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On Some Reversible Cubic Systems

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Abstract: We study three systems from the classification of cubic reversible systems given by Żołądek in 1994. Using affine transformations and elimination algorithms from these three families the six components of the center variety are derived and limit-cycle bifurcations in neighborhoods of the components are investigated. The invariance of the systems with respect to the generalized involutions introduced by Bastos, Buzzi and Torregrosa in 2021 is discussed. Computations are performed using the computer algebra systems MATHEMATICA and SINGULAR.

Keywords: centers; reversibility; cubic systems; involution; limit cycles

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1. Introduction

One of the long-standing problems in the theory of polynomial differential equations is the Poincaré center problem, which involves finding for which values of parameters α_{pq}, β_{pq} a given polynomial differential system of the form

$$\begin{aligned}\dot{x} &= -y + \sum_{p+q=2}^{n} \alpha_{pq} x^p y^q, \\ \dot{y} &= x + \sum_{p+q=2}^{n} \beta_{pq} x^p y^q\end{aligned} \tag{1}$$

has a center at the origin. A general approach to its study was proposed by Poincaré and Lyapunov [1,2]; however, this relies on checking an infinite number of conditions, which is difficult to verify in practice.

The problem has been studied for some fixed values of the degree n for more than a century by many authors. The only family completely investigated is the quadratic one ($n = 2$) [3–8]. Some partial results have been obtained for the cubic family (when in (1) $n = 3$), see e.g., [9–16] and references given there; however, it appears that the center problem for the cubic system is still far from resolved. Some partial classifications have also been obtained for higher-degree families; in particular, for systems in the form of a linear center perturbed by homogeneous polynomials of degree 4 and 5 [17,18].

By the Poincaré–Lyapunov theorem, the existence of a center at the origin of system (1) is equivalent to the existence of an analytic first integral of the form

$$\Phi(x,y) = x^2 + y^2 + \sum_{j+k>2} \phi_{j-1,k-1} x^j y^k \tag{2}$$

in a neighborhood of the origin. For system (1), one can always find a function of the form (2) such that

$$\dot{\Phi} = \frac{\partial \Phi}{\partial x}\dot{x} + \frac{\partial \Phi}{\partial y}\dot{y} = \sum_{k\geq 1} v_k\,(x^2+y^2)^{k+1} \tag{3}$$

where v_k are polynomials in the coefficients α_{pq}, β_{pq} of polynomials P and Q, called the *focus quantities* of the system (1). We denote the list of coefficients of the first equation of (1) by α and the list of coefficients of the second equation by β, so the polynomials v_k are polynomials in the variables (α, β), $v_k = v_k(\alpha, \beta)$, in the polynomial ring $\mathbb{R}[\alpha, \beta]$.

Clearly, system (1), with fixed coefficients (α^*, β^*), has a first integral (2) and, therefore, a center at the origin if and only if

$$v_k(\alpha^*, \beta^*) = 0 \quad \forall\, k \in \mathbb{N},$$

That is, (α^*, β^*) belongs to the variety $\mathbf{V}(\mathcal{V})$ of the ideal

$$\mathcal{V} = \langle v_1, v_2, v_3, \ldots \rangle.$$

The variety $\mathbf{V}(\mathcal{V})$ is called the center variety of system (1).

In studying the center problem for a given polynomial family of the form (1), one usually computes a few first focus quantities $v_1(\alpha,\beta), \ldots,\ v_m(\alpha,\beta)$ of the system, then finds the irreducible decomposition of the variety $\mathbf{V}(\mathcal{V}_m)$ of the ideal

$$\mathcal{V}_m = \langle v_1, v_2, v_3, \ldots v_m \rangle$$

and then for each component of the decomposition proves the existence of a local analytic first integral. In some sense the center problem for system (1) will be solved if all possible mechanisms of local integrability in systems (1) can be established, and algorithmic procedures allowing the proving of integrability for systems corresponding to the components of irreducible decomposition of the center variety would be proposed.

Up to now two main known mechanisms yielding integrability in polynomial systems are the Darboux integrability [19–24] and time reversibility [25–28]. In 1994, in the renowned work [29], Żołądek presented a classification of reversible centers of a cubic system. He gave 17 families of systems that are time reversible with respect to some rational transformations, and such that among systems of the families there are some that have a center. The reversible families of [29] contain systems with a center and systems that do not have a center, so the relation of the classification to the center variety is not discussed in [29].

This relation was investigated in [30,31], where the authors were looking for an affine transformation

$$\psi(x,y) = \left(\frac{-a_0 + x}{a_1}, \frac{b_1(a_0 - x) + a_1(-b_0 + y)}{a_1 b_2}\right) \tag{4}$$

whose inverse is

$$\psi(x,y)^{-1} = (a_0 + a_1 x, b_0 + b_1 x + b_2 y) \tag{5}$$

which brings the systems from Żołądek's families to the canonical forms

$$\begin{aligned} \dot{x} =& (1+Gx)(y + Hx^2 + Dxy + Ry^2), \\ \dot{y} =& -x + Ax^2 + 3Bxy + Cy^2 + Kx^3 + 3Lx^2y + Mxy^2 + Ny^3 \end{aligned} \tag{6}$$

and

$$\begin{aligned} \dot{x} =& y(1 + Dx + Px^2) + Hx^2 + Qx^3 + y^2(G + Vx), \\ \dot{y} =& -x + Ax^2 + 3Bxy + Cy^2 + Kx^3 + 3Lx^2y + Mxy^2 + Ny^3. \end{aligned} \tag{7}$$

However, for only 6 of 17 families of [29] has the study of [30,31] resulted in finding center conditions in terms of polynomial equalities in the coefficients of (6) and (7); that is, in finding components of the center variety (in this paper, speaking on the components of the center variety, we do not mean the proper components; they can simply be algebraic sets that are subsets of the center variety). For the other cases the conditions for the existence of a center in (6) and (7) were given in terms of elimination ideals.

The authors of [30,31] also did not consider all systems from the classifications of [29] that can be transformed to families (6) and (7). They only considered real systems transformed to (6) and (7) by real transformations. However, we will see below that some complex systems can be transformed to real systems (6) by complex affine transformations (4).

Recently, the following generalization of the notion of time reversibility has been introduced in [32].

Definition 1. *Let $U \subset \mathbb{R}^n$ be an open set, $\varphi : U \to U$ be an involution of class C^1, $\mathcal{X} : U \to \mathbb{R}^n$ be a vector field of class C^r and $F : U \to \mathbb{R}$ be a continuous function. It is said that $\mathcal{X}$ is orbitally φ-reversible if*

$$D_\varphi \cdot \mathcal{X} = F\mathcal{X} \circ \varphi.$$

The case $F \equiv -1$ corresponds to the classical notion of time reversibility.

The authors of [32] showed that all 17 families of [29] are orbitally φ-reversible and found the corresponding involutions.

In this paper we consider the families of [29] that are related to system (6) with $R = N = 0$; namely, the families denoted by CR_5^8, CR_7^9 and CR_8^{10} in [29]. That is,

$$\begin{aligned} \dot{X} =& X(l + p + mcX + (k+n)XY + mX^2 + qTX), \\ \dot{Y} =& -kXY^2 - lY + mX^2Y - (nXY + p + qTX)(2X + Y + c), \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{X} =& X(-(n+k) + (l-m)XY - (l+p)XT), \\ \dot{Y} =& nX + kY + nT + (m-l)X^2Y + pX^2T + mXYT + pXT^2 \end{aligned} \tag{9}$$

and

$$\begin{aligned} \dot{X} =& X(-k - 2nT + 2(l-m)XY - lXT), \\ \dot{Y} =& 2kY + nXT + nT^2 + (m-l)X^2Y + mXYT, \end{aligned} \tag{10}$$

respectively (above $T = x + y + c$).

Using the tools of computational algebra and algorithms of the elimination theory, we obtain from these three families six components of the center variety. For the obtained components we discuss the existence of orbital φ-reversibility and study bifurcations of small limit-cycles from the center at the origin.

Actually, from theoretical point of view, it is straightforward to find systems from families (8)–(10) that correspond to systems with a center at the origin in family (6) in an algorithmic way: in each of systems (8)–(10) one performs substitution (5), equates the coefficients of the corresponding terms in the obtained system and in system (6), and then eliminates from the obtained polynomial system the variables $l, p, m, c, k, n, q, a_0, a_1, b_0, b_1, b_2$ using the elimination theorem [14,33]. However, the algorithm requires computation of Groebner bases with respect to the lexicographic order and, therefore, it is extremely time- and memory-consuming. For this reason, the direct application of this computational approach is of limited use.

Our choice of the families mentioned above is due to the computational restrictions—the study involves laborious computations, so we chose the cases that look simpler from the computational point of view and where we were able to complete calculations using our computational facilities. Some results of the performed computations are large polynomial ideals that are not appropriate for the presentation in the paper due to their sizes, but we have made them available online at http://www.camtp.uni-mb.si/camtp/barbara/ (accessed on 6 March 2021).

2. Preliminaries

To study the center problem it is often computationally efficient to work, instead of real system (1), with its complexification obtained as follows. Using the substitution $X = x + iy$, we obtain from system (1) the complex differential equation

$$\dot{X} = iX + R(X, \overline{X}).$$

Then, we adjoin to this equation its complex conjugate and obtain the system

$$\dot{X} = iX + R(X, \overline{X}), \quad \dot{\overline{X}} = -i\bar{X} + \bar{R}(X, \overline{X}).$$

Now we denote $\overline{X}$ as a new variable Y and $\bar{R}$ as a new function obtaining the system of two complex differential equations, which we can write in the form

$$\begin{aligned} \dot{X} =& i(X - \sum_{p+q=1}^{n-1} a_{pq} X^{p+1} Y^q), \\ \dot{Y} =& -i(Y - \sum_{p+q=1}^{n-1} b_{qp} X^q Y^{p+1}), \end{aligned} \tag{11}$$

where $X, Y \in \mathbb{C}$, $p \geq -1$, $q > 0$, and in the case when $b_{qp} = \bar{a}_{pq}$ the system has a real preimage of the form (1).

For system (11) there is a function of the form

$$\Psi(X, Y) = XY - \sum_{j+k>2} \psi_{j-1,k-1} X^j Y^k, \tag{12}$$

such that

$$\dot{\Psi} = \frac{\partial \Psi}{\partial X} \dot{X} + \frac{\partial \Psi}{\partial Y} \dot{Y} = \sum_{k \in \mathbb{N}} g_{k,k}(a, b)(XY)^{k+1},$$

where a and b are parameters of the first and second equations in (11), respectively, and $g_{kk}(a, b)$ are polynomials of the ring $\mathbb{Q}[a, b]$, called the *focus quantities* of system (11). They form the ideal $\mathcal{B} = \langle g_{11}, g_{22}, g_{33} \ldots \rangle$, called the Bautin ideal of system (11). Its variety $\mathbf{V}(\mathcal{B})$ consists of all systems (11) admitting a local analytical first integral in a neighborhood of the origin. As is mentioned above by the Poincaré–Lyapunov theorem, the local integrability yields the existence of the center, so systems from $\mathbf{V}(\mathcal{B})$ that have a real preimage, have a center at the origin. For the cubic complex system (system (11) with $n = 3$) using the algorithm of ([14], Section 3.4), we have computed the first eight focus quantities $g_{11}, \ldots, g_{88}$ (available at http://www.camtp.uni-mb.si/camtp/barbara/FocusQuantitiesCubic8 (accessed on 6 March 2021)). Clearly, using these it is straightforward to obtain the focus quantities v_k of real systems (6) and (7).

We recall also a theorem related to parameterizations of affine varieties. Let $f_1, \ldots, f_n, g_1, \ldots, g_n$ be polynomials of the ring $k[t_1, \ldots, t_m]$, where k is an infinite field. Let W be the variety of $\langle g_1, \ldots, g_n \rangle$, and let $F : k^m \setminus W \longrightarrow k^n$ be the function defined by

$$F(t_1, \ldots, t_m) = \left(\frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)}, \ldots, \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)} \right).$$

The following statement is known as the rational implicitization theorem (see, for instance, ref. [33] for a proof). It can be used to check if a polynomial parametrization covers an affine variety.

Theorem 1. *Let k be an infinite field. Set $g = g_1 \cdots g_n$ and consider the ideal*

$$J = \langle f_1 - g_1 x_1, ..., f_n - g_n x_n, 1 - gy \rangle \subset k[y, t_1, ..., t_m, x_1, ..., x_n]$$

and its $m+1$st elimination ideal $J_{m+1} = J \cap k[x_1, ..., x_n]$. Then $V(J_{m+1})$ is the smallest variety in k^n containing $F(k^m \setminus W)$.

3. Some Center Conditions

In this section we look for components of the center variety of system (6) (the center conditions) associated to systems (8)–(10) and obtain the following result.

Theorem 2. *(1) System (6) with $N = R = 0$ has a center at the origin if its parameters belong to one of varieties $V_k = \mathbf{V}(I_k)$ ($k = 1, \ldots, 5$), where the ideals I_k are defined as follows:*

$I_1 = \langle A, D - G, C + G, G^2 + M, 3BC + 3L - CH \rangle$,

$I_2 = \langle G^2 + M, D - G, C + G, L^2 + H^2 M, GL - HM, GH + L, 3B + 2H, 2A^2 + 18H^2 + 9K \rangle$,

$I_3 = \langle H, CD + D^2 - DG - M, 3AB + 3BC + 3L, A + C + D, CK + 3DK - 2GK + CM + DM \rangle$,

$I_4 = \langle L, H, B, 36DK - 45GK - 2AM - 16DM + 24GM, 3DG + 2M, 2C + 2D + G \rangle$,

$I_5 = \langle 2C + D + 2G, 3B + 2H, 9DL - 8HM, 4GH + 3L, 3DG + 2M \rangle$,

or the variety V_6, which is parametrized as follows:

$$M = f_1, \quad L = \frac{f_2}{g_2}, \quad H = \frac{f_3}{g_2}, \quad K = \frac{f_4}{g_4}, \tag{13}$$

where

$$\begin{aligned}
f_1 =& BG(4A + 2C - D + 2G)2A - D,\\
f_2 =& BG(4A + 2C - D + 2G),\\
f_3 =& -3B(A + C + G),\\
f_4 =& -G(2A + C + G)(-108A^2B^2 + 12A^3C - 162AB^2C + 16A^2C^2 - 54B^2C^2\\
&- 12A^3D + 27AB^2D - 36A^2CD + 27B^2CD - 16AC^2D + 20A^2D^2 + 27ACD^2\\
&+ 4C^2D^2 - 11AD^3 - 6CD^3 + 2D^4 + 12A^3G - 162AB^2G + 32A^2CG - 108B^2CG\\
&- 36A^2DG + 27B^2DG - 32ACDG + 27AD^2G + 8CD^2G - 6D^3G + 16A^2G^2\\
&- 54B^2G^2 - 16ADG^2 + 4D^2G^2),\\
g_2 =& 2A - D,\\
g_4 =& (2A - D)^2 \tilde{g}_4,\\
\tilde{g}_4 =& 2AC + C^2 - 4AD - 4D^2 - 4AG - 6CG + 4DG - 7G^2
\end{aligned} \tag{14}$$

and all the above polynomials are defined on $\mathbf{V}(h)$, where $h = -18AB^2C + 4A^2C^2 - 9B^2C^2 + 36AB^2D - 12A^2CD + 27B^2CD - 4AC^2D + 8A^2D^2 - 18B^2D^2 + 12ACD^2 + C^2D^2 - 8AD^3 - 3CD^3 + 2D^4 - 72AB^2G + 8A^2CG - 54B^2CG - 12A^2DG + 45B^2DG - 8ACDG + 12AD^2G + 2CD^2G - 3D^3G + 4A^2G^2 - 45B^2G^2 - 4ADG^2 + D^2G^2$.

Moreover, systems $CR_7^{(9)}$ correspond to systems from $\mathbf{V}(I_1)$, systems $CR_5^{(8)}$ correspond to systems from $\mathbf{V}(I_1)$ and $\mathbf{V}(I_2)$, systems $CR_8^{(10)}$ correspond to systems from V_6, from $\mathbf{V}(I_3) \cap \mathbf{V}(h_1)$, $\mathbf{V}(I_4) \cap \mathbf{V}(h_2)$ where

$$\begin{aligned}
h_1 =& 9B^2C^2 + 4C^4 + 36B^2CD + 16C^3D + 27B^2D^2 + 24C^2D^2 + 16CD^3 + 4D^4 - 9B^2CG - 8C^3G +\\
& 9B^2DG - 24C^2DG - 24CD^2G - 8D^3G - 18B^2G^2 - 3C^2G^2 - 6CDG^2 - 3D^2G^2 + 5CG^3 + 5DG^3 + 2G^4,
\end{aligned} \tag{15}$$

$$h_2 = -675G^4K + 405G^4M - 720G^2KM + 360G^2M^2 - 192KM^2 + 80M^3 \tag{16}$$

and points of the variety $\mathbf{V}(I_5)$ for which

$$K = \frac{1}{18(6G-5D)}(20A^2D + 60AD^2 + 45D^3 - 24A^2G - 94ADG - 6D^2G + 36AG^2 + \\ -52DG^2 + 24G^3 + 180DH^2 - 216GH^2) \tag{17}$$

and

$$100A^2D^2 + 300AD^3 + 225D^4 - 160A^2DG - 320AD^2G - 120D^3G + 64A^2G^2 - 16ADG^2 + \\ -104D^2G^2 + 64AG^3 + 32DG^3 + 16G^4 + 900D^2H^2 - 2160DGH^2 + 1296G^2H^2 = 0. \tag{18}$$

(2) Systems corresponding to generic points of varieties $\mathbf{V}(I_1), \dots, \mathbf{V}(I_5)$ *and* V_6 *are Darboux integrable.*

Proof. (1) (a) Consider first system (8). We look for a transformation (5) that brings it to the canonical form (6). That is, we perform the substitution

$$(X, Y) = (a_0 + a_1x, b_0 + b_1x + b_2y) \tag{19}$$

and then we equate the transformed system with (6) obtaining that

$$R = N = 0$$

and the other coefficients satisfy an algebraic system of 16 equations.

Solving the suitable equations for l, a_1 and b_1 we then obtain two cases:

$$(i)\ \ 2a_0 + c = 0,$$
$$(ii)\ \ a_0b_0m - a_0b_0n - p - a_0^2q - a_0b_0q - a_0cq = 0.$$

Examining case (i), first we can express c and n, then eliminating with `Eliminate` of MATHEMATICA from the remaining equations $a_0, b_0, b_2, k, m, p, q$ we obtain the ideal

$$U = \langle A, 3BC - CH + 3L, -D - 2C - G, 2CG + G^2 + C^2, (C+G)H, M + C^2\rangle.$$

Computing with the routine `radical` of the compute algebra system SINGULAR [34] the radical of U we obtain the ideal I_1 given in the statement of the theorem.

Obviously (one also can use Theorem 1) a rational parametrization of $\mathbf{V}(I_1)$ is given by

$$A = 0,\ \ C = -G,\ \ D = G,\ \ L = \frac{3BG - GH}{3},\ \ M = -G^2. \tag{20}$$

Further computations show that for the coefficients of transformation (19) it holds:

$$a_1 = a_0G,\ \ b_1 = -Gb_0. \tag{21}$$

In case (ii) we first solve two of the remaining equations for p and n. Similarly as above, eliminating $a_0, b_0, b_2, c, k, m, q$ from the remaining equations we obtain the ideal

$$\langle -2A^2 - 18H^2 - 9K, 3B + 2H, -D - 2C - G, 2CG + G^2 + C^2, (C+G)H, L - CH, M + C^2\rangle \tag{22}$$

whose radical is the prime ideal I_2 given in the statement of the theorem. A rational parametrization of $\mathbf{V}(I_2)$ is given by

$$C = -G,\ \ D = G,\ \ H = -\frac{3}{2}B,\ \ K = -\frac{1}{18}(4A^2 + 81B^2),\ \ M = -G^2,\ \ L = \frac{3}{2}BG \tag{23}$$

and for this solution we find

$$a_1 = a_0 G, \quad b_1 = -\frac{2Ab_0G + 9Bb_2}{2(A+3G)}, \quad c = -\frac{a_0(2A+3G)}{A}. \tag{24}$$

(b) Consider now system (9). Applying transformation (19) and equating the obtained system with (6) we can calculate k and then again we have two cases:

$$(i) \;\; 2a_0 + c = 0,$$
$$(ii) \;\; a_0b_0l - a_0b_0m - n + a_0^2p - a_0b_0p - a_0cp = 0.$$

For (i), straightforward calculations yield p, n and $A = R = N = 0$. Now, the elimination of $a_0, a_1, b_0, b_2, l, m, n$ gives an ideal having the same radical as I_1. Using the parametrization (20) we find that

$$\begin{aligned} a_1 =& a_0G, \\ b_0 =& a_0(2G^2 - 2K - 3BGn + 2GHn - Gsn)/(2G^2 + 3BH - 2H^2 - 2K + Hs), \\ b_1 =& -Gb_0, \\ b_2 =& -4a_0G^2(H+Gn)/(2G^2 + 3BH - 2H^2 - 2K + Hs), \end{aligned} \tag{25}$$

where $s = (9B^2 - 12BH + 4H^2 + 8K)^{\frac{1}{2}}$.

For (ii) we calculate n, p, b_1 and get $R = N = 0$. After the elimination of $a_0, a_1, b_0, b_2, l, m, c$ we obtain ideal (22). Using the parametrization (23) of the variety of I_5 we find that

$$\begin{aligned} a_1 =& a_0G, \\ b_1 =& iG(2iA^2b_0 - 9ABb_0 + 27a_0BG)/(A(2A + 9iB + 6G)), \\ b_2 =& 6G^2(Aa_0 - Ab_0 + 3a_0G)/(A(2iA - 9B + 6iG)), \\ c =& -a_0(2A+3G)/A \end{aligned} \tag{26}$$

or

$$\begin{aligned} a_1 =& a_0G, \\ b_1 =& -iG(-2iA^2b_0 - 9ABb_0 + 27a_0BG)/(A(2A - 9iB + 6G)), \\ b_2 =& 6G^2(Aa_0 - Ab_0 + 3a_0G)/(A(-2iA - 9B - 6iG)), \\ c =& -a_0(2A+3G)/A. \end{aligned} \tag{27}$$

Thus, we see that in this case there is no real transformation of (9) to a system of the form (6).

(c) Consider now system (10). In this case performing transformation (19) and equating the corresponding terms we find k and l and then we have two possibilities:

$$(i) \;\; 2a_0 - b_0 + c = 0,$$
$$(ii) \;\; a_1b_0 + a_0^2b_0b_2m - a_0^2b_2n + a_0b_0b_2n - a_0b_2cn = 0.$$

Consider first case (i). Calculations give $R = N = 0$ and b_1. From the remaining equations we obtain the ideal V presented in Appendix A. For this case we were not able to perform elimination with MATHEMATICA, so we used the computer algebra system SINGULAR. Since it still was not possible to complete computations over the field of rational numbers we carried them out in the ring

$$\mathbb{Z}_{32,003}[A, B, C, D, G, H, K, L, M, w, c, a_0, a_1, b_2, n, m]$$

with the degree reverse lexicographic ordering. We first eliminated the variables $a_1, b_2, a_0, c, n, m, w$ from the ideal $\langle V, 1 - wa_1b_2a_0 \rangle$. Then with the routine `minAssGTZ` [35] of SINGULAR we computed the decomposition of the obtained ideal and found two components (output provided by Singular is presented at http://www.camtp.uni-mb.si/camtp/barbara/idealVed (accessed on 6 March 2021)). Performing the rational reconstruction of the first component with the algorithm of [35] we obtained the ideal $\hat{I}_3$ given in Appendix B.

Now taking from $\hat{I}_3$ polynomials generating the ideal I_3 from the statement of the theorem, we observe that the first eight focus quantities of system (6) vanish on $\mathbf{V}(I_3)$. Simple computations show that

$$\mathbf{V}(\hat{I}_3) = \mathbf{V}(I_3) \cap \mathbf{V}(h_1)$$

where h_1 is defined by (15). Thus, not all systems from $\mathbf{V}(I_3)$ correspond to (10) but we will see below that all systems from $\mathbf{V}(I_3)$ have a center at the origin.

The second component of the decomposition after the rational reconstruction gives a large ideal that we denote by I_6 (ideal I_6 is available at http://www.camtp.uni-mb.si/camtp/barbara/ideal6 (accessed on 6 March 2021)). Using `eliminate` of SINGULAR we computed the ideal

$$\tilde{I}_6 = \langle 1 - wg_2g_4, h, f_1 - M, f_2 - Lg_2, f_3 - Hg_2, f_4 - Kg_4 \rangle \cap \mathbb{Q}[A, B, C, D, G, H, K, L] \quad (28)$$

where $h, g_2, g_4, f_1, f_2, f_3, f_4$ are defined by (14) and then using the radical membership test [14,33] checked that all polynomials from I_6 vanish on the variety of $\tilde{I}_6$ and all polynomials from $\tilde{I}_6$ vanish on $\mathbf{V}(I_6)$. This means that

$$\mathbf{V}(I_6) = \mathbf{V}(\tilde{I}_6) = V_6 \quad (29)$$

and by Theorem 1 equality (28) means that (13) gives a rational parametrization of (29).

(ii) We solve suitable equations for c, b_1 and a_1. Then we observe that $N = R = 0$ and the polynomial system is defined by the ideal W given in Appendix C. We compute the center conditions with SINGULAR eliminating $w, a_0, a_1, b_2, n, m, b_0$ from the ideal

$$\langle W, 1 - wa_0b_2n(2a_0m + 3n) \rangle$$

in the ring $\mathbb{Q}[A, B, C, D, G, H, K, L, M, w, a_0, a_1, b_2, n, m, b_0]$. Then, computing the minimal associate primes of the obtained ideal with the `minAssGTZ` we obtain the ideals $\hat{I}_4$ and $\hat{I}_5$ given in Appendix D.

We observe that if we take the first six polynomials from the generators of $\hat{I}_4$ as the generators of the ideal I_4 from the statement of the theorem then the first 7 focus quantities v_i (defined by (3)) vanish on the variety $\mathbf{V}(I_4)$. In fact,

$$\mathbf{V}(\hat{I}_4) = \mathbf{V}(I_4) \cap \mathbf{V}(h_2),$$

where h_2 is defined by (16), but we will see below that all systems from $\mathbf{V}(I_4)$ have a center at the origin.

Similarly, we take certain polynomials from $\hat{I}_5$ such that the first seven focus quantities vanish on their variety and form the ideal I_5 of the statement of the theorem. Using the parametrization of $\mathbf{V}(I_5)$ given as

$$C = -\frac{1}{2}(D + 2G), \;\; B = -\frac{2}{3}H, \;\; L = -\frac{4}{3}GH, \;\; M = -\frac{3}{2}DG \quad (30)$$

we compute the coefficients of (4) and the parameters l, k, c of systems (10). Then using the `eliminate` of SINGULAR we eliminate from the remaining polynomial the variables m and n obtaining an ideal whose variety is defined by (17) and (18). This means that systems from family (10) correspond to systems from $\mathbf{V}(I_5)$ for which conditions (17) and (18) are fulfilled.

(2) We now show that all obtained systems are Darboux integrable. The following systems (31)–(36) correspond to the six components, presented in the statement of the theorem, respectively:

$$\begin{aligned}\dot{x} =&(1+Gx)(y+Hx^2+Gxy),\\ \dot{y} =&-x+Kx^3+3Bxy+G(3B-H)x^2y-Gy^2-G^2xy^2;\end{aligned} \tag{31}$$

$$\begin{aligned}\dot{x} =&(1+Gx)(y-\frac{3B}{2}x^2+Gxy),\\ \dot{y} =&-x+Ax^2-\frac{1}{18}(4A^2+81B^2)x^3+3Bxy+\frac{9BG}{2}x^2y-Gy^2-G^2xy^2;\end{aligned} \tag{32}$$

$$\begin{aligned}\dot{x} =&(1+Gx)(y+Dxy),\\ \dot{y} =&-x-(C+D)x^2-\frac{(C+D)(CD+D^2-DG)}{C+3D-2G}x^3+3Bxy+3BDx^2y+\\ &+Cy^2+(CD+D^2-DG)xy^2;\end{aligned} \tag{33}$$

$$\begin{aligned}\dot{x} =&(1+Gx)(y-\frac{2M}{3G}xy),\\ \dot{y} =&-x+\frac{-135G^2K+72G^2M-72KM+32M^2}{6GM}x^2+Kx^3+\frac{4M-3G^2}{6G}y^2+Mxy^2;\end{aligned} \tag{34}$$

$$\begin{aligned}\dot{x} =&(1+Gx)(y+Hx^2+Dxy),\\ \dot{y} =&-x+Ax^2+Kx^3-2Hxy-4GHx^2y+\frac{1}{2}(-D-2G)y^2-\frac{3}{2}DGxy^2;\end{aligned} \tag{35}$$

$$\begin{aligned}\dot{x} =&(1+Gx)(-\frac{3B(A+C+G)}{2A-D}x^2+y+Dxy),\\ \dot{y} =&-x+Ax^2+Kx^3+3Bxy+\frac{3BG(4A+2C-D+2G)}{2A-D}x^2y+Cy^2+G(C-D+G)xy^2,\end{aligned} \tag{36}$$

where for (36)

$$\begin{aligned}K =&-G(2A+C+G)(108A^2B^2-12A^3C+162AB^2C-16A^2C^2+54B^2C^2+12A^3D+\\ &-27AB^2D+36A^2CD-27B^2CD+16AC^2D-20A^2D^2-27ACD^2-4C^2D^2+11AD^3+\\ &+6CD^3-2D^4-12A^3G+162AB^2G-32A^2CG+108B^2CG+36A^2DG-27B^2DG+\\ &+32ACDG-27AD^2G-8CD^2G+6D^3G-16A^2G^2+54B^2G^2+16ADG^2-4D^2G^2)/\\ &((2A-D)^2(-2AC-C^2+4AD+4D^2+4AG+6CG-4DG+7G^2))\end{aligned}$$

and A, B, C, D, G satisfy the equation

$$\begin{aligned}&-18AB^2C+4A^2C^2-9B^2C^2+36AB^2D-12A^2CD+27B^2CD-4AC^2D+8A^2D^2-18B^2D^2+\\ &+12ACD^2+C^2D^2-8AD^3-3CD^3+2D^4-72AB^2G+8A^2CG-54B^2CG-12A^2DG+45B^2DG+\\ &-8ACDG+12AD^2G+2CD^2G-3D^3G+4A^2G^2-45B^2G^2-4ADG^2+D^2G^2=0.\end{aligned} \tag{37}$$

For each of these systems we have found a Darboux integral or an integrating factor as presented below. All six systems have the invariant line $1+Gx=0$ and consequently the Darboux factor

$$L_1 = 1+Gx. \tag{38}$$

Looking for other invariant curves we obtain the following.

System (31) has the Darboux integral

$$\Psi_1 = L_2 L_3^{\alpha_3},$$

where

$$\begin{aligned}L_2 =&1/2(2+3BHx^2-2(H^2+K)x^2-H\gamma x^2-2Hy-\gamma y-2GHxy-G\gamma xy-3B(1+Gx)y),\\ L_3 =&1/2(2+3BHx^2-2(H^2+K)x^2+H\gamma x^2-2Hy+\gamma y-2GHxy+G\gamma xy-3B(1+Gx)y),\\ \alpha_3 =&\frac{3B+2H+\gamma}{-3B-2H+\gamma}, \quad \gamma=((3B-2H)^2+8K)^{\frac{1}{2}}.\end{aligned}$$

System (32) is Hamiltonian with the Hamiltonian function

$$H_2 = \frac{1}{2}x^2 - \frac{A}{3}x^3 + \frac{1}{72}(4A^2+81B^2)x^4 - \frac{3B}{2}x^2y - \frac{3BG}{2}x^3y + \frac{1}{2}y^2 + Gxy^2 + \frac{1}{2}G^2x^2y^2.$$

System (33) has the Darboux integral

$$\Psi_3 = L_1 L_4^{\alpha_4} L_5^{\alpha_5},$$

where

$$\begin{aligned}
L_4 =& 2C\delta + 6D\delta - 4\delta G + 2C^2\delta x + 8CD\delta x + 6D^2\delta x - 4C\delta Gx - 4D\delta Gx + 2C^2D\delta x^2 + 4CD^2\delta x^2 + \\
&+ 2D^3\delta x^2 - 2CD\delta Gx^2 - 2D^2\delta Gx^2 - 3BC\delta y - 9BD\delta y + 6B\delta Gy + C\epsilon y + 3D\epsilon y - 2G\epsilon y + \\
&- 3BCD\delta xy - 9BD^2\delta xy + 6BD\delta Gxy + CD\epsilon xy + 3D^2\epsilon xy - 2DG\epsilon xy, \\
L_5 =& 2C\delta + 6D\delta - 4\delta G + 2C^2\delta x + 8CD\delta x + 6D^2\delta x - 4C\delta Gx - 4D\delta Gx + 2C^2D\delta x^2 + 4CD^2\delta x^2 + \\
&+ 2D^3\delta x^2 - 2CD\delta Gx^2 - 2D^2\delta Gx^2 - 3BC\delta y - 9BD\delta y + 6B\delta Gy - C\epsilon y - 3D\epsilon y + 2G\epsilon y + \\
&- 3BCD\delta xy - 9BD^2\delta xy + 6BD\delta Gxy - CD\epsilon xy - 3D^2\epsilon xy + 2DG\epsilon xy, \\
\alpha_4 =& -\frac{(\zeta + 3B\delta)G}{2(C+D)\zeta}, \quad \alpha_5 = -\frac{(\zeta - 3B\delta)G}{2(C+D)\zeta}, \quad \delta = (C+3D-2G)^{\frac{1}{2}}, \\
\epsilon =& (9B^2C + 4C^3 + 27B^2D + 12C^2D + 12CD^2 + 4D^3 - 18B^2G + \\
&- 12C^2G - 24CDG - 12D^2G + 8CG^2 + 8DG^2)^{\frac{1}{2}}, \\
\zeta =& (9B^2(C+3D-2G) + 4(C+D)(C+D-2G)(C+D-G))^{\frac{1}{2}}.
\end{aligned}$$

System (34) has the Darboux integral

$$\Psi_4 = \left(1 - \frac{2M}{3G}x\right) L_6^{\frac{1}{2}},$$

where

$$\begin{aligned}
L_6 =& 1 + (4M)/(3G)x + (4M^2(9G^2(9K-5M) + 5(9K-4M)M))/(9G^2(3G^2(9K-5M) + \\
&+ 5(3K-M)M))x^2 + (4KM^3x^3)/(3G\eta) + (20M^4y^2)/(9G^2\eta) + (20M^4xy^2)/(9G\eta), \\
\eta =& -27G^2K + 15G^2M - 15KM + 5M^2.
\end{aligned}$$

System (35) has the Darboux integral

$$\Psi_5 = L_1 L_7^{\frac{1}{2}},$$

where

$$\begin{aligned}
L_7 = \frac{1}{3K - 5AG - 10G^2}&(-5AG - 10G^2 + 3K + 10AG^2x + 20G^3x - 6GKx - 15AG^3x^2 + \\
&+ 9G^2Kx^2 - 12G^3Kx^3 + 60G^4Hx^2y + 30G^4y^2 + 30DG^4xy^2).
\end{aligned}$$

It is not easy to find a Darboux integral or an integrating factor for system (36) since we were not able to find a rational parametrization of the variety V_6. Solving Equation (37) for C one can obtain a parametrization of V_6 involving radicals of polynomial functions. However, computer algebra systems do not work efficiently with complicated expressions involving radicals, so we were not able to solve arising systems and find invariant curves with MATHEMATICA in this way. Instead, we use another approach. We first look for a curve on V_6 admitting a rational parametrization. As is well known, an algebraic curve admits a rational parameterization if and only if it is of genus zero (see e.g., [36]). Using the routine `genus` of the library `normal.lib` [34] of the computer algebra system SINGULAR we found that for

$$G = 9, \; D = 3, \; C = 5$$

the polynomial (37) defines a curve of genus zero. Then, with the routine `paraPlaneCurve` of the library `paraplanecurves.lib` [37] we obtained that the curve on the variety V_6 is defined parametrically in the following way

$$\begin{array}{lll} G=9, & A=\dfrac{44-240,223,725s^6}{31,255,875s^6}, & K=\dfrac{4(11+5,060,475s^6)(11-5,358,150s^6)}{40,516,875s^6}, \\ D=3, & B=\dfrac{574,215,075s^6-88}{99,225s^3}, & \\ C=5, & & L=\dfrac{179,498,025s^6-176}{11,025s^3}. \\ M=99, & H=\dfrac{44+197,358,525s^6}{33,075s^3}, & \end{array} \tag{39}$$

In this, case except for the Darboux factor (38), the system has the Darboux factor

$$\ell = 5+70x+297x^2+136,632,825s^6x^2-66,150s^3y-595,350s^3xy,$$

with the cofactor

$$\widetilde{\kappa} = 13,230s^3x+119,070s^3x^2+14y+126xy \tag{40}$$

yielding the integrating factor

$$\mu = L_1\ell^{-31/14}.$$

By analogy with (40) we look for a quadratic Darboux factor

$$L_8 = \sum_{j+k\leq 2} c_{jk}x^jy^k$$

with a cofactor of the form $\kappa = \kappa_1x+\kappa_2y+\kappa_3x^2+\kappa_4xy$. Equating the coefficients of similar monomials on both sides of

$$\mathcal{X}L_8 = \kappa\, L_8$$

we obtain an algebraic system. We look for a solution of the obtained system following the pattern arising for the system with parameters (39) and find the coefficients

$$\begin{aligned} \kappa_1 &= -\frac{1}{(A+C)(2A-D)}(3ABC+3BC^2+3ABG+6BCG+3BG^2+2A\kappa_3-D\kappa_3), \\ \kappa_2 &= C+G, \\ \kappa_4 &= CG+G^2 \end{aligned}$$

and c_{jk}. It is not possible to find κ_3 from the remaining equations. However, from the equation

$$m_1\mathcal{X}(L_1)/L_1+m_2\kappa+div(\dot{x},\dot{y})=0$$

where $div(\dot{x},\dot{y})$ is the divergence of the vector field (36), we find that

$$m_1=1,\ \ m_2=-2-\frac{D}{C+G},\ \ \kappa_3=3BG\frac{C+G}{D-2A}.$$

This means that system (36) has the cofactor

$$\kappa = \frac{3B(C+G)}{D-2A}x+(C+G)y+3BG\frac{C+G}{D-2A}x^2+(CG+G^2)xy$$

and the integrating factor

$$\mu_6 = L_1L_8^{-2-\frac{D}{C+G}},$$

where

$$
\begin{aligned}
L_8 =& \frac{1}{2(D-2A)^2}(-D^3(C+G)x^2 - 9B^2(C+G)^2x^2 + D^2(2+2Gx+C^2x^2+G^2x^2+ \\
&+2Cx(1+Gx)) + 4A^2(2+C^2x^2+G^2x^2+Gx(2-Dx)+Cx(2-Dx+2Gx))+ \\
&+3BD(C+G)(3Bx^2-2(y+Gxy)) + 2A(2D^2(C+G)x^2 - 2D(2+2Gx+C^2x^2+ \\
&+G^2x^2+2Cx(1+Gx)) + 3B(C+G)(-3Bx^2+2(y+Gxy))))
\end{aligned}
$$

with A, B, C, D, G satisfying Equation (37). Therefore, system (36) is Darboux integrable and, hence, has a center at the origin. □

Remark 1. *The ideal I_1 was also found in [30]. The variety V_6 is the same as the variety of the ideal obtained after the elimination of t from the ideal J_2 in [30].*

$$
\begin{aligned}
&J_2 = \langle 3AB+3BC+3BG+2AH-DH, 2A^3C+5A^2C^2+4AC^3+C^4-2A^3D-5A^2CD-4AC^2D-C^3D+ \\
&2A^3G+10A^2CG+12AC^2G+4C^3G-5A^2DG-8ACDG-3C^2DG+5A^2G^2+12ACG^2+6C^2G^2- \\
&4ADG^2-3CDG^2+4AG^3+4CG^3-DG^3+G^4-4A^2H^2-4ACH^2-C^2H^2+2ADH^2+CDH^2- \\
&4AGH^2-2CGH^2+DGH^2-G^2H^2-2A^2K-4ACK-2C^2K-4AGK-4CGK-2G^2K, \\
&4AGH+2CGH-DGH+2G^2H+3AL+3CL+3GL, CG-DG+G^2-M, \\
&A^2C^2+2AC^3+C^4-3A^2CD-6AC^2D-3C^3D+2A^2D^2+4ACD^2+2C^2D^2+2A^2CG+ \\
&6AC^2G+4C^3G-3A^2DG-12ACDG-9C^2DG+4AD^2G+4CD^2G+A^2G^2+6ACG^2+6C^2G^2- \\
&6ADG^2-9CDG^2+2D^2G^2+2AG^3+4CG^3-3DG^3+G^4-2ACH^2-C^2H^2+4ADH^2+3CDH^2- \\
&2D^2H^2-8AGH^2-6CGH^2+5DGH^2-5G^2H^2, 1-At-Ct-Gt\rangle.
\end{aligned}
$$

To check this with `eliminate` of SINGULAR we eliminated t from J_2 and then used the radical membership test [14,33].

Remark 2. *From the proof of the theorem we see that systems from family (8) (that is, CR_5^8) are transformed to systems (31) or to systems (32) by a real transformation (4) (with the coefficients defined by (21) and (24), respectively). Systems (31) also emerge from (9) (CR_7^9) via a real transformation (4) (defined by (25)), whereas systems (32) are obtained from systems (9) with complex parameters using complex transformation (4) (defined by (26)). In the case of system (10) (CR_8^{10}) transformations to systems (33) and (36) are real, whereas systems (34) and (35) emerge from systems (10) with complex coefficients via complex transformations.*

Remark 3. *For systems (8) and (9) similar calculations as above were performed in [30]; however, the authors of [30] limited their consideration to the case $2a_0 + c = 0$.*

4. Orbital Reversibility in Subfamilies of (6)

The following theorem is proved in [32].

Theorem 3. *Consider an involution φ defined on an open set $U \subset \mathbb{R}^2$ and denote the fixed points of involution φ by Fix φ. Let $\mathcal{X}$ be an orbitally φ-reversible vector field such that Fix $\varphi \cap U$ is a smooth manifold of dimension 1 and $p \in$ Fix $\varphi \cap U$ is an equilibrium point. The next properties hold:*

(1) If $F(p) = -1$ and $\det(D\mathcal{X}(p)) > 0$, then p is a center of $\mathcal{X}$.
(2) If $F(p) = -1$ and $\det(D\mathcal{X}(p)) < 0$, then p is a saddle of $\mathcal{X}$.
(3) If $F(p) = 1$, then Fix $\varphi \cap U$ is invariant under the flow of $\mathcal{X}$.

In [32] the following statement is also proved.

Lemma 1. *Let $\mathcal{X}$ be an orbitally φ-reversible vector field. If ψ is a change of coordinates, then the transformed vector field $\tilde{\mathcal{X}} = D_\psi \cdot \mathcal{X} \circ \psi^{-1}$ is an orbitally $\tilde{\varphi}$-reversible vector field, with $\tilde{\varphi} = \psi \circ \varphi \circ \psi^{-1}$ and $\tilde{F} = F \circ \psi^{-1}$ being the respective transformed involution and factor.*

We now discuss the orbital φ-reversibility of the obtained families.

Proposition 1. *Systems (31), (32) and (36) are orbitally φ-reversible. For systems (31) and (36) the curve $Fix\,\varphi$ passes through the center at the origin. For systems (32) the curve $Fix\,\varphi$ passes through a saddle point and the systems have two centers.*

Proof. To prove the proposition we use the results of [32] where involutions for reversible systems of [29] were found. In particular, it was shown in [32] that both systems (8) and (9) have the involution

$$\varphi(x,y) = \left(-x-c, -\frac{xy}{x+c}\right).$$

To all studied systems (6) we apply the change of coordinates (5).

Consider first system (31) (corresponding to $\mathbf{V}(I_1)$). In this case $c = -2a_0$. For parametrization (20) of the variety $\mathbf{V}(I_1)$ the values of transformation (5) are given by (21). After straightforward computations we find that

$$\tilde{\varphi} = \psi \circ \varphi \circ \psi^{-1} = \left(-x, -\frac{y(Gx+1)}{Gx-1}\right). \tag{41}$$

Then by Lemma 1 $\tilde{\varphi}$ defined above there is an involution of system (31) with the corresponding factor

$$\widetilde{F}_1 = \frac{Gx+1}{Gx-1}$$

(it can be also checked by direct computations that (41) is indeed a φ-involution of system (31)). From (41) we see that $Fix\,\tilde{\varphi}$ is the line $x = 0$.

Consider now system (32) (corresponding to $\mathbf{V}(I_2)$). Using (24) we find that the systems from $\mathbf{V}(I_2)$ are orbitally $\tilde{\varphi}_2$-reversible with

$$\tilde{F}_2(x,y) = \frac{A(1+Gx)}{-A-3G+AGx}$$

and

$$\tilde{\varphi}_2(x,y) = \left(\frac{3-Ax}{A}, \frac{9B(2Ax-3)-2A^2y(Gx+1)}{2A(A(Gx-1)-3G)}\right). \tag{42}$$

In this case $Fix\,\tilde{\varphi}$ is the line $x = \frac{3}{2A}$, which goes through the saddle point $\left(\frac{3}{2A}, \frac{27B}{4A(2A+3G)}\right)$. Since the line of symmetry does not pass through the center O at the origin, the singular point symmetric to O with respect to the line is also a center.

It was found in [32] that system (10) is orbitally φ-reversible with respect to the involution

$$\varphi = \left(\alpha(x,y), \frac{x(\alpha(x,y)+c+x)}{\alpha(x,y)}\right) \tag{43}$$

where $\alpha(x,y)$ is defined implicitly by the equation

$$\alpha(x,y)^2 + (x+c)\alpha(x,y) - xy = 0.$$

Therefore, according to Lemma 1, systems (36) are $\tilde{\varphi}$-reversible with respect to the involution

$$\tilde{\varphi} = \psi \circ \varphi \circ \psi^{-1}. \tag{44}$$

For involution (43) the fixed points are on the line $y = x + c$. The computations show that this line is mapped by (5) to a line passing through the origin. □

Remark 4. *Not all systems (33) correspond to systems (10), but only those whose coefficients satisfy the condition $h_1 = 0$, where h_1 is the polynomial defined by (15). Therefore, by Lemma 1, systems (33) whose coefficients satisfy the condition $h_1 = 0$ are $\tilde{\varphi}$-reversible with respect to (44).*

For systems (34) and (35) the situation is similar. However systems (34) and (35) correspond to complex systems from family (10), so in this case involution (44) is complex.

5. Bifurcations of Limit Cycles

In this section we study limit cycles that bifurcate from the origin under small perturbation of the centers of families given in Theorem 2.

Let $V = \mathbf{V}(I)$ be the variety of the ideal $I = \langle f_1, ..., f_m \rangle \subset k[x_1, ..., x_n]$ and let p be a point from V. By $T_p = p + \{v \mid J_p(I)v = 0\}$, where $J_p(I)$ is the Jacobian of polynomials $f_1, ..., f_m$, calculated at p, we denote the tangent space to V at p. Then $dim(T_p) = n - rank(J_p(I))$. By definition p is a smooth point of V if $dim(T_p) = dim(V_p)$.

Methods to obtain lower bounds for cyclicity using linear parts of focus quantities were proposed in [38,39]. In order to obtain the precise bound for cyclicity in some cases, they were further developed in [40].

As is mentioned in Section 2, the real polynomial system (1) is written in the complex form as

$$\dot{X} = i\left(X - \sum_{p+q=1}^{n-1} a_{pq} X^{p+1} \bar{X}^q\right), \tag{45}$$

where a_{pq} are complex parameters. Let g_{kk} be the focus quantities of system (11). Denote by $g_{kk}^{\mathbb{R}}$ the polynomials obtained after replacing b_{qp} with $\bar{a}_{pq}$ in g_{kk}. Then the center variety of real system (45) is the variety $V^{\mathbb{R}}$ of the ideal $\mathcal{B}^{\mathbb{R}} = \langle g_{11}^{\mathbb{R}}, g_{22}^{\mathbb{R}}, \ldots \rangle$. Let $\mathcal{B}_k^{\mathbb{R}}$ be the ideal $\langle g_{11}^{\mathbb{R}}, \ldots, g_{kk}^{\mathbb{R}} \rangle$. The following statement is a slightly reformulated Theorem 2.1 of [40].

Theorem 4. *Assume that for system (45) $p \in V^{\mathbb{R}}$ and rank $J_p(\mathcal{B}_k^{\mathbb{R}}) = k$. Then p lies on a component of $V^{\mathbb{R}}$ of codimension at least k and there are bifurcations of (45) that produce $k-1$ limit cycles locally from the center corresponding to the parameter value p.*

If, furthermore, p lies on a component C of $V^{\mathbb{R}}$ of codimension k, then p is a smooth point of the center variety, and the cyclicity of p at generic points of C is exactly $k-1$.

Thus, in some cases, the cyclicity of a generic point of a proper component of the center variety can be estimated if we know its dimension.

Proposition 2. *Dimensions of varieties V_i, $i = 1, 2, \ldots, 6$ given in the statement of Theorem 2 are 4, 4, 4, 2, 5 and 4, respectively.*

Proof. Consider the variety V_5. Using Theorem 1 we first check that (30), or more precisely,

$$\begin{array}{lll} D = t_1 = p_1(t_1), & A = t_4 = p_4(t_4), & C = -\dfrac{t_1}{2} - t_2 = p_7(t_1, t_2), \\ G = t_2 = p_2(t_2), & K = t_5 = p_5(t_5), & L = -\dfrac{4}{3} t_2 t_3 = p_8(t_2, t_3), \\ H = t_3 = p_3(t_3), & B = -\dfrac{2}{3} t_3 = p_6(t_3), & M = -\dfrac{3}{2} t_1 t_2 = p_9(t_1, t_2) \end{array} \tag{46}$$

is a parametrization of $\mathbf{V}(I_5)$. Indeed, computing the fifth elimination ideal of the ideal

$$\langle D - t_1, G - t_2, H - t_3, A - t_4, K - t_5, B + \frac{2}{3} t_3, C + \frac{t_1}{2} + t_2, L + \frac{4}{3} t_2 t_3, M + \frac{3}{2} t_1 t_2 \rangle$$

in the ring

$$\mathbb{Q}[t_1, t_2, t_3, t_4, t_5, A, B, C, D, G, H, K, L, M]$$

we obtain the ideal I_5. Computing now the rank of the matrix

$$\frac{\partial(p_1, p_2, \ldots, p_9)}{\partial(t_1, \ldots, t_5)}$$

at randomly taken points $t_1 = 2, t_2 = 1, t_3 = 3, t_4 = 4, t_5 = 12$ we see that it is equal to five. Therefore the dimension of the component is five.

The dimensions of the other components are computed similarly. □

In the next theorem we obtain estimations for the number of limit cycles bifurcating from the center at the origin for the components V_i, $i = 1, \ldots, 6$. Of course the number depends on the perturbed family. We consider the perturbations inside family (6) with $R = N = 0$, which we denote as A, the whole family (6) is denoted by B, and the complete family of cubic systems with a center at the origin; that is, the family

$$\dot{X} = i(X - \sum_{p+q=1}^{2} a_{pq} X^{p+1} \bar{X}^q), \tag{47}$$

where a_{pq} are complex parameters. We denote this family by C.

Theorem 5. *(1) The lower bounds for the number of limit cycles for systems corresponding to the generic points of the components V_i ($i = 1, \ldots, 5$) of the center variety under small perturbations in families* A, B *and* C *are given in the following table:*

	V_1	V_2	V_3	V_4	V_5
A	3	2	4	2	2
B	4	3	5	3	3
C	6	3	6	3	5

(2) There are systems from the component V_6 for which perturbations in families A, B *and* C *yield 4, 5 and 6 limit cycles, respectively.*

Proof. (1) Consider the component V_5. By Proposition 2 its dimension is 5. To treat the bifurcations inside the family A we substitute in the focus quantities v_i of system (6) $R = N = 0$. We first compute the Jacobian matrix of the polynomials v_1, v_2, v_3 with respect to the variables $A, B, C, D, G, H, K, L, M$ and then with the routine `MatrixRank` of MATHEMATICA we find that its rank on generic points of the surface (46) is 3. Computing the Jacobian matrix of the polynomials $v_1, \ldots, v_7$ we observe that its rank is also equal to 3. By Theorem 4 this means that two limit cycles can bifurcate from the origin after small perturbations in family A.

When we do not impose in system (6) the condition $N = R = 0$ and compute the Jacobian matrices with respect to the variables $A, B, C, D, G, H, K, L, M, N, R$, we obtain that the rank of the Jacobian matrix of $v_1, \ldots, v_4$ on the generic points of V_5 is 4 (and it remains 4 if we use more polynomials v_i); that is, 3 limit cycles can bifurcate from the origin in the case of perturbations inside family B.

To treat the family C we substitute in the focus quantities g_{kk} $a_{10} = A_{10} + iB_{10}, b_{01} = A_{10} - iB_{10}, a_{01} = A_{01} + iB_{01}, b_{10} = A_{01} - iB_{01}, \ldots$. Then we find that $rank J(\mathcal{B}_5^{\mathbb{R}}) = \cdots = rank J(\mathcal{B}_8^{\mathbb{R}}) = 6$ at generic points of V_5. By Theorem 4 this means that five limit cycles can bifurcate from the origin under small perturbations in the family C.

Since in all three cases the codimension of the component is greater than the rank of the Jacobians, we cannot get a sharp bound for cyclicity using Theorem 4.

The number of limit-cycles bifurcating from the origin for the other components V_i ($i = 1, \ldots, 4$) is determined similarly.

(2) For the component V_6 the calculations become too difficult since we do not have a rational parametrization of the component. For this reason, for the component V_6, we were not able to compute the rank at generic points; however, we checked the rank on some randomly chosen points of the component obtaining the estimations given in the statement of the theorem. □

6. Conclusions

Using three families of reversible systems of [29] we have found six components of the center variety. We do not know if they are proper components of the variety, but it was shown that all six families are Darboux integrable.

From the study we see that there is no one-to-one correspondence between the families of [29] and the components of the center variety. In fact, systems of the same family of [29] can correspond to different components of the center variety. Moreover, they not necessary correspond to whole components, but in three cases of the obtained six, they correspond to subcomponents of the Darboux integrable systems. In [29] only real systems were treated. However, if we consider them as complex systems, then some of the systems correspond to real systems (6) having a center via complex affine transformations. This indicates that to better understand the phenomenon of time reversibility and its relation to integrability, it is worthwhile to study it in a complex setting.

Checking the independence of focus quantities at the points of the obtained components of the center variety, we gave some estimations for the number of limit-cycles bifurcating from the center at the origin.

We also discussed the orbital φ-reversibility of the obtained families. For systems CR_5^8 and CR_7^9 it was shown in [32] that there are some systems in the families that are orbitally φ-reversible and have a center. Theorem 2 and Proposition 1 determine all systems in the families CR_5^8 and CR_7^9 with such properties. The existence of centers in the family CR_{10}^8 was not discussed in [32].

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Appendix A

$$
\begin{aligned}
&V = \langle 2a_1 - a_0 D - a_0 G + 2a_0 b_2 n, a_1^2 - a_0^2 DG + 2a_0 a_1 b_2 n, -a_1^2 - a_0 b_2 H + 2a_0^2 a_1 b_2 m + 2a_0 a_1 b_2 cm + \\
&8a_0^3 b_2^2 mn + 4a_0^2 b_2^2 cmn + 4a_0^2 b_2^2 n^2, -a_1^3 - a_0^2 b_2 GH + 2a_0^2 a_1^2 b_2 m + 2a_0 a_1^2 b_2 cm + 8a_0^3 a_1 b_2^2 mn + \\
&4a_0^2 a_1 b_2^2 cmn + 4a_0^2 a_1 b_2^2 n^2, 2a_1 b_2 m - M, -C + 2a_0 b_2 m + 2b_2 n, -8a_0 a_1^2 + a_0 b_2^2 - 4a_1^2 c + 8a_0^3 a_1 b_2 m + \\
&12a_0^2 a_1 b_2 cm + 4a_0 a_1 b_2 c^2 m + 16a_0^5 b_2^2 m^2 + 16a_0^4 b_2^2 cm^2 + 4a_0^3 b_2^2 c^2 m^2 + 4a_0^2 a_1 b_2 n + \\
&4a_0 a_1 b_2 cn + 16a_0^4 b_2^2 mn + 8a_0^3 b_2^2 cmn + 4a_0^3 b_2^2 n^2, -4a_1^2 - 3a_0 B b_2 + 2a_0 a_1 b_2 cm + \\
&16a_0^4 b_2^2 m^2 + 8a_0^3 b_2^2 cm^2 - 6a_0 a_1 b_2 n + 16a_0^3 b_2^2 mn + 4a_0^2 b_2^2 cmn + 4a_0^2 b_2^2 n^2, -2a_1^3 - \\
&3a_0^2 b_2 L - 4a_0^2 a_1^2 b_2 m - 2a_0 a_1^2 b_2 cm + 16a_0^4 a_1 b_2^2 m^2 + 8a_0^3 a_1 b_2^2 cm^2 - 6a_0 a_1^2 b_2 n - \\
&4a_0^2 a_1 b_2^2 cmn - 4a_0^2 a_1 b_2^2 n^2, -4a_0 a_1^3 - A a_0^2 b_2^2 - 2a_1^3 c - 12a_0^3 a_1^2 b_2 m - 6a_0^2 a_1^2 b_2 cm + \\
&24a_0^5 a_1 b_2^2 m^2 + 28a_0^4 a_1 b_2^2 cm^2 + 8a_0^3 a_1 b_2^2 c^2 m^2 + 32a_0^7 b_2^3 m^3 + 32a_0^6 b_2^3 cm^3 + \\
&8a_0^5 b_2^3 c^2 m^3 - 10a_0^2 a_1^2 b_2 n - 4a_0 a_1^2 b_2 cn + 4a_0^4 a_1 b_2^2 mn + 4a_0^3 a_1 b_2^2 cmn + 32a_0^6 b_2^3 m^2 n + \\
&16a_0^5 b_2^3 cm^2 n - 4a_0^3 a_1 b_2^2 n^2 + 8a_0^5 b_2^3 mn^2, -a_0 b_2 K - 4a_0 a_1^3 m - \\
&2a_1^3 cm - 8a_0^3 a_1^2 b_2 m^2 - 12a_0^2 a_1^2 b_2 cm^2 - 4a_0 a_1^2 b_2 c^2 m^2 + 32a_0^5 a_1 b_2^2 m^3 + \\
&32a_0^4 a_1 b_2^2 cm^3 + 8a_0^3 a_1 b_2^2 c^2 m^3 - 2a_1^3 n - 20a_0^2 a_1^2 b_2 mn - 12a_0 a_1^2 b_2 cmn - 16a_0^3 a_1 b_2^2 cm^2 n - \\
&8a_0^2 a_1 b_2^2 c^2 m^2 n - 8a_0 a_1^2 b_2 n^2 - 24a_0^3 a_1 b_2^2 mn^2 - 16a_0^2 a_1 b_2^2 cmn^2 - 8a_0^2 a_1 b_2^2 n^3 \rangle
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
&\hat{I}_3 = \langle KD^2 - \tfrac{GKD}{2} + \tfrac{GMD}{2} + \tfrac{M^2}{2} + \tfrac{KM}{2}, CK + 3DK - 2GK + CM + DM, D^2 + CD - GD - M,\\
&\tfrac{25KG^6}{27} - \tfrac{MG^6}{27} - \tfrac{50}{27}DKG^5 - \tfrac{16}{27}CMG^5 - \tfrac{32}{27}DMG^5 + \tfrac{25K^2G^4}{27} - \tfrac{16M^2G^4}{27} + B^2KG^4 + \tfrac{1}{3}B^2MG^4 - \tfrac{8}{27}C^2MG^4 +\\
&\tfrac{8}{27}D^2MG^4 + \tfrac{20}{27}KMG^4 + \tfrac{4}{27}DK^2G^3 - \tfrac{20}{27}CM^2G^3 - \tfrac{44}{27}DM^2G^3 - \tfrac{64}{27}DKMG^3 - \tfrac{8M^3G^2}{9} +\\
&\tfrac{1}{3}B^2M^2G^2 - \tfrac{4}{9}C^2M^2G^2 + \tfrac{4}{9}D^2M^2G^2 - \tfrac{16}{27}KM^2G^2 + \tfrac{31}{27}K^2MG^2 + \tfrac{4}{3}B^2KMG^2 - \tfrac{4}{27}CM^3G - \tfrac{4}{9}DM^3G -\\
&\tfrac{16}{27}DKM^2G - \tfrac{8M^4}{27} - \tfrac{4C^2M^3}{27} + \tfrac{4D^2M^3}{27} - \tfrac{8KM^3}{27} + \tfrac{8K^2M^2}{27} + \tfrac{1}{3}B^2KM^2, -\tfrac{25}{36}K^2G^5 + \tfrac{M^2G^5}{36} +\\
&\tfrac{10}{9}KMG^5 + \tfrac{25}{18}DK^2G^4 - \tfrac{4}{9}CM^2G^4 - \tfrac{4}{9}DM^2G^4 - \tfrac{5}{6}DKMG^4 - \tfrac{25K^3G^3}{36} - \tfrac{3}{4}B^2K^2G^3 - \tfrac{1}{4}B^2M^2G^3 +\\
&\tfrac{13}{4}KM^2G^3 - \tfrac{35}{18}K^2MG^3 - B^2KMG^3 - \tfrac{1}{9}DK^3G^2 - \tfrac{14}{9}CM^3G^2 - 2DM^3G^2 - \tfrac{32}{9}DKM^2G^2 +\\
&\tfrac{55}{18}DK^2MG^2 - \tfrac{4M^4G}{9} - \tfrac{1}{4}B^2M^3G - \tfrac{2}{9}C^2M^3G + \tfrac{2}{9}D^2M^3G - \tfrac{8}{9}KM^3G + \tfrac{10}{9}K^2M^2G - \tfrac{3}{2}B^2KM^2G -\\
&\tfrac{14}{9}K^3MG - \tfrac{7}{4}B^2K^2MG + \tfrac{2CM^4}{9} + \tfrac{2DM^4}{9} + \tfrac{4}{9}DKM^3 - \tfrac{4}{9}DK^2M^2 + \tfrac{1}{2}B^2DKM^2 + \tfrac{8}{9}DK^3M + B^2DK^2M,\\
&-\tfrac{25KG^4}{36} + \tfrac{MG^4}{36} + \tfrac{25}{18}DKG^3 + \tfrac{4}{9}CMG^3 + \tfrac{8}{9}DMG^3 - \tfrac{25K^2G^2}{36} + \tfrac{4M^2G^2}{9} - \tfrac{3}{4}B^2KG^2 - \tfrac{1}{4}B^2MG^2 + \tfrac{2}{9}C^2MG^2 -\\
&\tfrac{2}{9}D^2MG^2 + \tfrac{55}{36}KMG^2 + \tfrac{8}{9}DK^2G - \tfrac{4}{9}CM^2G - \tfrac{4}{9}DM^2G + B^2DKG + \tfrac{1}{2}B^2DMG - \tfrac{8}{9}DKMG + \tfrac{B^2M^2}{4} +\\
&\tfrac{2K^2M}{9} + \tfrac{1}{4}B^2KM, \tfrac{25KG^5}{18} - \tfrac{MG^5}{18} - \tfrac{25}{9}DKG^4 - \tfrac{8}{9}CMG^4 - \tfrac{16}{9}DMG^4 + \tfrac{25K^2G^3}{18} - \tfrac{8M^2G^3}{9} + \tfrac{3}{2}B^2KG^3 +\\
&\tfrac{1}{2}B^2MG^3 - \tfrac{4}{9}C^2MG^3 + \tfrac{4}{9}D^2MG^3 + \tfrac{5}{2}KMG^3 + \tfrac{2}{9}DK^2G^2 - \tfrac{16}{9}CM^2G^2 - \tfrac{32}{9}DM^2G^2 + B^2DMG^2 -\\
&\tfrac{16}{3}DKMG^2 - \tfrac{16M^3G}{9} + \tfrac{3}{2}B^2M^2G - \tfrac{8}{9}C^2M^2G + \tfrac{8}{9}D^2M^2G - \tfrac{32}{9}KM^2G + \tfrac{28}{9}K^2MG + \tfrac{7}{2}B^2KMG +\\
&\tfrac{8CM^3}{9} + \tfrac{8DM^3}{9} + \tfrac{16}{9}DKM^2 - \tfrac{16}{9}DK^2M - 2B^2DKM, \tfrac{MG^4}{18} + B^2DG^3 + DKG^3 + \tfrac{2}{9}CMG^3 + \tfrac{2}{3}DMG^3 +\\
&\tfrac{4M^2G^2}{9} + \tfrac{1}{2}B^2MG^2 + \tfrac{2}{9}C^2MG^2 - \tfrac{2}{9}D^2MG^2 + \tfrac{7}{18}KMG^2 + \tfrac{2}{9}CM^2G + \tfrac{2}{3}DM^2G + B^2DMG + \tfrac{8}{9}DKMG +\\
&\tfrac{4M^3}{9} + \tfrac{B^2M^2}{2} + \tfrac{2C^2M^2}{9} - \tfrac{2D^2M^2}{9} + \tfrac{4KM^2}{9}, -\tfrac{MG^3}{18} - B^2DG^2 - \tfrac{1}{2}DKG^2 - \tfrac{2}{9}CMG^2 - \tfrac{2}{3}DMG^2 + B^2D^2G -\\
&\tfrac{4M^2G}{9} - \tfrac{1}{2}B^2MG - \tfrac{2}{9}C^2MG + \tfrac{2}{9}D^2MG - \tfrac{8KMG}{9} + \tfrac{2CM^2}{9} + \tfrac{2DM^2}{9} + \tfrac{1}{2}B^2DM + \tfrac{4DKM}{9}, \tfrac{4MC^3}{9} +\\
&\tfrac{4}{9}GMC^2 + \tfrac{4M^2C}{9} + B^2MC + \tfrac{1}{9}G^2MC + 2B^2DG^2 - \tfrac{8DM^2}{9} + \tfrac{20GM^2}{9} + 2DG^2K + \tfrac{4D^3M}{9} + \tfrac{7}{3}DG^2M +\\
&B^2DM + 2B^2GM - \tfrac{16}{9}D^2GM - \tfrac{16DKM}{9} + \tfrac{32GKM}{9}, -\tfrac{G^4}{9} - \tfrac{CG^3}{3} + DG^3 + B^2G^2 - \tfrac{4D^2G^2}{3} + \tfrac{25KG^2}{9} + \tfrac{4MG^2}{3} +\\
&\tfrac{4C^3G}{9} + \tfrac{4D^3G}{9} + B^2CG - B^2DG - \tfrac{32DKG}{9} - \tfrac{20DMG}{9} - \tfrac{8M^2}{9} - B^2M - \tfrac{4C^2M}{9} + \tfrac{4D^2M}{9} - \tfrac{8KM}{9}, \tfrac{4C^4}{9} - \tfrac{4GC^3}{9} +\\
&B^2C^2 - \tfrac{G^2C^2}{3} + \tfrac{4MC^2}{3} + \tfrac{2G^3C}{9} - \tfrac{8GMC}{9} - \tfrac{4D^4}{9} + \tfrac{G^4}{9} - B^2D^2 - B^2G^2 - D^2G^2 + \tfrac{4D^3G}{3} + 4B^2DG + \tfrac{25G^2K}{9} -\\
&\tfrac{32DGK}{9} + 3B^2M + \tfrac{4D^2M}{9} - \tfrac{2G^2M}{9} - \tfrac{20DGM}{9} - \tfrac{8KM}{9}, AB + CB + \tfrac{2AH}{3} - \tfrac{DH}{3} + \tfrac{2GH}{3} + L, H, A + C + D\rangle
\end{aligned}
$$

Appendix C

$$
\begin{aligned}
&W = \langle -3D - 3G - 8a_0b_2m - 6b_2n, 8a_0^3b_0b_2m^3 - 27Hn - 24a_0^2b_0b_2m^2n - 36a_0^2b_2mn^2, -9DG + 16a_0^2b_2^2m^2 +\\
&24a_0b_2^2mn, -32a_0^4b_0b_2^2m^4 - 81GHn + 48a_0^3b_0b_2^2m^3n + 144a_0^3b_2^2m^2n^2 + 144a_0^2b_0b_2^2m^2n^2 + 216a_0^2b_2^2mn^3, 4a_0^4b_0^2m^4 -\\
&24a_0^4b_0m^3n + 48a_0^3b_0^2m^3n + 81n^2 + 36a_0^4m^2n^2 - 144a_0^3b_0m^2n^2 + 144a_0^2b_0^2m^2n^2, -C + 2a_0b_2m + 2b_2n,\\
&-16a_0^3b_0b_2m^3 - 81Bn + 48a_0^2b_0b_2m^2n + 72a_0^2b_2mn^2, -8a_0^2b_2^2m^2 - 3M - 12a_0b_2^2mn, -8a_0^5b_0^2b_2m^5 +\\
&192a_0^5b_0b_2m^4n - 243An^2 - 504a_0^5b_2m^3n^2 + 1224a_0^4b_0b_2m^3n^2 + 216a_0^3b_0^2b_2m^3n^2 - 1080a_0^4b_2m^2n^3 +\\
&2376a_0^3b_0b_2m^2n^3 - 432a_0^2b_0^2b_2m^2n^3, 128a_0^4b_0b_2^2m^4 - 243Ln - 192a_0^3b_0b_2^2m^3n - 576a_0^3b_2^2m^2n^2 - 576a_0^2b_0b_2^2m^2n^2 -\\
&864a_0^2b_2^2mn^3, -160a_0^6b_0^2b_2^2m^6 + 720a_0^5b_0^2b_2^2m^5n - 729KKn^2 + 1440a_0^6b_2^2m^4n^2 + 1440a_0^5b_0b_2^2m^4n^2 + 6480a_0^5b_2^2m^3n^3 -\\
&2160a_0^4b_0b_2^2m^3n^3 - 2160a_0^3b_0^2b_2^2m^3n^3 + 6480a_0^4b_2^2m^2n^4 - 6480a_0^3b_0b_2^2m^2n^4\rangle.
\end{aligned}
$$

Appendix D

$$
\begin{gathered}
\hat{I}_4 = \langle L, H, 2C + 2D + G, B, 36DK - 45GK - 2AM - 16DM + 24GM, 3DG + 2M, 24AD + 12D^2 - 30AG\\
-45G^2 - 8M, 48A^2 - 12D^2 + 6AG - 99G^2 + 216K - 56M, 135G^2K + 6AGM - 72G^2M + 72KM - 32M^2,\\
30AG^2 + 45G^3 + 16AM + 8DM + 8GM\rangle\\
\hat{I}_5 = \langle 2C + D + 2G, 3B + 2H, 9DL - 8HM, 4GH + 3L, 3DG + 2M, 60A^2D + 180AD^2 + 135D^3 - 72A^2G\\
+108AG^2 + 72G^3 + 540DH^2 + 270DK - 324GK + 486HL + 188AM + 12DM + 104GM, 20A^3 - 135AD^2\\
-135D^3 + 66A^2G - 84AG^2 - 56G^3 + 180AH^2 - 540DH^2 + 90AK + 288GK - 918HL - 144AM - 36DM\\
-72GM, 486A^2L^2 - 729AGL^2 - 486G^2L^2 + 4374H^2L^2 - 240A^2KM - 24AGKM + 48G^2KM - 2160H^2KM\\
+432AHLM + 224A^2M^2 + 24ADM^2 - 144D^2M^2 - 16AGM^2 - 64G^2M^2 - 288H^2M^2 + 2187KL^2\\
-1080K^2M - 2592L^2M + 1008KM^2 - 128M^3, 108A^2GL - 162AG^2L - 108G^3L - 80A^2HM - 240ADHM\\
-180D^2HM - 720H^3M + 486GKL - 729HL^2 - 360HKM - 282ALM - 156GLM - 16HM^2, 12,150D^2K^2\\
+7776G^2K^2 - 6075AGL^2 + 24,300G^2L^2 + 240A^2KM + 1080ADKM - 12,960D^2KM - 840AGKM\\
-6960G^2KM - 10,800H^2KM + 5400AHLM - 200A^2M^2 - 600ADM^2 + 3600D^2M^2 + 400AGM^2 + 1600G^2M^2\\
+28,800H^2M^2 - 10,935KL^2 + 14,040K^2M + 40,500L^2M - 13,680KM^2 + 3200M^3, 4050D^2HK + 1080AG^2L\\
+540G^3L + 80A^2HM + 1860ADHM + 180D^2HM - 3600H^3M - 1944GKL - 3645HL^2 + 4680HKM + 2010ALM\\
+1020GLM - 560HM^2, 240A^2GK + 24AG^2K - 48G^3K - 360A^2HL - 3240H^3L - 224A^2GM + 16AG^2M
\end{gathered}
$$

$$
\begin{gathered}
+64G^3M-960AH^2M-720DH^2M+1080GK^2-3240HKL-945AL^2+1890GL^2-1008GKM-288HLM+16AM^2 \\
-96DM^2+128GM^2, 675D^3K+270AD^2M+16A^2GM+136AG^2M+64G^3M-720DH^2M+720DKM \\
-216GKM-648HLM+256AM^2-96DM^2+128GM^2, 450AD^2K+288AG^2K+144G^3K-540AD^2M \\
-405D^3M-8A^2GM-308AG^2M-152G^3M-480AH^2M-1620DH^2M-405AL^2+1620GL^2+480AKM \\
-330DKM+348GKM-2466HLM-548AM^2-132DM^2-184GM^2, 240A^2H^2+720ADH^2+540D^2H^2 \\
+2160H^4+120A^2K+12AGK-24G^2K+2160H^2K+630AHL-112A^2M-12ADM+72D^2M \\
+8AGM+32G^2M+192H^2M+540K^2+945L^2-504KM+64M^2, 1440AG^3+720G^4-4050D^2K-2592G^2K \\
-80A^2M-1860ADM-180D^2M+2680AGM+1360G^2M+3600H^2M+3645L^2-4680KM+560M^2, 2880A^2G^2 \\
-720G^4-12,150D^2K+5184G^2K+1360A^2M-780ADM+3060D^2M+520AGM-80G^2M+25,200H^2M \\
+25,515L^2-6840KM+2000M^2, 3240AG^2L^2+1620G^3L^2+10,800DH^2KM+240A^2HLM-10,800H^3LM \\
+4960AH^2M^2+480DH^2M^2-5832GKL^2-10,935HL^3+14,040HKLM+6030AL^2M+3060GL^2M \\
-1680HLM^2, 23,328G^2K^2L-18,225AGL^3+72,900G^2L^3+32,400DHK^2M+720A^2KLM-2520AGKLM \\
-20,880G^2KLM-32,400H^2KLM+16,200AHL^2M+2880AHKM^2-34,560DHKM^2-600A^2LM^2+1200AGLM^2 \\
+4800G^2LM^2+86,400H^2LM^2-1600AHM^3+9600DHM^3-32,805KL^3+42,120K^2LM+121,500L^3M-41,040KLM^2 \\
+9600LM^3, 7776AG^2KL+3888G^3KL+10,800ADHKM-8640AG^2LM-4320G^3LM-12,960AH^2LM-160A^2HM^2 \\
-13,440ADHM^2-10,080D^2HM^2-40,320H^3M^2-10,935AL^3+43,740GL^3+12,960AKLM+10,368GKLM \\
-68,040HL^2M-8640HKM^2-15,360ALM^2-5280GLM^2-3200HM^3, 20,736G^3K^2+72,900G^3L^2-2304AG^2KM \\
-18,432G^3KM+54,000DH^2KM+2160A^2HLM-45,360H^3LM+64A^2GM^2+1024AG^2M^2+4096G^3M^2 \\
+27,360AH^2M^2+43,20DH^2M^2-58,320GKL^2-54,675HL^3-21,600DK^2M+34,560GK^2M+10,0440HKLM \\
+21,870AL^2M+118,260GL^2M-1920AKM^2+23,040DKM^2-33,792GKM^2-65,232HLM^2+1024AM^3-6144DM^3 \\
+8192GM^3, 129,600DH^2K^2M+2880A^2HKLM-129,600H^3KLM+64,800AH^2L^2M+11,520AH^2KM^2 \\
-138,240DH^2KM^2-2400A^2HLM^2+345,600H^3LM^2-6400AH^2M^3+38,400DH^2M^3-69,984GK^2L^2 \\
-131,220HKL^3+54,675AL^4-218,700GL^4+168,480HK^2LM+7560AKL^2M+62,640GKL^2M+48,6000HL^3M \\
-164,160HKLM^2-3600AL^2M^2-14,400GL^2M^2+38,400HLM^3, 32,400DH^3KM-38,880H^4LM+12,960AH^3M^2 \\
-7290AGL^3-3645G^2L^3-32,805H^2L^3-360A^2KLM-36AGKLM+72G^2KLM+35,640H^2KLM+16,200AHL^2M \\
+336A^2LM^2-24AGLM^2-96G^2LM^2-5616H^2LM^2+32AHM^3-192DHM^3+13,122KL^3-1620K^2LM \\
-9720L^3M+1512KLM^2-192LM^3, 32,400ADH^2KM-38,880AH^3LM-38,880ADH^2M^2-29,160D^2H^2M^2 \\
-116,640H^4M^2-17,496AGKL^2-8748G^2KL^2-32,805AHL^3+38,880AHKLM+19,440AGL^2M+9720G^2L^2M \\
-204,120H^2L^2M+240A^2KM^2+24AGKM^2-48G^2KM^2-21,600H^2KM^2-44,820AHLM^2-224A^2M^3-24ADM^3 \\
+144D^2M^3+16AGM^3+64G^2M^3-9216H^2M^3-98,415L^4-23,328KL^2M+1080K^2M^2+13,770L^2M^2-1008KM^3 \\
+128M^4, 188,956,800H^3KL^2M-78,732,000AH^2L^3M-1,728,000A^2HK^2M^2-155,520,000H^3K^2M^2-10,886,400AH^2KLM^2 \\
-446,148,000H^3L^2M^2+3,052,800A^2HKM^3+172,800ADHKM^3+160,185,600H^3KM^3+276,480AG^2LM^3 \\
+138,240G^3LM^3+5,184,000AH^2LM^3-1,323,520A^2HM^4+332,160ADHM^4+910,080D^2HM^4-40,665,600H^3M^4 \\
+85,030,560GK^2L^3+159,432,300HKL^4-66,430,125AL^5+265,720,500GL^5-188,956,800HK^2L^2M-5,248,800AKL^3M \\
-7,3483,200GKL^3M-590,490,000HL^4M-7,776,000HK^3M^2+129,600AK^2LM^2-259,200GK^2LM^2+16,7670,000HKL^2M^2 \\
+1,093,500AL^3M^2+15,309,000GL^3M^2+13,737,600HK^2M^3-21,600AKLM^3+63,936GKLM^3-3,203,7120HL^2M^3 \\
-5,771,520HKM^4+442,560ALM^4-26,880GLM^4+624,640HM^5, 3,149,280H^4L^2M-2,332,800H^4KM^2-1,049,760AH^3LM^2 \\
+590,490AGL^4+295,245G^2L^4+2,657,205H^2L^4+46,656AGKL^2M+23,328G^2KL^2M-314,9280H^2KL^2M \\
-1,312,200AHL^3M+14,400A^2K^2M^2+1440AGK^2M^2-2880G^2K^2M^2+129,600H^2K^2M^2-25,920AHKLM^2 \\
-38,880AGL^2M^2-19,440G^2L^2M^2+69,9840H^2L^2M^2-26,880A^2KM^3-1440ADKM^3+8640D^2KM^3 \\
-384AGKM^3+6528G^2KM^3-103,680H^2KM^3+21,600AHLM^3+12,544A^2M^4+1344ADM^4-8064D^2M^4 \\
-896AGM^4-3584G^2M^4-2304H^2M^4-1,062,882KL^4+787,320L^4M+64,800K^3M^2+155,520KL^2M^2 \\
-120,960K^2M^3-129,600L^2M^3+64,128KM^4-7168M^5, 116,640AH^3L^2M-86,400AH^3KM^2+349,920H^4LM^2+103,680AH^3M^3 \\
+77,760DH^3M^3+52,488AGKL^3+26,244G^2KL^3+98,415AHL^4-116,640AHKL^2M \\
-58,320AGL^3M-29,160G^2L^3M+612,360H^2L^3M-720A^2KLM^2-72AGKLM^2+ \\
144G^2KLM^2+64,800H^2KLM^2+134,460AHL^2M^2+672A^2LM^3-48AGLM^3 \\
-192G^2LM^3+27,648H^2LM^3+64AHM^4-384DHM^4+295,245L^5+69,984KL^3M \\
-3240K^2LM^2-41,310L^3M^2+3024KLM^3-384LM^4\rangle
\end{gathered}
$$

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