $S$ is the price process of an asset without a cash flow, we have $g(t, s, r, v) = r \cdot s$, since Equation (1) represents the dynamics under the martingale measure $\mathbb{Q}$. The general formulation of the function $g$ allows the asset to have a cash flow, which depends on the interest rate and the value process of the option.

We model the interest rate with a one-factor model with dynamics

$$\mathrm{d}r(t) = b\big(t, r(t)\big)\mathrm{d}t + \gamma\big(t, r(t)\big)\mathrm{d}W_2(t), \tag{2}$$

for deterministic functions $b : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ and $\gamma : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$. The extension to multi-factor models of the interest rate is straight forward. Let $B$ denote the money account with dynamics

$$\begin{aligned} \mathrm{d}B(t) &= r(t)B(t)\mathrm{d}t, \\ B(0) &= b_0 > 0. \end{aligned}$$

We let $P(\,\cdot\,, \bar{T})$ be the price process of a zero coupon bond with maturity $\bar{T} > T$, satisfying

$$P(t, \bar{T}) = \mathbb{E}^{\mathbb{Q}}\left[ e^{-\int_t^{\bar{T}} r(s)\mathrm{d}s} \,\middle|\, \mathcal{F}_t \right] = e^{-\int_t^{\bar{T}} f(t,s)\mathrm{d}s},$$

for $t \leq \bar{T}$. The existence of the zero coupon bond enables us to find the forward rates $f(t,s)$ for $0 \leq t \leq T$ and $t \leq s \leq T$, $f(t,t) = r(t)$.

The value of the option can be represented in the following way:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[ \int_t^T e^{-\int_t^\tau r(s)\mathrm{d}s} \phi\big(\tau, S(\tau), r(\tau), V(\tau)\big)\mathrm{d}\tau + e^{-\int_t^T r(s)\mathrm{d}s}\Phi\big(S(T)\big) \,\middle|\, \mathcal{F}_t \right].$$

The asset, the interest rate and the value process stipulate a forward–backward stochastic differential equation. We assume that the functions $g, \sigma, b, \gamma, \phi$ and $\Phi$ are sufficiently regular and refer to [12] for the existence and uniqueness of a solution. Since the functions $g, b, \sigma$ and $\gamma$ in the dynamics of the asset and the interest rate in Equations (1) and (2) are deterministic, $(S, r)$ is Markov. Hence, we write

$$\begin{aligned} V(t, S(t), r(t)) = \mathbb{E}^{\mathbb{Q}}\Bigg[ &\int_t^T e^{-\int_t^\tau r(s)\mathrm{d}s} \phi\big(\tau, S(\tau), r(\tau), V(\tau, S(\tau), r(\tau))\big)\mathrm{d}\tau \\ &+ e^{-\int_t^T r(s)\mathrm{d}s}\Phi\big(S(T)\big) \,\Big|\, S(t), r(t) \Bigg], \end{aligned} \tag{3}$$

with a slight misuse of notation since $V$ is now a function, $V : \mathbb{R}_+ \times \mathbb{R}^2 \mapsto \mathbb{R}$, and not the stochastic value process itself. Furthermore, we write

$$\mathrm{d}S(t) = g\big(t, S(t), r(t), V(t, S(t), r(t))\big)\mathrm{d}t + \sigma\big(t, S(t), r(t), V(t, S(t), r(t))\big)\mathrm{d}W_1(t). \tag{4}$$

In this model setup, there is a triangular interdependence between the underlying asset, $S$, the process of claims on the underlying, $\phi$ and $\Phi$, and the value process, $V$. The key motivating example is the corporate finance example from [1,13], elaborated in Example 1 below.

**Example 1.** *Consider a firm with a debt of K payable at time T with continuous coupon payments on the loan with continuous rate c paid until time T. The underlying assets of the firm are denoted by S. In corporate finance, a task is to calculate the equity value, V, of the firm. We assume that the firm pays out continuous dividends to its shareholders with rate δ. In the event that the firm cannot pay its debt at time T, the lending institution immediately takes over the firm. Then the equity value, V, the value of the debt, $V^d$, and the dynamics of the assets under the risk neutral measure $\mathbb{Q}$ are*

$$
\begin{aligned}
V(t,S(t),r(t)) = \mathbb{E}^{\mathbb{Q}}\Bigg[ & \int_t^T e^{-\int_t^\tau r(s)ds}\delta\big(\tau,S(\tau),r(\tau),V(\tau,S(\tau),r(\tau))\big)\mathrm{d}\tau \\
& + e^{-\int_t^T r(s)\mathrm{d}s}\max\big(S(T)-K,0\big)\ \Big|\ S(t),r(t)\Bigg], \\
V^d(t,S(t),r(t)) = \mathbb{E}^{\mathbb{Q}}\Bigg[ & \int_t^T e^{-\int_t^\tau r(s)ds}c\big(\tau,S(\tau),r(\tau),V(\tau,S(\tau),r(\tau))\big)\mathrm{d}\tau \\
& + e^{-\int_t^T r(s)\mathrm{d}s}\min\big(S(T),K\big)\ \Big|\ S(t),r(t)\Bigg], \\
\mathrm{d}S(t) = {} & r(t)S(t)\mathrm{d}t - \delta\big(t,S(t),r(t),V(t,S(t),r(t))\big)\mathrm{d}\,t \\
& - c\big(t,S(t),r(t),V(t,S(t),r(t))\big)\mathrm{d}t \\
& + \sigma\big(t,S(t),r(t),V(t,S(t),r(t))\big)\mathrm{d}W_1(t).
\end{aligned}
$$

*The equity value is the expected present value of future dividends plus the remaining part of the assets when the debt is paid at time T. The value of the debt is the expected present value of future coupon payments plus the debt payment at time T, which is the minimum of the assets and K. Dividends and coupon payments are withdrawn from the asset. It is sufficient to model either* $V(t,S(t),r(t))$ *or* $V^d(t,S(t),r(t)$ *since*

$$V(t,S(t),r(t)) + V^d(t,S(t),r(t)) = S(t).$$

*As an extension to [1], this setup allows the dividends, the coupon payments and the investment strategy of the asset, σ, to depend on the equity value (or similarly the value of the debt). To be consistent with [1], and for simplicity, we disregard taxation of dividends, including possible tax benefits from paying out dividends. Adding taxation would complicate the picture further as a third party beyond debt and equity holders is entitled to a tax cash flow, which may or may not be a function of current balance scheme entries. One of the benefits of this simplicity is that we can directly compare the methods presented in Sections 3 and 4 with a relatively simple numerically exact solution in Section 5.*

**Example 2.** *In with-profit life insurance, payments guaranteed in the insurance contract are based on prudent assumptions regarding future interest rate and mortality. This results in a surplus which, by legislation, is to be paid back to the policyholders as a bonus. The redistribution of the bonus contains certain degrees of freedom for the insurance company and depends on their dividend strategy. References [4,5] describe a projection model of the balance sheet of a with-profit life insurance company where a bonus is used to buy more insurance (spoken of as additional benefits). The model from [4,5] contains simplifying assumptions about the future dividend strategy, and an obvious extension of the model is to allow for a broader range of dividend strategies. A relevant extension of the model from [4,5] is to allow the future dividend strategy of the company to depend on the market reserve of future payments and future additional benefits and, in that case, a dependence structure similar to our model setup arises.*

*Let B denote the payment process of both the guaranteed payments and the additional benefits of a with-profit insurance contract. The payments are linked to states of the insured as, for instance, 'Active', 'Disable', and 'Dead', usually modelled by a Markov chain* $\big(Z(t)\big)_{t\geq 0}$ *on a finite state space* $\mathcal{J}$ *with corresponding counting processes* $N_{ij}(t)$, $i,j \in \mathcal{J}$, *counting jumps from state i to state j. Payments are divided in continuous payments during sojourn in state i,* $b^s(t,i)$, *and payments upon jump from state i to state j,* $b^j(t,i,j)$. *The market reserve of future payments from the insurance company to the insured is*

$$
\begin{aligned}
V(t) &= \mathbb{E}^{\mathbb{Q}}\left[ \int_t^T e^{-\int_t^\tau r(s)ds} \mathrm{d}B(\tau) \;\middle|\; \mathcal{F}(t) \right], \\
&= \mathbb{E}^{\mathbb{Q}}\left[ \int_t^T e^{-\int_t^\tau r(s)ds} \Big( b^s(\tau, Z(\tau))\mathrm{d}\tau \right. \\
&\quad \left. + \sum_{j:j\neq Z(\tau-)} b^j(t, Z(\tau-), j)\mathrm{d}N_{Z(\tau-)j}(\tau) \Big) \;\middle|\; \mathcal{F}(t) \right],
\end{aligned}
$$

*where $T$ is the termination of the insurance contract, $r$ is the stochastic interest rate, and $\mathcal{F}(t)$ is the formalized information available at time $t$. The market reserve above has a similar representation to that of the claim value from Equation (3), although in this example, we also model payments when there are jumps in the underlying Markov process $\big(Z(t)\big)_{t\geq 0}$.*

*In [4,5], the authors derive the dynamics of the so-called savings account and the surplus of a with-profit insurance contract (see Equations (8) and (9) in [4]), where the dividend strategy depends on the savings account and the surplus only. If we allow the dividend strategy to also depend on the market reserve, $V$, the dynamics of the savings account $X$ and the surplus $Y$ are in the form*

$$
\begin{aligned}
\mathrm{d}X(t) &= g_x\big(t, X(t), Y(t), r(t), V(t), Z(t)\big)\mathrm{d}t + \sigma_x\big(t, X(t), Y(t), r(t), V(t), Z(t)\big)\mathrm{d}W_1(t) \\
&\quad + \sum_{j:j\neq Z(t-)} h_x\big(t, X(t-), Y(t-), V(t-), Z(t-), j\big)\mathrm{d}N_{Z(t-)j}(t), \\
X(0) &= x_0 \in \mathbb{R}, \\
\mathrm{d}Y(t) &= g_y\big(t, X(t), Y(t), r(t), V(t), Z(t)\big)\mathrm{d}t + \sigma_y\big(t, X(t), Y(t), r(t), V(t), Z(t)\big)\mathrm{d}W_1(t) \\
&\quad + \sum_{j:j\neq Z(t-)} h_y\big(t, X(t-), Y(t-), V(t-), Z(t-), k\big)\mathrm{d}N_{Z(t-)j}(t), \\
Y(0) &= y_0 \in \mathbb{R}.
\end{aligned}
$$

*We choose not to write out the expressions for the functions $g_x, g_y, \sigma_x, \sigma_y, h_x$, and $h_y$. The dynamics of the savings account and the surplus above are in the form of the dynamics of the asset from Equation (1), except that the dynamics also include a $\mathrm{d}N_{ij}(t)$-term that models changes in the savings account and the surplus due to jumps of the state process $\big(Z(t)\big)_{t\geq 0}$. Since $V$ appears as an argument in the coefficients of $X$ and $Y$ and, further, the payment coefficients in the market reserve are themselves dependent on $X$ and $Y$, we find that the ultimate computational challenge of a forward–backward system is exactly the same in this example as in the previous one. That challenge is, however, here better hidden behind generalized notation, jump risk, and a conceptual world specific to the domain of with-profit insurance.*

*The with-profit insurance example is here proven to be both conceptually and notationally cumbersome, and we choose not to continue with this application in focus. The structure and the idea of our approach are much easier to comprehend in a version of the problem where the jump processes and the domain specific notions of profit-sharing are peeled away. What is important is that the forward–backward dependence structure that we study in this paper arises as a natural and practically relevant extension of the model from [4,5], when allowing for dividends to depend on the market reserve.*

The feed-back of the value process into the dynamics of the underlying is challenging from a simulation point of view. We disentangle the simulation of the underlying and the estimation of the value process in three parts. First, in Section 3, we set up a deterministic world where asset prices, processes of claims, and the value process are represented by their intrinsic values. In this deterministic as-if market, the forward–backward element of the problem is solved by an iteration procedure. Second, based on the assumption that there is a parametric relation between the value process and its intrinsic value, we perform a standard forward Monte Carlo simulation in Section 4 for a given parameter. Third,

in Section 4.2, the forward Monte Carlo simulation is performed a number of times for the iterated determination of the parameter value.

There is a separate issue of existence and uniqueness to the system formalized by the feed-back construction of the problem. The question is about for which specifications of $\phi$, $\Phi$, $g$ and $\sigma$ there exist a unique solution to our value problem. While the answer to that question is of obvious relevance, our contribution is completely different. This paper contains only a financial engineering approach to the construction of an approximation technique based on the intrinsic value deterministic basis.

## 3. Intrinsic Value

### *3.1. Definition of Intrinsic Value*

Our definition of intrinsic values is based on the idea of setting up a hypothetical market at a fixed time point $t$ with an underlying process, a process of claims on the underlying, and prices as-if the market were deterministic. The hypothetical market is fixed at time $t$, consistent with all market prices and therefore measurable with respect to $\mathcal{F}_t$. Thereby, the interest rate $r$ can be replaced by the forward interest rate $f(t,\cdot)$ everywhere in the hypothetical market, and the dynamics of the asset $S$ are replaced by the dynamics of the intrinsic value of the asset $S^{\mathrm{IV}}$. The intrinsic value of the value process is defined as the value of the claim in this as-if deterministic market.

**Definition 1.** *We define the intrinsic value of the asset and the value process, $S^{IV}$ and $V^{IV}$, at time $u$ calculated at a fixed time $t \leq u$ by interchanging the interest rate with the forward interest rate calculated at time $t$ and by eliminating uncertainty from the financial market from time $t$ onwards, hence:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV}}(t,u) &= g\big(u, S^{\mathrm{IV}}(t,u), f(t,u), V^{\mathrm{IV}}(u,S(t),r(t))\big),\\ S^{\mathrm{IV}}(t,t) &= S(t),\end{aligned}$$

$$\begin{aligned}V^{\mathrm{IV}}(u,S(t),r(t)) = &\int_u^T e^{-\int_u^\tau f(t,s)\mathrm{d}s}\phi\big(\tau, S^{\mathrm{IV}}(t,\tau), f(t,\tau), V^{\mathrm{IV}}(\tau,S(t),r(t))\big)\mathrm{d}\tau\\ &+ e^{-\int_u^T f(t,s)\mathrm{d}s}\Phi\big(S^{\mathrm{IV}}(t,T)\big),\end{aligned}$$

*with the functions $g$, $\phi$, and $\Phi$ as defined in the previous section.*

The intrinsic value of the asset $u \mapsto S^{\mathrm{IV}}(t,u)$ and the value process $u \mapsto V^{\mathrm{IV}}(u,S(t),r(t))$ are measurable with respect to $\mathcal{F}_t$.

It is appropriate to relate the intrinsic value definition above with the conventional intuition of the intrinsic value being the option value if the option is exercised now. There, one must carefully distinguish European options from American options. For a European option, the decision to exercise now means, for example in the case of a call option, that you must decide now whether you want to buy, at maturity, the stock at strike price $K$. Consider the classic Black–Scholes model; since you can buy the stock today at price $S(t)$ and the strike price $K$ at price $e^{-r(T-t)}K$, the intrinsic value of the option becomes $(S(t) - e^{-r(T-t)}K)^+$. This intrinsic value of a European call option conforms with our intrinsic value definition. In the European call case it is not an option of when to buy, only whether to buy. The intrinsic value of a European put option becomes $(e^{-r(T-t)}K - S(t))^+$.

For the American option, things are more complicated. The conventional intrinsic value of an American call (put) is $(S(t)-K)^+$ $((K-S(t))^+)$. That is the value if the option is exercised today, and exercise here means both choosing to buy and actually buying. We did not even speak of exercise timing options in our definition, but the natural generalization is to maximize the intrinsic value over exercise times in an as-if market. Then, in the Black–Scholes model with $r > 0$, our intrinsic value of the American call option becomes $(S(t) - e^{-r(T-t)}K)^+$, since it maximizes the intrinsic value to exercise at time $T$, even in the

as-if market. Note that our intrinsic values of the European and American calls coincide but that our intrinsic value of the American call does not conform with the conventional definition. In the case of an American put, the intrinsic value is maximized by exercising now in the as-if market such that our intrinsic value becomes $(K - S(t))^+$. Note that our intrinsic values for the European and American puts do not coincide but that our intrinsic value of the American put does conform with the conventional definition. The difference between the call and the put is of course related to the fact that the American call should not be exercised prematurely, not even in the as-if market, whereas the American put should, even in the as-if market.

A different way to think of our intrinsic value is that it is the value of the option in the original stochastic market in the case where the decision about whether to buy or not to buy (in the case of a call) is made on the basis of the information one has today. Our intrinsic value is the value of the option as-if one does not learn more about the realization of $S$ before deciding to exercise the option to buy (in the case of a call).

We now consider a decomposition of the value process in its intrinsic value and time value

$$V(t, S(t), r(t)) = V^{\mathrm{IV}}(t, S(t), r(t)) + V^{\mathrm{TV}}(t, S(t), r(t)). \tag{5}$$

Such a decomposition is standard. Following our definition of the intrinsic value, the time value represents the value added from being allowed to base optional decisions in the future on future values and not just on the current values. Thus, time value is the value of information added over time. For the conventional intrinsic value (see above), the time value can be thought of as the value of not having to exercise necessarily today but being able to time the exercise better. For plain vanilla options, this has a clear meaning. For more general options, this idea may be more difficult to generalize, whereas our definition can be directly generalized.

Obviously, one can write

$$V^{\mathrm{TV}}(t, S(t), r(t)) = \alpha(t, S(t), r(t)) V^{\mathrm{IV}}(t, S(t), r(t)),$$

for some function $\alpha$. We are going to work with an approximation of the time value where we disregard some of the arguments in the function $\alpha$, for example, not allowing for a stochastic $\alpha$ means approximating, for a deterministic and possibly parametric function $\alpha$, by

$$V^{\mathrm{TV}}(t, S(t), r(t)) \approx \alpha(t) V^{\mathrm{IV}}(t, S(t), r(t)).$$

This approximation reflects the idea that options lose their time value over time. This is obviously true when the time value reflects the value added by information added in the future. However, it is obviously an approximation to assume that the function is independent of the $(S(t), r(t))$. Even simpler is the approximation

$$V^{\mathrm{TV}}(t, S(t), r(t)) \approx \alpha V^{\mathrm{IV}}(t, S(t), r(t)),$$

for a constant $\alpha$. This implies

$$V(t, S(t), r(t)) \approx (1 + \alpha) V^{\mathrm{IV}}(t, S(t), r(t)). \tag{6}$$

Note that

$$V(T, S(T), r(T)) = \Phi\big(S(T)\big) = V^{\mathrm{IV}}(T, S(T), r(T)).$$

Therefore, if approximated by a function $\alpha$, one would prefer a function fulfilling $\alpha(T) = 0$. The approximation by a constant $\alpha \neq 0$ is also necessarily inaccurate at time $T$.

In the following, we approximate the value via a parametric relation between the claim value, $V$, and the intrinsic value of the claim value, $V^{\mathrm{IV}}$. The assumption is that the value process is linear in its intrinsic value. This does not mean that we truly believe that the value is linear in the intrinsic value. This is instead a first approximation that can be intuitively thought of as being based on a first order expansion of the value around the

intrinsic value. Thus, we know that we already lose accuracy at this state. Other more involved parametric forms could be proposed and the steps of our method, as explained in the next section, could be properly adapted to any parametric form. We stress that our way of working with the intrinsic value, as a key to break down the original and utterly complicated problem into a series of solvable sub-problems, does not as such depend on the special case of linearity. That is merely chosen as a simple case of demonstration, and its merits and drawbacks are made visible in Section 5.

**Remark 1.** *Assume that the interest rate is deterministic and that the drift term in the dynamics of the asset and the claim processes are linear in the underlying and the value process in the sense that the functions $g$, $\phi$ and $\Phi$ are in the form*

$$\begin{aligned} g(t,s,v) &= g_0(t) + g_1(t)\cdot s + g_2(t)\cdot v, \\ \phi(t,s,v) &= \phi_0(t) + \phi_1(t)\cdot s + \phi_2(t)\cdot v, \\ \Phi(s) &= \Phi_0 + \Phi_1 \cdot s, \end{aligned}$$

*for deterministic functions $g_i$ and $\phi_i$, $i = 0,1,2$ and $\Phi_0, \Phi_1 \in \mathbb{R}$. Then the value process is given by*

$$V(t,S(t)) = h_0(t) + h_1(t)\cdot S(t),$$

*and the intrinsic value of the value process is given by*

$$V^{\mathrm{IV}}(u,S(t)) = h_0(u) + h_1(u)\cdot S^{\mathrm{IV}}(t,u),$$

*for functions $h_0$ and $h_1$ that solve a system of ordinary differential equations. Hence, in the case with a deterministic interest rate and full linearity,*

$$V(t,S(t)) = V^{\mathrm{IV}}(t,S(t)),$$

*and the time value of the value process is equal to zero. See Chapter 3 in [14] for the derivation of the system of ordinary differential equations for $h_0$ and $h_1$.*

Appendix A investigates the quality of the intrinsic value approximation for $\phi = 0$.

### *3.2. Calculation of the Intrinsic Value*

In this section, we solve the forward–backward element by an iteration in a deterministic as-if market. We study how to calculate the intrinsic value of the underlying and the intrinsic value of the value process. From Definition 1, we see that in order to calculate $V^{\mathrm{IV}}(t,S(t),r(t))$, we must solve the following system of differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV}}(t,u) &= g\big(u, S^{\mathrm{IV}}(t,u), f(t,u), V^{\mathrm{IV}}(u,S(t),r(t))\big), \\ S^{\mathrm{IV}}(t,t) &= S(t), \\ \frac{\mathrm{d}}{\mathrm{d}u} V^{\mathrm{IV}}(u,S(t),r(t)) &= f(t,u)V^{\mathrm{IV}}(u,S(t),r(t)) \\ &\quad - \phi\big(u, S^{\mathrm{IV}}(t,u), f(t,u), V^{\mathrm{IV}}(u,S(t),r(t))\big), \\ V^{\mathrm{IV}}(T,S(t),r(t)) &= \Phi\big(S^{\mathrm{IV}}(t,T)\big). \end{aligned} \tag{7}$$

The intrinsic value of the underlying and the intrinsic value of the value process satisfy a deterministic forward–backward system of ordinary differential equations given by (7). We propose two iteration methods to solve the forward–backward system of ordinary differential equations. The starting point of both methods is to suppress the entanglement of $S$ and $V$ that prevents us from solving the system of differential equations. The iteration procedures are performed in the hypothetical market set up at fixed time $t \leq T$ and are

measurable with respect to $\mathcal{F}_t$, thus the price process of the asset $S$, the interest rate $r$, and the price of the process of the zero coupon bond $P(\cdot, \bar{T})$ are known up to and including time $t$.

The first method is a perturbation argument, where the forward–backward nature of the equations is preserved but the equations are decoupled. The second method is a shooting method where the boundary conditions are modified but we preserve a system of coupled equations. In both methods, the modification takes place in the first iteration to trigger the iteration procedure. We describe the first and the $k$th iteration for $k \geq 2$ in both methods. The objective of both methods is to solve the system of differential equations in Equation (7) in order to calculate the function $u \mapsto V^{\mathrm{IV}}(u, S(t), r(t))$.

3.2.1. Method I: Perturbation Method

The modification in the perturbation argument is a substitution in the differential equation of $S^{\mathrm{IV}}(t, u)$ in Equation (7), where we substitute the unknown $V^{\mathrm{IV}}(u, S(t), r(t))$ with a known function, which is measurable with respect to $\mathcal{F}_t$. We denote the function $u \mapsto v(u, S(t), r(t))$.

Iteration 1

In the first iteration, the intrinsic value of the stock index satisfies the differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV},(1)}(t,u) &= g\big(u, S^{\mathrm{IV},(1)}(t,u), f(t,u), v(u,S(t),r(t))\big), \\ S^{\mathrm{IV},(1)}(t,t) &= S(t), \end{aligned}$$

and the intrinsic value of the value process satisfies the differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} V^{\mathrm{IV},(1)}(u,S(t),r(t)) &= f(t,u) V^{\mathrm{IV},(1)}(u,S(t),r(t)) \\ &\quad - \phi\big(u, S^{\mathrm{IV},(1)}(t,u), f(t,u), V^{\mathrm{IV},(1)}(u,S(t),r(t))\big), \\ V^{\mathrm{IV},(1)}(T,S(t),r(t)) &= \Phi\big(S^{\mathrm{IV},(1)}(t,T)\big). \end{aligned}$$

This is a solvable system of differential equations. In the numerical study in Section 5, we choose $v(u, S(t), r(t)) = 0$.

Iteration $k$

We use the fact that we know the intrinsic value of the value process from the previous iteration and insert this into the differential equation of $S^{\mathrm{IV}}$. The intrinsic value of the stock index satisfies the differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV},(k)}(t,u) &= g\big(u, S^{\mathrm{IV},(k)}(t,u), f(t,u), V^{\mathrm{IV},(k-1)}(u,S(t),r(t))\big), \\ S^{\mathrm{IV},(k)}(t,t) &= S(t). \end{aligned}$$

The intrinsic value of the value process satisfies the differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} V^{\mathrm{IV},(k)}(u,S(t),r(t)) &= f(t,u) V^{\mathrm{IV},(k)}(u,S(t),r(t)) \\ &\quad - \phi\big(u, S^{\mathrm{IV},(k)}(t,u), f(t,u), V^{\mathrm{IV},(k)}(u,S(t),r(t))\big), \\ V^{\mathrm{IV},(k)}(T,S(t),r(t)) &= \Phi\big(S^{\mathrm{IV},(k)}(t,T)\big). \end{aligned}$$

This is a solvable system of differential equations.

Stopping Criteria

We suggest the stopping criteria

$$\min_{k\geq 2}\left\{\left|V^{\mathrm{IV},(k)}(t,S(t),r(t))-V^{\mathrm{IV},(k-1)}(t,S(t),r(t))\right|<\varepsilon\right\},$$

for $\varepsilon>0$. Another criteria is to fix the number of iterations. Let $\kappa$ be the resulting number of iterations. With the perturbation method, we estimate the solution to the system of the differential equations given by Equation (7), and the resulting estimate of the intrinsic value of the value process is

$$u\mapsto V^{\mathrm{IV},(\kappa)}(u,S(t),r(t)).$$

3.2.2. Method II: Shooting Method

The modification in the shooting method is the assumption that we know the boundary condition at time $t$ in the differential equation of the intrinsic value of the value process in Equation (7). We assume that $V^{\mathrm{IV}}(t,S(t),r(t))=\tilde{V}(t,S(t),r(t))$ for a known function $\tilde{V}$, which is measurable with respect to $\mathcal{F}_t$. In the numerical study in Section 5, we choose $\tilde{V}(t,S(t),r(t))=0$.

Iteration 1

We assume that

$$V^{\mathrm{IV},(1)}(t,S(t),r(t))=\tilde{V}(t,S(t),r(t)). \tag{8}$$

We solve the following system of forward differential equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV},(1)}(t,u)&=g\big(u,S^{\mathrm{IV},(1)}(t,u),f(t,u),V^{\mathrm{IV},(1)}(u,S(t),r(t))\big),\\ S^{\mathrm{IV},(1)}(t,t)&=S(t),\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}V^{\mathrm{IV},(1)}(u,S(t),r(t))&=f(t,u)V^{\mathrm{IV},(1)}(u,S(t),r(t))\\ &\quad-\phi\big(u,S^{\mathrm{IV},(1)}(t,u),f(t,u),V^{\mathrm{IV},(1)}(u,S(t),r(t))\big),\\ V^{\mathrm{IV},(1)}(t,S(t),r(t))&=\tilde{V}(t,S(t),r(t)).\end{aligned}$$

This is a solvable system of differential equations.

If we solve the differential equations from Equation (7) with the boundary condition in Equation (8), we obtain

$$\begin{aligned}V^{\mathrm{IV},(1)}(T,S(t),r(t))&=\tilde{V}(t,S(t),r(t))e^{\int_t^T f(t,s)\mathrm{d}s}\\ &\quad-\int_t^T e^{\int_\tau^T f(t,s)\mathrm{d}s}\phi\big(\tau,S^{\mathrm{IV},(1)}(t,\tau),f(t,\tau),V^{\mathrm{IV},(1)}(\tau,S(t),r(t))\big)\mathrm{d}\tau.\end{aligned}$$

The boundary condition in Equation (7) states that

$$V^{\mathrm{IV}}(T,S(t),r(t))=\Phi\big(S^{\mathrm{IV}}(t,T)\big).$$

The difference $V^{\mathrm{IV},(1)}(T,S(t),r(t))-\Phi\big(S^{\mathrm{IV},(1)}(t,T)\big)$ is an estimate of how wrong our assumption is that $V^{\mathrm{IV},(1)}(t,S(t),r(t))=\tilde{V}(t,S(t),r(t))$. We use the estimate to adjust the boundary condition of the intrinsic value of the value process at time $t$ in the next iteration, such that we, in the second iteration, assume that

$$
\begin{aligned}
&V^{\mathrm{IV},(2)}(t,S(t),r(t)) \\
&= \tilde{V}(t,S(t),r(t)) - e^{-\int_t^T f(t,s)\mathrm{d}s}\Big(V^{\mathrm{IV},(1)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(1)}(t,T)\big)\Big) \\
&= \int_t^T e^{-\int_t^\tau f(t,s)\mathrm{d}s}\phi\big(\tau, S^{\mathrm{IV},(1)}(t,\tau), f(t,\tau), V^{\mathrm{IV},(1)}(\tau,S(t),r(t))\big)\mathrm{d}\tau \\
&\quad + e^{-\int_t^T f(t,s)\mathrm{d}s}\Phi\big(S^{\mathrm{IV},(1)}(t,T)\big),
\end{aligned}
$$

which is the solution to the differential equation of $V^{\mathrm{IV}}$ from Equation (7) with the boundary condition $V^{\mathrm{IV}}(T,S(t),r(t)) = \Phi\big(S^{\mathrm{IV},(1)}(t,T)\big)$.

Iteration $k$

In the $k$th iteration, we assume that

$$
\begin{aligned}
&V^{\mathrm{IV},(k)}(t,S(t),r(t)) \\
&= \int_t^T e^{-\int_t^\tau f(t,s)\mathrm{d}s}\phi\big(\tau, S^{\mathrm{IV},(k-1)}(t,\tau), f(t,\tau), V^{\mathrm{IV},(k-1)}(\tau,S(t),r(t))\big)\mathrm{d}\tau \\
&\quad + e^{-\int_t^T f(t,u)\mathrm{d}u}\Phi\big(S^{\mathrm{IV},(k-1)}(t,T)\big).
\end{aligned}
$$

We solve the forward system of differential equations with the boundary condition above

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV},(k)}(t,u) &= g\big(u, S^{\mathrm{IV},(k)}(t,u), f(t,u), V^{\mathrm{IV},(k)}(u,S(t),r(t))\big), \\
S^{\mathrm{IV},(k)}(t,t) &= S(t),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}V^{\mathrm{IV},(k)}(u,S(t),r(t)) &= f(t,u)V^{\mathrm{IV},(k)}(u,S(t),r(t)) \\
&\quad - \phi\big(u, S^{\mathrm{IV},(k)}(t,u), f(t,u), V^{\mathrm{IV},(k)}(u,S(t),r(t))\big).
\end{aligned}
$$

Hopefully, we have that

$$
\Big|V^{\mathrm{IV},(k)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(k)}(t,T)\big)\Big| < \Big|V^{\mathrm{IV},(k-1)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(k-1)}(t,T)\big)\Big|,
$$

such that we in the $k$th iteration are closer to the true value of the intrinsic value of the value process at time $T$ than in the previous iteration.

Stopping Criteria

We suggest the stopping criteria

$$
\begin{aligned}
\min_k \Bigg\{ \Bigg| &\Big|V^{\mathrm{IV},(k)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(k)}(t,T)\big)\Big| \\
&- \Big|V^{\mathrm{IV},(k-1)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(k-1)}(t,T)\big)\Big| \Bigg| < \varepsilon \Bigg\},
\end{aligned}
$$

for $\varepsilon > 0$. Another criteria is to fix the number of iterations. Let $\kappa$ be the resulting number of iterations. The resulting estimate of the intrinsic value of the value process is

$$
u \mapsto V^{\mathrm{IV},(\kappa)}(u,S(t),r(t)).
$$

## 4. Intrinsic Value Monte Carlo

We perform a standard forward Monte Carlo simulation based on the parametric relation between the value process and the intrinsic value of the value from Equation (6) for

a given parameter $\alpha \in \mathbb{R}$. The objective is to estimate the value process $V$ from Equation (3), when the price process of the underlying asset has the dynamics in Equation (4). Solutions to the system of ordinary differential equations in the deterministic as-if market from Section 3.2 are used as input in the Monte Carlo simulation.

Valuation of the value process in the setup from Section 2 is beyond a standard Monte Carlo simulation since the asset itself depends on the value process. The intrinsic value approximation of the value process enables us to simulate the assets despite the dependence of the value process.

Our objective is to use a Monte Carlo simulation to estimate the value process at a fixed time $t_0$

$$\begin{aligned}\hat{V}(t_0, S(t_0), r(t_0)) = \frac{1}{N}\sum_{i=1}^{N}\Bigg(\int_{t_0}^{T} e^{-\int_{t_0}^{\tau} r_i(u)\mathrm{d}u}\phi\big(\tau, S_i(\tau), r_i(\tau), V(\tau, S_i(\tau), r_i(\tau))\big)\mathrm{d}\tau \\ + e^{-\int_{t_0}^{T} r_i(u)\mathrm{d}u}\Phi\big(S_i(T)\big)\Bigg),\end{aligned} \tag{9}$$

for independent realizations $\big(S_1(T), (r_1(u))_{u\in[t_0,T]}\big), ..., \big(S_N(T), (r_N(u))_{u\in[t_0,T]}\big)$ of the asset and the interest rate. How to simulate the asset is not obvious when its dynamics depend on the unknown value process, nor is how to calculate the continuous payments $\phi$ since they also depend on the unknown value process.

### *4.1. Simulation of the Asset*

We divide the interval $[t_0, T]$ in $M$ equidistant subintervals

$$\begin{aligned}&t_0 < t_1 < ... < t_M = T,\\ &t_j - t_{j-1} = \Delta.\end{aligned}$$

We simulate $N$ paths of the interest rate according to a Euler scheme based on its dynamics from Equation (2)

$$r_i(t_{j+1}) = r_i(t_j) + b\big(t_j, r_i(t_j)\big)\Delta + \gamma\big(t_j, r_i(t_j)\big)\sqrt{\Delta}Y_{i,j},$$

for i.i.d. $Y_{i,j} \sim \mathcal{N}(0,1)$ for $i = 1, ..., N$ and $j = 0, ..., M-1$. The forward interest calculated at time $t_j$ for path $i$ is based on $r_i(t_j)$ is $u \mapsto f_i(t_j, u)$ for $u \geq t_j$. We assume that the forward interest rates are known based on the simulated interest rates.

We simulate the underlying asset according to a Euler scheme based on its dynamics from Equation (4), and simulate $N$ paths in the grid $(t_0, t_1, ..., t_M)$. The Euler scheme is

$$\begin{aligned}S_i(t_{j+1}) = S_i(t_j) + g\big(t_j, S_i(t_j), r_i(t_j), V(t_j, S_i(t_j), r_i(t_j))\big)\Delta \\ + \sigma\big(t_j, S_i(t_j), r_i(t_j), V(t_j, S_i(t_j), r_i(t_j))\big)\sqrt{\Delta}Z_{i,j},\end{aligned} \tag{10}$$

for i.i.d. $Z_{i,j} \sim \mathcal{N}(0,1)$ for $i = 1, ..., N$ and $j = 0, ..., M-1$. We cannot simulate from this Euler scheme since it depends on the unknown value process. Our solution is to use the intrinsic value approximation of the value process from Equation (6).

The method is an iteration procedure, where we iterate over the value of $\alpha$ in the intrinsic value approximation in Equation (6). We also suggested two iteration methods for calculating the intrinsic value of the value process in Section 3.2. We denote the iteration over $\alpha$ as the outer iteration and the iteration to calculate $V^{\mathrm{IV}}$ as the inner iteration. The outer iteration procedure is described in Section 4.2. For now, we assume that the value of $\alpha$ is fixed, and we describe the simulation of the asset for a general value of $\alpha$.

Inserting the intrinsic value approximation of the value process from Equation (10) in the Euler scheme yields

$$\begin{aligned} S_i(t_{j+1}) = S_i(t_j) &+ g\big(t_j, S_i(t_j), r_i(t_j), (1+\alpha)V^{\mathrm{IV}}(t_j, S_i(t_j), r_i(t_j))\big)\Delta \\ &+ \sigma\big(t_j, S_i(t_j), r_i(t_j), (1+\alpha)V^{\mathrm{IV}}(t_j, S_i(t_j), r_i(t_j))\big)\sqrt{\Delta}Z_{i,j}. \end{aligned}$$

We see in Section 3.2 above that we need an iteration procedure (denoted as the inner iteration) to calculate the intrinsic value of the value process, $V^{\mathrm{IV}}(t, S(t), r(t))$, for fixed $t$ and $S(t)$ known. This implies that, with the Euler scheme above, we need to perform the inner iteration procedure $N \times (M-1)+1$ times in order to simulate $N$ independent realizations of $S(T)$. Therefore, we implement the possibility of calculating the intrinsic value of the value process in a looser partition of the interval $[t_0, T]$ than in the partition in the Euler scheme above. Let $L \leq M-1$ and choose

$$\begin{aligned} &t_0 = \tau_0 < \tau_1 < ... < \tau_L < T, \\ &(\tau_0, \tau_1, ..., \tau_L) \subseteq (t_0, t_1, ..., t_{M-1}). \end{aligned}$$

We define the mapping

$$h(t) = \max_{l=0,...,L} \Big\{ \tau_l \mid \tau_l \leq t \Big\}, \quad \text{for } t \in [t_0, T].$$

Then, $h(t)$ returns the largest $\tau_l$ less than $t$. This results in the Euler scheme

$$\begin{aligned} S_i(t_{j+1}) = S_i(t_j) &+ g\big(t_j, S_i(t_j), r_i(t_j), (1+\alpha)V^{\mathrm{IV}}(t_j, S_i(h(t_j)), r_i(h(t_j)))\big)\Delta \\ &+ \sigma\big(t_j, S_i(t_j), r_i(t_j), (1+\alpha)V^{\mathrm{IV}}(t_j, S_i(h(t_j)), r_i(h(t_j)))\big)\sqrt{\Delta}Z_{i,j}. \end{aligned} \tag{11}$$

If $t_j = \tau_l$ for some $l = 0, ..., L$, we insert the intrinsic value of the value process at time $t_j$ calculated at time $t_j$, $V^{\mathrm{IV}}(t_j, S_i(t_j), r_i(t_j))$, which depends on $S_i(t_j)$ and $r_i(t_j)$. If instead $t_j \neq \tau_l$ for any $l = 0, ..., L$, we insert the intrinsic value of the value process at time $t_j$ calculated at the largest $\tau_l$ less than $t_j$, $V^{\mathrm{IV}}(t_j, S_i(h(t_j)), r_i(h(t_j)))$, which does not depend on $S_i(t_j)$ and $r_i(t_j)$, since we have not updated the calculation of $V^{\mathrm{IV}}$ with new information about the asset and the interest rate. We need $N \times L + 1$ calculations of $V^{\mathrm{IV}}$ with the inner iteration procedure to simulate $N$ independent realizations of the asset if we simulate from the Euler scheme in Equation (11).

In the case $L = 0$, we only estimate the intrinsic value of the value process once, using one of the iteration procedures (the perturbation argument or the shooting method) based on $S(t_0)$ and insert $V^{\mathrm{IV}}(t_j, S(t_0), r(t_0))$ into the Euler scheme. This is the choice of $L$ with the lowest computation time, since we only need one inner iteration procedure.

By solving the Euler scheme, we obtain $N$ independent paths of the asset and the intrinsic value of the value process. We calculate a Monte Carlo estimate of the value process at time $t_0$ as

$$\begin{aligned} &\hat{V}(t_0, S(t_0), r(t_0)) \\ &= \frac{1}{N}\sum_{i=1}^{N}\Bigg( \int_{t_0}^{T} e^{-\int_{t_0}^{\tau} r_i(u)\mathrm{d}u} \phi\big(\tau, S_i(\tau), r_i(\tau), (1+\alpha)V^{\mathrm{IV}}(\tau, S_i(\tau), r_i(\tau))\big)\mathrm{d}\tau \\ &\qquad + e^{-\int_{t_0}^{T} r_i(u)\mathrm{d}u}\Phi\big(S_i(T)\big)\Bigg), \end{aligned}$$

with appropriate approximations of the integrals.

### *4.2. The Choice of α*

In this section, we describe the outer iteration procedure used in the determination of the parameter value in the intrinsic value approximation of the value process. Simulation

of the asset from the Euler scheme in Equation (11) depends on the choice of $\alpha$ in the intrinsic value approximation of the value process from Equation (6).

The intention is to choose $\alpha$ such that the approximation is exact at time $t_0$

$$V(t_0, S(t_0), r(t_0)) = (1+\alpha)V^{\mathrm{IV}}(t_0, S(t_0), r(t_0)).$$

With the intrinsic value Monte Carlo method described above, we need the value of $\alpha$ to estimate $V(t_0, S(t_0), r(t_0))$, and therefore, we iterate over the value of $\alpha$. This is denoted by the outer iteration.

In the first outer iteration, we choose $\alpha = \alpha^{(\mathrm{I})}$. Based on this value of $\alpha$, we simulate the asset according to the Euler scheme in Equation (11), which includes $N \times L + 1$ inner iterations of either the perturbation argument or the shooting method, and calculate a Monte Carlo estimate of the value process at time $t_0$:

$$\begin{aligned}&\hat{V}^{(\mathrm{I})}(t_0, S(t_0), r(t_0))\\ &= \frac{1}{N}\sum_{i=1}^{N}\Bigg(\int_{t_0}^{T} e^{-\int_{t_0}^{\tau} r_i(u)\mathrm{d}u}\phi\big(\tau, S_i(\tau), r_i(\tau), (1+\alpha^{(\mathrm{I})})V^{\mathrm{IV}}(\tau, S_i(\tau), r_i(\tau))\big)\mathrm{d}\tau\\ &\qquad + e^{-\int_{t_0}^{T} r_i(u)\mathrm{d}u}\Phi\big(S_i(T)\big)\Bigg),\end{aligned}$$

with appropriate approximations of the integrals. For the second outer iteration, we choose $\alpha$ such that

$$\hat{V}^{(\mathrm{I})}(t_0, S(t_0), r(t_0)) = (1+\alpha^{(\mathrm{II})})V^{\mathrm{IV}}(t_0, S(t_0), r(t_0)),$$

and estimate the value process at time $t_0$ based on this value of $\alpha$. We continue the outer iteration procedure until $\alpha$ (or equivalently the estimate of the value process) reaches a fixed point.

### *4.3. Relations to Least-Squares Monte Carlo and Nested Simulation*

Simulation of the asset from the Euler scheme from Equation (10) requires that we know or can estimate the value process at time $t_j > t_0$, $V(t_j, S_i(t_j), r_i(t_j))$. The value process from Equation (3) is a conditional expectation. The estimation of future conditional expectations also appears in the valuation of American options to establish when to exercise the option. The authors of [6] proposed least-squares Monte Carlo for estimating future conditional expectations when valuing American options. Another solution is nested simulation.

The main difference between our intrinsic value Monte Carlo method and [6] is that, in [6], they are able to simulate the underlying whereas in our setup, the underlying depends on the conditional expectation since we feed back the value process into the dynamics of the underlying, which complicates the simulation. A perturbation argument, as described in Section 3.2.1, on the dynamics of the asset from Equation (4), combined with the least-squares Monte Carlo method, is an alternative to the intrinsic value Monte Carlo method in order to estimate the value process in Equation (3), but the clarification of this is beyond the scope of this paper. If the feedback of the value process into the underlying is eliminated and the dependence of the value process $V$ only appears in the process of claims, $\phi$, classical least-squares Monte Carlo can be used to estimate the value process.

## 5. Numerical Study

In this section, we illustrate our approximation methods with an example. Specifically, we consider the corporate finance example (Example 1) with the underlying assets of a firm, $S(t)$, governed by a Black–Scholes like market with dividends that depend on the value of the equity, $V(t, S(t))$, which again depends on the assets in the future. For simplicity, we choose the dividends to be linear in the equity value. In this situation, Equation (1) reduces to

$$\mathrm{d}S(t) = \big(rS(t) - \delta V(t, S(t))\big)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \tag{12}$$

with the constant interest rate $r$, the dividend yield $\delta V(t, S(t))$ for $\delta \in \mathbb{R}$, which depend on the value of the equity, and $\sigma > 0$ is the volatility. When the interest rate is constant, $f(t, u) = r$ for all $t \geq 0$ and $u \geq t$.

For the example model, the Euler scheme, Equation (10), reduces to

$$S_i(t_{j+1}) = S_i(t_j) + \big(rS_i(t_j) - \delta V(t_j, S_i(t_j))\big)\Delta + \sigma S_i(t_j)\sqrt{\Delta} Z_{i,j}, \tag{13}$$

and the Monte Carlo simulation of the option value, Equation (9), reduces to

$$\hat{V}(t_0, S(t_0)) = \frac{1}{N}\sum_{i=1}^{N}\left(\int_{t_0}^{T} e^{-(\tau - t_0)r}\delta V(\tau, S_i(\tau))\mathrm{d}\tau + e^{-(T-t_0)r}\Phi\big(S_i(T)\big)\right), \tag{14}$$

with $\Phi(s) = \big(s - K\big)^+$, where $K$ is the debt of the firm payable at time $T$. Equations (13) and (14) illustrate the problem that we aim to solve. The two equations depend on each other, and the asset is governed by a forward equation while the value is governed by a backward equation. This entanglement is solved by the approximation methods described in this paper.

### *5.1. Approximation Method Details*

The approximation methods consist of a combination of two iteration schemes, which we call the inner and the outer iterations. We describe the schemes separately below and note that we have sketched the numerical schemes in Appendix B.

#### 5.1.1. The Outer Iteration

The outer iteration is based on the intrinsic value Monte Carlo method described in Section 4. The method is to approximate the value $V(t_j, S_i(t_j))$ in the Euler scheme in Equation (13) by assuming that it is proportional to the intrinsic value defined in Section 3. The Euler scheme is then given by

$$S_i(t_{j+1}) = S_i(t_j) + \big(rS_i(t_j) - \delta(1+\alpha)V^{\mathrm{IV}}(t_j, S_i(t_j))\big)\Delta + \sigma S_i(t_j)\sqrt{\Delta} Z_{i,j}. \tag{15}$$

This way, the asset dynamics are disentangled from the value process. The proportionality factor $1 + \alpha$ is then determined in an iterative way such that the value of the factor in the next iteration would make the intrinsic value approximation correct.

The calculation of the intrinsic value is the subject of the next section.

#### 5.1.2. The Inner Iteration

The calculation of the intrinsic value involves solving a forward–backward system of equations. Although the equations are deterministic, this is not a trivial task in itself. In the paper, we have presented two approximation methods and, since both of the methods are iterative, we call this step the inner iteration.

In this example, the intrinsic value of the asset and the value process at a fixed time $t$ satisfy the differential equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV}}(t,u) &= rS^{\mathrm{IV}}(t,u) - \delta V^{\mathrm{IV}}(u, S(t)),\\
S^{\mathrm{IV}}(t,t) &= S(t),\\
\frac{\mathrm{d}}{\mathrm{d}u}V^{\mathrm{IV}}(u, S(t)) &= (r-\delta)V^{\mathrm{IV}}(u, S(t)),\\
V^{\mathrm{IV}}(T, S(t)) &= \Phi\big(S^{\mathrm{IV}}(t,T)\big).
\end{aligned} \tag{16}$$

In the present case, the intrinsic value can actually be calculated analytically, but this is not always the case. Instead, we use the two approximation methods described in Sections 3.2.1 and 3.2.2.

Perturbation Method

In the perturbation method, the forward–backward nature of the equations is preserved, but the equations are decoupled. This is done by initially assuming that, to the lowest order, the dividend is small such that it is negligible in the differential equation of the intrinsic value of asset, which can then be used to calculate an approximation of the intrinsic value of the equity, which then in turn can be used to calculate a better approximation of the intrinsic value of the asset and so on. That is, the $k$'th iteration of Equation (16) can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV},(k)}(t,u) &= rS^{\mathrm{IV},(k)}(t,u) - \delta V^{\mathrm{IV},(k-1)}(u,S(t)),\\
S^{\mathrm{IV},(k)}(t,t) &= S(t),\\
\frac{\mathrm{d}}{\mathrm{d}u} V^{\mathrm{IV},(k)}(u,S(t)) &= (r-\delta)V^{\mathrm{IV},(k)}(u,S(t)),\\
V^{\mathrm{IV},(k)}(T,S(t)) &= \Phi\big(S^{\mathrm{IV},(k)}(t,T)\big),
\end{aligned}$$

for $V^{\mathrm{IV},(0)}(u,S(t)) = 0$ for all $u \geq t$.

Shooting Method

In the shooting method, the forward–backward nature of the equations is removed while preserving a system of coupled equations. This is done by guessing an initial value of the value process and then solving the coupled forward equations. The result is checked against the actual final value of the value process, then the initial value is adjusted, and the system of equations is solved again. That is, the $k$'th iteration of Equation (16) can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV},(k)}(t,u) &= rS^{\mathrm{IV},(k)}(t,u) - \delta V^{\mathrm{IV},(k)}(u,S(t)),\\
S^{\mathrm{IV},(k)}(t,t) &= S(t),\\
\frac{\mathrm{d}}{\mathrm{d}u} V^{\mathrm{IV},(k)}(u,S(t)) &= (r-\delta)V^{\mathrm{IV},(k)}(u,S(t)),\\
V^{\mathrm{IV},(k)}(t,S(t)) &= e^{-(r-\delta)(T-t)}\Phi\big(S^{\mathrm{IV},(k-1)}(t,T)\big),
\end{aligned}$$

for $\Phi\big(S^{\mathrm{IV},(0)}(t,T)\big) = 0$.

### *5.2. Finite Difference Method*

For our example model in Equation (12), an exact numerical solution of the value, $V$, can be obtained by numerically solving the partial differential equation (PDE) given by the corresponding Feynmann–Kac formula

$$\frac{\partial V}{\partial t} = -rs\frac{\partial V}{\partial s} + \delta V\frac{\partial V}{\partial s} - \frac{1}{2}\sigma^2 s^2\frac{\partial^2 V}{\partial s^2} + rV - \delta V,$$

with the boundary condition given by the payoff $V(T,s) = \Phi(s)$.

We solve the PDE on a rectangular grid for the spot and time dimensions. In the time dimension, we transform to a forward equation and use the explicit Euler method for the time stepping. For the spot dimension, we make the usual transformation into a log spot grid, $x = \ln s$, for numerical stability and choose a central difference scheme. Other numerical schemes can be used, but we remind the reader to ensure numerical stability when choosing a solution method.

### 5.3. Numerical Results

After having described the different numerical schemes, we are now ready to present the resulting values calculated by the schemes. We calculate the value as a function of the initial spot, both in units of $K$. The numerical parameters are described in the caption of Figure 1, where we show the value, $V(S_0, t_0)$, as a function of initial spot, $S_0$, as calculated using the approximation methods and compared to the numerically exact solution of the PDE. The left plots show the perturbation method and the right plots show the shooting method. The value is calculated for different dividend rates $\delta$ from 0.0, corresponding to no dividends (lower plots) and up to 0.5 corresponding to paying half the value in dividends (top plots). In each plot, the solid graph is the numerically exact value as calculated by the PDE while the values given by the approximation methods are calculated with either one, two, or five updates of the intrinsic value in the Monte Carlo simulation. All the plots have the same axis for better visual comparison. Let us carefully walk through what can be seen.

Let us first notice that the two approximation methods (the perturbation method and the shooting method) yield very similar results and, for this reason, we need not distinguish between these. Let us next look at the lower plots corresponding to no dividend, that is, a pure Black–Scholes market. In this case, the market is decoupled from the value and the outer iteration reduces to a Monte Carlo simulation of the Black–Scholes market, see Equation (15). For this reason, the accuracy of the approximation methods are independent of the number of updates of the intrinsic value, $N_t$. Lastly, as we increase the dividend ratio (middle and top plots), the market and the value are no longer decoupled and simple Monte Carlo no longer suffices. Therefore, we need the methods presented in this paper. If we turn our attention to the top plots where the dividend rate is largest, the value calculated with only one update of the intrinsic value overestimates the actual value, but as we increase the number of updates of the intrinsic value, the approximation methods converge to the exact value. Specifically, we note that the methods reach the accuracy of the Monte Carlo simulation with five updates of the intrinsic value.

An interesting feature of the intrinsic value methods is that it is directly possible to find the time value of the value process from Equation (5), which represents the value added from being allowed to base optimal decisions in the future on future values, since $\alpha$ is the ratio between the time value and the intrinsic value. This is shown in Figure 2 for $\delta = 0.5$ for both methods and with a different number of updates of the intrinsic value, $N_t$. We note three observations. First, the two methods yield similar $\alpha$, as they should. Second, when the intrinsic value is not updated frequently enough, the approximation methods overestimate the time value in the present example. Third, $\alpha$ decreases as we are increasingly in-the-money. This is also what we would expect since stochasticity has less influence on the value far from the strike.

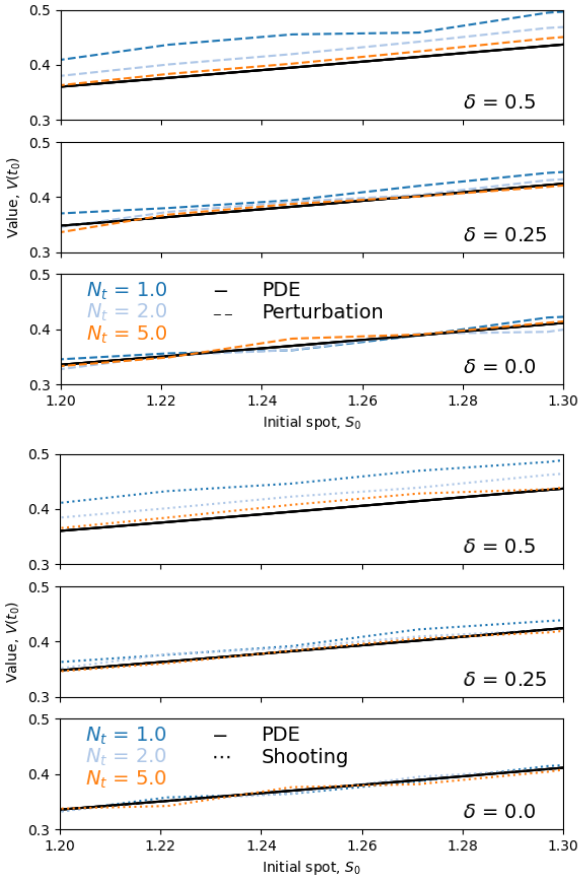



**Figure 1.** The value as a function of the initial spot as calculated using the perturbation and shooting methods for different dividend rates, $\delta$, and number of updates of the intrinsic value, $N_t$. The calculated values are compared to numerically exact values calculated by the PDE. The PDE is calculated on a grid with $10^2$ spot and $10^3$ time points. The intrinsic value methods are calculated with $10^4$ Monte–Carlo paths that have $10^2$ time points. The inner and outer iterations are stopped after 5 iterations.

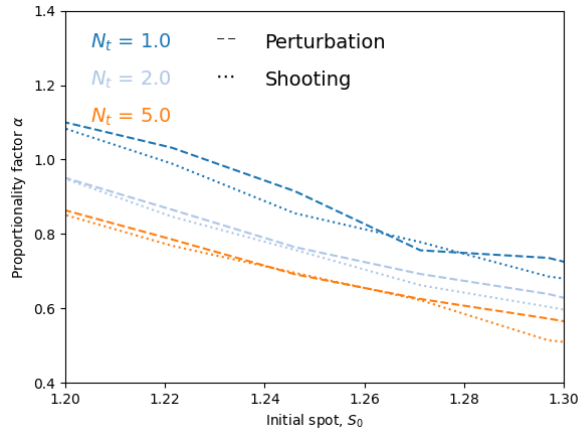



**Figure 2.** The proportionality factor, $\alpha$, as a function of the initial spot, $S_0$, calculated using the perturbation and shooting methods for a dividend rate of $\delta = 0.5$ and for various numbers of updates of the intrinsic value, $N_t$.

Lastly, in Figure 3, we show the value (top plot) and $\alpha$ (bottom plot) as a function of the initial spot, $S_0$, including the crossover from being in-the-money to being out-of-the-money at $S_0 = 1$ for different dividends. When out-of-the-money, the intrinsic value, $V^{\text{IV}}$, is equal

to zero and the outer iteration in Equation (15) becomes independent of $\alpha$ and reduces to a Monte Carlo simulation in a pure Black–Scholes market for all values of the dividend rate $\delta$. Therefore, our calculations for the three values of $\delta$ coincide when out-of-the-money. A significant part of the value is caused by the time value when the initial spot is close to the strike, and therefore the value of $\alpha$ increases when the initial spot approaches the strike. The consequence is that our method is unstable close to the strike around $S_0 = 1$. When in-the-money, the method is stable and we saw already in Figure 1 that it converges fast and properly to the true equity value.

In this section, we have discussed the performance and quality of the approximation method proposed in this paper. We have seen that our method estimates quite accurately the equity value of a firm in a corporate finance setup with dividend payments linear in the equity value itself. This is clearly a numerically challenging problem but in our example it is possible to disentangle, without really losing accuracy, the forward–backward equation for the intrinsic value and the stochastic simulation via an assumption of the time value relation to the intrinsic value. The specific numerical illustration relies on the choice of dividend function (here linearity), the choice of parametric relation (here linearity), and terminal claim (here the standard call option payoff). Although both the parametric relation and the terminal claim are here chosen naturally, the lack of correctness when out-of-the-money and the instability at-the-money are a consequence of their relation. It is left to future works to better understand how sensitive the high quality of the approximation is to these assumptions and how the parametric relation might be improved to avoid problems when not in-the-money. However, for this particular corporate value example, one may note the following. When not in-the-money, that is, when the asset value does not exceed the value of the debt, a delicate equity valuation is predominantly relevant for areas such as credit risk assessment. Then, the firm is probably bankrupt by solvency rules that do not rely on the market valuation performed here. So, within that context, we live with our method being approximately accurate only when in-the-money. Of course, introducing premature bankruptcy in the arguments calls for introducing premature bankruptcy within the model. This is also left for future studies.

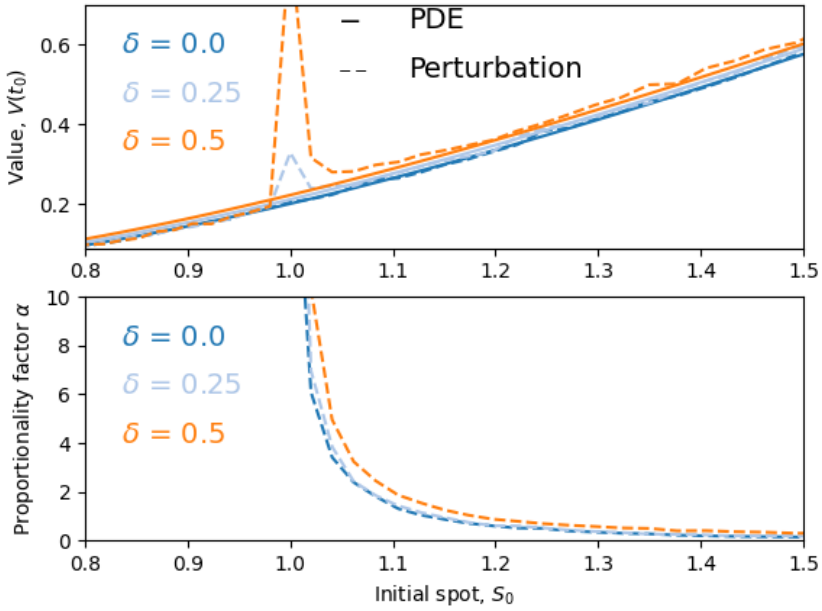



**Figure 3.** The value and the proportionality factor, $\alpha$, as a function of the initial spot, $S_0$, on an interval including the cross over from being out-of-the-money to being in-the-money at $S_0 = 1$ for different dividends rates, $\delta$, and with 5 updates of the intrinsic value, $N_t = 5$.

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## Appendix A. Quality of the Intrinsic Value Approximation

We study the quality of the approximation in Equation (6) under the assumption that there is no continuous process of claims, $\phi = 0$, and that the claim $\Phi$ is infinitely differentiable. The Taylor series of $\Phi\big(S(T)\big)$ around $S^{\mathrm{IV}}(t,T)$ is

$$\Phi\big(S(T)\big) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{(n)}\big(S^{\mathrm{IV}}(t,T)\big)\Big(S(T) - S^{\mathrm{IV}}(t,T)\Big)^n,$$

where $\Phi^{(n)}(s) = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\Phi(s)$. With the Taylor series above, we may write

$$\begin{aligned}
&V(t,S(t),r(t)) \\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)\mathrm{d}s}\Phi\big(S(T)\big) \;\Big|\; S(t),r(t)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)\mathrm{d}s}\sum_{n=0}^{\infty}\frac{1}{n!}\Phi^{(n)}\big(S^{\mathrm{IV}}(t,T)\big)\Big(S(T) - S^{\mathrm{IV}}(t,T)\Big)^n \;\Big|\; S(t),r(t)\Big] \\
&= \Bigg(1 + \underbrace{\sum_{n=1}^{\infty}\frac{\Phi^{(n)}\big(S^{\mathrm{IV}}(t,T)\big)}{n!\Phi\big(S^{\mathrm{IV}}(t,T)\big)}\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T (r(s)-f(t,s))\mathrm{d}s}\Big(S(T) - S^{\mathrm{IV}}(t,T)\Big)^n \;\Big|\; S(t),r(t)\Big]}_{=\alpha(t,S(t),r(t))}\Bigg) \\
&\quad \times V^{\mathrm{IV}}(t,S(t),r(t)).
\end{aligned}$$

Hence, the quality of the approximation in Equation (6) relies on how well we can approximate $\alpha(t,S(t),r(t))$ with a constant $\alpha$. We study the approximation analytically in the classic Black-Scholes market.

**Example A1.** *Let the asset have dynamics*

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t),$$

*for a deterministic interest rate $r \in \mathbb{R}$ and volatility $\sigma > 0$, and where $W$ is a Brownian motion under the risk neutral measure $\mathbb{Q}$. Since the interest rate is deterministic, the forward interest rate is equal to $r$. The intrinsic value of the asset is*

$$S^{\mathrm{IV}}(t,u) = S(t)e^{r(u-t)},$$

*and*

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\Big[S(T) - S^{\mathrm{IV}}(t,T) \;\Big|\; S(t)\Big] &= 0, \\
\mathbb{E}^{\mathbb{Q}}\Big[\big(S(T) - S^{\mathrm{IV}}(t,T)\big)^2 \;\Big|\; S(t)\Big] &= S^{\mathrm{IV}}(t,T)^2\big(e^{\sigma^2(T-t)} - 1\big).
\end{aligned}$$

*We consider a first order polynomial claim*

$$\Phi(s) = a + b\cdot s \quad \Rightarrow \quad \alpha(t,S(t)) = 0.$$

*Hence, for a first order polynomial claim, the approximation in Equation (6) is exact for $\alpha = 0$ in the classic Black-Scholes market.*

*For a second order polynomial claim, we have that*

$$\Phi(s) = a + b\cdot s + c\cdot s^2 \quad \Rightarrow \quad \alpha(t,S(t)) = \frac{2\cdot c\cdot S^{\mathrm{IV}}(t,T)^2\big(e^{\sigma^2(T-t)} - 1\big)}{2\big(a + b\cdot S^{\mathrm{IV}}(t,T) + c\cdot S^{\mathrm{IV}}(t,T)^2\big)}.$$

*For $a = b = 0$, the function $\alpha$ depends solely on time to maturity and the volatility. Therefore, if we allow for a time dependent $\alpha$ in the approximation in Equation (6), the approximation is exact with the claim $\Phi(s) = c \cdot s^2$.*

*In general, for a polynomial claim in the form $\Phi(s) = c \cdot s^M$, $M \in \mathbb{N}$, in the classic Black-Scholes market, the approximation in Equation (6) is exact if we allow for an $\alpha$ which depends on time.*

In general, the quality of the intrinsic value approximation of the value process depends on how well $S^{\mathrm{IV}}(t,T)$ approximates $S(T)$ and the behaviour of $\Phi$ and its derivatives. The size of the terms $\mathbb{E}^{\mathbb{Q}}\left[ \left(S(T) - S^{\mathrm{IV}}(t,T)\right)^n \,\middle|\, S(t) \right]$, $n \in \mathbb{N}$, in the expression of $\alpha(t, S(t))$ depends on the volatility function $\sigma$ and a time to maturity $T - t$, since we disregard the volatility in the interval $[t, T]$ when we define $S^{\mathrm{IV}}(t,T)$.

## Appendix B. Sketch of Algorithms

### *Appendix B.1. Outer Iteration*

The algorithm for the outer iteration can be sketched as follows

```
alpha[1] = alpha_0
for n_alpha = 1 : N_alpha
    S[:,1] = S_0
    V_IV[1:N_t] = CalculateIntrinsicValue(t[1:N_t], S_0)
    for i = 1 : N_MC
        for j = 1 : N_t-1
            if j in t_IV
              V_IV[j:N_t] = CalculateIntrinsicValue(t[j:N_t], S[i,j])
            end
       S[i,j+1] = Euler(t[j], S[i,j], (1+alpha[n_alpha])*V_IV[j:N_t])
        end
    end
    V = CalculateMonteCarloValue(t[:], S[:,:])
    if V_IV[1] != 0
        alpha[n_alpha + 1] = V/V_IV[1] - 1
    else
        alpha[n_alpha + 1] = alpha[n_alpha]
    end
end
```

### *Appendix B.2. Inner Iteration*

#### Appendix B.2.1. Perturbation Method

The algorithm for the perturbation method iteration can be sketched as follows

```
V_IV[:] = 0
for k = 1 : N_k
    S_IV[1] = S_0
    for j = 1 : N_t-1
        S_IV[j+1] = Euler(t[j], S_IV[j], V_IV[j])
    end
    V_IV[N_t] = Phi(S:IV[N_t])
    for j = N_t : 1
        V_IV[j-1] = BackwardEuler(t[j], V_IV[j], S_IV[j])
    end
end
```

Appendix B.2.2. Shooting Method

The algorithm for the shooting method iteration can be sketched as follows

```
V_IV[1] = V_0
for k = 1 : N_k
    S_IV[1] = S_0
    for j = 1 : N_t-1
        S_IV[j+1] = Euler(t[j], S_IV[j], V_IV[j])
        V_IV[j] = Euler(t[j], S_IV[j])
    end
    V_IV[1] = Discount(t[1], t[Nt], S_IV[Nt])
end
```

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