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A Study on Fuzzy Order Bounded Linear Operators in Fuzzy Riesz Spaces

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Abstract: This paper aims to study fuzzy order bounded linear operators between two fuzzy Riesz spaces. Two lattice operations are defined to make the set of all bounded linear operators as a fuzzy Riesz space when the codomain is fuzzy Dedekind complete. As a special case, separation property in fuzzy order dual is studied. Furthermore, we studied fuzzy norms compatible with fuzzy ordering (fuzzy norm Riesz space) and discussed the relation between the fuzzy order dual and topological dual of a locally convex solid fuzzy Riesz space.

Keywords: fuzzy order bounded operators; fuzzy order dual spaces; fuzzy norm riesz spaces; locally convex-solid fuzzy riesz spaces



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1. Introduction

The theory of fuzzy relation by generalizing concepts of reflexivity, antisymmetry, and transitivity was initiated by Zadeh [1]. Then, Venugopalan [2] developed a structure of fuzzy ordered sets. Since then, many authors have studied fuzzy relations and ordering by using different approaches [3–7].

Vector spaces are used with other tools such as fuzzy topology, fuzzy norm and fuzzy metric. Felbin [8] studied the finite-dimensional fuzzy normed linear spaces. Saadati and Vaezpour [9] investigated the theory of fuzzy Banach spaces. Cheng and Mordeson [10] introduced the concept of a fuzzy linear operator. Harisha et al. [11] gave the concept of degree sequence of graph operator for some standard graphs. Xaio and Zhu [12] discussed the completeness of fuzzy norm space of a linear operator. Then, Bag and Samanta [13,14] defined and studied the fuzzy bounded linear operators and their properties. Binzar et al. [15] studied the boundedness in fuzzy normed linear spaces. Recently, Kim and Lee [16] investigated the theory of approximation properties in Felbin-Fuzzy normed spaces. Furthermore, vector space with fuzzy order is known as fuzzy Riesz space. In a series of papers Beg and Islam [17–21] investigated the basic theory of fuzzy Riesz spaces. Hong [22] introduced the concepts of fuzzy Riesz subspaces, fuzzy ideals, fuzzy bands and fuzzy band projections. Park et al. [23] defined and studied the Riesz fuzzy norm spaces. Iqbal et al. [24] defined and characterized the notion of unbounded fuzzy order convergence in fuzzy Riesz spaces. Iqbal and Bashir [25] gave the notion of Dedekind completion of Archimedean fuzzy Riesz spaces. Recently, Cheng et al. [26,27] modified the notion of fuzzy Riesz homomorphisms. Our references for classical Riesz spaces are [28–30].

Bag [20] defined the fuzzy positive operator between two fuzzy Riesz spaces (K, μ) and (H, η) and studied its extension. In this paper, we study the space of all fuzzy order bounded positive linear operators denoted $\mathcal{L}_b(K, H)$. We show that it is fuzzy Dedekind

complete when (H, η) is fuzzy Dedekind complete by defining suitable fuzzy lattice operations. The set of all fuzzy order (σ -order) continuous bounded linear operators denoted $\mathcal{L}_n(K, H)(\mathcal{L}_c(K, H))$ are fuzzy bands of $\mathcal{L}_b(K, H)$ when H is fuzzy Dedekind complete and $\mathcal{L}_n(K, H) \subseteq \mathcal{L}_c(K, H)$. As a special case, other related concepts such as separation properties, fuzzy order continuous dual and σ -fuzzy order continuous dual on K are studied.

Fuzzy normed spaces are significant with applications in many areas such as dynamical systems, engineering, and fluid mechanics. We study fuzzy ordering in fuzzy norm spaces and define the notion of fuzzy norm Riesz space. Fuzzy norm with compatible fuzzy order is more fruitful as ordering is very common in real-life scenarios. Defining suitable fuzzy ordering in important norm spaces such as L^p spaces is a challenging problem. We define a fuzzy order in $L^\infty[0, 1]$ by using a probability measure. We develop the basic framework of fuzzy norm Riesz spaces and prove many related results. We also study the connections between the topological structure and fuzzy lattice structure of a fuzzy Riesz space when the induced topology of the fuzzy norm is locally convex-solid. Towards the end of this paper, we prove that the topological dual of a locally-convex solid fuzzy Riesz space is always a fuzzy Riesz space, and in fact, it is a fuzzy ideal in its fuzzy order dual.

Together with the novel notions and many proven results, this paper significantly contributes to the theory of fuzzy Riesz space of all fuzzy order bounded operators, which not only helped us to solve the problem of fuzzy order dual spaces, fuzzy norm Riesz space and locally convex-solid fuzzy Riesz spaces but can also be used in future to explore fuzzy Riesz space in other directions.

The contents of the paper are organized as follows. In Section 2, some basic definitions and results are recalled. Section 3 discusses the space of all fuzzy order bounded linear operators and their properties. Section 4 is devoted to define and study the fuzzy Banach lattice and locally convex-solid fuzzy Riesz spaces. In the end, some concluding remarks for possible future lines are given in Section 5.

2. Preliminaries

A fuzzy order μ on a set K is a fuzzy set on $K \times K$ with the understanding that k precedes g if and only if $\mu(k, g) > 1/2$ for $k, g \in K$ and the following conditions are also satisfied:

- (i) $\forall k \in K \mu(k, k) = 1$ (reflexivity);
- (ii) for $k, g \in K \mu(k, g) + \mu(g, k) > 1$ implies $k = g$ (antisymmetric);
- (iii) for $k, h \in K \mu(k, h) \geq \bigvee_{g \in K} [\mu(k, g) \wedge \mu(g, h)]$ (transitivity).

The tuple (K, μ) is called *fuzzy ordered set (FOS)*. For $C \subseteq K$, the two fuzzy sets $U(C)$ and $L(C)$ are defined as follows.

$$U(C)(g) = \begin{cases} 0, & \text{if } (\uparrow k)(g) \leq 1/2 \text{ for some } k \in C; \\ (\bigcap_{k \in C} \uparrow k)(g), & \text{otherwise.} \end{cases}$$

$$L(C)(g) = \begin{cases} 0, & \text{if } (\downarrow k)(g) \leq 1/2 \text{ for some } k \in C; \\ (\bigcap_{k \in C} \downarrow k)(g), & \text{otherwise.} \end{cases}$$

Let $(C)^u$ denotes the set of all upper bounds of C and $k \in (C)^u$ if $U(C)(k) > 0$. Analogously, $(C)^l$ denotes the set of all lower bounds and $k \in (C)^l$ if $L(C)(k) > 0$. Furthermore, $d \in K$ is known as *supremum* of C in K if (i) $d \in (C)^u$ (ii) $g \in (C)^u$ implies $g \in (d)^u$. Infimum is defined analogously. A subset C is said to be *fuzzy order bounded* if $(C)^u$ and $(C)^l$ are non-empty.

A real vector space K with fuzzy order μ is known as *fuzzy ordered vector space (FOVS)* if μ satisfies:

- (i) for $k, g \in K$ if $\mu(k, g) > 1/2$ then $\mu(k, g) \leq \mu(k + h, g + h)$ for all $h \in K$;
- (ii) for $k, g \in K$ if $\mu(k, g) > 1/2$ then $\mu(k, g) \leq \mu(\lambda k, \lambda g)$ for all $0 \leq \lambda \in \mathbb{R}$.

$k \in K$ is known as *positive* if $\mu(0, k) > 1/2$, and *negative* if $\mu(k, 0) > 1/2$. Furthermore, K^+ as set of all positive elements in K , i.e., $K^+ = \{k \in K : \mu(0, k) > 1/2\}$ is referred to as the *fuzzy positive cone*. $A \subseteq K$ is called *directed upwards* (*directed downward*) if for each finite subset B of A we have $A \cap (B)^u \neq \emptyset$ ($A \cap (B)^l \neq \emptyset$), respectively. Furthermore, for a net $(k_\lambda)_{\lambda \in \Lambda}$ $k_\lambda \uparrow k$ reads as the net (k_λ) directed upwards to k if $\lambda_0 \leq \lambda_1$ we have $\mu(k_{\lambda_0}, k_{\lambda_1}) > 1/2$ and $\sup\{k_\lambda\} = k$. $k_\lambda \downarrow k$ is defined analogously.

A FOVS (K, μ) is said to be *Archimedean* if $\mu(nk, g) > 1/2$ for all $n \in \mathbb{N}$ implies that $\mu(k, 0) > 1/2$ for all $k, g \in K$. Therefore, $\{\frac{k}{n}\} \downarrow 0$ and $\{nk\}$ is unbounded from above for all $0 \neq k \in K^+$.

A FOVS (K, μ) is said to be *fuzzy Riesz space* (FRS) if $k \vee g = \sup\{k, g\}$ and $k \wedge g = \inf\{k, g\}$ exist in K for all $k, g \in K$.

For $k \in K$, $k^+ = k \vee 0$ and $k^- = (-k) \vee 0$ are known to be the positive and negative parts of k , respectively, moreover, the absolute value of k is known as $|k| = (-k) \vee k$. $k_1, k_2 \in K$ are called *orthogonal or disjoint* if $k_1 \wedge k_2 = 0$ and written as $k_1 \perp k_2$. Furthermore, $C_1, C_2 \subseteq K$ are called *disjoint* and denoted by $C_1 \perp C_2$ if $k_1 \perp k_2 = 0$ for each $k_1 \in C_1$ and $k_2 \in C_2$. For $\emptyset \neq C \subseteq K$ its *disjoint complement* is defined as $C^d = \{k \in K : k \perp g \text{ for each } g \in C\}$.

An FRS (K, μ) is called *fuzzy Dedekind complete* if each non-empty subset of K which is bounded from above has a supremum in K and *σ -fuzzy Dedekind complete*, if each nonempty countable subset of K bounded from above has a supremum in K . Let $k, g \in K$ with $\mu(k, g) > 1/2$ then the *fuzzy order interval* $[k, g] \subseteq K$ is defined by

$$[k, g] = \{h \in K : \mu(k, h) > 1/2 \text{ and } \mu(h, g) > 1/2\}.$$

A fuzzy positive operator P between two FRSs is a linear map $P : K \rightarrow H$ such that $P(k) \in H^+$ for all $k \in K^+$.

Proposition 1. If k and g are elements of an FRS (K, μ) then:

- (i) $k = k^+ - k^-$;
- (ii) $k^+ \wedge k^- = 0$;
- (iii) $|k| = k^+ + k^-$;
- (iv) $\mu(|k| - |g|, |k - g|) > 1/2$.

A net $(k_\lambda)_{\lambda \in \Lambda}$ in an FRS (K, μ) is said to be *fuzzy order convergent* to $k \in K$ denoted $k_\lambda \xrightarrow{fo} k$ if there exists another net $(g_\lambda)_{\lambda \in \Lambda}$ in K^+ directed downwards to zero such that $\mu(|k_\lambda - k|, g_\lambda) > 1/2$ for each $\lambda \in \Lambda$. A subset C of an FRS (K, μ) is called *fuzzy solid* if $\mu(|k|, |g|) > 1/2$ and $g \in C$ implies $k \in C$. Fuzzy solid vector subspaces are called *fuzzy ideals*. Moreover, C is said to be *fuzzy order closed* if $(k_\lambda) \subseteq C$ and $k_\lambda \xrightarrow{fo} k$ imply $k \in C$. A fuzzy order closed ideal is known as *fuzzy band*.

Let K be a non-empty set and \star a continuous t-norm on $[0, 1]$. A mapping $N : K \times (0, \infty) \rightarrow [0, 1]$ is said to be *fuzzy norm* on K iff $k, g \in K$ and $\lambda \in \mathbb{F}$ the following conditions are satisfied:

- (i) for each $t \in \mathbb{R}$ with $t > 0$ $N(k, t) = 1 \Leftrightarrow k = 0$;
- (ii) for each $t \in \mathbb{R}$ with $t > 0$;

$$N(\lambda k, t) = N(k, \frac{t}{\lambda}) \text{ if } \lambda \neq 0;$$

- (iii) for each $t, s \in \mathbb{R}$ with $t, s > 0$;

$$N(k + g, t + s) \geq N(k, t) \star N(g, s);$$

- (iv) $N(k, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
- (v) $\lim_{t \rightarrow \infty} N(k, t) = 1$.

The triple (K, N, \star) is called fuzzy norm space.

A family τ of fuzzy sets of K is said to be fuzzy topology if

- (i) τ contain all constant fuzzy sets in K ;
- (ii) for $\{C_i\}_{i \in \Delta} \in \tau$ we have $\sup_{i \in \Delta} C_i \in \tau$;
- (iii) if $C, D \in \tau$ then $C \wedge D \in \tau$.

The pair (K, τ) is called *fuzzy topological space (for short FTS)*. A fuzzy set C in (K, τ) is a *neighborhood* of a point $k \in K$ iff there is $V \in \tau$ such that $V \leq C$ and $V(k) = C(k) > 0$. A map f from a FTS K to a FTS H is called *continuous* at some $k \in K$ if $f^{-1}(V)$ is a neighborhood of k in K for each neighborhood V of $f(k)$ in H .

A net $(k_\lambda)_{\lambda \in \Lambda}$ in a fuzzy topological space (K, τ) converges to a point k denoted $k_\lambda \xrightarrow{f\tau} k$ if given a neighborhood V of k , there exists a $\lambda_0 \in \Lambda$ such that $k_\lambda \in V$ whenever $\lambda \geq \lambda_0$.

A fuzzy set $C \subseteq K$ is said to be *convex* if $\lambda C + (1 - \lambda)C \leq C$ for each $\lambda \in [0, 1]$, *balanced* if $\lambda C \leq C$ for each scalar λ with $|\lambda| \leq 1$ and *absorbing* if $\sup_{\lambda > 0} \lambda C = 1$.

A fuzzy linear topology on a vector space K over field \mathbb{R} is a fuzzy topology τ on K such that the two mappings

- (i) $+: K \times K \rightarrow K, (k, g) \mapsto k + g$
- (ii) $\cdot: \mathbb{R} \times K \rightarrow K, (k, g) \mapsto \lambda k$

are continuous when \mathbb{R} has the usual fuzzy topology and $\mathbb{R} \times K, K \times K$ the corresponding product fuzzy topologies. The pair (K, τ) is called *fuzzy topological vector space*. For further details, we refer to [18,22,24,31–39].

3. Fuzzy Positive Linear Operators

In order to study $\mathcal{L}_b(K, H)$ we start our work in $\mathcal{L}(K, H)$ the set of all linear operators between (K, μ) and (H, η) . Of course, $\mathcal{L}(K, H)$ is a vector space with point wise operations. However, the natural point wise ordering, i.e., $P \leq V$ if $\eta(P(k), V(k)) > 1/2$ for each $k \in K$, does not induce lattice structure on $\mathcal{L}(K, H)$. Thus, in order to define proper lattice operation on $\mathcal{L}_b(K, H)$ first we work on the modulus of positive linear operator.

Definition 1. A fuzzy positive operator P between two FRSs (K, μ) and (H, η) possesses a modulus if $|P| = P \vee (-P)$. The modulus of P means the supremum of the set $\{-P, P\}$ in $\mathcal{L}(K, H)$.

The existence of the modulus of a fuzzy positive operator is given in the following proposition.

Proposition 2. If P is a fuzzy positive operator between two FRSs (K, μ) and (H, η) such that $\sup\{|Pg| : \mu(|g|, k) > 1/2\}$ exists in H for all $k \in K^+$ then modulus of P exists and

$$|P|(k) = \sup\{|Pg| : \mu(|g|, k) > 1/2\}.$$

Proof. Suppose $V : K^+ \rightarrow H^+$ is defined by $V(k) = \sup\{|Pg| : \mu(|g|, k) > 1/2\}$ for $k \in K^+$. Since $\mu(|g|, k) > 1/2$ implies $|\pm g| = |g|$, $\mu(|\pm g|, k) > 1/2$. Now, we show that V is additive.

Let $h, l \in K^+$. If $\mu(|g|, h) > 1/2$ and $\mu(|r|, l) > 1/2$ then

$$\mu(|g + r|, |g| + |r|) > 1/2 \text{ and } \mu(|g| + |r|, h + l) > 1/2.$$

Thus,

$$P(g) + P(r) = P(g + r) \text{ and } \eta(P(g + r), V(h + l)) > 1/2.$$

Therefore, $\eta(V(h) + V(l), V(h + l)) > 1/2$. Conversely, if $\mu(|g|, h + l) > 1/2$, then by Theorem 4.12 in [17] there exists g_1 and g_2 with $\mu(|g_1|, h) > 1/2$, $\mu(|g_2|, l) > 1/2$ and $g = g_1 + g_2$. Then

$$P(g) = P(g_1) + P(g_2) \text{ and } \eta(P(g_1) + P(g_2), V(h) + V(l)) > 1/2,$$

so $\eta(V(h + l), V(h) + V(l)) > 1/2$. Hence, V is additive. By Lemma 2.4 in [20], V extends uniquely as a fuzzy positive operator from K to H .

It is left to show that V is a supremum of $\{-P, P\}$. Observe that $P \leq V$ and $-P \leq V$. Assume that $\pm P \leq U$. Clearly, U is a fuzzy positive operator. Fix $k \in K^+$. If $\mu(|g|, k) > 1/2$ then

$$Pg = Pg^+ - Pg^- \text{ and } \eta(Pg^+ - Pg^-, Ug^+ + Ug^-) > 1/2.$$

Therefore,

$$Ug^+ + Ug^- = U|g| \text{ and } \eta(U|g|, Uk) > 1/2.$$

Thus, $\eta(V(k), U(k)) > 1/2$ for $k \in K^+$. Hence $V = P \vee (-P)$ in $\mathcal{L}(K, H)$. \square

Remark 1. A fuzzy positive linear operator P between two FRSs (K, μ) and (H, η) is said to be fuzzy order bounded if $P(C) \subseteq H$ is fuzzy order bounded whenever $C \subseteq K$ is fuzzy order bounded. Now, we know that $\sup\{|Pg| : \mu(|g|, k) > 1/2\}$ exists when H is fuzzy Dedekind complete. Thus, with the help of Proposition 2 we define the fuzzy lattice operations in $\mathcal{L}_b(K, H)$ for fuzzy Dedekind complete (H, η) .

Theorem 1. If (K, μ) and (H, η) are FRSs with H fuzzy Dedekind complete, then the fuzzy order vector space $\mathcal{L}_b(K, H)$ is a fuzzy Dedekind complete Riesz space with the following fuzzy lattice operations,

$$|P|(k) = \sup\{|Pg| : \mu(|g|, k) > 1/2\},$$

$$[V \vee P](k) = \sup\{V(g) + P(h) : g, h \in K^+ \text{ and } g + h = k\},$$

$$[V \wedge P](k) = \inf\{V(g) + P(h) : g, h \in K^+ \text{ and } g + h = k\}$$

for each $V, P \in \mathcal{L}_b(K, H)$ and $k \in K^+$. In addition, $P_\lambda \downarrow 0$ in $\mathcal{L}_b(K, H)$ iff $P_\lambda(k) \downarrow 0$ in H for all $k \in K^+$.

Proof. Since P is fuzzy order bounded,

$$\sup\{|Pg| : \mu(|g|, k) > 1/2\} = \sup\{Pg : \mu(|g|, k) > 1/2\} = \sup P[-k, k]$$

exists in H for $k \in K^+$. By Proposition 2, the modulus of P exists and also

$$|P|(k) = \sup\{Pg : \mu(|g|, k) > 1/2\}.$$

Now, we show that $\mathcal{L}_b(K, H)$ is an FRS. Let $V, P \in \mathcal{L}_b(K, H)$ and $k \in K^+$ satisfying $g + h = k$ iff there exists some $l \in K$ such that $\mu(|l|, k) > 1/2$ with $g = 1/2(k + l)$ and $h = 1/2(k - l)$ for $g, h \in K^+$. It follows from Theorem 4.11 in [17] that

$$\begin{aligned} [V \vee P](k) &= 1/2[V(k) + P(k) + |V - P|(k)] \\ &= 1/2[V(k) + P(k) + \sup\{(V - P)(l) : \mu(|l|, k) > 1/2\}] \\ &= 1/2 \sup\{V(k) + V(l) + P(k) - P(l) : \mu(|l|, k) > 1/2\} \\ &= \sup\{V(1/2(k + l)) + P(1/2(k - l)) : \mu(|l|, k) > 1/2\} \\ &= \sup\{V(g) + P(h) : g, h \in K^+ \text{ and } g + h = k\}. \end{aligned}$$

$[V \wedge P]$ can be proven analogously.

Now, we have to show that $\mathcal{L}_b(K, H)$ is fuzzy Dedekind complete. Let $P_\lambda \uparrow P$ in $\mathcal{L}_b(K, H)$. Assume that $V(k) = \sup\{P_\lambda(k)\}$ implies that $P_\lambda(k) \uparrow V(k)$ for all $k \in K^+$. As

$P_\lambda(k+g) = P_\lambda(k) + P_\lambda(g)$, it follows that $V : K^+ \rightarrow H^+$ is additive, then V defines a fuzzy positive operator from K to H . Clearly, $P_\lambda \uparrow V$ in $\mathcal{L}_b(K, H)$. \square

Remark 2. Theorem 1 yields that if (K, μ) and (H, η) are FRSS with H fuzzy Dedekind complete then every fuzzy order bounded operator $P : K \rightarrow H$ satisfies

$$P^+(k) = \sup\{Pg : \mu(g, k) > 1/2\}$$

$$P^-(k) = \sup\{-Pg : \mu(g, k) > 1/2\}$$

for each $k \in K^+$ and we have $P = P^+ - P^-$. In order to derive some formulas for fuzzy positive operators, we first prove the approximation properties of fuzzy positive operators which are discussed as follows.

Lemma 1. If $P : K \rightarrow H$ is a fuzzy positive operator between two FRSS with H σ -fuzzy Dedekind complete then there exists a fuzzy positive operator $V : K \rightarrow H$ for each $k \in K^+$ such that:

- (i) $V \leq P$;
- (ii) $V(k) = P(k)$;
- (iii) $V(g) = 0$ for each $g \perp k$.

Proof. The proof is essentially the same as for Proposition 2 with the use of Lemma 2.4 in [20]. \square

Now, we use Lemma 1 to prove the following theorem.

Theorem 2. If $P : K \rightarrow H$ is a fuzzy positive operator between two FRSS with H σ -fuzzy Dedekind complete, then for each $k \in K$ we have:

- (i) $P(k^+) = \max\{V(k) : V \in \mathcal{L}(K, H) \text{ and } V \leq P\}$;
- (ii) $P(k^-) = \max\{-V(k) : V \in \mathcal{L}(K, H) \text{ and } V \leq P\}$;
- (iii) $P(|k|) = \max\{V(k) : V \in \mathcal{L}(K, H) \text{ and } -P \leq V \leq P\}$.

Proof. (i) Fix $k \in K$. By Lemma 1, there exists a fuzzy positive operator $V : K \rightarrow H$ such that $V \leq P$, $V(k^+) = P(k^+)$ and $V(k^-) = 0$. If $U \in \mathcal{L}(K, H)$ with $U \leq P$ then

$$\eta(U(k), U(k^+)) > 1/2 \text{ and } \eta(U(k^+), P(k^+)) > 1/2.$$

- (ii) The proof of this part can be obtained from (i) by using identity $k^- = (-k)^+$.
- (iii) Suppose that operator $U : K \rightarrow H$ satisfies $-P \leq U \leq P$, then $U(k) = U(k^+) - U(k^-)$ such that

$$\eta(U(k^+) - U(k^-), P(k^+) + P(k^-)) > 1/2 \text{ and } P(k^+) + P(k^-) = P(|k|).$$

Hence, $U(k) = P(|k|)$.

On the contrary, by Lemma 1, there exist two fuzzy positive operators $V_1, V_2 : K \rightarrow H$ such that

$$V_1(k^+) = P(k^+) \text{ and } V_1(k^-) = 0.$$

Furthermore,

$$V_2(k^-) = P(k^-) \text{ and } V_2(k^+) = 0.$$

Therefore, $U = V_1 - V_2$ satisfies $-P \leq U \leq P$ and $U(k) = P(|k|)$.

\square

Definition 2. A fuzzy positive operator P between two FRSS (K, μ) and (H, η) is said to be:

- (i) fuzzy order continuous if $k_\lambda \xrightarrow{f_o} 0$ in K implies $P(k_\lambda) \xrightarrow{f_o} 0$ in H ;
- (ii) fuzzy σ -order continuous if $(k_n)_{n \in \mathbb{N}} \xrightarrow{f_o} 0$ in K implies $P(k_n)_{n \in \mathbb{N}} \xrightarrow{f_o} 0$ in H .

The notion of fuzzy order continuous operators have nice characterizing conditions.

Theorem 3. If P is a fuzzy order continuous operator between two FRSS (K, μ) and (H, η) with H fuzzy Dedekind complete then following statements are equivalent:

- (i) P is fuzzy order continuous ;
- (ii) if $k_\lambda \downarrow 0$ in K then $P(k_\lambda) \downarrow 0$ in H ;
- (iii) if $k_\lambda \downarrow 0$ in K then $\inf(P(k_\lambda)) = 0$ in H ;
- (iv) P^+ and P^- are both fuzzy order continuous;
- (v) $|P|$ is fuzzy order continuous.

Proof. (i) \rightarrow (ii) and (ii) \rightarrow (iii) are obvious.

(iii) \rightarrow (iv) Let $k_\lambda \downarrow 0$ in K and $P(k_\lambda) \downarrow h$ in H for $h \in H^+$. We have to show that $h = 0$. Fixed some γ and put $k = k_\gamma$. Take $k, g \in K^+$ such that $\mu(g, k) > 1/2$ and for each $\lambda \succeq \gamma$ we have

$$g - g \wedge k_\lambda = g \wedge g - g \wedge k_\lambda \text{ and } \mu(g - g \wedge k_\lambda, k - k_\lambda) > 1/2.$$

Therefore,

$$P(g) - P(g \wedge k_\lambda) = P(g - g \wedge k_\lambda), \eta(P(g - g \wedge k_\lambda), P^+(k - k_\lambda)) > 1/2$$

and

$$P^+(k - k_\lambda) = P^+(k) - P^+(k_\lambda) \text{ implies } \eta(P(g - g \wedge k_\lambda), P^+(k) - P^+(k_\lambda)) > 1/2.$$

It follows that

$$\eta(h, P^+(k_\lambda)) > 1/2 \text{ and } \eta(P^+(k_\lambda), P^+(k) + P(g \wedge k_\lambda) - P(g)) > 1/2. \quad (1)$$

Since $\mu(g, k) > 1/2$ we have $g \wedge k_\lambda \downarrow 0$ for each $\lambda \succeq \gamma$. It follows from (iii) that $\inf(P(g \wedge k_\lambda)) = 0$. Therefore, from (3.1) $\eta(h, P^+(k) - P(g)) > 1/2$ for each $\mu(g, k) > 1/2$. Thus, $P^+(k) = \sup\{P(g) : \mu(g, k) > 1/2\}$ implies that $h = 0$.

(iv) \rightarrow (v) is straightforward.

(v) \rightarrow (i) immediately follows from $\eta(|P(k)|, |P|(|k|)) > 1/2$.

□

Remark 3. The set of all fuzzy order continuous operator of $\mathcal{L}_b(K, H)$ are denoted by $\mathcal{L}_n(K, H)$, i.e.,

$$\mathcal{L}_n(K, H) = \{P \in \mathcal{L}_b(K, H) : P \text{ is fuzzy order continuous}\}.$$

Analogously, $\mathcal{L}_c(K, H)$ denotes the set of all σ -fuzzy order continuous operator, i.e.,

$$\mathcal{L}_c(K, H) = \{P \in \mathcal{L}_b(K, H) : P \text{ is } \sigma\text{-fuzzy order continuous}\}.$$

Both $\mathcal{L}_c(K, H)$ and $\mathcal{L}_n(K, H)$ are vector subspaces of $\mathcal{L}_b(K, H)$. Furthermore, $\mathcal{L}_n(K, H) \subseteq \mathcal{L}_c(K, H)$. The following proposition shows that both $\mathcal{L}_n(K, H)$ and $\mathcal{L}_c(K, H)$ are fuzzy bands of $\mathcal{L}_b(K, H)$.

Proposition 3. If (K, μ) and (H, η) are FRSS with H fuzzy Dedekind complete, then $\mathcal{L}_n(K, H)$ and $\mathcal{L}_c(K, H)$ are both fuzzy bands of $\mathcal{L}_b(K, H)$.

Proof. If $|V| \leq |P|$ in $\mathcal{L}_b(K, H)$ with $P \in \mathcal{L}_b(K, H)$ then by Theorem 3 $V \in \mathcal{L}_b(K, H)$. Thus, $\mathcal{L}_n(K, H)$ are fuzzy ideal of $\mathcal{L}_b(K, H)$.

Now, we show that $\mathcal{L}_n(K, H)$ is a fuzzy band. Let $(P_\gamma)_{\gamma \in \Gamma} \in \mathcal{L}_n(K, H)$ and $P_\gamma \uparrow P$ in $\mathcal{L}_b(K, H)$. Let $k_\lambda \uparrow k$ in K^+ . For fixed γ , we have

$$\eta(P(k - k_\lambda), (P - P_\gamma)(k) + P_\gamma(k - k_\lambda)) > 1/2,$$

and $k - k_\lambda \downarrow 0$. As $P_\gamma \in \mathcal{L}_n(K, H)$ implies that

$$\eta(\inf(P(k - k_\lambda)), (P - P_\gamma)(k)) > 1/2$$

for each γ . Thus, $(P - P_\gamma) \downarrow 0$. Therefore, $\inf(P(k - k_\lambda)) = 0$ and so $P(k_\lambda) \uparrow P(k)$. Hence $P \in \mathcal{L}_n(K, H)$.

$\mathcal{L}_c(K, H)$ can be proved analogously. \square

Fuzzy Order Dual Spaces

A fuzzy positive linear functional u between an FRS (K, μ) and \mathbb{R} is a linear map $u : K \rightarrow \mathbb{R}$ such that $u(k) \in \mathbb{R}^+$ for all $k \in K^+$. The vector space K^\sim of all fuzzy order bounded linear functionals on K is said to be *fuzzy order dual* of K , i.e., $K^\sim = \mathcal{L}_b(K, \mathbb{R})$. Furthermore, $(K^\sim)^+$ is the set of all fuzzy order bounded positive linear functionals. By Theorem 1, K^\sim is a fuzzy Dedekind complete Riesz space. Furthermore, according to Theorem 1, the following fuzzy lattice operations hold for K^\sim .

Proposition 4. If K^\sim is a fuzzy order dual of an FRS (K, μ) , then $u, v \in K^\sim$ and $k \in K^+$ the following statements are true:

- (i) $u^+(k) = \sup\{u(g) : g \in K^+ \text{ and } \mu(g, k) > 1/2\}$;
- (ii) $u^-(k) = \sup\{-u(g) : g \in K^+ \text{ and } \mu(g, k) > 1/2\}$;
- (iii) $|u|(k) = \sup\{|u(g)| : \mu(|g|, k) > 1/2\}$;
- (iv) $[u \vee v](k) = \sup\{u(g) + v(h) : g, h \in K^+ \text{ and } g + h = k\}$;
- (v) $[u \wedge v](k) = \inf\{u(g) + v(h) : g, h \in K^+ \text{ and } g + h = k\}$.

Now, we discuss the FRSs whose fuzzy order dual separates the points of the spaces.

Definition 3. The fuzzy order dual K^\sim of an FRS (K, μ) separates the points of K if for all $0 \neq k \in K^+$ there exists $0 \neq u \in (K^\sim)^+$ with $u(k) \neq 0$.

Proposition 5. If K^\sim separates the points of an FRS (K, μ) , then $k \in K^+$ iff $u(k) \geq 0$ holds for all $u \in (K^\sim)^+$.

Proof. The forward implication is obvious.

Conversely, let $k \in K$ satisfies $u(k) \geq 0$ for each $u \in (K^\sim)^+$. If $u \in (K^\sim)^+$ is fixed, then Theorem 2 yields that there exists some $v \in (K^\sim)^+$ such that $u(k^-) = -v(k)$. As $v(k) \geq 0$ implies that $-v(k) \leq 0$. Thus, $u(k^-) = 0$. Therefore, K^\sim separates the points of K , we have $k^- = 0$. Hence $k = k^+ - k^- = k^+$, i.e., $\mu(0, k^+) > 1/2$. \square

In addition to the fuzzy order dual of an FRS, one can consider the fuzzy bands of fuzzy order continuous and σ -fuzzy order continuous linear functionals.

Remark 4. The set of all fuzzy order continuous linear functionals $\mathcal{L}_n(K, \mathbb{R})$ is denoted by K_n^\sim , i.e., $K_n^\sim := \mathcal{L}_n(K, \mathbb{R})$. Analogously, σ -fuzzy order continuous linear functionals is denoted by $K_c^\sim := \mathcal{L}_c(K, \mathbb{R})$. Where K_n^\sim and K_c^\sim are called fuzzy order continuous dual and σ -fuzzy order continuous dual of K , respectively. By Proposition 3 both K_n^\sim and K_c^\sim are fuzzy bands of K^\sim .

Definition 4. If $u \in K^\sim$ then:

- (i) the null fuzzy ideal of u denoted N_u and defined as $N_u := \{k \in K : |u|(|k|) = 0\}$;
- (ii) the disjoint complement of null fuzzy ideal denoted $C_u = N_u^d$ is said to be fuzzy carrier of u and is defined as $C_u := \{k \in K : k \perp N_u\}$. Note that a fuzzy carrier is indeed a fuzzy band.

One can easily prove that the null fuzzy ideal is a fuzzy band if the fuzzy order bounded linear functional is fuzzy order continuous. The following proposition shows that the two fuzzy linear functionals are disjoint if and only if their fuzzy carriers are disjoint.

Proposition 6. If (K, μ) is an Archimedean FRS then $u, v \in K_n^\sim$, then the following statements are equivalent:

- (i) $u \perp v$;
- (ii) $C_u \subseteq N_v$
- (iii) $C_v \subseteq N_u$
- (iv) $C_u \perp C_v$.

Proof. Without loss of generality, assume that positive $u, v \in K_n^\sim$.

(i) \rightarrow (ii) Let positive $c \in C_u = N_u^d$ and $\epsilon \in (0, 1)$. Since $u \wedge v = 0$, there exists a sequence (k_n) in K^+ satisfying $k_n \uparrow k$ and $u(k_n) + v(k - k_n) < \epsilon$ for each n .

Take $g_n = \bigwedge_{i=1}^n k_i$ and $g_n \downarrow 0$ in K^+ . Indeed, if $\mu(g, g_n) > 1/2$ then $\eta(u(g), u(g_n)) > 1/2$ and $u(g_n) < \epsilon$ implies $u(g) = 0$. Therefore, $g \in C_n \cap N_u = \{0\}$ implies $g = 0$.

Since $v \in (K_n^\sim)^+$, we have $v(k - g_n) \uparrow v(k)$. However,

$$v(k - g_n) = v\left(\bigvee_{i=1}^n (k - k_i)\right),$$

$$\eta\left(v\left(\bigvee_{i=1}^n (k - k_i)\right), \sum_{i=1}^n v(k - k_i)\right) > 1/2 \text{ and } \sum_{i=1}^n v(k - k_i) < \epsilon.$$

It follows that positive $v(k) \leq \epsilon$ for each $\epsilon \in (0, 1)$. Therefore, $v(k) = 0$ and hence $C_u \subseteq N_v$.

(ii) \rightarrow (iii) Since N_u is a fuzzy band, so $C_u = N_u^d$. By Theorem 5.8 in [22] $C_v = N_v^d \subseteq N_v^{dd} = N_v$.

(iii) \rightarrow (iv) $C_v \subseteq N_u$ and $N_u \perp C_u$, we have $C_v \subseteq N_u \perp C_u$ implies $C_v \perp C_u$.

(iv) \rightarrow (i) Suppose $C_v \perp C_u$. If $k = g + h \in N_v \oplus C_v$ then

$$[u \wedge v](k) = [u \wedge v](g) + [u \wedge v](h),$$

$$\eta([u \wedge v](g) + [u \wedge v](h), v(g) + u(h)) > 1/2 \text{ and } v(g) + u(h) = 0.$$

Therefore, by Theorem 4.7(ii) in [22] $u \wedge v = 0$ holds for fuzzy order dense ideal $N_v \oplus C_v$. Hence, $u \perp v$. \square

4. Fuzzy Normed Riesz Spaces

In the current section, we study the fuzzy norm in view of fuzzy ordering. Later on, we study the fuzzy topological dual, which is space of all fuzzy continuous linear functionals on a locally convex-solid fuzzy Riesz space K is a vector subspace of its fuzzy order dual.

Definition 5. Let (K, μ) be an FRS. A fuzzy norm N on K is said to be fuzzy Riesz norm if $\mu(|k|, |g|) > 1/2$ implies $N(k, t) \geq N(g, t)$ for each $k, g \in K$ and $0 < t \in \mathbb{R}$. If N is a fuzzy Riesz norm on K , then (K, N, μ) is called fuzzy normed Riesz space. If a fuzzy norm Riesz space is also fuzzy norm complete, then it is called fuzzy Banach lattice.

Example 1. Let $K = \mathbb{R}^2$ be a linear space over field \mathbb{R} , $k = (k_1, k_2) \in K$. Define $\mu : K \times K \rightarrow [0, 1]$ by

$$\mu(k, h) = \begin{cases} 1, & \text{if } k = h; \\ 3/4, & \text{if } k_1 \leq h_1 \text{ and } k_2 \leq h_2 \text{ and } k \neq h; \\ 2/3, & \text{if } k_1 < h_1 \text{ and } k_2 > h_2 \text{ and } k \neq h; \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

(K, μ) is an FRS. Fuzzy norm on K , $N : K \times \mathbb{R} \rightarrow [0, 1]$ is defined as

$$N(k, t) = \begin{cases} \frac{t^2}{(t+|k_1|)(t+|k_2|)} & \text{for } t > 0; \\ 0, & \text{for } t \leq 0, \end{cases} \quad (3)$$

Clearly, fuzzy norm is compatible with fuzzy ordering, hence K is a fuzzy normed Riesz spaces.

Example 2. Let $K = L^\infty[0, 1]$ space of all essentially bounded functions. Let m be a probability measure. Consider

$$\mu(f, g) = m(\{x \in [0, 1] : f(x) \leq g(x)\}) \quad (4)$$

for all $f, g \in K$. Then, (K, μ) is a fuzzy Riesz space. Clearly, (K, μ) with essential supremum norm is a Fuzzy normed Riesz space.

The following result shows that a closed unit ball in a fuzzy norm Riesz space is fuzzy solid.

Proposition 7. Let a fuzzy norm on an FRS (K, μ) is fuzzy Riesz norm iff its closed unit ball with radius r

$$B_N = \{k \in K, N(k, t) \geq 1 - r \forall 0 < t \in \mathbb{R}\}$$

is a fuzzy solid subset.

Proof. If N is a fuzzy Riesz norm then clearly B_N is a fuzzy solid subset.

Conversely, if B_N is a fuzzy solid and $\mu(|k|, |g|) > 1/2$ in K then

$$\mu(|\frac{1}{N(k, t) + \epsilon}k|, |\frac{1}{N(k, t) + \epsilon}g|) > 1/2$$

for all $\epsilon \in (0, 1)$. As $\frac{1}{N(k, t) + \epsilon}g \in B_N$ implies $\frac{1}{N(k, t) + \epsilon}k \in B_N$. Therefore, $N(g, t) \leq N(k, t) + \epsilon$ for all ϵ . Hence $N(g, t) \leq N(k, t)$. \square

It should be obvious that in a fuzzy norm Riesz space (K, N, μ) , $N(|k|, t) = N(k, t)$ for each $k \in K$ and $0 < t \in \mathbb{R}$. A few properties of fuzzy norm Riesz space are given in the following proposition.

Proposition 8. If (K, N, μ) is a fuzzy normed Riesz space then following statements are true:

- (i) K is an Archimedean FRS;
- (ii) the fuzzy lattice operations $(k, g) \mapsto k \vee g, (k, g) \mapsto k \wedge g, k \mapsto k^+, k \mapsto k^-$ and $k \mapsto |k|$ are fuzzy continuous from $K \times K$ (or from K , resp.) into K ;
- (iii) the positive cone is fuzzy norm closed;
- (iv) the closure of a fuzzy ideal is a fuzzy ideal;
- (v) the closure of a fuzzy Riesz subspace is a fuzzy Riesz subspace;
- (vi) every fuzzy band is closed.

Proof. (i) For $k, g \in K$ and $\mu(nk, g) > 1/2$ for each $n \in \mathbb{N}$, it follows that $\mu(nk^+, g^+) > 1/2$ and $N(g^+, t) \leq N(nk^+, t)$ for each $0 < t \in \mathbb{R}$. Thus, $k^+ = 0$. Hence, $\mu(k, 0) > 1/2$.

(ii) Suppose $k_\lambda \xrightarrow{f_n} k$ in K . Then for each $0 < t \in \mathbb{R}$

$$N(|k_\lambda| - |k|, t) \geq N(|k_\lambda - k|, t) = N(k_\lambda - k, t) = 1.$$

Therefore, $|k_\lambda| \xrightarrow{f_n} |k|$. Hence, the modulus operation is fuzzy continuous. Since all fuzzy lattice operations can be expressed by the modulus, these lattice operations are fuzzy continuous as well.

(iii) Let $K^+ = \{k, k^- = 0\}$ be the fuzzy positive cone. Therefore, it is the inverse image of fuzzy closed set $\{0\}$ with respect to fuzzy continuous map $k \mapsto k^-$.

(iv) Suppose C is a fuzzy ideal of K . Then $\mu(|k|, |g|) > 1/2$ we have $g \in \bar{C}$. Take a sequence $(g_n)_{n \in \mathbb{N}} \subseteq C$ with $g_n \xrightarrow{f_n} g$. Define $k_n^+ := k^+ \wedge |g_n|$ and $k_n^- := k^- \wedge |g_n|$ for each n , we have $k_n = k_n^+ - k_n^- \in C$. Clearly, $\mu(k_n, g_n) > 1/2$ for each n . Since C is a fuzzy ideal, we have $k_n \xrightarrow{f_n} k$. By (ii), hence $k \in \bar{C}$.

- (v) It is the immediate consequences of (ii) and (iv).
 (vi) Let $k_n \uparrow k$ in K , i.e., $k = \sup_n k_n$. Suppose C is a σ -fuzzy ideal in K and $(h_n) \subseteq C$ such that $h_n \xrightarrow{f_n} h$ in K . Let $g_n := |h_n| \wedge |h|$ we have $(g_n) \subseteq C$. By (ii) (g_n) fn-converges to $|h|$. Defining $k_n := \sup_n g_n$ in C and satisfying $\mu(g_n, k_n) > 1/2$ and $\mu(k_n, |h|) > 1/2$. Thus, we have for each $0 < t \in \mathbb{R}$

$$N(k_n - |h|, t) \geq N(g_n - |h|, t)$$

which shows that (k_n) fn-converges to $|h|$ in K . However, $k_n \uparrow |h|$. Since C is a σ -fuzzy ideal, we have $|h| \in C$ implies that $h \in C$. Hence C is closed.

□

Lemma 2. Let (K, N, μ) be a fuzzy normed Riesz space. If a net $k_\lambda \uparrow k$ and $\lim_\lambda k_\lambda = k$ then $k = \sup_\lambda k_\lambda$.

Proof. For fixed γ , take $\gamma \leq \lambda$, we have $\mu(k_\gamma, k_\lambda) > 1/2$. Since K^+ is fuzzy closed, it follows that $\mu(k_\gamma, k) > 1/2$. Thus, k is an upper bound of (k_λ) . If $g \in K$ is another upper bound, then $\mu(k_\gamma, g) > 1/2$ implies $\mu(k, g) > 1/2$. Hence $k = \sup_\lambda k_\lambda$. □

An essential result about continuity of fuzzy positive operators between fuzzy Banach lattices is given as follow.

Proposition 9. If P is a fuzzy positive operator between fuzzy Banach lattice (K, N_1, μ) to fuzzy normed Riesz space (H, N_2, η) then it is fuzzy continuous.

Proof. Let $P : K \rightarrow H$ is a fuzzy positive operator. Suppose on contrary that, P is not fuzzy continuous then it must be fuzzy unbounded. Therefore, there exists a sequence $(k_n)_{n \in \mathbb{N}}$ in K such that $k_n \downarrow 0$ satisfying $N_1(k_n, t) = 1$ and $N_2(P(k_n), t) \leq M$ for each $0 < t \in \mathbb{R}$ and $M \in (0, 1)$. Since K is fuzzy norm complete then $g := \sum_n k_n$ exists in K . Clearly, $\mu(k_n, g) > 1/2$ for each n . Thus, $\eta(P(k_n), P(g)) > 1/2$ and so

$$N_2(P(g), t) \leq N_2(P(k_n), t) \leq M$$

for each n , a contradiction. □

With the induced topology τ of fuzzy norms (K, τ) is a topological vector space. Furthermore, a topological vector space is said to be *locally convex* if it has a local base at zero consisting of convex sets. We aim to study the relationship between the fuzzy lattice structure of K under the fuzzy order μ and the topological structure of K . To construct a relation between topological dual denoted K' (set of all continuous linear functionals with respect to topology τ) and fuzzy order dual, we adopt a general approach by considering a locally convex topology τ on FRS (K, μ) generated by a family of complete fuzzy Riesz norms on K . We call the triple (K, μ, τ) *locally convex-solid fuzzy Riesz space*. Every locally convex-solid topology on a fuzzy Riesz space makes fuzzy lattice operations continuous functions.

Proposition 10. If (K, μ, τ) is a locally convex-solid fuzzy Riesz space then following statements are true:

- (i) $k \mapsto k^+$ from K to K is continuous;
- (ii) $k \mapsto k^-$ from K to K is continuous;
- (iii) $k \mapsto |k|$ from K to K is continuous;
- (iv) $(k, g) \mapsto k \vee g$ from $K \times K$ to K is continuous;
- (v) $(k, g) \mapsto k \wedge g$ from $K \times K$ to K is continuous.

Proof. It is an immediate consequence of Theorem 4.11 in [17]. □

The following result shows some essential characterization of locally convex-solid fuzzy Riesz space.

Proposition 11. *If (K, μ, τ) is a locally convex-solid fuzzy Riesz space then following statements are true:*

- (i) *the FRS K is a fuzzy Archimedean;*
- (ii) *the fuzzy positive cone K^+ is a τ -closed;*
- (iii) *τ -closure of fuzzy solid subset C of K is also a fuzzy solid;*
- (iv) *τ -closure of fuzzy Riesz subspace of an FRS is a fuzzy Riesz subspace;*
- (v) *every fuzzy band is a τ -closed;*
- (vi) *if $k_\lambda \xrightarrow{\tau} k$ in K then $k_\lambda \uparrow k$ in K ;*
- (vii) *if two nets (k_λ) and (g_λ) satisfied $\mu(k_\lambda, g_\lambda) > 1/2$ and $k_\lambda - g_\lambda \xrightarrow{\tau} 0$ then $k_\lambda \downarrow k$ iff $g_\lambda \downarrow k$.*

Proof. (i) Let $\mu(nk, g) > 1/2$ for each $n \in \mathbb{N}$ and $k, g \in K^+$. As $\mu(k, \frac{1}{n}g) > 1/2$ and $\frac{1}{n}g \xrightarrow{\tau} 0$ implies that $k = 0$.

(ii) By Proposition 10 $k \mapsto k^-$ is continuous. Therefore, $K^+ = \{k \in K : k^- = 0\}$. Hence K^+ is τ -closed.

(iii) Let C be a fuzzy solid subset of K . Then $\mu(|k|, |g|) > 1/2$ we have $g \in \overline{C}$. Take a net $(g_\lambda) \subseteq C$ with $g_\lambda \xrightarrow{\tau} g$. Define $k_\lambda := (k \wedge g_\lambda) \vee (| - g_\lambda|)$ for each λ . Clearly, $\mu(k_\lambda, g_\lambda) > 1/2$ for each λ . Therefore, $(k_\lambda) \subseteq C$ such that $k_\lambda \xrightarrow{\tau} k$. Hence, $k \in \overline{C}$.

(iv) Suppose L is a fuzzy Riesz subspace of K . Clearly, \overline{L} is a vector subspace of K . If $g \in \overline{L}$ therefore, there exists a net (g_λ) in L such that $g_\lambda \xrightarrow{\tau} g$. By Proposition 10 $(g_\lambda)^+ \subseteq L$ such that $g_\lambda^+ \xrightarrow{\tau} g^+$. Hence $g^+ \in \overline{L}$.

(v) Take a $\emptyset \neq C \subseteq K$ with disjoint complement $C^d = \{k \in K, |k| \wedge |g| = 0, \forall g \in C\}$ is τ -closed. Indeed, if $k \in \overline{C^d}$ then $k_\lambda \xrightarrow{\tau} k$ for some $(k_\lambda) \subseteq C$. Thus, $0 = |g| \wedge |k_\lambda| \xrightarrow{\tau} |g| \wedge |k|$ for each $g \in C$. Since $|k| \wedge |g| = 0$ implies that $k \in C^d$. Hence, C^d is τ -closed. Since K fuzzy Archimedean then by Theorem 5.8 in [22] every fuzzy band satisfies $C = C^{dd}$. Hence, C is τ -closed.

(vi) For fixed $\lambda, \gamma \succeq \lambda$ we have $\mu(k \vee k_\lambda - k, k \vee k_\gamma - k) > 1/2$ and $\mu(k_\gamma - k, |k_\gamma - k|) > 1/2$. Thus, $|k_\gamma - k| \xrightarrow{\tau} 0$. Therefore, $k \vee k_\lambda - k = 0$, i.e., $\mu(k_\lambda, k) > 1/2$ for each λ . Suppose there exists some $g \in K$ such that $\mu(k_\lambda, g) > 1/2$ for each λ . Thus, $g - k_\lambda \xrightarrow{\tau} g - k$ implies that $g - k \in K^+$, i.e., $\mu(k, g) > 1/2$. Hence, $k_\lambda \uparrow k$ in K .

(vii) Suppose $k_\lambda \downarrow k$ and $\mu(k, g) > 1/2, \mu(g, g_\lambda) > 1/2$ for each λ . Then $\mu((g - k_\lambda)^+, g_\lambda - k_\lambda) > 1/2$ for each λ . Thus, $(g - k_\lambda)^+ \xrightarrow{\tau} 0$. Therefore, $(g - k_\lambda)^+ \uparrow (g - k)^+ = g - k$. By (vi) we have $g - k = 0$ implies that $g_\lambda \downarrow k$. Conversely, suppose that $g_\lambda \downarrow k$ in K . Then $\mu(k - k \wedge k_\lambda, g_\lambda - k_\lambda) > 1/2$, therefore, $k - k \wedge k_\lambda \xrightarrow{\tau} 0$. So $k - k \wedge k_\lambda \uparrow$ by (vi) $k - k \wedge k_\lambda \uparrow 0$ implies that $k - k \wedge k_\lambda = 0$ for each λ , thus $\mu(k, k_\lambda) > 1/2$. Hence, $k_\lambda \downarrow k$ in K .

□

The topological dual K' of K is a vector space consisting of all fuzzy continuous linear functionals on K . The topological dual K' of a locally convex-solid fuzzy Riesz space is a fuzzy ideal in its fuzzy order dual.

Theorem 4. *If (K, μ, τ) is a locally convex-solid fuzzy Riesz space, then the topological dual K' (is fuzzy Dedekind complete in its own right) is a fuzzy ideal of the fuzzy order dual K^\sim . Moreover, if for each $u, v \in K'$ and $h \in K^+$ then*

$$[u \vee v](h) = \sup\{u(k) + v(g) : k, g \in K^+ \text{ and } k + g = h\}$$

and

$$[u \wedge v](h) = \inf\{u(k) + v(g) : k, g \in K^+ \text{ and } k + g = h\}.$$

Proof. We show that K' is a vector subspace of K^\sim . Suppose on contrary that some $u \in K'$ and u does not exist in K^\sim there exist some $k \in K^+$ and a sequence $(k_n)_{n \in \mathbb{N}} \subseteq [0, k]$ satisfying $u(k_n) \geq n$ for each n . Now $\mu(\frac{1}{n}k_n, \frac{1}{n}k) > 1/2$ and $\frac{1}{n}k \xrightarrow{f\tau} 0$. Therefore, $\lim u(\frac{1}{n}k_n) = 0$, a contradiction.

Now, we show that K' is a fuzzy ideal of K^\sim . Suppose $\psi(|u|, |v|) > 1/2$ in K^\sim with $v \in K'$. Let $k_\lambda \xrightarrow{f\tau} 0$ and $\epsilon \in (0, 1)$. The Theorem 1 yields that there exists a net (g_λ) with $\mu(g_\lambda, k_\lambda) > 1/2$ and $|v|(|k_\lambda|) \leq |v(g_\lambda)| + \epsilon$ for each λ . Clearly, $g_\lambda \xrightarrow{f\tau} 0$. Thus,

$$\psi(|u(k_\lambda)|, |u|(|k_\lambda|)) > 1/2 \text{ and } |u|(|k_\lambda|) \leq |v|(|k_\lambda|).$$

Therefore, $|u(k_\lambda)| \leq |v(g_\lambda)| + \epsilon$ then $\limsup |u(k_\lambda)| \leq \epsilon$ for each $\epsilon \in (0, 1)$. Therefore, $\limsup |u(k_\lambda)| = 0$ and $\lim u(k_\lambda) = 0$. Hence, $u \in K'$. The fuzzy lattice operations are acquired from Theorem 1. \square

5. Conclusions

In the present paper, we have defined fuzzy lattice operations on the space of all fuzzy order bounded linear operators between two fuzzy Riesz spaces to make it fuzzy Riesz space when the range is fuzzy Dedekind complete. As a particular case, we studied the separation property of fuzzy order dual. As a future research line, we plan to work on concrete norm spaces such as L^p spaces; one can also define and explore the notions of fuzzy Riesz orthomorphisms, unbounded fuzzy norm convergence and unbounded fuzzy norm topology in fuzzy Banach lattices.

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