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Article

Definite Integral of Logarithmic Functions in Terms of the Incomplete Gamma Function

Robert Reynolds *,† and Allan Stauffer †

Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, ON M3J1P3, Canada; stauffer@yorku.ca

* Correspondence: milver@my.yorku.ca; Tel.: +1-416-319-8383

† These authors contributed equally to this work.

Abstract: In this article we derive some entries and errata for the book of Gradshteyn and Ryzhik which were originally published by Bierens de Haan. We summarize our results using tables for easy reading and referencing.

Keywords: entries of Gradshteyn; Bierens de Haan; incomplete gamma function; definite integral; Cauchy's integral

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1. Introduction

In 1867, David Bierens de Haan [1] published his famous volume containing a vast number of interesting definite integrals. In this work, we focus on deriving entries from his book whose closed forms are in terms of the incomplete gamma function. We will apply our contour integral method [2] and derive a definite logarithmic integral in terms of the incomplete gamma function. Specifically, we will derive the equations given by

$$\int_0^\infty (x+1)^{-n} \log^k(a(x+1))dx \tag{1}$$

$$\int_0^1 \left(\frac{1}{x}\right)^{2-n} \log^k\left(\frac{a}{x}\right)dx \tag{2}$$

$$\int_0^1 \left(\frac{1}{x}\right)^{2-n} (a+\log(x))^{-k}dx \tag{3}$$

in terms of the incomplete gamma function where the proof is carried out in Section 5. We then use this formulae to derive several entries in [1,3] and summarize our results in Tables 1 and 2. The derivation of the definite integral follows the method used by us in [2], which involves Cauchy's generalized integral formula. The generalized Cauchy integral formula is given by

$$\frac{\Gamma(k+1)}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw = y^k \tag{4}$$

or equivalently

$$\frac{\Gamma(k+1)}{2\pi i}\int_C \frac{e^{wy}}{(yw)^{k+1}}ydw = 1 \tag{5}$$

By using the substitution $z = wy$, we see that an equality of the form

$$\frac{\Gamma(k+1)}{2\pi i}\int_C \frac{e^z}{z^{k+1}}dz = \lim_{R\to\infty}\frac{\Gamma(k+1)}{2\pi i}\int_{\omega-iR}^{\omega+iR} \frac{e^z}{z^{k+1}}dz = 1 \tag{6}$$

with $\omega > 0$ should be established. Consider the situation that $0 < k < 1$. Then by replacing C with a Hankel loop or, what amounts to the same, a keyhole curve, and by invoking Cauchy's integral theorem, the following equalities can be deduced:

$$\begin{aligned}\frac{\Gamma(k+1)}{2\pi i}\int_C \frac{e^z}{z^{k+1}}dz &= \frac{\Gamma(k+1)}{2\pi i}\int_C \frac{e^z-1}{z^{k+1}}dz \\ &= \frac{\Gamma(k+1)}{2\pi i}\int_0^\infty \frac{1-e^t}{t^{k+1}}dt(e^{ik\pi}-e^{-ik\pi}) \\ &= \frac{\Gamma(k+1)\sin(k\pi)}{\pi}\int_0^\infty \frac{1-e^{-t}}{t^{k+1}}dt.\end{aligned} \tag{7}$$

For $0 < k < 1$ and $y \geq 0$, put

$$f(k,y) = \int_0^\infty \frac{1-e^{-yt}}{t^{k+1}}dt \tag{8}$$

The, n $f(k,0) = 0$ and

$$\frac{\partial}{\partial y}f(k,y) = \int_0^\infty t^{-k}e^{-ty}dt = y^{k-1}\int_0^\infty t^{-k}e^{-t}dt = y^{k-1}\Gamma(1-k). \tag{9}$$

Hence, for $0 < k < 1$ and $y > 0$, the next equalities show up

$$f(k,y) = \int_0^\infty \frac{1-e^{-yt}}{t^{k+1}}dt = \frac{y^k}{k}\Gamma(1-k). \tag{10}$$

The equalities in (7) and (10) entail

$$\begin{aligned}&\frac{\Gamma(k+1)}{2\pi i}\int_C \frac{e^z}{z^{k+1}}dz = \frac{\Gamma(k+1)\sin(k\pi)}{\pi}\int_0^\infty \frac{1-e^{-t}}{t^{k+1}}dt \\ &= \frac{\Gamma(k+1)\sin(k\pi)}{\pi}f(k,1) = \frac{\Gamma(k+1)\sin(k\pi)}{\pi}\frac{\Gamma(1-k)}{k}\end{aligned} \tag{11}$$

applying the reflection formula for the Gamma function in combination with $\Gamma(k+1) = k\Gamma(k) = 1$. From the analyticity in k, it follows that the first expression in (11) is equal to 1 for $Re(k) > 0$. By interpreting the complex integral in this first expression as a symmetric limit, it seems that the equality is also valid for $Re(k) > -1$.

By writing, for $0 < k < 1$,

$$\begin{aligned}1 = \frac{\Gamma(k+1)}{2\pi i}\int_C \frac{e^z}{z^{k+1}}dz &= \frac{\Gamma(k+1)}{2\pi i}\int_C \frac{e^z+e^{-z}-2}{z^{k+1}}dz \\ &= \frac{\Gamma(k+1)}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^z+e^{-z}-2}{z^{k+1}}dz \\ &= \frac{2\Gamma(k+1)\sin(k\pi/2)}{\pi}\int_0^\infty \frac{1-\cos(t)}{t^{k+1}}dt.\end{aligned} \tag{12}$$

where C is, in general, an open contour in the complex plane with a bilinear concomitant [2] that has the same value at the end points of the contour and is such that this integral makes sense. More precisely, the equality in Equation (5) is valid when k belongs to $\mathbb{N} = 0, 1, 2, 3, \ldots$ and $C = C_{\omega,R}$ takes the of form $t \to \omega + it$, $-R \leq t \leq R$, where $\omega > 0$ is fixed and $R \to \infty$. If $\omega < 0$ is fixed and $R \to \infty$, then this integral vanishes. An application of Cauchy's residue calculus yields the equality in Formula (5). In item (5), some of these problems are averted by assuming that k is complex. In appropriate domains, the corresponding functions are analytic. See Equation (18) as well.

This method involves using a form of Equation (5) and then multiplying both sides by a function, then taking a definite integral of both sides. This yields a definite integral in terms of a contour integral. A second formulae for the incomplete gamma function in terms of the contour integral is derived by multiplying Equation (5) by a different function and performing some substitutions and integrating so that the contour integrals are the same.

Table 1. Table of definite integrals in work of Gradshteyn and Ryzhik.

$f(x)$	$\int_0^1 f(x)dx$	*Restrictions*
$\log(a-\log(x))$	$\log(a)+e^a\Gamma(0,a)$	$a \leq 0$
$\log(a+\log(x))$	$\log(a)+e^{-a}\Gamma(0,-a)$	$a \geq 0$
$(a+\log(x))^{-n}$	$e^{-a}(-1)^{-n}\Gamma(1-n,-a)$	$Im(a) > 0$
$\frac{1}{a+\log(x)}$	$-e^{-a}\Gamma(0,-a)$	$a \geq 0$
$\frac{1}{a-\log(x)}$	$e^a\Gamma(0,a)$	$a \leq 0$
$\frac{1}{(a+\log(x))^2}$	$e^{-a}\Gamma(-1,-a)$	$a \geq 0$
$\frac{1}{(\log(x)-a)^2}$	$e^a\Gamma(-1,a)$	$a \leq 0$
$\frac{\log(x)}{(a+\log(x))^2}$	$\frac{1}{a}+$ $e^{-a}\left(\log\left(e^{-a}\right)+1\right)\Gamma(-1,-a)$	$a \geq 0$
$\frac{\log(x)}{(a-\log(x))^2}$	$-\frac{1}{a}+ae^a\Gamma(-1,a)+e^a\Gamma(-1,a)$	$a \leq 0$
$(a-\log(x))^{-k}$	$e^a\Gamma(1-k,a)$	$a \leq 0, k > 0$
$\frac{1}{a^2+\log^2(x)}$	$\frac{ie^{-ia}\left(e^{2ia}\Gamma(0,ia)-\Gamma(0,-ia)\right)}{2a}$	$a \leq 0$
$\frac{1}{a^2-\log^2(x)}$	$-\frac{e^{-a}\Gamma(0,-a)-e^a\Gamma(0,a)}{2a}$	$a \leq 0$
$\frac{\log(x)}{a^2+\log^2(x)}$	$\frac{1}{2}\left(-e^{-ia}\Gamma(0,-ia)-e^{ia}\Gamma(0,ia)\right)$	$a \leq 0$
$\frac{\log(x)}{a^2-\log^2(x)}$	$\frac{1}{2}\left(e^{-a}\Gamma(0,-a)+e^a\Gamma(0,a)\right)$	$a \leq 0$

Table 2. Table of definite integrals in Gradshteyn and Ryzhik.

$f(x)$	$\int_0^1 f(x)dx$	*Restrictions*
$(a-\log(x))^{-k}$	$e^a\Gamma(1-k,a)$	$a = 0, k > 0$
$\frac{1}{a^2+\log^2(x)}$	$\frac{ie^{-ia}\left(e^{2ia}\Gamma(0,ia)-\Gamma(0,-ia)\right)}{2a}$	$a \leq 0$
$\frac{1}{a^2-\log^2(x)}$	$-\frac{e^{-a}\Gamma(0,-a)-e^a\Gamma(0,a)}{2a}$	$a \leq 0$
$\frac{\log(x)}{a^2+\log^2(x)}$	$\frac{1}{2}\left(-e^{-ia}\Gamma(0,-ia)-e^{ia}\Gamma(0,ia)\right)$	$a \leq 0$
$\frac{\log(x)}{a^2-\log^2(x)}$	$\frac{1}{2}\left(e^{-a}\Gamma(0,-a)+e^a\Gamma(0,a)\right)$	$a \leq 0$
$\frac{1}{\left(a^2+\log^2(x)\right)^2}$	$\frac{e^{-ia}\left((a-i)\Gamma(0,-ia)+(a+i)e^{2ia}\Gamma(0,ia)\right)}{4a^3}$	$a \leq 0$
$\frac{1}{\left(a^2-\log^2(x)\right)^2}$	$\frac{e^{-a}\left((-a-1)\Gamma(0,-a)-(a-1)e^{2a}\Gamma(0,a)\right)}{4a^3}$	$a \leq 0$
$\frac{\log(x)}{\left(a^2+\log^2(x)\right)^2}$	$-\frac{iae^{-ia}\Gamma(0,-ia)-iae^{ia}\Gamma(0,ia)+2}{4a^2}$	$a \leq 0$
$\frac{\log(x)}{\left(a^2-\log^2(x)\right)^2}$	$-\frac{-\frac{2}{a}-e^{-a}\Gamma(0,-a)+e^a\Gamma(0,a)}{4a}$	$a \leq 0$
$\frac{1}{a^4-\log^4(x)}$	$\frac{-e^{-a}\Gamma(0,-a)-ie^{-ia}\Gamma(0,-ia)+ie^{ia}\Gamma(0,ia)+e^a\Gamma(0,a)}{4a^3}$	$a \leq 0$
$\frac{\log(x)}{a^4-\log^4(x)}$	$\frac{e^{-a}\Gamma(0,-a)-e^{-ia}\left(\Gamma(0,-ia)+e^{2ia}\Gamma(0,ia)\right)+e^a\Gamma(0,a)}{4a^2}$	$a \leq 0$
$\frac{\log^2(x)}{a^4-\log^4(x)}$	$-\frac{e^{-a}\Gamma(0,-a)-ie^{-ia}\Gamma(0,-ia)+ie^{ia}\Gamma(0,ia)-e^a\Gamma(0,a)}{4a}$	$a \leq 0$
$\frac{\log^3(x)}{a^4-\log^4(x)}$	$\frac{1}{4}\left(e^{-a}\Gamma(0,-a)+e^{-ia}\Gamma(0,-ia)+e^{ia}\Gamma(0,ia)+e^a\Gamma(0,a)\right)$	$a \leq 0$

2. Definite Integral of the Contour Integral

We used the method in [2]. Using Equation (5), we replace y by $\log(a(x+1))$ and then multiply by $(x+1)^{-n}$. Next, we take the infinite integral over $x \in [0, \infty)$ to obtain

$$\begin{aligned} &\frac{1}{k!}\int_0^\infty (x+1)^{-n}\log^k(a(x+1))dx \\ &= \frac{1}{2\pi i}\int_0^\infty \int_C a^w w^{-k-1}(x+1)^{w-n}dwdx \\ &= \frac{1}{2\pi i}\int_C \int_0^\infty a^w w^{-k-1}(x+1)^{w-n}dxdw \\ &= \frac{1}{2\pi i}\int_C \frac{a^w w^{-k-1}}{n-w-1}dw \end{aligned} \tag{13}$$

where $Re(n-w) > 1$. We are able to switch the order of integration over x and w using Fubini's theorem since the integrand is of bounded measure over the space $C \times \mathbb{R}$.

3. The Incomplete Gamma Function

The incomplete gamma functions [4], $\gamma(a,z)$ and $\Gamma(a,z)$, are defined by

$$\gamma(a,z) = \int_0^z t^{a-1}e^{-t}dt \tag{14}$$

and

$$\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt \tag{15}$$

where $Re(a) > 0$. The incomplete gamma function has a recurrence relation given by

$$\gamma(a,z) + \Gamma(a,z) = \Gamma(a) \tag{16}$$

where $a \neq 0, -1, -2, \ldots$ The incomplete gamma function is continued analytically by

$$\gamma(a, ze^{2m\pi i}) = e^{2\pi m i a}\gamma(a,z) \tag{17}$$

and

$$\Gamma(a, ze^{2m\pi i}) = e^{2\pi m i a}\Gamma(a,z) + (1-e^{2\pi m i a})\Gamma(a) \tag{18}$$

where $m \in \mathbb{Z}$, $\gamma^*(a,z) = \frac{z^{-a}}{\Gamma(a)}\gamma(a,z)$ is entire in z and a. When $z \neq 0$, $\Gamma(a,z)$ is an entire function of a and $\gamma(a,z)$ is meromorphic with simple poles at $a = -n$ for $n = 0, 1, 2, \ldots$ with residue $\frac{(-1)^n}{n!}$. These definitions are listed in Section 8.2(i) and (ii) in [4].

4. Incomplete Gamma Function in Terms of the Contour Integral

In this section, we will once again use Cauchy's generalized integral formula, Equation (5), and take the infinite integral to derive equivalent sum representations for the contour integrals. We proceed using Equation (5) and replace y by $\log(a) + y$ and multiply both sides by $e^{(1-n)y}$ and simplify to obtain

$$\frac{e^{y-ny}(\log(a)+y)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C a^w w^{-k-1}e^{y(-n+w+1)}dw \tag{19}$$

Next, we take the definite integral over $y \in [0, \infty)$ and simplify it in terms of the incomplete gamma function to obtain

$$\begin{aligned}
&\frac{a^{n-1}(n-1)^{-k-1}\Gamma(k+1,(n-1)\log(a))}{\Gamma(k+1)} \\
&= \frac{1}{2\pi i}\int_0^\infty \int_C a^w w^{-k-1} e^{y(-n+w+1)} dw dy \\
&= \frac{1}{2\pi i}\int_C \int_0^\infty a^w w^{-k-1} e^{y(-n+w+1)} dy dw \\
&= -\frac{1}{2\pi i}\int_C \frac{a^w w^{-k-1}}{-n+w+1} dw
\end{aligned} \tag{20}$$

from Equation (3.351.2) in [3] where $Re(w-n+1) < 1$. Note that Equation (20) can be derived from the simple change in variable $t = (n-1)(y+\log(a))$ after integrating Equation (19) as suggested by the reviewer.

5. Definite Integral in Terms of the Incomplete Gamma Function

In this section, we derive definite integrals in terms of the incomplete gamma function. In Equations (21) and (22), we experience difficulties when k is a negative integer and a is a positive integer and $(n-1)\log(a) \leq 0$. This is also true for Equation (23) when k is a positive integer and $a(1-n) \leq 0$. In these derivations, we employ analytic continuation to extend the domain of the evaluations of the given analytic functions.

Theorem 1. *For all $k, a \in \mathbb{C}, Re(n) > 1$,*

$$\int_0^\infty (x+1)^{-n} \log^k(a(x+1)) dx = a^{n-1}(n-1)^{-k-1}\Gamma(k+1,(n-1)\log(a)) \tag{21}$$

Proof. Since the right-hand side of Equations (13) and (20) are equal, we can equate the left-hand sides to yield the stated result. □

Corollary 1.

$$\int_0^1 \left(\frac{1}{t}\right)^{2-n} \log^k\left(\frac{a}{t}\right) dt = a^{n-1}(n-1)^{-k-1}\Gamma(k+1,(n-1)\log(a)) \tag{22}$$

Proof. Use Equation (21) with the transformation $t = \frac{1}{1+x}$ and change the integration limits to $t \in [0,1]$. □

Corollary 2.

$$\int_0^1 \left(\frac{1}{t}\right)^{2-n} (a+\log(t))^{-k} dt = (-1)^{-k}\left(e^{-a}\right)^{n-1}(n-1)^{k-1}\Gamma(1-k, a-an) \tag{23}$$

Proof. Use Equation (22) and set $k = -k, a = e^{-a}$ factor and simplify the left-hand side. □

Note: the equations in Theorem 1 may also be written in the forms given by

$$\int_0^\infty \frac{(\log(a(x+1)))^k}{(x+1)^n} dx = \frac{a^n}{n^{k+1}}\Gamma(k+1, n\log(a))$$

$$\int_0^1 x^{n-1}\left(\log\frac{a}{x}\right)^k dx = \frac{a^n}{n^{k+1}}\Gamma(k+1, n\log(a))$$

$$\int_0^1 x^{n-1}\left(a+\log\frac{1}{x}\right)^{-k} dx = e^{na} n^{k-1}\Gamma(-k+1, na)$$

6. Derivation of a Few Entries in Table 4.229 in Work of Gradshteyn and Ryzhik

In this section, we will derive a few entries in terms of the incomplete gamma function.

6.1. Derivation of Entry 4.229.5 in Gradshteyn and Ryzhik

Use Equation (22) and set $a = e^{-a}$, then take the first partial derivative with to respect k, then set $k = 0, n = 2$ and simplify to obtain

$$\int_0^1 \log(a + \log(x))dx = \log(a) + e^{-a}\Gamma(0, -a) \tag{24}$$

where $a \in \mathbb{C}$.

6.2. Derivation of Entry 4.229.6 in Gradshteyn and Ryzhik

Use Equation (22) and set $a = e^{a}$, then take the first partial derivative with respect to k, then set $k = 0, n = 2$ and simplify to obtain

$$\int_0^1 \log(a - \log(x))dx = \log(a) + e^{a}\Gamma(0, a) \tag{25}$$

where $a \in \mathbb{C}$.

7. Derivation of Table 4.212 in Gradshteyn and Ryzhik

In this section, we will derive a few entries in terms of the incomplete gamma function.

7.1. Derivation of Entry 4.212.1 in Work of Gradshteyn and Ryzhik

Use Equation (23) and set $k = 1, n = 2$ and simplify to obtain

$$\int_0^1 \frac{1}{a + \log(x)}dx = -e^{-a}\Gamma(0, -a) \tag{26}$$

where $Im(a) \neq 0$. Note that when $a < 0$, this equality experiences difficulty in evaluation.

7.2. Derivation of Entry 4.212.2 in Gradshteyn and Ryzhik

Use Equation (23) and set $k = 1, n = 2, a = -a$ and simplify to obtain

$$\int_0^1 \frac{1}{a - \log(x)}dx = e^{a}\Gamma(0, a) \tag{27}$$

where $a \in \mathbb{C}$.

7.3. Derivation of Entry 4.212.3 in Gradshteyn and Ryzhik

Use Equation (23) and set $k = n = 2$ and simplify to obtain

$$\int_0^1 \frac{1}{(a + \log(x))^2}dx = e^{-a}\Gamma(-1, -a) \tag{28}$$

where $Im(a) \neq 0$.

7.4. Derivation of Entry 4.212.4 in Gradshteyn and Ryzhik

Use Equation (23) and set $k = n = 2, a = -a$ and simplify to obtain

$$\int_0^1 \frac{1}{(\log(x) - a)^2}dx = e^{a}\Gamma(-1, a) \tag{29}$$

where $a \in \mathbb{C}$. Note that when $a < 0$, this equality experiences difficulty in evaluation.

7.5. Derivation of Entry 4.212.5 in Gradshteyn and Ryzhik

Use Equation (23) and take the first partial derivative with respect to n then set $k = n = 2, a = a$ and simplify to obtain

$$\int_0^1 \frac{\log(x)}{(a+\log(x))^2}dx = \frac{1}{a} + e^{-a}\left(\log\left(e^{-a}\right)+1\right)\Gamma(-1,-a) \tag{30}$$

where $Im(a) \neq 0$.

7.6. Derivation of Entry 4.212.6 in Gradshteyn and Ryzhik

Use Equation (23) and take the first partial derivative with respect to n then set $k = n = 2, a = a$ and simplify to obtain

$$\int_0^1 \frac{\log(x)}{(a-\log(x))^2}dx = -\frac{1}{a} + ae^a\Gamma(-1,a) + e^a\Gamma(-1,a) \tag{31}$$

where $a \in \mathbb{C}$. Note that when $a < 0$, this equality experiences difficulty in evaluation.

7.7. Derivation of Entry 4.212.8 in Gradshteyn and Ryzhik

Use Equation (23) and set $n = 2, k = n$ and simplify to obtain

$$\int_0^1 (a+\log(x))^{-n}dx = e^{-a}(-1)^{-n}\Gamma(1-n,-a) \tag{32}$$

where $a \in \mathbb{C}$. Note that $(-1)^{-k} = e^{-i\pi k}$ when k is not an integer.

7.8. Derivation of Entry 4.212.9 in Gradshteyn and Ryzhik

Use Equation (23) and set $n = 2, a = -a$ and simplify to obtain

$$\int_0^1 (a-\log(x))^{-k}dx = e^a\Gamma(1-k,a) \tag{33}$$

where $k, a \in \mathbb{C}$.

8. Derivation of Entries for Table 4.213 in Work of Gradshteyn and Ryzhik

In this section, we will derive a few entries in terms of the incomplete gamma function. We will derive two integrals that will be used in this section.

We use Equation (23) and set $k = 1$ to form the first equation. Using Equation (23) and setting $k = 1, a = e^{-a}$ to form the second equation. We take the difference of these two equations to obtain

$$\int_0^1 \frac{x^{n-2}}{a^2-\log^2(x)}dx = \frac{(e^a)^{n-1}\Gamma(0,a(n-1)) - (e^{-a})^{n-1}\Gamma(0,a-an)}{2a} \tag{34}$$

and add them to obtain

$$\int_0^1 \frac{x^{n-2}\log(x)}{a^2-\log^2(x)}dx = \frac{1}{2}\left(\left(e^{-a}\right)^{n-1}\Gamma(0,a-an) + (e^a)^{n-1}\Gamma(0,a(n-1))\right) \tag{35}$$

Note that when $a \in \mathbb{R}$ this equality experiences difficulty in evaluation.

8.1. Derivation of Entry 4.213.1 in Gradshteyn and Ryzhik

Use Equation (34) and set $n = 2, a = ai$ and simplify to obtain

$$\int_0^1 \frac{1}{a^2+\log^2(x)}dx = \frac{ie^{-ia}\left(e^{2ia}\Gamma(0,ia) - \Gamma(0,-ia)\right)}{2a} \tag{36}$$

where $a \in \mathbb{C}$. Note that when a is purely imaginary this equality experiences difficulty in evaluation.

8.2. Derivation of Entry 4.213.2 in Gradshteyn and Ryzhik

Use Equation (34) and set $n = 2$ and simplify to obtain

$$\int_0^1 \frac{1}{a^2 - \log^2(x)} dx = -\frac{e^{-a}\Gamma(0, -a) - e^{a}\Gamma(0, a)}{2a} \tag{37}$$

where $a \in \mathbb{C}$. Note that when $a \in \mathbb{R}$, this equality experiences difficulty in evaluation.

8.3. Derivation of Entry 4.213.3 in Gradshteyn and Ryzhik

Use Equation (35) and set $n = 2, a = ai$ and simplify to obtain

$$\int_0^1 \frac{\log(x)}{a^2 + \log^2(x)} dx = \frac{1}{2}\left(-e^{-ia}\Gamma(0, -ia) - e^{ia}\Gamma(0, ia)\right) \tag{38}$$

where $a \in \mathbb{C}$. Note that when a is purely imaginary, this equality experiences difficulty in evaluation.

8.4. Derivation of Entry 4.213.4 in Gradshteyn and Ryzhik

Use Equation (38) and set $a = ai$ and simplify to obtain

$$\int_0^1 \frac{\log(x)}{a^2 - \log^2(x)} dx = \frac{1}{2}\left(e^{-a}\Gamma(0, -a) + e^{a}\Gamma(0, a)\right) \tag{39}$$

where $a \in \mathbb{C}$. Note that when $a \in \mathbb{R}$, this equality experiences difficulty in evaluation.

8.5. Derivation of Entry 4.213.5 in Gradshteyn and Ryzhik

Use Equation (36) and take the first partial derivative with respect to a and simplify to obtain

$$\int_0^1 \frac{1}{\left(a^2 + \log^2(x)\right)^2} dx = \frac{e^{-ia}\left((a - i)\Gamma(0, -ia) + (a + i)e^{2ia}\Gamma(0, ia)\right)}{4a^3} \tag{40}$$

where $a \in \mathbb{C}$. Note that when a is purely imaginary, this equality experiences difficulty in evaluation.

8.6. Derivation of Entry 4.213.6 in Gradshteyn and Ryzhik

Use Equation (39) and take the first partial derivative with respect to a and simplify to obtain

$$\int_0^1 \frac{1}{\left(a^2 - \log^2(x)\right)^2} dx = \frac{e^{-a}\left((-a - 1)\Gamma(0, -a) - (a - 1)e^{2a}\Gamma(0, a)\right)}{4a^3} \tag{41}$$

where $Im(a) \neq 0$. Note that when $a \in \mathbb{R}$, this equality experiences difficulty in evaluation.

8.7. Derivation of Entry 4.213.7 in Gradshteyn and Ryzhik

Use Equation (38) and take the first partial derivative with respect to a and simplify to obtain

$$\int_0^1 \frac{\log(x)}{\left(a^2 + \log^2(x)\right)^2} dx = -\frac{iae^{-ia}\Gamma(0, -ia) - iae^{ia}\Gamma(0, ia) + 2}{4a^2} \tag{42}$$

where $a \in \mathbb{C}$. Note that when a is purely imaginary, this equality experiences difficulty in evaluation.

8.8. Derivation of Entry 4.213.8 in Gradshteyn and Ryzhik

Use Equation (39) and take the first partial derivative with respect to a and simplify to obtain

$$\int_0^1 \frac{\log(x)}{\left(a^2 - \log^2(x)\right)^2} dx = -\frac{-\frac{2}{a} - e^{-a}\Gamma(0,-a) + e^a\Gamma(0,a)}{4a} \tag{43}$$

where $a \in \mathbb{C}$. Note that when $a \in \mathbb{R}$, this equality experiences difficulty in evaluation.

9. Derivation of Table 4.214 in Work of Gradshteyn and Ryzhik

9.1. Derivation of Entry 4.214.1 in Gradshteyn and Ryzhik

Use Equations (40) and (41) and simplify to obtain

$$\int_0^1 \frac{1}{a^4 - \log^4(x)} dx = \frac{-e^{-a}\Gamma(0,-a) - ie^{-ia}\Gamma(0,-ia) + ie^{ia}\Gamma(0,ia) + e^a\Gamma(0,a)}{4a^3} \tag{44}$$

where $Re(a) \neq 0, Im(a) \neq 0$.

9.2. Derivation of Entry 4.214.2 in Gradshteyn and Ryzhik

Use Equations (42) and (43) and simplify to obtain

$$\int_0^1 \frac{\log(x)}{a^4 - \log^4(x)} dx = \frac{e^{-a}\Gamma(0,-a) - e^{-ia}\left(\Gamma(0,-ia) + e^{2ia}\Gamma(0,ia)\right) + e^a\Gamma(0,a)}{4a^2} \tag{45}$$

where $Re(a) \neq 0, Im(a) \neq 0$.

9.3. Derivation of Entry 4.214.3 in Gradshteyn and Ryzhik

Use Equations (40) and (41) and take their difference and simplify to obtain

$$\int_0^1 \frac{\log^2(x)}{a^4 - \log^4(x)} dx = -\frac{e^{-a}\Gamma(0,-a) - ie^{-ia}\Gamma(0,-ia) + ie^{ia}\Gamma(0,ia) - e^a\Gamma(0,a)}{4a} \tag{46}$$

where $Re(a) \neq 0, Im(a) \neq 0$.

9.4. Derivation of Entry 4.214.4 in Gradshteyn and Ryzhik

Use Equations (42) and (43) and take their difference and simplify to get

$$\int_0^1 \frac{\log^3(x)}{a^4 - \log^4(x)} dx = \frac{1}{4}\left(e^{-a}\Gamma(0,-a) + e^{-ia}\Gamma(0,-ia) + e^{ia}\Gamma(0,ia) + e^a\Gamma(0,a)\right) \tag{47}$$

where $Re(a) \neq 0, Im(a) \neq 0$.

10. Derivation of Table 125 in Bierens de Haan in Terms of the Incomplete Gamma Function

In this section, we note that difficulties in evaluation arise when $q > 0, q \in \mathbb{R}$ or q is purely imaginary.

10.1. Derivation of Entry BI(125)(1) in Bierens de Haan

Bierens de Haan Use Equation (23) and set $k = 1, a = q, n = p + 1$ and simplify to obtain

$$\int_0^1 \frac{x^{p-1}}{q + \log(x)} dx = -\left(e^{-q}\right)^p \Gamma(0,-pq) \tag{48}$$

where $p, q \in \mathbb{C}$.

10.2. Derivation of Entry BI(125)(2) in Bierens de Haan

Use Equation (23) and set $k=1, a=-q, n=p+1$ and simplify to obtain

$$\int_0^1 \frac{x^{p-1}}{q-\log(x)}dx = (e^q)^p\Gamma(0,pq) \tag{49}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.3. Derivation of Entry BI(125)(3) in Bierens de Haan

Use Equations (48) and (49) and add together, simplifying the left-hand side to obtain

$$\int_0^1 \frac{x^{p-1}}{q^2+\log^2(x)}dx = \frac{i\left(\left(e^{iq}\right)^p\Gamma(0,ipq)-\left(e^{-iq}\right)^p\Gamma(0,-ipq)\right)}{2q} \tag{50}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.4. Derivation of Entry BI(125)(4) in Bierens de Haan

Use Equations (48) and (49) and taking their difference simplifying the left-hand side to obtain

$$\int_0^1 \frac{x^{p-1}\log(x)}{q^2+\log^2(x)}dx = \frac{1}{2}\left(\left(e^{-iq}\right)^p(-\Gamma(0,-ipq))-\left(e^{iq}\right)^p\Gamma(0,ipq)\right) \tag{51}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.5. Derivation of Entry BI(125)(5) in Bierens de Haan

Use Equation (50) and set $q = qi$ and simplify to obtain

$$\int_0^1 \frac{x^{p-1}}{q^2-\log^2(x)}dx = \frac{(e^q)^p\Gamma(0,pq)-(e^{-q})^p\Gamma(0,-pq)}{2q} \tag{52}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.6. Derivation of Entry BI(125)(6) in Bierens de Haan

Use Equation (52) and take the first partial derivative with respect to p and simplify to obtain

$$\begin{aligned}&\int_0^1 \frac{x^{p-1}\log(x)}{q^2-\log^2(x)}dx \\ &= \frac{1}{2pq}\left(e^{pq}(e^{-q})^p - e^{-pq}(e^q)^p - p\left(e^{-q}\right)^p\log\left(e^{-q}\right)\Gamma(0,-pq) + p(e^q)^p\log(e^q)\Gamma(0,pq)\right)\end{aligned} \tag{53}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.7. Derivation of Entry BI(125)(7) in Bierens de Haan

Use Equations (50) and (52) and take their difference to obtain

$$\begin{aligned}&\int_0^1 \frac{x^{p-1}}{q^4-\log^4(x)}dx \\ &= -\frac{(e^{-q})^p\Gamma(0,-pq)+i\left(e^{-iq}\right)^p\Gamma(0,-ipq)-i\left(e^{iq}\right)^p\Gamma(0,ipq)-(e^q)^p\Gamma(0,pq)}{4q^3}\end{aligned} \tag{54}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.8. Derivation of Entry BI(125)(8) in Bierens de Haan

Use Equation (54) and take the first partial derivative with respect to p and simplify to obtain

$$\begin{aligned}
&\int_0^1 \frac{x^{p-1}\log(x)}{q^4-\log^4(x)}dx \\
&= \frac{e^{pq}(e^{-q})^p}{4pq^3} + \frac{ie^{ipq}(e^{-iq})^p}{4pq^3} - \frac{ie^{-ipq}(e^{iq})^p}{4pq^3} - \frac{e^{-pq}(e^q)^p}{4pq^3} - \frac{(e^{-q})^p\log(e^{-q})\Gamma(0,-pq)}{4q^3} \\
&\quad - \frac{i(e^{-iq})^p\log(e^{-iq})\Gamma(0,-ipq)}{4q^3} + \frac{i(e^{iq})^p\log(e^{iq})\Gamma(0,ipq)}{4q^3} + \frac{(e^q)^p\log(e^q)\Gamma(0,pq)}{4q^3}
\end{aligned} \tag{55}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.9. Derivation of Entry BI(125)(9) in Bierens de Haan

Use Equations (50) and (52) and add together to obtain

$$\begin{aligned}
&\int_0^1 \frac{x^{p-1}\log^2(x)}{\log^4(x)-q^4}dx \\
&= \frac{1}{4q}\left((e^{-q})^p\Gamma(0,-pq) - i\left(e^{-iq}\right)^p\Gamma(0,-ipq) + i\left(e^{iq}\right)^p\Gamma(0,ipq) - (e^q)^p\Gamma(0,pq)\right)
\end{aligned} \tag{56}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.10. Derivation of Entry BI(125)(10) in Bierens de Haan

Use Equation (56) and take the first partial derivative with respect to p and simplify to obtain

$$\begin{aligned}
\int_0^1 \frac{x^{p-1}\log^3(x)}{q^4-\log^4(x)}dx &= \frac{e^{pq}(e^{-q})^p}{4pq} - \frac{ie^{ipq}(e^{-iq})^p}{4pq} + \frac{ie^{-ipq}(e^{iq})^p}{4pq} - \frac{e^{-pq}(e^q)^p}{4pq} \\
&\quad - \frac{(e^{-q})^p\log(e^{-q})\Gamma(0,-pq)}{4q} + \frac{i(e^{-iq})^p\log(e^{-iq})\Gamma(0,-ipq)}{4q} \\
&\quad - \frac{i(e^{iq})^p\log(e^{iq})\Gamma(0,ipq)}{4q} + \frac{(e^q)^p\log(e^q)\Gamma(0,pq)}{4q}
\end{aligned} \tag{57}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.11. Derivation of Entry BI(125)(11) in Bierens de Haan

Use Equation (23) and set $k = 2, a = q, n = p+1$ to obtain

$$\int_0^1 \frac{x^{p-1}}{(q+\log(x))^2}dx = p(e^{-q})^p\Gamma(-1,-pq) \tag{58}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.12. Derivation of Entry BI(125)(12) in Bierens de Haan

Use Equation (58) and take the first partial derivative with respect to p and simplify to obtain

$$\begin{aligned}
&\int_0^1 \frac{x^{p-1}\log(x)}{(q+\log(x))^2}dx \\
&\qquad = \frac{e^{pq}(e^{-q})^p}{pq} + (e^{-q})^p\Gamma(-1,-pq) + p(e^{-q})^p\log(e^{-q})\Gamma(-1,-pq)
\end{aligned} \tag{59}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.13. Derivation of Entry BI(125)(13) in Bierens de Haan

Use Equation (23) and set $k = 2, a = -q, n = p + 1$ to obtain

$$\int_0^1 \frac{x^{p-1}}{(q - \log(x))^2} dx = p(e^q)^p \Gamma(-1, pq) \tag{60}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.14. Derivation of Entry BI(125)(14) in Bierens de Haan

Use Equation (60) and take the first partial derivative with respect to p and simplify to obtain

$$\int_0^1 \frac{x^{p-1}\log(x)}{(q - \log(x))^2} dx = -\frac{e^{-pq}(e^q)^p}{pq} + (e^q)^p \Gamma(-1, pq) + p(e^q)^p \log(e^q) \Gamma(-1, pq) \tag{61}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.15. Derivation of Entry BI(125)(15) in Bierens de Haan

Use Equation (50) and take the first partial derivative with respect to q simplify to obtain

$$\begin{aligned}\int_0^1 \frac{x^{p-1}}{\left(q^2 + \log^2(x)\right)^2} dx = &-\frac{ie^{ipq}\left(e^{-iq}\right)^p}{4q^3} + \frac{ie^{-ipq}\left(e^{iq}\right)^p}{4q^3} + \frac{p\left(e^{-iq}\right)^p \Gamma(0, -ipq)}{4q^2} \\ &- \frac{i\left(e^{-iq}\right)^p \Gamma(0, -ipq)}{4q^3} + \frac{p\left(e^{iq}\right)^p \Gamma(0, ipq)}{4q^2} + \frac{i\left(e^{iq}\right)^p \Gamma(0, ipq)}{4q^3}\end{aligned} \tag{62}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.16. Derivation of Entry BI(125)(16) in Bierens de Haan

Use Equation (62) and take the first partial derivative with respect to p simplify to obtain

$$\begin{aligned}&\int_0^1 \frac{x^{p-1}\log(x)}{\left(q^2 + \log^2(x)\right)^2} dx \\ &= \frac{\left(e^{-iq}\right)^p E_1(-ipq)}{4q^2} + \frac{\left(e^{iq}\right)^p E_1(ipq)}{4q^2} + \frac{p\left(e^{-iq}\right)^p \log\left(e^{-iq}\right) E_1(-ipq)}{4q^2} \\ &- \frac{i\left(e^{-iq}\right)^p \log\left(e^{-iq}\right) E_1(-ipq)}{4q^3} + \frac{p\left(e^{iq}\right)^p \log\left(e^{iq}\right) E_1(ipq)}{4q^2} \\ &+ \frac{i\left(e^{iq}\right)^p \log\left(e^{iq}\right) E_1(ipq)}{4q^3} + \frac{ie^{ipq}\left(e^{-iq}\right)^p}{4pq^3} - \frac{ie^{-ipq}\left(e^{iq}\right)^p}{4pq^3} - \frac{ie^{ipq}\left(e^{-iq}\right)^p \log\left(e^{-iq}\right)}{4q^3} \\ &+ \frac{ie^{-ipq}\left(e^{iq}\right)^p \log\left(e^{iq}\right)}{4q^3}\end{aligned} \tag{63}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.17. Derivation of Entry BI(125)(17) in Bierens de Haan

Use Equation (62) and take the second partial derivative with respect to p simplify to obtain

$$\begin{aligned}
&\int_0^1 \frac{x^{p-1}\log^2(x)}{\left(q^2+\log^2(x)\right)^2}dx \\
&\quad= \frac{p\left(e^{-iq}\right)^p \log^2\left(e^{-iq}\right) E_1(-ipq)}{4q^2} - \frac{i\left(e^{-iq}\right)^p \log^2\left(e^{-iq}\right) E_1(-ipq)}{4q^3} \\
&\quad+ \frac{p\left(e^{iq}\right)^p \log^2\left(e^{iq}\right) E_1(ipq)}{4q^2} + \frac{i\left(e^{iq}\right)^p \log^2\left(e^{iq}\right) E_1(ipq)}{4q^3} \\
&\quad+ \frac{\left(e^{-iq}\right)^p \log\left(e^{-iq}\right) E_1(-ipq)}{2q^2} + \frac{\left(e^{iq}\right)^p \log\left(e^{iq}\right) E_1(ipq)}{2q^2} \\
&\quad- \frac{ie^{ipq}\left(e^{-iq}\right)^p}{4p^2q^3} + \frac{ie^{-ipq}\left(e^{iq}\right)^p}{4p^2q^3} - \frac{e^{ipq}\left(e^{-iq}\right)^p}{2pq^2} - \frac{e^{-ipq}\left(e^{iq}\right)^p}{2pq^2} \\
&\quad- \frac{ie^{ipq}\left(e^{-iq}\right)^p \log^2\left(e^{-iq}\right)}{4q^3} + \frac{ie^{-ipq}\left(e^{iq}\right)^p \log^2\left(e^{iq}\right)}{4q^3} \\
&\quad+ \frac{ie^{ipq}\left(e^{-iq}\right)^p \log\left(e^{-iq}\right)}{2pq^3} - \frac{ie^{-ipq}\left(e^{iq}\right)^p \log\left(e^{iq}\right)}{2pq^3}
\end{aligned} \tag{64}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.18. Derivation of Entry BI(125)(18) in Bierens de Haan

Use Equation (62) and set $q = qi$ simplify to obtain

$$\begin{aligned}
&\int_0^1 \frac{x^{p-1}}{\left(q^2-\log^2(x)\right)^2}dx \\
&\quad= -\frac{p(e^{-q})^p E_1(-pq)}{4q^2} - \frac{(e^{-q})^p E_1(-pq)}{4q^3} - \frac{p(e^{q})^p E_1(pq)}{4q^2} + \frac{(e^{q})^p E_1(pq)}{4q^3} \\
&\quad- \frac{e^{pq}(e^{-q})^p}{4q^3} + \frac{e^{-pq}(e^{q})^p}{4q^3}
\end{aligned} \tag{65}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.19. Derivation of Entry BI(125)(19) in Bierens de Haan

Use Equation (65) and take the first partial derivative with respect to p and simplify to obtain

$$\begin{aligned}
&\int_0^1 \frac{x^{p-1}\log(x)}{\left(q^2-\log^2(x)\right)^2}dx \\
&\quad= -\frac{(e^{-q})^p E_1(-pq)}{4q^2} - \frac{(e^{q})^p E_1(pq)}{4q^2} - \frac{p(e^{-q})^p \log(e^{-q}) E_1(-pq)}{4q^2} \\
&\quad- \frac{(e^{-q})^p \log(e^{-q}) E_1(-pq)}{4q^3} - \frac{p(e^{q})^p \log(e^{q}) E_1(pq)}{4q^2} + \frac{(e^{q})^p \log(e^{q}) E_1(pq)}{4q^3} \\
&\quad+ \frac{e^{pq}(e^{-q})^p}{4pq^3} - \frac{e^{-pq}(e^{q})^p}{4pq^3} - \frac{e^{pq}(e^{-q})^p \log(e^{-q})}{4q^3} + \frac{e^{-pq}(e^{q})^p \log(e^{q})}{4q^3}
\end{aligned} \tag{66}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.20. Derivation of Entry BI(125)(20) in Bierens de Haan

Use Equation (65) and take the second partial derivative with respect to p simplify to obtain

$$\int_0^1 \frac{x^{p-1}\log^2(x)}{\left(q^2-\log^2(x)\right)^2}dx$$
$$= -\frac{p(e^{-q})^p\log^2(e^{-q})E_1(-pq)}{4q^2} - \frac{(e^{-q})^p\log^2(e^{-q})E_1(-pq)}{4q^3} - \frac{p(e^q)^p\log^2(e^q)E_1(pq)}{4q^2}$$
$$+ \frac{(e^q)^p\log^2(e^q)E_1(pq)}{4q^3} - \frac{(e^{-q})^p\log(e^{-q})E_1(-pq)}{2q^2} - \frac{(e^q)^p\log(e^q)E_1(pq)}{2q^2}$$
$$- \frac{e^{pq}(e^{-q})^p}{4p^2q^3} + \frac{e^{-pq}(e^q)^p}{4p^2q^3} + \frac{e^{pq}(e^{-q})^p}{2pq^2} + \frac{e^{-pq}(e^q)^p}{2pq^2} - \frac{e^{pq}(e^{-q})^p\log^2(e^{-q})}{4q^3}$$
$$+ \frac{e^{-pq}(e^q)^p\log^2(e^q)}{4q^3} + \frac{e^{pq}(e^{-q})^p\log(e^{-q})}{2pq^3} - \frac{e^{-pq}(e^q)^p\log(e^q)}{2pq^3} \tag{67}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.21. Derivation of Entry BI(125)(21) in Bierens de Haan

Use Equation (23) and set $k = a, a = q, n = p + 1$ to obtain

$$\int_0^1 x^{p-1}(q+\log(x))^{-a}dx = (-1)^{-a}p^{a-1}(e^{-q})^p\Gamma(1-a, q-(p+1)q) \tag{68}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

10.22. Derivation of Entry BI(125)(22) in Bierens de Haan

Use Equation (23) and set $k = a, a = -q, n = p + 1$ to obtain

$$\int_0^1 x^{p-1}(q-\log(x))^{-a}dx = p^{a-1}(e^q)^p\Gamma(1-a, pq) \tag{69}$$

where $0 < Re(p) < 1, q \in \mathbb{C}$.

11. Discussion

In this article, we demonstrate an exercise in integration theory using our contour integral method [2] to derive definite integrals of Bierens De Haan [1] in terms of the incomplete gamma function. We were able to provide errata and extend the range of computation through analytic continuation of the incomplete gamma function. We will be applying our method to other integrals tabled in [1] to derive other known and new integral forms in terms of other special functions.

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