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New Approximations to Bond Prices in the Cox–Ingersoll–Ross Convergence Model with Dynamic Correlation

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Abstract: We study a particular case of a convergence model of interest rates. The bond prices are given as solutions of a parabolic partial differential equation and we consider different possibilities of approximating them, using approximate analytical solutions. We consider an approximation already suggested in the literature and compare it with a newly suggested one for which we derive the order of accuracy. Since the two formulae use different approaches and the resulting leading terms of the error depend on different parameter sets of the model, we propose their combination, which has a higher order of accuracy. Finally, we propose one more approach, which leads to higher accuracy of the resulting approximation formula.

Keywords: short rate model; bond price; approximate analytical solution

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1. Introduction

The time value of money is expressed through interest rates. Therefore, their modelling is necessary in all situations in which cash flows from different times are considered. Short rate models constitute one class of interest rate models. In short rate models, the instantaneous interest rate (so called short rate) is modelled as a stochastic quantity, using a stochastic differential equation or a system of such equations. Interest rates with other maturities are then obtained from the bond prices, whose prices (similarly to other interest rate derivatives) are solutions to a parabolic partial differential equation. We refer the reader to books [1,2] that deal with modelling interest rates in detail.

In particular, our interest lies in convergence models, which are used to model interest rates in a country that is going to enter into a monetary union, or in a country that has interest rates strongly influenced by rates in a different country. The system of stochastic differential equations governing the evolution of the domestic short rate therefore consists of the equation for the union rate and the equation for the domestic rate, whose drift is a function also of the union rate. The randomness is incorporated by Wiener processes, which model random shocks in the behaviour of the short rates. The increments of the Wiener processes can be correlated. The first model of this kind was proposed in [3] and extended in different ways afterwards, for example in [4] to estimate the model using noparametric techniques; in [5,6] to generalize the original system of stochastic differential equations, etc. These generalizations can include dynamic correlation, as suggested in [7]. A dynamic correlation in financial markets was studied previously in [8,9]. In this approach, the correlation between increments of Wiener processes is then modelled as a deterministic function of time. In the context of convergence models, this can be used to describe another source of convergence, when, besides the trend part of the domestic short process being influenced by the union rate, also the random shocks are getting more correlated.

Closed form solutions of the partial differential equation for the bond prices can be obtained only in some special cases. In more complicated models, the partial differential equation can be solved numerically, and the prices may be obtained by Monte Carlo simulations (using an alternative expression for the bond prices, using expected values) or approximated using approximate analytical solutions, which can be implemented in different ways (see, for example, [10–13].

We consider a particular convergence model with nonconstant volatilities and a dynamic correlation. In Section 2 we review the relevant models and the bond pricing partial differential equation, which needs to be solved. Section 3 deals with approximate analytical solutions to this PDE. We recall a known approximation formula for a more general model, which is based on the original model [3] with constant volatilities and correlation. Afterwards, we suggest approximating the desired bond price by the price in a model with the same volatilities and zero correlation. We derive its accuracy and show that the order is the same as in the case of the first approximation. However, the leading term of the error depends on different parameters of the model. This is a motivation for creating a combination of these two approximations, which has a higher accuracy. Finally, we propose one more approach leading to higher accuracy in the last part of the Section 3. Finally, in Section 4, we end the paper with concluding remarks and ideas for future research.

2. Cox–Ingersoll–Ross Model with Dynamic Correlation

The model, which we consider for the domestic short rate r_d and the short rate in the monetary union r_u (in what follows, we refer to the interest rates that influence the domestic rates as the rates in the monetary union, using, for simplicity, this concrete application of the model), is given by the following system of stochastic differential equations specified under the physical measure:

$$dr_d = (a + \kappa_d(r_u - r_d))dt + \sigma_d\sqrt{r_d}dw_d, \quad (1)$$

$$dr_u = \kappa_u(\theta - r_u)dt + \sigma_u\sqrt{r_u}dw_u, \quad (2)$$

where the increments of the Wiener processes w_d and w_u are correlated. The correlation takes the form $cor(dw_d dw_u) = \rho(t)$, where $\rho(t)$ is a deterministic function of time with values in the interval $(-1, 1)$. The parameters $\kappa_d, \kappa_u, \theta, \sigma_d, \sigma_u$ of the model are positive numbers, a is non-negative.

We make several observations about the model. The short rate in the monetary union given by (2) follows a classical Cox–Ingersoll–Ross model (CIR hereafter) from [14]. It is the mean reverting process, reverting to the mean level θ; its characteristic feature is the volatility proportional to $\sqrt{r_u}$. We note that a more general form of the model, allowing any positive power r_d^γ to take place instead of the square root, was suggested by Chan, Karolyi, Longstaff and Sanders (CKLS hereafter) in [15]. The equation for the domestic short rate r_d is an analogy to the equation from the pioneering model in [3], with a constant volatility replaced by the CIR-type multiple of the square root $\sqrt{r_d}$. The drift function describes the reverting to the current level of the union rate, with a possible minor divergence given by a. This parameter was suggested in [3], but was found to be insignificant in the statistical estimation. We kept it in the equation, as it does not provide any complication to subsequent computations and can be kept at the zero level, if desired. Originally, the correlation ρ was constant in [3]; here, we introduce a dynamic correlation, according to the motivation given in the introduction.

In order to find the bond prices, further quantities, so-called market prices of risk, $\lambda_d(r_d, r_u, t), \lambda_u(r_d, r_u, t)$ have to be specified. By analogy to the one-factor CIR model and the two-factor model with constant correlation (cf. [16,17]), we take them to be proportional to square roots of the respective short rates, i.e., $\lambda_d(r_d, r_u, t) = \lambda_d\sqrt{r_d}$ and $\lambda_u(r_d, r_u, t) = \lambda_u\sqrt{r_u}$. Then (see, for example, [18] for the derivation), denoting by $P(r_d, r_u, t)$, the price of the domestic bond at time t when the short rates are equal to r_d and r_u, is a solution to the following partial differential equation:

$$\begin{aligned} \frac{\partial P}{\partial t} + (a + \kappa_d(r_u - r_d)) - \lambda_d \sigma_d r_d) \frac{\partial P}{\partial r_d} + (\kappa_u(\theta - r_u) - \lambda_u \sigma_u r_u) \frac{\partial P}{\partial r_u} & \\ + \frac{1}{2}\sigma_d^2 r_d \frac{\partial^2 P}{\partial r_d^2} + \frac{1}{2}\sigma_u^2 r_u \frac{\partial^2 P}{\partial r_u^2} + \rho(t)\sigma_d \sigma_u \sqrt{r_d r_u} \frac{\partial^2 P}{\partial r_d \partial r_u} - r_d P &= 0 \end{aligned} \quad (3)$$

satisfied by all $r_d, r_u > 0$ and for all $t \in (0, T)$, where T is the maturity of the bond. The terminal condition is given by $P(r_d, r_u, T) = 1$ for all r. The bond prices in the monetary union are given by a closed form formula, see [14], which can be done in the CKLS framework only in the CIR case of $\gamma_u = 1/2$ and the case of a constant volatility, i.e., $\gamma_u = 0$.

We remark that using the market prices of risk, it is possible to rewrite the system of stochastic differential Equations (1) and (2) into the following so-called risk neutral form:

$$dr_d = (a_1 + a_2 r_d + a_3 r_u)dt + \sigma_d \sqrt{r_d} d\tilde{w}_d, \quad (4)$$

$$dr_u = (b_1 + b_2 r_u)dt + \sigma_u \sqrt{r_u} d\tilde{w}_u, \quad (5)$$

where $\tilde{w}_d$ and $\tilde{w}_u$ are Wiener processes in the so-called risk neutral measure, which is a probability measure equivalent to the original one in which the model is observed. The parameters are given by the following:

$$a_1 = a, a_2 = -\kappa_d - \lambda_d \sigma_d, a_3 = \kappa_d, b_1 = \kappa_u \theta, b_2 = -\kappa_u - \lambda_u \sigma_u. \quad (6)$$

The correlation structure is preserved, so $cor(d\tilde{w}_d d\tilde{w}_u) = \rho(t)$. Again, we refer the reader to more details to [18] for more details on the risk neutral measure.

Using this parametrization and the change in time variable $\tau = T - t$, the partial differential Equation (3) transforms into the equation $P(r_d, r_u, \tau)$ given by the following:

$$\begin{aligned} \frac{\partial P}{\partial t} + (a_1 + a_2 r_d + a_3 r_u) \frac{\partial P}{\partial r_d} + (b_1 + b_2 r_u) \frac{\partial P}{\partial r_u} & \\ + \frac{1}{2}\sigma_d^2 r_d \frac{\partial^2 P}{\partial r_d^2} + \frac{1}{2}\sigma_u^2 r_u \frac{\partial^2 P}{\partial r_u^2} + \rho(T - \tau)\sigma_d \sigma_u \sqrt{r_d r_u} \frac{\partial^2 P}{\partial r_d \partial r_u} - r_d P &= 0 \end{aligned} \quad (7)$$

for $r_d, r_u > 0$ and $\tau \in (0, T)$ with initial condition $P(r_d, r_u, 0) = 1$ for $r_d, r_u > 0$. This is the equation for which we are trying to find an analytical approximation solution.

3. Approximating the Bond Prices in the Cox–Ingersoll–Ross Convergence Model with Dynamic Correlation

In this section, we present our results concerning approximate analytical solutions of Equation (7). The first subsection uses a known result concerning a more general convergence model. The approximation formula is based on the convergence model with constant volatilities and a constant nonzero correlation. Therefore, it might be seen as an approximation disregarding the nonconstant nature of volatility but taking the correlation into account. The second subsection uses a different approach, which can be seen as a complement. We suggest to approximate the bond price by the solution to the bond pricing PDE for the model with the same CIR-type nonconstant volatilities but with zero correlation. We derive the order of accuracy for this approximation. Since the orders of accuracy for these two approximations turn out to be the same, we are able to combine them and obtain an approximation with a higher order of accuracy. In the last subsection, we propose another approach for increasing the accuracy of the approximation and derive one more formula, approximating the bond prices.

We note that all of these approximations have the order of accuracy given in the form of the difference between logarithms of the exact and approximate bond price. It is derived that this difference is $O(\tau^k)$ for a certain natural number k as $\tau \to 0^+$. It means that the difference is guaranteed to be small for small times to maturity τ. However, it is often the case that approximations of this kind provide a very accurate approximation for τ up to several years; see, for example, ref. [7] for a numerical comparison in the case of a convergence model with constant volatilities and dynamic correlation.

3.1. A Known Approximation for a CKLS-Type Convergence Model

In [7] a dynamic correlation was introduced into a convergence model with non-constant CKLS-type volatilities, i.e., the domestic short rate r_d and the European (or, in general, the leading) short rate r_u follow the system of stochastic differential equations in the following risk-neutral measure:

$$dr_d = (a_1 + a_2 r_d + a_3 r_u)dt + \sigma_d r_d^{\gamma_d} dw_d, \quad (8)$$

$$dr_u = (b_1 + b_2 r_u)dt + \sigma_u r_u^{\gamma_u} dw_u, \quad (9)$$

where the correlation between increments of the Wiener processes w_d, w_u is a deterministic function of time $\rho(t)$. The paper proposes an approximate analytical solution to the bond prices, which is based on the exact solution in case of constant volatilities (i.e., $\gamma_d = \gamma_u = 0$) and a constant correlation ρ, which was derived in the original pioneering paper [3] by Corzo and Schwartz. The bond price $P^{cs}(r_d, r_u, \tau)$ in the Corzo and Schwartz model has the following form:

$$\log P^{cs}(r_d, r_u, \tau) = A^{cs}(\tau) - D^{cs}(\tau) r_d - U^{cs}(\tau) r_u, \quad (10)$$

where, in the generic case $a_2 \neq b_2$, the functions A^{cs}, D^{cs}, U^{cs} are given by the following:

$$D^{cs}(\tau) = \frac{e^{a_2\tau} - 1}{a_2}, U^{cs}(\tau) = \frac{a_3\left(-a_2(e^{b_2\tau} - 1) + b_2(e^{a_2\tau} - 1)\right)}{a_2 b_2 (a_2 - b_2)}, \quad (11)$$

$$A^{cs}(\tau) = \int_0^\tau -a_1 D(s) - b_1 U(s) + \frac{\sigma_d^2}{2} D^2(s) + \frac{\sigma_u^2}{2} U^2(s) + \rho\sigma_d\sigma_u D(s)U(s) ds. \quad (12)$$

The function A^{cs} can be written in the close form, too. We omit evaluating the integral for the sake of brevity.

The approximation proposed in [7] is given by substituting instantaneous volatilities $\sigma_d r_d^{\gamma_d}$ and $\sigma_u r_u^{\gamma_u}$ in place of constant volatilities in the model and the correlation at the time of bond maturity $\rho(T)$, in place of the constant correlation, i.e., the following:

$$\sigma_d^2 \mapsto \sigma_d^2 r_d^{2\gamma_d}, \ \sigma_u^2 \mapsto \sigma_u^2 r_u^{2\gamma_u}, \ \rho \mapsto \rho(T), \quad (13)$$

in the solution from the model by Corzo and Schwartz, given by (10)–(12).

It was shown in [7] that if we denote the exact solution of the bond price in the CKLS model with dynamic correlation by $P^{ex,ckls}$ and the proposed approximation by $P^{ap,ckls}$, then the difference of their logarithms is of the order τ^4 as $\tau \to 0^+$. In particular, we have (cf. [7]) the following:

$$\begin{aligned} \log P^{ap,ckls}(r_d, r_u, \tau) - \log P^{ex,ckls}(r_d, r_u, \tau) &= -\frac{1}{24}\sigma_d^2 \gamma_d r_d^{2\gamma_d - 1} \Big[2(a_1 + a_2 r_d + a_3 r_u) \\ &\quad - r_d^{2\gamma_d - 1}\sigma_d^2(2\gamma_d - 1)\Big]\tau^4 + O(\tau^5) \end{aligned} \quad (14)$$

as $\tau \to 0^+$. It is worth noting that the accuracy formula has the leading term that does not depend on the correlation function ρ.

In our case of the CIR model, when $\gamma_d = 1/2$, $\gamma_u = 1/2$, we, therefore, have the following theorem.

Theorem 1. *Let $P^{ex}(r_d, r_u, \tau)$ be the exact solution of the partial differential Equation (7) and $P^{ap,cir}(r_d, r_u, \tau)$ be the approximation obtained by substituting the following:*

$$\sigma_d^2 \mapsto \sigma_d^2 r_d,\ \sigma_u^2 \mapsto \sigma_u^2 r_u,\ \rho \mapsto \rho(T) \tag{15}$$

in (10)–(12). Then,

$$\log P^{ap,cir}(r_d, r_u, \tau) - \log P^{ex,cir}(r_d, r_u, \tau) = -\frac{1}{24}\sigma_d^2(a_1 + a_2 r_d + a_3 r_u)\tau^4 + O(\tau^5) \tag{16}$$

as $\tau \to 0^+$.

3.2. Approximation by Using the Cox–Ingersoll–Ross Model with Zero Correlation

As outlined at the beginning of this section, we propose to approximate the solution of Equation (7) by the solution obtained for $\rho = 0$. In this case, it is known that the solution has a separated form and its calculation can be transformed into solving a system of ordinary differential equations, cf. [16,17]. Unlike the approximation from the previous subsection, this approximation is not a closed form formula. However, a numerical solution of the resulting system can be easily performed, using a selected numerical method. This is a significantly easier problem from a numerical point of view, compared to solving the partial differential equation for the original case of a dynamic correlation. We use this special form of the solution for the zero correlation case in the proof of the following theorem, giving the accuracy of this approximation.

Theorem 2. *Let $P^{ex}(r_d, r_u, \tau)$ be the solution of the partial differential Equation (7), and let $P^{0,cir}(r_d, r_u, \tau)$ be the solution of the same equation with ρ identically equal to zero. Then,*

$$\log P^{0,cir}(r_d, r_u, \tau) - \log P^{ex}(r_d, r_u, \tau) = -\frac{1}{8}a_3\sigma_d\sigma_u\sqrt{r_d r_u}\rho(T)\tau^4 + O(\tau^5) \tag{17}$$

as $\tau \to 0^+$.

Proof. Denoting the logarithm of the exact bond price by $f(\tau, r_d, r_u)$, we can write the partial differential equation, which it satisfies, in the following form:

$$\begin{aligned} -\frac{\partial f}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_u)\frac{\partial f}{\partial r_d} + (b_1 + b_2 r_u)\frac{\partial f}{\partial r_u} + \frac{\sigma_d^2 r_d}{2}\left[\frac{\partial^2 f}{\partial r_d^2} + \left(\frac{\partial f}{\partial r_d}\right)^2\right] & \\ + \frac{\sigma_u^2 r_u}{2}\left[\frac{\partial^2 f}{\partial r_u^2} + \left(\frac{\partial f}{\partial r_u}\right)^2\right] + \rho(T-\tau)\sigma_d\sigma_u\sqrt{r_d r_u}\left[\frac{\partial f}{\partial r_d}\frac{\partial f}{\partial r_u} + \frac{\partial^2 f}{\partial r_d \partial r_u}\right] - r_d &= 0 \end{aligned} \tag{18}$$

for $\tau \in (0, T)$, where T is the maturity of the bond, and for $r_d, r_u > 0$. If we substitute the logarithm of the approximate solution denoted by $f^{ap} = \log P^{0,cir}$ to the left hand side of (18), we obtain a nontrivial right hand side, which we denote by $h(r_d, r_u, \tau)$. As can be verified by substitution into the PDR, the function f^{ap} has the following form:

$$f^{ap}(r_d, r_u, \tau) = A(\tau) - D(\tau)r_d - U(\tau)r_u$$

and the functions $A(\tau), D(\tau), U(\tau)$ satisfy the following system of ordinary differential equations:

$$\dot{D}(\tau) = 1 + a_2 D(\tau) - \frac{1}{2}\sigma_d^2 D^2(\tau), \tag{19}$$

$$\dot{U}(\tau) = a_3 D(\tau) + b_2 U(\tau) - \frac{1}{2}\sigma_u^2 U^2(\tau), \tag{20}$$

$$\dot{A}(\tau) = -a_1 D(\tau) - b_1 U(\tau), \tag{21}$$

with initial conditions $A(0) = D(0) = U(0) = 0$, cf. [16,17]. Using the system (19)–(21) we can write the following:

$$D(\tau) = \tau + O(\tau^2), U(\tau) = \frac{1}{2}a_3\tau^2 + O(\tau^3), A(\tau) = -\frac{1}{2}a_1\tau^2 + O(\tau^3). \tag{22}$$

Using the leading terms of the functions $D(\tau)$, $U(\tau)$, $A(\tau)$ given by (22) together with the following:

$$\rho(T-\tau) = \rho(T) + O(\tau) \tag{23}$$

we obtain the following expression for the function h:

$$h(r_d, r_u, \tau) = \frac{1}{2}a_3\sigma_d\sigma_u\sqrt{r_d r_u}\rho(T)\tau^3 + O(\tau^4). \tag{24}$$

Introducing the function $g = f^{ap} - f^{ex}$, we can write the following PDE, which it satisfies:

$$\begin{aligned}
&-\frac{\partial g}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_u)\frac{\partial g}{\partial r_d} + (b_1 + b_2 r_u)\frac{\partial g}{\partial r_u} + \frac{1}{2}\sigma_d^2 r_d\left[\left(\frac{\partial g}{\partial r_d}\right)^2 + \frac{\partial^2 g}{\partial r_d^2}\right] \\
&\quad + \frac{1}{2}\sigma_u^2 r_u\left[\left(\frac{\partial g}{\partial r_u}\right)^2 + \frac{\partial^2 g}{\partial r_u^2}\right] + \rho(T-\tau)\sigma_d\sigma_u\sqrt{r_d r_u}\left(\frac{\partial g}{\partial r_d}\frac{\partial g}{\partial r_u} + \frac{\partial^2 g}{\partial r_u \partial r_d}\right) \\
&= h(r_d, r_u, \tau) + \sigma_d^2 r_d\left[\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_d}\right] + \sigma_u^2 r_u\left[\left(\frac{\partial f^{ex}}{\partial r_u}\right)^2 - \frac{\partial f^{ex}}{\partial r_u}\frac{\partial f^{ap}}{\partial r_u}\right] \\
&\quad + \rho(T-\tau)\left(2\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_u} - \frac{\partial f^{ex}}{\partial r_u}\frac{\partial f^{ap}}{\partial r_d} - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_u}\right) \qquad (25) \\
&= h(r_d, r_u, \tau) + \sigma_d^2 r_d\left[\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 + \frac{\partial f^{ex}}{\partial r_d}D\right] + \sigma_u^2 r_u\left[\left(\frac{\partial f^{ex}}{\partial r_u}\right)^2 + \frac{\partial f^{ex}}{\partial r_u}U\right] \\
&\quad + \rho(T-\tau)\left(2\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_u} + \frac{\partial f^{ex}}{\partial r_u}D + \frac{\partial f^{ex}}{\partial r_d}U\right). \qquad (26)
\end{aligned}$$

Since we are interested in the order of the function g as $\tau \to 0^+$, we write it in the form $g(r_d, r_u, \tau) = \sum_{i=\omega}^{\infty} c_k(r_d, r_u)\tau^\omega$. Logarithms of both the exact and approximate bond prices have zero value; therefore, $g(r_d, r_u, 0) = 0$ and $\omega \geq 1$. The lowest order term on the left hand side of (26) is, therefore, $(-\omega)c_\omega(r_d, r_u)\tau^{\omega-1}$ and we match it with the lowest order term on the right hand side.

Since $f^{ex}(r_d, r_u, \tau) = O(\tau)$, also its partial derivatives with respect to r_d and r_u are $O(\tau)$. Together with (22)–(24), we see that the right hand side of Equation (26) is at least of the order $O(\tau^2)$. It follows that $\omega \geq 3$. This implies that $f^{ex}(r_d, r_u, \tau) = f^{ap}(r_d, r_u, \tau) + O(\tau^3)$ and consequently, $\partial f^{ex}/\partial r = \partial f^{ex}/\partial r + O(\tau^3)$ for $r = r_d$ and $r = r_u$. Using the orders of the functions D and U given by (22), we have the following estimates for the terms on the right hand side of (26):

$$\begin{aligned}
\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 + \frac{\partial f^{ex}}{\partial r_d}D &= D^2 + O(\tau^4) + (-D + O(\tau^3))D = O(\tau^4), \\
\left(\frac{\partial f^{ex}}{\partial r_u}\right)^2 - \frac{\partial f^{ex}}{\partial r_u}U &= (U^2 + O(\tau^5) - (-U + O(\tau^3))(-U) = O(\tau^5), \\
2\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_u} + \frac{\partial f^{ex}}{\partial r_u}D + \frac{\partial f^{ex}}{\partial r_d}U &= 2(-D + O(\tau^3))(-U + O(\tau^3)) + (-U + O(\tau^3))D \\
&\quad + (-D + O(\tau^3))U = O(\tau^4).
\end{aligned}$$

Recalling (23), we can, therefore, conclude that the right hand side of Equation (26) has the order $O(\tau^3)$, and the only $O(\tau^3)$ term comes from the expansion of the function h given by (24). Finally, it means that $\omega = 4$ and

$$-4c_4(r_d, r_u, \tau) = \frac{1}{2} a_3 \sigma_d \sigma_u \sqrt{r_d r_u} \rho(T),$$

from which the claim of the theorem follows. □

3.3. Combination of the Approximations

Let us note the different features of the approximations from the previous subsections and their different leading terms in error estimates.

The first approximation keeps some of the information about correlation (instead of dynamic correlation it assumes it being constant) and disregards the nonconstant behaviour of the volatilities (they are taken to be constant, equal to their present value). The leading term in the error is a multiple of the risk neutral drift of the domestic short rate; the constant depends on its volatility parameter.

On the other hand, the latter approximation keeps the volatilities unchanged but disregards the correlation completely (it is taken to be zero). The leading term of the error is a multiple of the product of the instantaneous volatilities of both domestic and union short rates. The constant depends on the correlation at maturity and one of the coefficients from the risk neutral drift of the domestic short rate. We can also notice that in the usual case of positive correlation, the leading term is positive (recall the transformation of the parameters (6) and Equation (1) in the original probability measure).

This different character of the approximations motivates us to use both of them by creating their suitable combination. Recall that we have the following:

$$\begin{aligned} \log P^{ap,cir}(r_d, r_u, \tau) - \log P^{ex}(r_d, r_u, \tau) &= \tilde{c}(r_d, r_u)\tau^4 + O(\tau^5), \\ \log P^{0,cir}(r_d, r_u, \tau) - \log P^{ex}(r_d, r_u, \tau) &= c(r_d, r_u)\tau^4 + O(\tau^5) \end{aligned}$$

with

$$\tilde{c}(r_d, r_u) = -\frac{1}{24}\sigma_d^2(a_1 + a_2 r_d + a_3 r_u), c(r_d, r_u) = -\frac{1}{8} a_3 \sigma_d \sigma_u \sqrt{r_d r_u} \rho(T).$$

Therefore, the following theorem holds.

Theorem 3. *Let $P^{ex}(r_d, r_u, \tau)$ be the solution of the partial differential Equation (7) and let $P^{ap,new}(r_d, r_u, \tau)$ be a new approximation given by the following:*

$$\log P^{ap,new}(r_d, r_u, \tau) = \alpha(r_d, r_u) \log P^{ap,cir}(r_d, r_u, \tau) + (1 - \alpha(r_d, r_u)) \log P^{0,cir}(r_d, r_u, \tau)$$

with

$$\alpha(r_d, r_u) = \frac{c(r_d, r_u)}{c(r_d, r_u) - \tilde{c}(r_d, r_u)} = \frac{3a_3 \sigma_d \sigma_u \sqrt{r_d r_u} \rho(T)}{3a_3 \sigma_d \sigma_u \sqrt{r_d r_u} \rho(T) - \sigma_d^2(a_1 + a_2 r_d + a_3 r_u)},$$

where $P^{ap,cir}$ is the approximation from Theorem 1 and $P^{0,cir}$ is the approximation from Theorem 2. Then,

$$P^{ap,new}(r_d, r_u, \tau) - \log P^{ex}(r_d, r_u, \tau) = O(\tau^5).$$

The new approximation uses information from both approaches and it has a higher order of accuracy. The weights are not necessarily from the interval $(0, 1)$. We remark that this property is, however, guaranteed if the risk neutral drift of the domestic short rate is negative and the correlation at the bond's maturity is positive. In such a case, the leading terms of the errors have opposite signs, and, as expected, the new improved approximation lies between them. In general, this does not need to be the case.

3.4. A Different Substitution of Parameters in the Constant Volatility Model

In this subsection, we provide a different approach to increasing the accuracy of the approximation from Section 3.1. Instead of combining it with a different approximation as we did in the previous subsection, we change the values, which are substituted into the closed form solution for the bond price in the model with constant volatilities. Recall that the approximation from Section 3.1 consists of substituting the terms in the solution by Corzo and Schwartz, according to (15). Now, we look for a modified substitution of the following form:

$$\sigma_d^2 \mapsto \sigma_d^2 r_d + d_1(r_d, r_u)\tau,\ \sigma_u^2 \mapsto \sigma_u^2 r_u,\ \rho \mapsto \rho(T), \tag{27}$$

where our choice of the function $d_1(r_d, r_u)$ leads to an approximation formula with a higher order of accuracy. Let us remark that our motivation for altering the σ_d^2 term comes from the fact that in computation of the function $h(r_d, r_u, \tau)$, it is this term that gets multiplied by functions of the lowest order.

In order to see that the function h is indeed the crucial term of (25), when computing the order of accuracy, we show that the remaining terms on the right hand never enter the matching of the right hand side with the $O(\tau^{\omega-1})$ term of the left hand side when $g(r_d, r_u, \tau) = \sum_{k=\omega}^{\infty} c_k \tau^k$. We saw that this was the case when the approximation was given by the solution to the model with zero correlation, when the terms $\partial f^{ap}/\partial r_d$ and $\partial f^{ap}/\partial r_u$ were equal to functions characterized by ordinary differential equation, which allowed us to determine their order. In general, we can write the following:

$$\begin{aligned}
\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_d} &= -\frac{\partial f^{ex}}{\partial r_d}\left(\frac{\partial f^{ap}}{\partial r_d} - \frac{\partial f^{ex}}{\partial r_d}\right) = -\frac{\partial f^{ex}}{\partial r_d}\frac{\partial g}{\partial r_d} \\
&= O(\tau)O(\tau^{\omega}) = O(\tau^{\omega+1}), \\
\left(\frac{\partial f^{ex}}{\partial r_u}\right)^2 - \frac{\partial f^{ex}}{\partial r_u}\frac{\partial f^{ap}}{\partial r_u} &= -\frac{\partial f^{ex}}{\partial r_u}\left(\frac{\partial f^{ap}}{\partial r_u} - \frac{\partial f^{ex}}{\partial r_u}\right) = -\frac{\partial f^{ex}}{\partial r_u}\frac{\partial g}{\partial r_u} \\
&= O(\tau)O(\tau^{\omega}) = O(\tau^{\omega+1}), \\
2\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_u} - \frac{\partial f^{ex}}{\partial r_u}\frac{\partial f^{ap}}{\partial r_d} - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_u} &= -\frac{\partial f^{ex}}{\partial r_u}\left(\frac{\partial f^{ap}}{\partial r_d} - \frac{\partial f^{ex}}{\partial r_d}\right) - \frac{\partial f^{ex}}{\partial r_d}\left(\frac{\partial f^{ap}}{\partial r_u} - \frac{\partial f^{ex}}{\partial r_u}\right) \\
&= -\frac{\partial f^{ex}}{\partial r_u}\frac{\partial g}{\partial r_d} - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial g}{\partial r_u} \\
&= O(\tau)O(\tau^{\omega}) + O(\tau)O(\tau^{\omega}) = O(\tau^{\omega+1}).
\end{aligned}$$

Since the terms multiplying the expressions above in (25) are $O(1)$, being either independent of τ or an $O(1)$ function $\rho(T-\tau)$, their products cannot be matched with $O(\tau^{\omega-1})$ term on the left hand side. Therefore, the term being matched is $h(r_d, r_u, \tau)$, and if this function is $O(\tau^{\mu})$, then $\omega = \mu + 1$.

Now, using (27) and substituting the resulting f^{ap} into the left hand side of (18) in place of f leads to the function $h(r_d, r_u, \tau)$ given in the closed form, which can be expanded as the following:

$$h(r_d, r_u, \tau) = \frac{1}{6}\Big(-(a_1 + a_2 r_d + a_3 r_u)\sigma_d^2 + 4d_1(r_d, r_u)\Big)\tau^3 + O(\tau^4).$$

Therefore, in order to cancel the $O(\tau^3)$ term, we need to take the following:

$$d_1(r_d, r_u) = \frac{\sigma_d^2}{4}(a_1 + a_2 r_d + a_3 r_u) \tag{28}$$

in the substitution (27). With this choice, we obtain $h(r_d, r_u, \tau) = O(\tau^4)$ and therefore, the approximation f^{ap} satisfies $f^{ap} - f^{ex} = O(\tau^5)$.

We can summarize the computations in this subsection in the following theorem.

Theorem 4. *Let $P^{ex}(r_d, r_u, \tau)$ be the exact solution of the partial differential Equation (7), and let $P^{ap}(r_d, r_u, \tau)$ be the approximation obtained by substituting the following:*

$$\sigma_d^2 \mapsto \sigma_d^2 r_d + \frac{\sigma_d^2}{4}(a_1 + a_2 r_d + a_3 r_u)\tau,\ \sigma_u^2 \mapsto \sigma_u^2 r_u,\ \rho \mapsto \rho(T) \tag{29}$$

in (10)–(12). Then

$$\log P^{ap}(r_d, r_u, \tau) - \log P^{ex,cir}(r_d, r_u, \tau) = O(\tau^5) \tag{30}$$

as $\tau \to 0^+$.

4. Conclusions

In this paper, we considered several ways of approximating the bond prices in a Cox–Ingersoll–Ross model convergence model with dynamic correlation. A closed form approximation formula, which was already suggested, is complemented by an alternative method with the same order of accuracy, requiring a numerical solution of a system of ordinary differential equations. This is not a closed form approximation but it is still a numerically uncomplicated problem. Moreover, it uses different information from the original model and leads to an error, which has the leading term depending on different set of parameters. These two approximations can be combined to form yet another approximation, which has a higher order of precision. An alternative way of obtaining a higher order of accuracy is modifying the substitution made in the original formula.

In the future, it might be useful to look for simple approximation formulae, also in the case of other, more complicated models. The model considered in the paper assumes a dynamic correlation, while the other parameters of the model are kept constant. This is not necessarily true in reality, which leads to models with time-dependent parameters. They may be dynamic (given by a deterministic function of time) or stochastic (governed by a stochastic differential equation). We mention, for example, a cyclical model [19], with the volatility and equilibrium short rate modelled as periodic functions of time. A review of stochastic volatility models for interest rates can be found in [20]. In addition, Wiener processes may not be sufficient to capture the stochastic character of the interest rates. A popular class of models uses Lévy processes to model jumps; we refer the reader to [21–23] for examples of interest rate models of this form.

A second path of future research is the experimental part of the study concerned with approximation formulae, which includes an analysis of real data. A simple approximate solution of the bond pricing equation is useful in many applications, where an evaluation of the bond prices is necessary. We plan to focus our future work on using these approximations in inverse problems, similar to those in [24–26]. In particular, we are interested in estimating the implied correlation from the market data. This will bring interesting information about the nature of convergence of interest rates.

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