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Formulas and Properties, Their Links and Characteristics

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Abstract: In this work, we study links between first-order formulas and arbitrary properties for families of theories, classes of structures and their isomorphism types. Possibilities for ranks and degrees for formulas and theories with respect to the given properties are described. Characteristics of generic sentences and generic theories with respect to these properties are described and characterized.

Keywords: formula; property; rank; degree; generic sentence; generic theory

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1. Introduction

First-order formulas are used to express semantic and syntacticdefinable properties. They are broadly used in mathematics and other applications to study and to classify various aspects of reality. Definability is connected with the decidability, computability and complexity of mathematical objects [1–3]. This tool has been used to solve a serious of famous mathematical problems including the quantifier elimination and the decidability of an algebraically closed field for real closed fields [4–6], etc. Reducibilities for definable sets and their properties are studied and described in [7].

Since in general there are more naturalthan definable properties, these formulas can express them in a partial way. In the present paper, we study the links between formulas and arbitrary properties, considering characteristics reflecting measures of their correspondence.

The paper is organized as follows. Preliminary notions, notations and results are represented in Section 1. In Section 2, we consider the links between formulas and properties for semantic and syntactic families. We study and characterize these links with respect to constructions of formulas, set-theoretic operations and closures. In Section 3, we study, characterize and describe rank values and degree values for formulas with respect to the given properties. In Section 4, we describe spectra for cardinalities of definable properties. In Section 5, generic sentences and theories with respect to properties are introduced, and their links and ranks and described. Some illustrations of the considered links between formulas and properties are considered in Section 6.

Throughout this work, we use the standard terminology in mathematical logic [6], as well as the concepts, notations and results of [8–10].

2. Preliminaries

Let Σ be a language. If Σ is relational, we denote by $\mathcal{T}_\Sigma$ the family of all theories of the language Σ. If Σ contains functional symbols f, then $\mathcal{T}_\Sigma$ is the family of all theories of the language Σ', which is obtained by replacements of all n-ary symbols f with $(n+1)$-ary predicate symbols R_f interpreted by $R_f = \{(\bar{a}, b) \mid f(\bar{a}) = b\}$.

Following [8], we define the *rank* $\mathrm{RS}(\cdot)$ for families $\mathcal{T} \subseteq \mathcal{T}_\Sigma$, similar to the Morley rank for a fixed theory, and a hierarchy with respect to these ranks in the following way.

By $F(\Sigma)$, we denote the set of all formulas in the language Σ, and by $\mathrm{Sent}(\Sigma)$, we denote the set of all sentences in $F(\Sigma)$.

For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$, we denote by $\mathcal{T}_\varphi$ the set of all theories $T \in \mathcal{T}$ with $\varphi \in T$.

Any set $\mathcal{T}_\varphi$ is called the *φ-neighborhood*, or simply a *neighborhood*, for $\mathcal{T}$, or the (*φ-*)*definable* subset of $\mathcal{T}$. The set $\mathcal{T}_\varphi$ is also called (*formula* or *sentence-*)*definable* (by the sentence φ) with respect to $\mathcal{T}$, (*sentence-*)$\mathcal{T}$*-definable*, or simply *s-definable*.

Definition 1 ([8]). *For the empty family $\mathcal{T}$, we put the rank* $\mathrm{RS}(\mathcal{T}) = -1$; *for finite nonempty families $\mathcal{T}$, we put* $\mathrm{RS}(\mathcal{T}) = 0$; *and for infinite families, $\mathcal{T}$ —* $\mathrm{RS}(\mathcal{T}) \geq 1$. *For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$, we put*$\mathrm{RS}(\mathcal{T}) \geq \alpha$ *if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that* $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$. *If α is a limit ordinal, then* $\mathrm{RS}(\mathcal{T}) \geq \alpha$ *if* $\mathrm{RS}(\mathcal{T}) \geq \beta$ *for any $\beta < \alpha$. We set* $\mathrm{RS}(\mathcal{T}) = \alpha$ *if* $\mathrm{RS}(\mathcal{T}) \geq \alpha$ *and* $\mathrm{RS}(\mathcal{T}) \ngeq \alpha + 1$. *If* $\mathrm{RS}(\mathcal{T}) \geq \alpha$ *for any α, we put* $\mathrm{RS}(\mathcal{T}) = \infty$.

*A family $\mathcal{T}$ is called e-*totally transcendental, *or* totally transcendental, *if* $\mathrm{RS}(\mathcal{T})$ *is an ordinal.*

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, *we define the* degree $\mathrm{ds}(\mathcal{T})$ *of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that* $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Definition 2 ([11]). *An infinite family $\mathcal{T}$ is called e-*minimal *if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.*

According to this definition, a family $\mathcal{T}$ is *e*-minimal iff $\mathrm{RS}(\mathcal{T}) = 1$ and $\mathrm{ds}(\mathcal{T}) = 1$ [8] and iff $\mathcal{T}$ has a unique accumulation point [11].

In [12]. the notion of *E*-closure was introduced and characterized as follows:

Proposition 1. *If $\mathcal{T} \subseteq \mathcal{T}_\Sigma$ is an infinite set and $T \in \mathcal{T}_\Sigma \setminus \mathcal{T}$, then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure* Cl_E*) if and only if for any sentence $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

The following theorem characterizes the property of *e*-total transcendency for countable languages.

Theorem 1 ([8]). *For any family $\mathcal{T}$ with $|\Sigma(\mathcal{T})| \leq \omega$, the following conditionsare equivalent:*

(1) $|\mathrm{Cl}_E(\mathcal{T})| = 2^\omega$;
(2) $e\text{-}\mathrm{Sp}(\mathcal{T}) = 2^\omega$;
(3) $\mathrm{RS}(\mathcal{T}) = \infty$.

Theorem 2 ([9]). *For any language Σ, either* $\mathrm{RS}(\mathcal{T}_\Sigma)$ *is finite, if Σ consists of finitely many 0-ary and unary predicates, and finitely many constant symbols, or* $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$ *otherwise.*

For a language Σ, we denote by $\mathcal{T}_{\Sigma,n}$ the family of all theories in $\mathcal{T}_\Sigma$ having n-element models, $n \in \omega$, as well as by $\mathcal{T}_{\Sigma,\infty}$ the family of all theories in $\mathcal{T}_\Sigma$ having infinite models.

Theorem 3 ([9]). *For any language Σ, either* $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = 0$, *if Σ is finite, or $n = 1$ and Σ has finitely many predicate symbols, or* $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = \infty$ *otherwise.*

Theorem 4 ([9]). *For any language Σ, either* $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty})$ *is finite, if Σ is finite and without predicate symbols of arities $m \geq 2$ as well as without functional symbols of arities $n \geq 1$, or* $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty}) = \infty$ *otherwise.*

According to this definition, the families $\mathcal{T}_\Sigma$, $\mathcal{T}_{\Sigma,n}$, $\mathcal{T}_{\Sigma,\infty}$ are *E*-closed. Thus, combining Theorem 1 with Theorems 2–4, we obtain the following possibilities of cardinalities for the families $\mathcal{T}_\Sigma$, $\mathcal{T}_{\Sigma,n}$, $\mathcal{T}_{\Sigma,\infty}$ depending on Σ and $n \in \omega$:

Proposition 2. *For any language Σ, either$\mathcal{T}_\Sigma$ is countable, if Σ consistsof finitely many 0-ary and unary predicates and finitely many constant symbols, or $|\mathcal{T}_\Sigma| \geq 2^\omega$ otherwise.*

Proposition 3. *For any language Σ, either $\mathcal{T}_{\Sigma,n}$ is finite, if Σ is finite or $n = 1$ and Σ has finitely many predicate symbols, or $|\mathcal{T}_{\Sigma,n}| \geq 2^\omega$ otherwise.*

Proposition 4. *For any language Σ, either $\mathcal{T}_{\Sigma,\infty}$ is at most countable, if Σ is finite and without predicate symbols of arities $m \geq 2$ as well as without functional symbols of arities $n \geq 1$, or $|\mathcal{T}_{\Sigma,\infty}| \geq 2^\omega$ otherwise.*

Definition 3 ([10]). *If $\mathcal{T}$ is a family of theories and Φ is a set of sentences, then we put $\mathcal{T}_\Phi = \bigcap\limits_{\varphi\in\Phi} \mathcal{T}_\varphi$ and the set $\mathcal{T}_\Phi$ is called (type) or (diagram-)definable (by the set Φ) with respect to $\mathcal{T}$, (diagram-)$\mathcal{T}$ -definable or simply d-definable.*

Clearly, finite unions of d-definable sets are again d-definable. Considering infinite unions $\mathcal{T}'$ of d-definable sets $\mathcal{T}_{\Phi_i}$, $i \in I$, one can represent them by sets of sentences with infinite disjunctions $\bigvee\limits_{i\in I} \varphi_i$, $\varphi_i \in \Phi_i$. We call these unions $\mathcal{T}' d_\infty$*-definable* sets.

Definition 4 ([10]). *Let $\mathcal{T}$ be a family of theories, Φ be a set of sentences and α be an ordinal $\leq \mathrm{RS}(\mathcal{T})$ or -1. The set Φ is called an α-ranking for $\mathcal{T}$ if $\mathrm{RS}(\mathcal{T}_\Phi) = \alpha$. A sentence φ is called an α-ranking for $\mathcal{T}$ if $\{\varphi\}$ is an α-ranking for $\mathcal{T}$.*

The set Φ (the sentence φ) is called a *ranking* for $\mathcal{T}$ if it is an α-ranking for $\mathcal{T}$ with some α.

Proposition 5 ([10]). *For any ordinals $\alpha \leq \beta$, if $\mathrm{RS}(\mathcal{T}) = \beta$, then $\mathrm{RS}(\mathcal{T}_\varphi) = \alpha$ for some (α-ranking) sentence φ. Moreover, there are $\mathrm{ds}(\mathcal{T})$ pairwise $\mathcal{T}$-inconsistent β-ranking sentences for $\mathcal{T}$, and if $\alpha < \beta$, then there are infinitely many pairwise $\mathcal{T}$-inconsistent α-ranking sentences for $\mathcal{T}$.*

Theorem 5 ([10]). *Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, and let α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then, there is a d_∞-definable subfamily $\mathcal{T}^* \subset \mathcal{T}$ such that $\mathrm{RS}(\mathcal{T}^*) = \alpha$ and $\mathrm{ds}(\mathcal{T}^*) = n$.*

Theorem 6 ([13]). *For any two disjoint subfamilies $\mathcal{T}_1$ and $\mathcal{T}_2$ of an E-closed family $\mathcal{T}$, the following conditions are equivalent:*

(1) $\mathcal{T}_1$ and $\mathcal{T}_2$ are separated by some sentence φ: $\mathcal{T}_1 \subseteq \mathcal{T}_\varphi$ and $\mathcal{T}_2 \subseteq \mathcal{T}_{\neg\varphi}$;
(2) E-closures of $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint in $\mathcal{T}$: $\mathrm{Cl}_E(\mathcal{T}_1) \cap \mathrm{Cl}_E(\mathcal{T}_2) \cap \mathcal{T} = \emptyset$;
(3) E-closures of $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint: $\mathrm{Cl}_E(\mathcal{T}_1) \cap \mathrm{Cl}_E(\mathcal{T}_2) = \emptyset$.

Definition 5 ([12]). *Let $\mathcal{T}_0$ be a family of theories. A subset $\mathcal{T}_0' \subseteq \mathcal{T}_0$ is said to be generating if $\mathcal{T}_0 = \mathrm{Cl}_E(\mathcal{T}_0')$. The generating set $\mathcal{T}_0'$ (for $\mathcal{T}_0$) is minimal if $\mathcal{T}_0'$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}_0'$ is least if $\mathcal{T}_0'$ is contained in each generating set for $\mathcal{T}_0$.*

Theorem 7 ([12]). *If $\mathcal{T}_0'$ is a generating set for a E-closed set $\mathcal{T}_0$, then the following conditions are equivalent:*

(1) $\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;
(2) $\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;
(3) Any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$; i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_\varphi = \{T\}$;
(4) Any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_\varphi$; i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.

3. Relations between Formulas and Properties

Definition 6. *Let Σ be a language, $\varphi \rightleftharpoons \varphi(\overline{x})$ be a formula in $F(\Sigma)$ and P_s be a subclass of the class $K(\Sigma)$ of all structures $\mathcal{A}$ in the language Σ. We say that $\varphi(\overline{x})$* partially *(respectively,* totally*)* satisfies *P_s, denoted by $\varphi \rhd_{\mathrm{ps}} P_s$ or $\varphi \rhd_s^{\exists} P_s$ ($\varphi \rhd_{\mathrm{ts}} P_s$ or $\varphi \rhd_s^{\forall} P_s$), if there are $\mathcal{A} \in P_s$ and $\overline{a} \in A$ (for any $\mathcal{A} \in P_s$ there is $\overline{a} \in A$) such that $\mathcal{A} \models \varphi(\overline{a})$.*

If P_{is} is a subclass of the class $\mathrm{ITK}(\Sigma)$ of isomorphism types for the class $K(\Sigma)$, then we say that $\varphi(\overline{x})$ partially *(respectively,* totally*)* satisfies *P_{its}, denoted by $\varphi \rhd_{\mathrm{pits}} P_{\mathrm{its}}$ or $\varphi \rhd_{\mathrm{its}}^{\exists} P_{\mathrm{its}}$ ($\varphi \rhd_{\mathrm{tits}} P_{\mathrm{its}}$ or $\varphi \rhd_{\mathrm{its}}^{\forall} P_{\mathrm{its}}$) if $\varphi \rhd_{\mathrm{ps}} P_s$ ($\varphi \rhd_{\mathrm{ts}} P_s$), where P_s consists of allstructures whose isomorphism types belong to P_{its}.*

If P_t is a subset of the set $\mathcal{T}_\Sigma$ of all complete theories in the language Σ, then we say that $\varphi(\overline{x})$ partially *(respectively,* totally*)* satisfies *P_t, denoted by $\varphi \rhd_{\mathrm{pt}} P_t$ or $\varphi \rhd_t^{\exists} P_t$ ($\varphi \rhd_{\mathrm{tt}} P_t$ or $\varphi \rhd_t^{\forall} P_t$), if there are $T \in P_t$, $\mathcal{M} \models T$, and $\overline{a} \in M$ (for any $T \in P_t$ there are $\mathcal{M} \models T$ and $\overline{a} \in M$) such that $\mathcal{M} \models \varphi(\overline{a})$.*

We write $\not\rhd_\xi$ if a $\rhd_\xi$-relation does not hold.

Remark 1. *According to this definition, we have the following obvious properties:*

1. *If $P_s \neq \emptyset$ and $\varphi \rhd_{\mathrm{ts}} P_s$, then $\varphi \rhd_{\mathrm{ps}} P_s$. Similarly, if $\varphi \rhd_{\mathrm{tits}} P_{\mathrm{its}}$ for nonempty P_{its}, then $\varphi \rhd_{\mathrm{pits}} P_{\mathrm{its}}$, and if $\varphi \rhd_{\mathrm{tt}} P_t$ for nonempty P, then $\varphi \rhd_{\mathrm{pt}} P_t$.*
2. *For any singleton P_s, $\varphi \rhd_{\mathrm{ps}} P_s$ implies $\varphi \rhd_{\mathrm{ts}} P_s$. Similarly, $\varphi \rhd_{\mathrm{pits}} P_{\mathrm{its}}$ implies $\varphi \rhd_{\mathrm{tits}} P_{\mathrm{its}}$ for any singleton P_{its}, and $\varphi \rhd_{\mathrm{pt}} P_t$ implies $\varphi \rhd_{\mathrm{tt}} P$ for any singleton P_t.*
3. *If $\varphi \rhd_{\mathrm{ps}} \{\mathcal{A}\}$ and $\mathcal{A} \equiv \mathcal{B}$, then $\varphi \rhd_{\mathrm{ps}} \{\mathcal{B}\}$. This implies that the relations $\varphi \rhd_{\mathrm{pt}} P_t$ and $\varphi \rhd_{\mathrm{tt}} P_t$ do not depend on the choice of models $\mathcal{M} \models T$ for $T \in P_t$.*
4. *(Reflexivity) For any sentence φ and a (nonempty) family $\mathcal{T}_\varphi \subseteq \mathcal{T}_\Sigma$, we have $\varphi \rhd_{\mathrm{tt}} \mathcal{T}_\varphi$ (and $\varphi \rhd_{\mathrm{pt}} \mathcal{T}_\varphi$).*
5. *(Monotony) If $\varphi \rhd_{\mathrm{ps}} P_s$, $\varphi \vdash \psi$ and $P_s \subseteq P'_s \subseteq K(\Sigma)$ then $\psi \rhd_{\mathrm{ps}} P'_s$. If $\varphi \rhd_{\mathrm{ts}} P_s$, $\varphi \vdash \psi$ and $P_s \supseteq P'_s$, then $\psi \rhd_{\mathrm{ps}} P'_s$. If $\varphi \rhd_{\mathrm{pits}} P_{\mathrm{its}}$, $\varphi \vdash \psi$ and $P_{\mathrm{its}} \subseteq P'_{\mathrm{its}} \subseteq \mathrm{ITK}(\Sigma)$ then $\psi \rhd_{\mathrm{pits}} P'_{\mathrm{its}}$. If $\varphi \rhd_{\mathrm{tits}} P_{\mathrm{its}}$, $\varphi \vdash \psi$ and $P_{\mathrm{its}} \supseteq P'_{\mathrm{its}}$ then $\psi \rhd_{\mathrm{ps}} P'_{\mathrm{its}}$. If $\varphi \rhd_{\mathrm{pt}} P_t$, $\varphi \vdash \psi$ and $P_t \subseteq P'_t \subseteq \mathcal{T}_\Sigma$, then $\psi \rhd_{\mathrm{ps}} P'_t$. If $\varphi \rhd_{\mathrm{tt}} P_t$, $\varphi \vdash \psi$ and $P_t \supseteq P'_t$ then $\psi \rhd_{\mathrm{tt}} P'_t$.*

For a property P_s, we denote by $\mathrm{ITK}(P_s)$ the class of isomorphism types for structures in P_s and by $\mathrm{Th}(P_s)$ the set $\{T \in \mathcal{T}_\Sigma \mid \mathcal{A} \models T \text{ for some } \mathcal{A} \in P_s\}$.

For a property P_{its}, we denote by $K(P_{\mathrm{its}})$ the class of all structures whose isomorphism types are represented in P_{its} and by $\mathrm{Th}(P_{\mathrm{its}})$ the set $\mathrm{Th}(K(P_{\mathrm{its}}))$.

For a property P_t, we denote by $K(P_t)$ the class of all models of theories in P_t and by $\mathrm{ITK}(P_t)$ the class $\mathrm{ITK}(K(P_t))$.

In terms of these notations, by definition, we have the following natural links between semantic properties P_s and P_{its} and syntactic properties P_t:

Proposition 6. *For any formula $\varphi \in F(\Sigma)$ and properties P_s, P_{its} and P_t, the following conditions hold:*

(1) $\varphi \rhd_{\mathrm{ps}} P_s$ *iff* $\varphi \rhd_{\mathrm{pits}} \mathrm{ITK}(P_s)$, *and iff* $\varphi \rhd_{\mathrm{pt}} \mathrm{Th}(P_s)$;
(2) $\varphi \rhd_{\mathrm{ts}} P_s$ *iff* $\varphi \rhd_{\mathrm{tits}} \mathrm{ITK}(P_s)$, *and iff* $\varphi \rhd_{\mathrm{tt}} \mathrm{Th}(P_s)$;
(3) $\varphi \rhd_{\mathrm{pits}} P_{\mathrm{its}}$ *iff* $\varphi \rhd_{\mathrm{ps}} K(P_{\mathrm{its}})$, *and iff* $\varphi \rhd_{\mathrm{pt}} \mathrm{Th}(P_{\mathrm{its}})$;
(4) $\varphi \rhd_{\mathrm{tits}} P_{\mathrm{its}}$ *iff* $\varphi \rhd_{\mathrm{ts}} K(P_{\mathrm{its}})$, *and iff* $\varphi \rhd_{\mathrm{tt}} \mathrm{Th}(P_{\mathrm{its}})$;
(5) $\varphi \rhd_{\mathrm{pt}} P_t$ *iff* $\varphi \rhd_{\mathrm{ps}} K(P_t)$, *and iff* $\varphi \rhd_{\mathrm{pits}} \mathrm{ITK}(P_t)$;
(6) $\varphi \rhd_{\mathrm{tt}} P_t$ *iff* $\varphi \rhd_{\mathrm{ts}} K(P_t)$, *and iff* $\varphi \rhd_{\mathrm{tits}} \mathrm{ITK}(P_t)$.

In items (3) *and* (4), *the class $K(P_{\mathrm{its}})$ can be replaced by a subclass K' such that $\mathrm{ITK}(K') = P_{\mathrm{its}}$. Similarly, in items* (5) *and* (6), *the class $K(P_t)$ can be replaced by a subclass K' such that $\mathrm{Th}(K') = P_t$, and independently $\mathrm{ITK}(P_t)$ can be replaced by a subclass K'' such that $\mathrm{Th}(K'') = P_t$.*

According to Proposition 6, semantic properties P_s and P_{its} can be naturally transformed into syntactic ones P_t and vice versa. This means that natural model-theoretic properties such as ω-categoricity, stability, simplicity, etc. can be formulated for theories, for structures and for their isomorphism types.

The links between $\rhd$-relations highlighted in Proposition 6 allow us to reduce our consideration to the relations $\rhd_{\text{pt}}$ and $\rhd_{\text{tt}}$. Besides, for simplicity, we principally consider sentences φ instead of formulas in general. Reductions of formulas $\psi(\overline{x})$ to sentences use the operators $\psi(\overline{x}) \mapsto \forall\overline{x}\,\psi(\overline{x})$ and $\psi(\overline{x}) \mapsto \exists\overline{x}\,\psi(\overline{x})$.

Proposition 7. *For any sentences $\varphi, \psi \in \text{Sent}(\Sigma)$ and properties $P_t, P'_t \subseteq \mathcal{T}_\Sigma$, the following conditions hold:*

(1) If $(\varphi \wedge \psi) \rhd_{\text{pt}} (P_t \cap P'_t)$, then $\varphi \rhd_{\text{pt}} P_t$ and $\psi \rhd_{\text{pt}} P'_t$; the converse implication does not hold. There are $\varphi', \psi' \in \text{Sent}(\Sigma)$ and $P''_t \in \mathcal{T}_\Sigma$ such that $\varphi' \rhd_{\text{pt}} P''_t$, $\psi' \rhd_{\text{pt}} P''_t$, and $(\varphi' \wedge \psi') \not\rhd_{\text{pt}} P''_t$;
(2) $(\varphi \wedge \psi) \rhd_{\text{tt}} P_t$ iff $\varphi \rhd_{\text{tt}} P_t$ and $\psi \rhd_{\text{tt}} P_t$;
(3) If $\varphi \rhd_{\text{tt}} P_t$ and $\psi \rhd_{\text{tt}} P'_t$, then $(\varphi \wedge \psi) \rhd_{\text{tt}} (P_t \cap P'_t)$; the converse implication does not hold. There are $\varphi', \psi' \in \text{Sent}(\Sigma)$ and $P''_t, P'''_t \in \mathcal{T}_\Sigma$ such that $(\varphi \wedge \psi) \rhd_{\text{tt}} (P''_t \cap P'''_t)$, whereas $\varphi' \not\rhd_{\text{tt}} P''_t$ and $\psi' \not\rhd_{\text{tt}} P'''_t$.

Proof.

(1) If $(\varphi \wedge \psi) \rhd_{\text{pt}} (P_t \cap P'_t)$, then there is $T \in P_t \cap P'_t$ with $(\varphi \wedge \psi) \in T$. Since $\varphi, \psi \in T$, $T \in P_t$ and $T \in P'_t$, we obtain $\varphi \rhd_{\text{pt}} P_t$ and $\psi \rhd_{\text{pt}} P'_t$. Therefore, it is sufficient to note for (1) that sentences φ', ψ' asserting distinct finite cardinalities m and n for universes partially satisfy a property P''_t containing a theory T_1 with an m-element model and a theory T_2 with an n-element model. At the same time, $(\varphi' \wedge \psi') \not\rhd_{\text{pt}} P''_t$ since $(\varphi' \wedge \psi')$ is inconsistent.
(2) If $(\varphi \wedge \psi) \rhd_{\text{tt}} P_t$, then $(\varphi \wedge \psi)$, and so φ and ψ belong to all theories in P_t implying $\varphi \rhd_{\text{tt}} P_t$ and $\psi \rhd_{\text{tt}} P_t$. Conversely, if $\varphi \rhd_{\text{tt}} P_t$ and $\psi \rhd_{\text{tt}} P_t$, then φ, ψ, and so $(\varphi \wedge \psi)$ belongs to all theories in P_t, implying $(\varphi \wedge \psi) \rhd_{\text{tt}} P_t$.
(3) If $\varphi \rhd_{\text{tt}} P_t$ and $\psi \rhd_{\text{tt}} P'_t$, then $\varphi \in \bigcap P_t$ and $\psi \in \bigcap P'_t$, implying $(\varphi \wedge \psi) \in \bigcap P_t \cap \bigcap P'_t$; i.e., $(\varphi \wedge \psi) \rhd_{\text{tt}} (P_t \cap P'_t)$. Finally, if P''_t and P'''_t are nonempty with $P''_t \cap P'''_t = \emptyset$ and φ' and ψ' are inconsistent sentences, then $(\varphi \wedge \psi) \rhd_{\text{tt}} (P''_t \cap P'''_t)$ and $\varphi' \not\rhd_{\text{pt}} P''_t$ and $\psi' \not\rhd_{\text{pt}} P'''_t$, implying $\varphi' \not\rhd_{\text{tt}} P''_t$ and $\psi' \not\rhd_{\text{tt}} P'''_t$.

□

Proposition 8. *For any sentences $\varphi, \psi \in \text{Sent}(\Sigma)$ and properties $P_t, P'_t \subseteq \mathcal{T}_\Sigma$ the following conditions hold:*

(1) If $\varphi \rhd_{\text{pt}} P_t$ or $\psi \rhd_{\text{pt}} P'_t$, then $(\varphi \vee \psi) \rhd_{\text{pt}} (P_t \cup P'_t)$; the converse implication does not hold. There are $\varphi', \psi' \in \text{Sent}(\Sigma)$ and $P''_t \in \mathcal{T}_\Sigma$ such that $(\varphi' \vee \psi') \rhd_{\text{pt}} P''_t$, and $\varphi' \not\rhd_{\text{pt}} P''_t$ or $\psi' \not\rhd_{\text{pt}} P''_t$;
(2) $(\varphi \vee \psi) \rhd_{\text{pt}} P_t$ iff $\varphi \rhd_{\text{pt}} P_t$ or $\psi \rhd_{\text{pt}} P_t$;
(3) If $\varphi \rhd_{\text{tt}} P_t$ and $\psi \rhd_{\text{tt}} P'_t$, then $(\varphi \vee \psi) \rhd_{\text{tt}} (P_t \cup P'_t)$; the converse implication does not hold. There are $\varphi', \psi' \in \text{Sent}(\Sigma)$ and $P''_t \in \mathcal{T}_\Sigma$ such that $(\varphi' \vee \psi') \rhd_{\text{tt}} (P''_t)$ whereas $\varphi' \not\rhd_{\text{tt}} P''_t$ and $\psi' \not\rhd_{\text{tt}} P''_t$.

Proof.

(1) If $\varphi \rhd_{\text{pt}} P_t$ or $\psi \rhd_{\text{pt}} P'_t$, then $\varphi \in T$ for some $T \in P_t$ or $\psi \in T'$ for some $T' \in P'_t$. Therefore T or T' witness that $(\varphi \vee \psi) \rhd_{\text{pt}} (P_t \cup P'_t)$. If φ' is a tautology and ψ' is an inconsistent sentence, then for any nonempty $P''_t \subseteq \mathcal{T}_\Sigma$, we have $(\varphi' \vee \psi') \rhd_{\text{pt}} P''_t$, $\varphi' \rhd_{\text{pt}} P''_t$, and $\psi' \not\rhd_{\text{pt}} P''_t$.
(2) It holds that a sentence $(\varphi \vee \psi)$ belongs to a complete theory T if and only if $\varphi \in T$ or $\psi \in T$.
(3) If $\varphi \rhd_{\text{tt}} P_t$ and $\psi \rhd_{\text{tt}} P'_t$, then $\varphi \in \bigcap P_t$ and $\psi \in \bigcap P'_t$. Therefore, $(\varphi \vee \psi) \in \bigcap P_t$ and $(\varphi \vee \psi) \in \bigcap P'_t$, implying $(\varphi \vee \psi) \in \bigcap P_t \cap \bigcap P'_t$; thus, $(\varphi \vee \psi) \rhd_{\text{tt}} (P_t \cup P'_t)$.

Now, let $P_t'' = \mathcal{T}_\Sigma$, φ' be a sentence belonging to some but not all theories in P_t''. For the sentence $\psi' = \neg\varphi'$, we have $(\varphi' \vee \psi') \rhd_{\mathrm{tt}} (P_t'')$ since $(\varphi \vee \psi)$ is a tautology, $\varphi' \not\rhd_{\mathrm{tt}} P_t''$ and $\psi' \not\rhd_{\mathrm{tt}} P_t''$ by the choice of φ'. □

Proposition 9. *For any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P_t \subseteq \mathcal{T}_\Sigma$, the following conditions hold:*

(1) $\varphi \rhd_{\mathrm{pt}} P_t$ *iff* $\neg\varphi \not\rhd_{\mathrm{tt}} P_t$;
(2) $\varphi \rhd_{\mathrm{tt}} P_t$ *iff* $\neg\varphi \not\rhd_{\mathrm{pt}} P_t$.

Proof.

(1) If $\varphi \rhd_{\mathrm{pt}} P_t$, then there is $T \in P_t$ such that $\varphi \in T$. Since T is complete, then $\neg\varphi \notin T$, implying $\neg\varphi \not\rhd_{\mathrm{tt}} P_t$. Conversely, if $\neg\varphi \not\rhd_{\mathrm{tt}} P_t$, then $\neg\varphi$ does not belong to some theory $T \in P_t$. Since T is complete, then $\varphi \in T$, implying $\varphi \rhd_{\mathrm{pt}} P_t$.
(2) The second proposition immediately follows from (1).
□

Proposition 10. *For any formula $\varphi \in F(\Sigma)$ and a property $P_t \subseteq \mathcal{T}_\Sigma$, the following conditions hold:*

(1) *If* $\varphi = \forall x\psi$ *and* $\varphi \rhd_{\mathrm{pt}} P_t$, *then* $\psi \rhd_{\mathrm{pt}} P_t$;
(2) *If* $\varphi = \forall x\psi$ *and* $\varphi \rhd_{\mathrm{tt}} P_t$, *then* $\psi \rhd_{\mathrm{tt}} P_t$;
(3) *If* $\varphi = \exists x\psi$ *and* $\psi \rhd_{\mathrm{pt}} P_t$, *then* $\psi \rhd_{\mathrm{pt}} P_t$;
(4) *If* $\varphi = \exists x\psi$ *and* $\psi \rhd_{\mathrm{tt}} P_t$, *then* $\psi \rhd_{\mathrm{tt}} P_t$.

Proof.

(1) Let $\varphi = \forall x\psi$ and $\varphi \rhd_{\mathrm{pt}} P_t$. Then, there are $T \in P_t$, $\mathcal{M} \models T$, $\overline{a} \in M$ such that $\mathcal{M} \models \varphi(\overline{a})$. This implies $\mathcal{M} \models \forall x\psi(x, \overline{a})$; therefore, there is $b \in M$ with $\mathcal{M} \models x\psi(b, \overline{a})$—i.e., $\psi \rhd_{\mathrm{pt}} \{T\}$, and thus $\psi \rhd_{\mathrm{pt}} P_t$.
(2) We repeat the arguments presented in (1), replacing $T \in P_t$ with an arbitrary value.
(3) Let $\varphi = \exists x\psi$ and $\psi \rhd_{\mathrm{pt}} P_t$. Then, there are $T \in P_t$, $\mathcal{M} \models T$, $\overline{a}, b \in M$ such that $\mathcal{M} \models \psi(b, \overline{a})$. This implies $\mathcal{M} \models \exists x\psi(x, \overline{a})$—i.e., $\varphi \rhd_{\mathrm{pt}} \{T\}$, and thus $\psi \rhd_{\mathrm{pt}} P_t$.
(4) As for (2), we repeat the arguments presented in (3), replacing $T \in P_t$ with an arbitrary value.
□

The following two theorems assert that the relations $\rhd_{\mathrm{pt}}$ and $\rhd_{\mathrm{tt}}$ are preserved under E-closures.

Theorem 8. *For any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P_t \subseteq \mathcal{T}_\Sigma$, the following conditions are equivalent:*

(1) $\varphi \rhd_{\mathrm{pt}} P_t$;
(2) $\varphi \rhd_{\mathrm{pt}} \mathrm{Cl}_E(P_t)$;
(3) $\varphi \rhd_{\mathrm{pt}} P_t'$ *for any/some* P_t' *with* $\mathrm{Cl}_E(P_t') = \mathrm{Cl}_E(P_t)$.

Proof.

(1) ⇒ (2) holds in view of $P_t \subseteq \mathrm{Cl}_E(P_t)$ and the monotony of the relation $\rhd_{\mathrm{pt}}$.
(2) ⇒ (3). It suffices to show that $\varphi \rhd_{\mathrm{pt}} P_t'$ for any P_t' with $\mathrm{Cl}_E(P_t') = \mathrm{Cl}_E(P_t)$. Since $\varphi \rhd_{\mathrm{pt}} \mathrm{Cl}_E(P_t)$, there is a theory $T \in \mathrm{Cl}_E(P_t)$ with $\varphi \in T$. If $T \in P't$, we have $\varphi \rhd_{\mathrm{pt}} P_t'$. Otherwise, T is an accumulation point of P_t', implying, in view of Proposition 1.1, that *varphi* belongs to infinitely many theories in P_t'. Therefore, $\varphi \rhd_{\mathrm{pt}} P_t'$.
(3) ⇒ (1) is obvious.
□

Theorem 9. *For any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P_t \subseteq \mathcal{T}_\Sigma$, the following conditions are equivalent:*

(1) $\varphi \rhd_{\mathrm{tt}} P_t$;

(2) $\varphi \rhd_{\mathrm{tt}} \mathrm{Cl}_E(P_t)$;
(3) $\varphi \rhd_{\mathrm{tt}} P'_t$ *for any/some* P'_t *with* $\mathrm{Cl}_E(P'_t) = \mathrm{Cl}_E(P_t)$.

Proof.

$(1) \Rightarrow (2)$. Let $\varphi \rhd_{\mathrm{tt}} P_t$. If P_t is finite, then $\mathrm{Cl}_E(P_t) = P_t$, and we have $\varphi \rhd_{\mathrm{tt}} \mathrm{Cl}_E(P_t)$. If P_t is finite, then by Proposition 1.1, $\mathrm{Cl}_E(P_t)$ consists of theories in P_t and of theories $T \in \mathcal{T}_\Sigma \setminus P_t$ such that for any sentence $\psi \in T$, the set $(P_t)_\psi$ is infinite. Since $\varphi \rhd_{\mathrm{tt}} P_t$, φ belongs to each such theory T. Thus, $\varphi \rhd_{\mathrm{tt}} \mathrm{Cl}_E(P_t)$;
$(2) \Rightarrow (1)$ and $(2) \Rightarrow (3)$ are obvious;
$(3) \Rightarrow (2)$ follows assuming $\varphi \rhd_{\mathrm{tt}} P'_t$ for any/some P'_t with $\mathrm{Cl}_E(P'_t) = \mathrm{Cl}_E(P_t)$ repeating the arguments presented in $(1) \Rightarrow (2)$.
□

For a property $P_t \subseteq \mathcal{T}_\Sigma$, we denote by $\nabla(P_t)$ the set of all sentences $\varphi \in \mathrm{Sent}(\Sigma)$ with $\varphi \rhd_{\mathrm{pt}} P_t$ and by $\triangle(P_t)$ the set of all sentences $\varphi \in \mathrm{Sent}(\Sigma)$ with $\varphi \rhd_{\mathrm{tt}} P_t$.

According to this definition, $\nabla(\emptyset) = \emptyset$, $\triangle(\emptyset) = \mathrm{Sent}(\Sigma)$, $\nabla(\mathcal{T}_\Sigma)$ consists of all consistent sentences $\varphi \in \mathrm{Sent}(\Sigma)$ and $\triangle(\mathcal{T}_\Sigma)$ consists of all tautologies $\varphi \in \mathrm{Sent}(\Sigma)$.

Proposition 11. *For any property* $P_t \subseteq \mathcal{T}_\Sigma$ *the following conditions hold:*

(1) $\nabla(P_t) = \bigcup P_t$;
(2) $\nabla(P_t)$ *is consistent iff* $|P_t| \leq 1$, *and* $\nabla(P_t)$ *is a complete theory iff* P_t *is a singleton;*
(3) $\triangle(P_t) = \bigcap P_t$;
(4) $\triangle(P_t)$ *is a consistent theory iff* $P_t \neq \emptyset$, *and* $\triangle(P_t)$ *is complete iff* P_t *is a singleton;*
(5) *For any* $P_t \neq \emptyset$, $\nabla(P_t) \supseteq \triangle(P_t)$, *and* $\nabla(P_t) = \triangle(P_t)$ *iff* P_t *is a singleton.*

Proof.

(1) If $\varphi \in \nabla(P_t)$, then $\varphi \rhd_{\mathrm{pt}} P_t$ and $\varphi \in T$ for some $T \in P_t$, implying $\nabla(P_t) \subseteq \bigcup P_t$. Conversely, if $\varphi \in \bigcup P_t$, then $\varphi \in T$ for some $T \in P_t$, implying $\varphi \rhd_{\mathrm{pt}} P_t$ and therefore $\varphi \in \nabla(P_t)$;
(2) Since $\nabla(\emptyset) = \emptyset$, it is consistent. If $P_t = \{T\}$, then $\nabla(P_t) = T$; i.e., $\nabla(P_t)$ is consistent and complete. If P_t contains two distinct theories T_1 and T_2, then $T_1 \cup T_2 \subseteq \nabla(P_t)$, implying that $\nabla(P_t)$ is inconsistent as there are sentences ψ such that $\psi \in T_1$ and $\neg\psi \in T_2$;
(3) If $\varphi \in \triangle(P_t)$, then $\varphi \rhd_{\mathrm{tt}} P_t$ and $\varphi \in T$ for any $T \in P_t$, implying $\triangle(P_t) \subseteq \bigcap P_t$. Conversely, if $\varphi \in \bigcap P_t$, then $\varphi \in T$ for any $T \in P_t$, implying $\varphi \rhd_{\mathrm{tt}} P_t$ and therefore $\varphi \in \triangle(P_t)$;
(4) Since $\triangle(\emptyset) = \mathrm{Sent}(\Sigma)$, it is inconsistent. If $P_t \neq \emptyset$, then $\triangle(P_t) = \bigcap P_t$ by (3), implying that $\triangle(P_t)$ is a consistent theory as an intersection of complete theories. If $\triangle(P_t)$ is complete, then P_t is both nonempty and does not contain two distinct theories; i.e., P_t is a singleton. Conversely, if $P_t = \{T\}$ then $\triangle(P_t) = T$ which is a complete theory;
(5) If $P_t \neq \emptyset$, then by (1) and (3), we have $\nabla(P_t) = \bigcup P_t \supseteq \bigcap P_t = \triangle(P_t)$. If $P_t = \{T\}$, then $\triangle(P_t) = T = \nabla(P_t)$. If $T_1, T_2 \in P_t$ for some $T_1 \neq T_2$, then $\nabla(P_t) = \bigcup P_t \supseteq T_1 \cup T_2 \supsetneq T_1 \cap T_2 \supseteq \bigcap P_t = \triangle(P_t)$.
□

Theorems 6, 8 and 9 and Proposition 11 immediately imply the following corollary on the separability of properties with respect to the relations $\rhd_{\mathrm{pt}}$ and $\rhd_{\mathrm{tt}}$.

Corollary 1. *For any properties* $P_1, P_2 \subseteq \mathcal{T}_\Sigma$, *the following conditions hold:*

(1) *There exists* $\varphi \in \mathrm{Sent}(\Sigma)$ *such that* $\varphi \rhd_{\mathrm{pt}} P_1$ *and* $\neg\varphi \rhd_{\mathrm{pt}} P_2$ *iff* P_1 *and* P_2 *are nonempty and* $|P_1 \cup P_2| \geq 2$; *in particular, there exists* $\varphi \in \mathrm{Sent}(\Sigma)$ *such that* $\varphi \rhd_{\mathrm{pt}} P_1$ *and* $\neg\varphi \rhd_{\mathrm{pt}} P_1$ *iff* $|P_1| \geq 2$;
(2) *There exists* $\varphi \in \mathrm{Sent}(\Sigma)$ *such that* $\varphi \rhd_{\mathrm{tt}} P_1$ *and* $\neg\varphi \rhd_{\mathrm{tt}} P_2$ *iff* $\mathrm{Cl}_E(P_1) \cap \mathrm{Cl}_E(P_2) = \emptyset$.

Corollary 2. *For any nonempty property $P_t \subseteq \mathcal{T}_\Sigma$ the following conditions hold:*

(1) The set $\triangle(P_t)$ forms a filter $\triangle(P_t)/\equiv$ on $\{\equiv(\varphi) \mid \varphi \in \mathrm{Sent}(\Sigma)\}$ with respect to $\vdash$;
(2) The filter $\triangle(P_t)/\equiv$ is principal iff $\bigcap P_t$ is forced by some sentence; i.e., $\bigcap P_t$ is a finitely axiomatizable theory, which is incomplete for $|P_t| \geq 2$;
(3) The filter $\triangle(P_t)/\equiv$ is an ultrafilter iff P_t is a singleton.

Proof.

(1) The first proposition holds by monotony and Proposition 7, (2);
(2) The second immediately follows from Proposition 11, (3);
(3) The third is satisfied in view of Proposition 11, (4).

□

4. Ranks of Sentences and Spectra with Respect to Properties

In this section, we introduce a measure of complexity for sentences satisfying a property using the RS-rank for families of theories [8–10].

Definition 7. *For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P = P_t \subseteq \mathcal{T}_\Sigma$, we put $\mathrm{RS}_P(\varphi) = \mathrm{RS}(P_\varphi)$, and $\mathrm{ds}_P(\varphi) = \mathrm{ds}(P_\varphi)$ if $\mathrm{ds}(P_\varphi)$ is defined.*

If $P = \mathcal{T}_\Sigma$, then we omit P and write $\mathrm{RS}(\varphi)$, $\mathrm{ds}(\varphi)$ instead of $\mathrm{RS}_P(\varphi)$ and $\mathrm{ds}_P(\varphi)$, respectively.

Clearly, if $P \subseteq P' \subseteq \mathcal{T}_\Sigma$ and $\varphi \in \mathrm{Sent}(\Sigma)$, then $\mathrm{RS}_P(\varphi) \leq \mathrm{RS}_{P'}(\varphi)$, and if $\mathrm{RS}_P(\varphi) = \mathrm{RS}_{P'}(\varphi) \in \mathrm{Ord}$, then $\mathrm{ds}_P(\varphi) \leq \mathrm{ds}_{P'}(\varphi)$.

Proposition 12.

(1) $\varphi \rhd_{\mathrm{tt}} P$ iff $\mathrm{RS}_P(\neg\varphi) = -1$.
(2) $\varphi \rhd_{\mathrm{pt}} P$ iff $\mathrm{RS}_P(\varphi) \geq 0$.

Definition 8. *For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P \subseteq \mathcal{T}_\Sigma$, we say that φ is P-*totally transcendental *if $\mathrm{RS}_P(\varphi)$ is an ordinal.*

A sentence φ is co-(P)-totally transcendental *if $\neg\varphi$ is P-totally transcendental.*

We omit P and consider totally transcendental and co-totally transcendental sentences if $P = \mathcal{T}_\Sigma$.

According to this definition, each sentence $\varphi \in \mathrm{Sent}(\Sigma)$ obtains the characteristics $\mathrm{RS}_P(\varphi)$ and $\mathrm{RS}_P(\neg\varphi)$, considering that φ is (co)-rich enough with respect to the property P. The characteristics $\mathrm{ds}_P(\varphi)$ and $\mathrm{ds}_P(\neg\varphi)$, if they are defined, give additional information regarding the "P-richness" of φ.

For instance, if $\mathrm{RS}_P(\varphi) = 0$ and $\mathrm{ds}_P(\varphi) = n$, then φ is *P-finite*, exactly satisfying n theories in P. Respectively, if $\mathrm{RS}_P(\neg\varphi) = 0$ and $\mathrm{ds}_P(\neg\varphi) = n$, then φ is *P-cofinite*; i.e., it does not satisfy exactly n theories in P.

Clearly, φ is both P-finite and P-cofinite iff P is nonempty and is finite.

Theorem 10. *For a language Σ, there is a totally transcendental sentence $\varphi \in \mathrm{Sent}(\Sigma)$ iff Σ has finitely many predicate symbols.*

Proof. If Σ has finitely many predicate symbols, we choose a sentence φ, assuming that the universe is a singleton. Since functional symbols have unique interpretations and there are finitely many possibilities for (non)empty language predicates, we obtain $\mathrm{RS}(\varphi) = 0$; that is, φ is totally transcendental.

Conversely, if Σ has infinitely many predicate symbols, then each consistent sentence φ obtains a 2-tree with respect to (non)empty predicates $Q \in \Sigma \setminus \Sigma(\varphi)$. This 2-tree evinces that $\mathrm{RS}(\varphi) = \infty$; i.e., φ is not totally transcendental. □

Remark 2. *If Σ is finite, then for the proof of Theorem 10, it suffices to choose a sentence φ assuming that a universe is finite, since there are finitely manypossibilities, up to isomorphism, for interpretations of language symbols implying* $\mathrm{RS}(\varphi) = 0$.

The following definition introduces values for the richness of a sentence with respect to a property.

Definition 9. *For a language Σ, a property $P \subseteq \mathcal{T}_\Sigma$, an ordinal α and a natural number $n \geq 1$, a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called (P, α, n)-(co-)*rich *if $\mathrm{RS}_P(\varphi) = \alpha$ and $\mathrm{ds}_P(\varphi) = n$ (respectively, $\mathrm{RS}_P(\neg\varphi) = \alpha$ and $\mathrm{ds}_P(\neg\varphi) = n$).*

*A sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called (P, ∞)-(co-)*rich *if $\mathrm{RS}_P(\varphi) = \infty$ (respectively, $\mathrm{RS}_P(\neg\varphi) = \infty$).*

If $P = \mathcal{T}_\Sigma$, we write that φ is (α, n)-(co-)rich instead of (P, α, n)-(co-)rich and ∞-(co-)rich instead of (P, ∞)-(co-)rich.

If for a property P there is a $(P, *)$-(co-)rich sentence φ, we say that P has a $(P, *)$-(co-)rich sentence, *where $* = \alpha, n$ or $\alpha = \infty$.*

According to this definition, if a sentence φ is (P, α, n)-rich, then $\mathrm{RS}(P_\varphi) = \alpha$, $\mathrm{ds}(P_\varphi) = n$.

Theorem 11.

(1) If a property $P \subseteq \mathcal{T}_\Sigma$ has a (P, α, m)-rich sentence φ which is (P, β, n)-co-rich, then $\mathrm{RS}(P) = \max\{\alpha, \beta\}$, $\mathrm{ds}(P) = m$ for $\alpha > \beta$, $\mathrm{ds}(P) = n$ for $\alpha < \beta$, and $\mathrm{ds}(P) = m + n$ for $\alpha = \beta$.

(2) If for a property $P \subseteq \mathcal{T}_\Sigma$, $\mathrm{RS}(P) = \alpha$ and $\mathrm{ds}(P) = n$, then for each sentence $\varphi \in \mathrm{Sent}(\Sigma)$, the following assertions hold:

- *(i)* $\mathrm{RS}_P(\varphi) \leq \alpha$;
- *(ii) If $\mathrm{RS}_P(\varphi) = \alpha$, then φ is (P, α, m)-rich for some $m \leq n$, and for $m = n$, either $\varphi \rhd_{\mathrm{tt}} P$ or φ is (P, β, k)-co-rich for some $\beta < \alpha$ and $k \in \omega$, and if $m < n$, then φ is $(P, \alpha, n - m)$-co-rich.*

Proof.

(1) For the sentence φ, we have $\mathrm{RS}_P(\varphi) = \alpha$, $\mathrm{ds}_P(\varphi) = m$, $\mathrm{RS}_P(\neg\varphi) = \beta$, $\mathrm{ds}_P(\neg\varphi) = n$. This means that P is divided into two disjointed parts P_φ and $P_{\neg\varphi}$ with given characteristics $\mathrm{RS}(P_\varphi) = \alpha$, $\mathrm{ds}(P_\varphi) = m$, $\mathrm{RS}(P_{\neg\varphi}) = \beta$, $\mathrm{ds}(P_{\neg\varphi}) = n$;

If $\mathrm{RS}(P) = 0$, then $|P| = \mathrm{ds}(P)$, $\mathrm{RS}_P(\varphi) = \mathrm{RS}_P(\neg\varphi) = 0$, $\mathrm{ds}(P) = |P_\varphi| + |P_{\neg\varphi}| = m + n$;
If $\mathrm{RS}(P) > 0$, then a tree evincing the value $\mathrm{RS}(P) = \gamma$ can be transformed step-by-step using theories either in P_φ or in $P_{\neg\varphi}$: in each step evincing $\mathrm{RS}(P) = \gamma$, there are infinitely many branches of previous values related to P_φ or to $P_{\neg\varphi}$.

In the first case, related to P_φ, we have $\gamma = \alpha$, and in the second case, related to $P_{\neg\varphi}$, $\gamma = \beta$. If $\alpha > \beta$, a tree for P_φ shows that $l = m$. If $\alpha < \beta$, a tree for $P_{\neg\varphi}$ shows that $l = n$. If $\alpha = \beta$, then both trees for P_φ and for $P_{\neg\varphi}$ show that $\gamma = \alpha$ and $l = m + n$, since there are exactly $l + m$ s-definable subsets of P having the rank γ and the degree 1.

(2) For (i), we can see that by the monotony of rank (if $P_1 \subseteq P_2$ then $\mathrm{RS}(P_1) \leq \mathrm{RS}(P_2)$) and inclusion, $P_\varphi \subseteq P$. For (ii), we can see that it holds by the monotony of degree for a fixed rank (if $P_1 \subseteq P_2$ and $\mathrm{RS}(P_1) = \mathrm{RS}(P_2) \in \mathrm{Ord}$ then $\mathrm{ds}(P_1) \leq \mathrm{ds}(P_2)$) and the inclusion that $P_\varphi \subseteq P$. Besides, if $\mathrm{ds}(P_\varphi) = m = n = \mathrm{ds}(P)$, then P can not have a tree in $P_{\neg\varphi} = P \setminus P_\varphi$, showing that $\mathrm{RS}(P_{\neg\varphi}) = \alpha$, since otherwise $\mathrm{ds}(P)$ should be more than n. Therefore, either $\neg\varphi$ is P-inconsistent—i.e., $\varphi \rhd_{\mathrm{tt}} P$—or φ is (P, β, k)-co-rich for some $\beta < \alpha$ and $k \in \omega$. If $m < n$, then φ is $(P, \alpha, n - m)$-co-rich in view of (1).

□

By Theorem 11 for any e-totally transcendental property P and any $\alpha \leq \mathrm{RS}(P)$, there are s-definable subfamilies P_φ with $\mathrm{RS}(P_\varphi) = \alpha$. Similarly, all values $m \leq \mathrm{ds}(P)$ are also realized by appropriate s-definable subfamilies.

Thus, the *spectrum* $\mathrm{Sp}_{\mathrm{Rd}}(P)$ for the pairs $(\mathrm{RS}_P(\varphi), \mathrm{ds}_P(\varphi))$ with nonempty P_φ forms the set

$$\{(\mathrm{RS}(P), m) \mid 1 \leq m \leq \mathrm{ds}(P)\} \cup \{(\alpha, m) \mid \alpha < \mathrm{RS}(P), m \in \omega \setminus \{0\}\}, \tag{1}$$

which is an initial segment $O[(\beta, n)]$ consisting of all pairs $(\alpha, m) \in \mathrm{Ord} \times (\omega \setminus 0)$ with $\alpha \leq \beta$ and $m \leq n$ for $\alpha = \beta$, $\mathrm{RS}(P) = \beta$, $\mathrm{ds}(P) = n$.

Remark 3. *If* $\mathrm{RS}(P) = \infty$, *then s-definable subfamilies* P_φ *can have only values* $\mathrm{RS}(P_\varphi) = \infty$ *or both the value* $\mathrm{RS}(P) = \infty$ *and pairs, forming some initial segment* $O[(\beta, n)]$.

Indeed, let P be a family of theories in a language Σ of independent 0-ary predicates Q_l, $l \in \lambda$, $\lambda \geq \omega$, such that each sentence $Q_{i_1}^{\delta_1} \wedge \ldots \wedge Q_{i_k}^{\delta_k}$, $i_1 < \ldots < i_k < \lambda$, $\delta_1, \ldots, \delta_k \in \{0, 1\}$, $k \in \omega$, is P-consistent. Each P-consistent sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is divided into 2-trees, showing that $\mathrm{RS}_P(\varphi) = \infty$. In such a case, we say that the spectrum $\mathrm{Sp}_{\mathrm{Rd}}(P)$ equals $\{\infty\}$.

The family P above can be extended by a family P' with dependent predicates Q_l producing a given RS-rank and ds-degree for a subfamily with, e.g., $Q_0 \leftrightarrow Q_1$. Therefore, the arguments for Theorem 11 produce an initial segment $O[(\beta, n)]$ for the spectrum $\mathrm{Sp}_{\mathrm{Rd}}(P')$ of the s-definable family P'. Thus, $\mathrm{Sp}_{\mathrm{Rd}}(P \cup P') = O[(\beta, n)] \cup \{\infty\}$.

Since each nonempty s-definable subfamily has a spectrum of the form $O[(\beta, n)]$, or $\{\infty\}$, or $O[(\beta, n)] \cup \{\infty\}$, initial segments are well-ordered with respect to the relation $\subseteq$, and the ordinal $\mathrm{RS}(P)$-ranks are bounded by $|\mathcal{T}_\Sigma| \leq 2^{\max\{|\Sigma|, \omega\}}$, all values $\mathrm{Sp}_{\mathrm{Rd}}(P)$, for nonempty properties $P \subseteq \mathcal{T}_\Sigma$, are exhausted by these three possibilities, and we obtain the following:

Theorem 12. *For any nonempty property* $P \subseteq \mathcal{T}_\Sigma$, *one of the following possibilities holds for some* $\beta \in \mathrm{Ord}$ *and* $n \in \omega \setminus \{0\}$:

(1) $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)]$;
(2) $\mathrm{Sp}_{\mathrm{Rd}}(P) = \{\infty\}$;
(3) $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)] \cup \{\infty\}$.

All possibilities above are realized by appropriate languages Σ *and properties* $P \subseteq \mathcal{T}_\Sigma$.

Theorem 13. *Any value* $\mathrm{Sp}_{\mathrm{Rd}}(P)$ *can be naturally extended until* $\overline{\mathrm{Sp}}_{\mathrm{Rd}}(P) = \mathrm{Sp}_{\mathrm{Rd}}(P) \cup \{-1\}$, *corresponding to the value* $\mathrm{RS}(\varnothing) = -1$ *of the empty subfamily of* $\mathcal{T}_\Sigma$. *It is also natural to put* $\overline{\mathrm{Sp}}_{\mathrm{Rd}}(P) = \{-1\}$ *for an empty* $P \subset \mathcal{T}_\Sigma$. *In view of Theorem 3.6, we obtain the following description of extended spectra* $\overline{\mathrm{Sp}}_{\mathrm{Rd}}(P)$:

(i) $\overline{\mathrm{Sp}}_{\mathrm{Rd}}(P) = \{-1\}$;
(ii) $\overline{\mathrm{Sp}}_{\mathrm{Rd}}(P) = O[(\beta, n)] \cup \{-1\}$;
(iii) $\overline{\mathrm{Sp}}_{\mathrm{Rd}}(P) = \{-1, \infty\}$;
(iv) $\overline{\mathrm{Sp}}_{\mathrm{Rd}}(P) = O[(\beta, n)] \cup \{-1, \infty\}$.

Theorem 14 ([10]). *Let* $\mathcal{T}$ *be a family of a countable language* Σ, *and with* $\mathrm{RS}(\mathcal{T}) = \infty$, *let* α *be a countable ordinal,* $n \in \omega \setminus \{0\}$. *Then, there is a* d_∞*-definable subfamily* $\mathcal{T}^* \subset \mathcal{T}$ *such that* $\mathrm{RS}(\mathcal{T}^*) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}^*) = n$.

Theorems 12 and 14 immediately imply the following:

Corollary 3. *Let* $\mathcal{T}$ *be a family of a countable language* Σ, *and with* $\mathrm{RS}(\mathcal{T}) = \infty$, *let* α *be a countable ordinal,* $n \in \omega \setminus \{0\}$. *Then, there is a* d_∞*-definable property* $P \subset \mathcal{T}$ *such that* $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\alpha, n)]$.

5. Spectra for Cardinalities of Definable Subproperties

In this section, we study some refinements of the relation $\rhd_{\mathrm{pt}}$.

For a cardinality $\lambda \geq 1$, a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P \subseteq \mathcal{T}_\Sigma$, we write $\varphi \rhd^{\lambda}_{\mathrm{pt}} P$ if φ satisfies exactly λ theories in P; i.e., $|P_\varphi| = \lambda$.

According to this definition, if $P \neq \emptyset$ and $\varphi \rhd_{\mathrm{tt}} P$, then $\varphi \rhd^{|P|}_{\mathrm{pt}} P$, and conversely $\varphi \rhd^{|P|}_{\mathrm{pt}} P$ implies $\varphi \rhd_{\mathrm{tt}} P$ for finite P. For an infinite P, the converse implication may fail. Moreover, since infinite sets can be divided into two parts of same cardinality, one can easily introduce an expansion P' of P by a 0-ary predicate Q such that $Q \rhd^{|P'|}_{\mathrm{pt}} P'$ and $\neg Q \rhd^{|P'|}_{\mathrm{pt}} P'$, implying that $Q \not\rhd_{\mathrm{tt}} P'$.

For a property P, we denote by $\mathrm{Sp}_{\mathrm{pt}}(P)$ the set $\{\lambda \mid \varphi \rhd^{\lambda}_{\mathrm{pt}} P \text{ for some sentence } \varphi\}$. This set is called the pt-*spectrum* of P.

According to this definition, $|P| \in \mathrm{Sp}_{\mathrm{pt}}(P)$ for any nonempty P and $\lambda \leq |P|$ for any $\lambda \in \mathrm{Sp}_{\mathrm{pt}}(P)$; i.e., $\mathrm{sup}\mathrm{Sp}_{\mathrm{pt}}(P) = ||P| + 1|$.

A natural question arises regarding the description of pt-spectra.

This question is easily answered for finite P, since in such a case, all subsets of theories are separated as s-definable singletons from their complements, and we obtain the following:

Proposition 13. *For any finite property $P \subseteq \mathcal{T}_\Sigma$, $\mathrm{Sp}_{\mathrm{pt}}(P) = (|P| + 1) \setminus \{0\}$.*

The following assertion generalizes Proposition 13 in terms of isolated points due to Theorem 7:

Proposition 14. *If P has exactly $n \in \omega \setminus \{0\}$ isolated points, then $\mathrm{Sp}_{\mathrm{pt}}(P) \cap \omega = (n+1) \setminus \{0\}$.*

Proof. Let $T_1, \ldots, T_n$ be isolated points in P. If$k \in \mathrm{Sp}_{\mathrm{pt}}(P) \cap \omega$—i.e., there is a sentence φ with $\varphi \rhd^{k}_{\mathrm{pt}} P$—then P_φ consists of isolated points $T_{i_1}, \ldots, T_{i_k}$ since elements of the finite set P_φ are separated as s-definable singletons from their complements in P_φ. Then, $k \in (n+1) \setminus \{0\}$. Conversely, if $k \in \omega \setminus \mathrm{Sp}_{\mathrm{pt}}(P)$, then $k > n$, since each $k \leq n$ equals $|P_\varphi|$ for some sentence φ, implying $k \in \mathrm{Sp}_{\mathrm{pt}}(P)$. □

Proposition 15. *For any nonempty property $P \subseteq \mathcal{T}_\Sigma$, either $\mathrm{Sp}_{\mathrm{pt}}(P) \cap \omega$ equals an initial segment $(n+1) \setminus \{0\}$ for some $n \in \omega \setminus \{0\}$ or $\mathrm{Sp}_{\mathrm{pt}}(P) \cap \omega = \omega \setminus \{0\}$.*

Proof. If there are sentences φ with a finite P_φ, then either there is a P_φ with the greatest finite cardinality, implying $\mathrm{Sp}_{\mathrm{pt}}(P) \cap \omega = (n+1) \setminus \{0\}$ by the arguments of Proposition 14, or the finite cardinalities $|P_\varphi|$ are unbounded, which means $\mathrm{Sp}_{\mathrm{pt}}(P) \cap \omega = \omega \setminus \{0\}$. □

Proposition 16. *For any infinite property $P \subseteq \mathcal{T}_\Sigma$, there is a nonempty set $Y \subseteq |P|$ of infinite cardinalities such that either there is $n \in \omega \setminus \{0\}$ with $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup (n+1) \setminus \{0\}$, $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup \omega \setminus \{0\}$ or $\mathrm{Sp}_{\mathrm{pt}}(P) = Y$. All values $Y \cup (n+1) \setminus \{0\}$, $Y \cup \omega \setminus \{0\}$ and Y, for a nonempty set Y of infinite cardinalities and $n \in \omega \setminus \{0\}$, are realized as $\mathrm{Sp}_{\mathrm{pt}}(P)$ for appropriate properties P.*

Proof. Since $\mathrm{sup}\mathrm{Sp}_{\mathrm{pt}}(P) \leq |P|$ and Proposition 15 describes all possibilities for $\mathrm{Sp}_{\mathrm{pt}}(P) \cap \omega$, it suffices, for a nonempty set Y of infinite cardinalities and $n \in \omega \setminus \{0\}$, to find a property P_1 with $\mathrm{Sp}_{\mathrm{pt}}(P_1) = Y \cup (n+1) \setminus \{0\}$, a property P_2 with $\mathrm{Sp}_{\mathrm{pt}}(P_2) = Y \cup \omega \setminus \{0\}$ and a property P_3 with $\mathrm{Sp}_{\mathrm{pt}}(P_2) = Y$. For the property P_1, one can take a finite n-element family Y_n, expanded by a 0-ary predicate Q_0 marking all theories in Y_n, and extend Y_n by families Y_λ, for each $\lambda \in Y$, of λ theories of λ independent 0-ary predicates Q^λ_i expanded by a 0-ary predicate Q_λ marking all theories in Y_λ. Any P_1-consistent sentence φ satisfies either $k \leq n$ theories in Y_n or λ many theories in Z_λ and possibly μ many theories in Y_μ for finitely many $\mu < \lambda$. This means that the cardinalities $|(P_1)_\varphi|$ evince the equality $\mathrm{Sp}_{\mathrm{pt}}(P_1) = Y \cup (n+1) \setminus \{0\}$.

For the property P_2, we repeat the process for P_1, replacing the part Y_n with an e-minimal family Y^λ_0 consisting of some $\lambda \in Y$ theories, all of which are marked by the new

0-ary predicate Q_0. Realizing this process, we find that s-definable sets are finite, cofinite or consist of λ theories for $\lambda \in Y$. Thus, $\mathrm{Sp}_{\mathrm{pt}}(P_1) = Y \cup \omega \setminus \{0\}$.

For the property P_3, we repeat the process for P_1 without the part Y_n, obtaining $\mathrm{Sp}_{\mathrm{pt}}(P_1) = Y$. □

Summarizing Propositions 13–16, we obtain the following theorem describing tje pt-spectra of properties.

Theorem 15. *For any nonempty property $P \subseteq \mathcal{T}_\Sigma$, one of the following conditions holds:*

(1) $\mathrm{Sp}_{\mathrm{pt}}(P) = (n+1) \setminus \{0\}$ *for some $n \in \omega \setminus \{0\}$; it is satisfied iff P is finite with $|P| = n$;*

(2) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup (n+1) \setminus \{0\}$ *for some nonempty set $Y \subseteq |P|$ of infinite cardinalities and $n \in \omega \setminus \{0\}$;*

(3) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup \omega \setminus \{0\}$ *for some nonempty set $Y \subseteq |P|$ of infinite cardinalities;*

(4) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y$ *for some nonempty set $Y \subseteq |P|$ of infinite cardinalities.*

All values $(n+1) \setminus \{0\}$, $Y \cup (n+1) \setminus \{0\}$, $Y \cup \omega \setminus \{0\}$ and Y, for a nonempty set Y of infinite cardinalities and $n \in \omega \setminus \{0\}$, are realized as $\mathrm{Sp}_{\mathrm{pt}}(P)$ for an appropriate property P.

The following assertion shows that Y in Theorem 15 is finite for a property P with $\mathrm{RS}(P) = 1$.

Proposition 17. *If $\mathrm{RS}(P) = 1$ then $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup \omega \setminus \{0\}$ for some finite nonempty set $Y \subseteq |P|$ of infinite cardinalities with $|Y| \leq \mathrm{ds}(P)$.*

Proof. Since $\mathrm{RS}(P) = 1$, P is divided into $k = \mathrm{ds}(P)$ infinite s-definable e-minimal parts $P_1, \ldots, P_k$, each part P_i has only finite and cofinite s-definable subsets producing $\mathrm{Sp}_{\mathrm{pt}}(P_i) = \{|P_i|\} \cup \omega \setminus \{0\}$. Since each s-definable subset of P is a Boolean combination of s-definable subsets of P_i and $|Z_i \cup Z_j| = \max\{|Z_i|, |Z_j|\}$ for infinite s-definable $Z_i \subseteq P_i$, $Z_j \subseteq P_J$, $i, j \leq k$, we obtain $\mathrm{Sp}_{\mathrm{pt}}(P) = \{|P_1|, \ldots, |P_k|\} \cup \omega \setminus \{0\}$. □

Remark 4. *Describing possibilities for* pt*-spectra $\mathrm{Sp}_{\mathrm{pt}}(P)$, we admit that properties P may not be E-closed. If we assume that P is infinite and E-closed, then we have two cases: either P is e-totally transcendental with the least generating set of a cardinality $\mu_1 \leq \max\{|\Sigma|, \omega\}$ and with $\mu_2 \leq \max\{|\Sigma|, \omega\}$ accumulation points, or $\mathrm{RS}(P) = \infty$ with $|P| \geq 2^\omega$ by Theorem 1.2. In the first case, values for $\lambda \in \mathrm{Sp}_{\mathrm{pt}}(P)$ are exhausted by all cardinalities in $\omega \setminus \{0\}$ and by some infinite cardinalities $\leq \max\{|\Sigma|, \omega\}$. In particular, for a countable Σ, since $\mathrm{RS}(P)$ is a countable ordinal, we have $\mathrm{Sp}_{\mathrm{pt}}(P) = (\omega+1) \setminus \{0\}$. In the second case, values for $\lambda \in \mathrm{Sp}_{\mathrm{pt}}(P)$ are exhausted by cardinalities in $\omega \setminus \{0\}$ or by cardinalities of part of its initial segment, depending on tje existence of the least generating set for P, and by some infinite cardinalities $\leq \max\{|\Sigma|, \omega\}$ and cardinalities $\geq 2^\omega$. In particular, for a countable Σ, $\mathrm{Sp}_{\mathrm{pt}}(P)$ includes 2^ω, and possibly ω, depending on the existence of an infinite, totally transcendental, definable part. Thus, in Theorem 4.5, for an E-closed P, some cases are not realized: $\{\omega\} \cup (n+1) \setminus \{0\}$, $\{\omega\}$, $\{\omega_1\}$, $\{\omega, \omega_1\}$, etc.*

From Remark 4 and using Theorem 1, we have the following:

Theorem 16. *For any nonempty E-closed property $P \subseteq \mathcal{T}_\Sigma$ with at most the countable language Σ, one of the following possibilities holds:*

(1) $\mathrm{Sp}_{\mathrm{pt}}(P) = (n+1) \setminus \{0\}$ *for some $n \in \omega \setminus \{0\}$, if P is finite with $|P| = n$;*

(2) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{2^\omega\} \cup (n+1) \setminus \{0\}$ *for some $n \in \omega$, if P is infinite and has n isolated points;*

(3) $\mathrm{Sp}_{\mathrm{pt}}(P) = (\omega+1) \setminus \{0\}$, *if P is infinite and totally transcendental;*

(4) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{\omega, 2^\omega\} \cup \omega \setminus \{0\}$, *if P has an infinite, totally transcendental, definable subfamily but P itself is not totally transcendental;*

(5) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{2^\omega\} \cup \omega \setminus \{0\}$, *if P has infinitely many isolated points but does not have infinite, totally transcendental, definable subfamilies.*

Remark 5. *Possibilities in 16 give low bounds forthe correspondening cases in uncountable languages.*

6. *P*-Generic Sentences and *P*-Generic Theories

Definition 10 ([14–16]). *For a property $P \subseteq \mathcal{T}_\Sigma$, a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called* P-generic *if* $\mathrm{RS}_P(\varphi) = \mathrm{RS}(P)$, *and* $\mathrm{ds}_P(\varphi) = \mathrm{ds}(P)$ *if* $\mathrm{ds}(P)$ *is defined.*
If $P = \mathcal{T}_\Sigma$, then we omit P, and a P-generic sentence is called generic.

According to this definition, we have the following:

Proposition 18. *Any P-generic sentence φ is $(P, \mathrm{RS}(P), \mathrm{ds}(P))$-rich if $\mathrm{RS}(P)$ is an ordinal and (P, ∞)-rich if $\mathrm{RS}(P) = \infty$. In contrast, each $(P, \mathrm{RS}(P), \mathrm{ds}(P))$-rich sentence, for an ordinal $\mathrm{RS}(P)$, is P-generic, and each (P, ∞)-rich sentence, for $\mathrm{RS}(P) = \infty$, is P-generic.*

Proposition 19. *If $\varphi \rhd_{\mathrm{tt}} P$, then φ is P-generic.*

In view of Proposition 19, any property $P \subseteq \mathcal{T}_\Sigma$ has P-generic sentences.

Corollary 4. *If a property $P \subseteq \mathcal{T}_\Sigma$ is finite and $\varphi \in \mathrm{Sent}(\Sigma)$, then $\varphi \rhd_{\mathrm{tt}} P$ iff φ is P-generic.*

Proof. For Proposition 19, it suffices to show that if φ is P-generic, then $\varphi \rhd_{\mathrm{tt}} P$. If $P = \emptyset$, then both $\mathrm{RS}(P) = \mathrm{RS}_P(\varphi) = -1$ and $\varphi \rhd_{\mathrm{tt}} P$. If P consists of $n \geq 1$ theories, then $\mathrm{RS}(P) = 0$, $\mathrm{ds}(P) = n$. Assuming that φ is P-generic, we have $\mathrm{RS}(P_\varphi) = 0$, $\mathrm{ds}(P_\varphi) = n$, implying that φ belongs to all n theories in P and $\varphi \rhd_{\mathrm{tt}} P$. □

Remark 6. *In view of Corollary 4, the converse implication for Proposition 19 holds iff P is finite. Indeed, if $P = \mathcal{T}_\Sigma$ for a countable language Σ with $\mathrm{RS}(P) = \infty$, which is characterized in Theorem 1, then we can construct a 2-tree of sentences φ_δ, $\delta \in {}^{<\omega}2$, showing the value of $\mathrm{RS}(P)$. This means that P is divided into two disjointed definable parts P_{φ_0} and P_{φ_1} with $\mathrm{RS}(P_{\varphi_0}) = \mathrm{RS}(P_{\varphi_1}) = \infty$. Thus, φ_0 and φ_1 are generic, whereas $\varphi_0 \not\rhd_{\mathrm{tt}} P$ and $\varphi_1 \not\rhd_{\mathrm{tt}} P$. Moreover, this effect works both for an arbitrary property P with $\mathrm{RS}(P) = \infty$ and for an arbitrary property P with $\mathrm{RS}(P) \in \mathrm{Ord}$ and $\mathrm{RS}(P) \geq 1$. In the latter case, we can remove a branch in the tree, resulting in the values $\mathrm{RS}(P)$ and $\mathrm{ds}(P)$ when only considering a sentence $\varphi_1 \wedge \neg\psi$, where φ_1 is a tautology and P_ψ is nonempty with $\mathrm{RS}(P_\psi) < \mathrm{RS}(P)$. In such a case, $\varphi_1 \wedge \neg\psi$ is P-generic and $(\varphi_1 \wedge \neg\psi) \not\rhd_{\mathrm{tt}} P$.*

In view of Remark 6, we have the following:

Proposition 20. *For a property $P \subseteq \mathcal{T}_\Sigma$, there is a P-generic sentence $\varphi \in \mathrm{Sent}(\Sigma)$ with minimal/least P_φ iff P is finite. If that φ exists, then $P_\varphi = P$.*

By Proposition 20 for a property $P \subseteq \mathcal{T}_P$ with $\mathrm{RS}(P) \geq 1$, P-generic sentences φ produce infinite descending chains of s-definable subfamilies P_φ.

Proposition 21. *If for a property $P \subseteq \mathcal{T}_\Sigma$, $\mathrm{RS}(P) = \alpha \in \mathrm{Ord}$, $\mathrm{ds}(P) = 1$, then for any sentence $\varphi \in \mathrm{Sent}(\Sigma)$, either $\mathrm{RS}_P(\varphi) = \alpha$ and $\mathrm{ds}_P(\varphi) = 1$, or $\mathrm{RS}_P(\neg\varphi) = \alpha$ and $\mathrm{ds}_P(\neg\varphi) = 1$.*

Proof. According to the conjecture for P and the monotony for pairs of values of RS and ds, we have $\mathrm{RS}_P(\varphi) \leq \alpha$ and $\mathrm{RS}_P(\neg\varphi) \leq \alpha$ for any $\varphi \in \mathrm{Sent}(\Sigma)$, and if the RS-rank equals α, then the ds-degree equals 1. We cannot obtain $\mathrm{RS}_P(\varphi) < \alpha$ and $\mathrm{RS}_P(\neg\varphi) < \alpha$ for $\varphi \in \mathrm{Sent}(\Sigma)$ by Theorem 11. Thus, $\mathrm{RS}_P(\varphi) = \alpha$ and $\mathrm{ds}_P(\varphi) = 1$, or $\mathrm{RS}_P(\neg\varphi) = \alpha$ and $\mathrm{ds}_P(\neg\varphi) = 1$. The latter conditions cannot be satisfied simultaneously, as otherwise $\mathrm{ds}(P) \geq 2$. □

Since any property $P \subseteq \mathcal{T}_\Sigma$ with $\mathrm{RS}(P) = \alpha \in \mathrm{Ord}$ is represented as a disjointed union of $\mathrm{ds}(P)$ s-definable subfamilies $P_1, \ldots, P_{\mathrm{ds}(P)}$ with $\mathrm{RS} = \alpha$ and $\mathrm{ds} = 1$, Proposition 21 immediately implies the following:

Corollary 5. *For any property $P \subseteq \mathcal{T}_\Sigma$ with $\mathrm{RS}(P) = \alpha \in \mathrm{Ord}$ and any sentence $\varphi \in \mathrm{Sent}(\Sigma)$, either φ is P-generic or $\neg\varphi$ is P-generic, or, for $\mathrm{ds}(P) > 1$ with non-P-generic φ and $\neg\varphi$, φ is represented as a disjunction of k $(P, \alpha, 1)$-rich sentences and $\neg\varphi$ is represented as a disjunction of m $(P, \alpha, 1)$-rich sentences such that $k + m = \mathrm{ds}(P)$, $k > 0$, $m > 0$.*

Remark 7. *By Proposition 21 for any property $P \subseteq \mathcal{T}_\Sigma$ with $\mathrm{RS}(P) = \alpha \in \mathrm{Ord}$ and $\mathrm{ds}(P) = 1$, there is a unique ultrafilter U_P consisting of P-generic sentences. By Proposition 20, this ultrafilter is principal if and only if P is finite; i.e., in such a case, it is a singleton. In any case, U_P produces a theory $T \in \mathcal{T}_\Sigma$ consisting of P-generic sentences only. This theory T is called* P-generic.

If P is infinite, then T belongs to the E-closure $\mathrm{Cl}_E(P)$ [8,12] of P as a unique element of the α-th Cantor–Bendixson derivative of $\mathrm{Cl}_E(P)$; i.e., an element of $\mathrm{Cl}_E(P)$ having the Cantor–Bendixson rank $\mathrm{CB}(\mathrm{Cl}_E(P)) = \alpha$ [8].

If $\mathrm{ds}(P) > 1$, we can divide P into $\mathrm{ds}(P)$ s-definable parts P_i with $\mathrm{RS}(P_i) = \mathrm{RS}(P)$ and $\mathrm{ds}(P_i) = 1$, each of which has a unique P_i-generic theory T_i. The set $\{T_1, \ldots, T_{\mathrm{ds}(P)}\}$ is called the set of P-generic theories.

Thus, we have the following:

Proposition 22. *Each e-totally transcendental property P has finitely many—exactly $\mathrm{ds}(P) \geq 1$—P-generic theories. These theories have the Cantor–Bendixson rank $\mathrm{CB}(\mathrm{Cl}_E(P)) = \alpha = \mathrm{RS}(P)$.*

Now, we extend the results above to generic sentences and theories for properties P with $\mathrm{RS}(P) = \infty$.

Proposition 23. *If for a property $P \subseteq \mathcal{T}_\Sigma$, $\mathrm{RS}(P) = \infty$, then for any sentence $\varphi \in \mathrm{Sent}(\Sigma)$, either $\mathrm{RS}_P(\varphi) = \infty$ or $\mathrm{RS}_P(\neg\varphi) = \infty$.*

Proof. Assume that for a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ we have $\mathrm{RS}_P(\varphi) < \infty$ and $\mathrm{RS}_P(\neg\varphi) < \infty$. We can suppose that φ and $\neg\varphi$ are both P-consistent. Then, φ is (P, α, m)-rich and (P, β, n)-co-rich for some $\alpha, \beta \in \mathrm{Ord}$, $m, n \in \omega$. Applying Theorem 3.4, we obtain $\mathrm{RS}(P) \in \mathrm{Ord}$, contradicting $\mathrm{RS}(P) = \infty$. □

Proposition 23 immediately implies the following:

Corollary 6. *For any property $P \subseteq \mathcal{T}_\Sigma$ with $\mathrm{RS}(P) = \infty$ and any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ either φ is P-generic or $\neg\varphi$ is P-generic.*

Remark 8. *By Proposition 23, for any property $P \subseteq \mathcal{T}_\Sigma$ with $\mathrm{RS}(P) = \infty$, there is an ultrafilter U_P consisting of P-generic sentences. Since the condition $\mathrm{RS}(P) = \infty$ implies the existence of a 2-tree of Σ-sentences, there is at least a continuum of these ultrafilters, and by Theorem 1, there are exactly a continuum of ultrafilters for the language Σ with $|\Sigma| \leq \omega$. Each U_P produces a theory $T \in \mathcal{T}_\Sigma$ consisting of P-generic sentences only. This theory T is called* P-generic.

The P-generic theories form the perfect kernel with respect to Cantor–Bendixson derivatives of $\mathrm{Cl}_E(P)$; i.e., the set of elements of $\mathrm{Cl}_E(P)$ having the Cantor–Bendixson rank $\mathrm{CB}(\mathrm{Cl}_E(P)) = \infty$.

Applying Theorem 12 and summarizing Remarks 7 and 8, we obtain the following:

Theorem 17.

(1) For any nonempty property $P \subseteq \mathcal{T}_\Sigma$, there are $\mathrm{ds}(P)$ P-generic theories if P is totally transcendental, and at least a continuum if P is not totally transcendental. In the latter

case, either all theories in P are P-generic if $\mathrm{Sp}_{\mathrm{Rd}}(P) = \{\infty\}$, or P has at least $\beta \cdot \omega + n$ non-P-generic theories if $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)] \cup \{\infty\}$.

(2) *The* CB*-rank of each P-generic theory equals* $\mathrm{RS}(P)$.

Definition 11 ([11]). *For a property $P \subseteq \mathcal{T}_\Sigma$, a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called* P-complete *if φ isolates a unique theory T in P; i.e., P_φ is a singleton. In such a case, the theory $T \in P_\varphi$ is called* P-finitely axiomatizable *(by the sentence φ).*

Since P-finitely axiomatizable theories are isolated points, we obtain the following:

Proposition 24. *For any nonempty property $P \subseteq \mathcal{T}_\Sigma$, a P-finitely axiomatizable theory T is P-generic iff P is finite.*

7. Illustrations

The proposed approach connecting formulas, properties and their characteristics can be applied by classifying various classes of structures and their theories. For instance, with a natural quantifier elimination and a description by invariants, we can reduce the characteristics of families of structures and their theories to possibilities of invariant values by evaluating the complexity of given class and its generating set. This approach produces a natural description for families of theories of abelian groups via Szmielew invariants [17–19] and for families of cubic theories via numbers and dimensions of connected components [20], etc.

Example 1. *Let $P \subseteq \mathcal{T}_\Sigma$ be a property, where Σ consists of finitely many 0-ary and unary predicates and finitely many constant symbols. Using Theorem 2 for sentences $\varphi \in \mathrm{Sent}(\Sigma)$, we have finitely many possibilities for neighborhoods P_φ. Thus, $\mathrm{RS}_P(\varphi) = -1$ or $\mathrm{RS}_P(\varphi) = 0$, producing finite initial segments for the spectra.*

Similarly, using Theorem 3, we have the aforementioned possibilities for $P \subseteq \mathcal{T}_{\Sigma,n}$, where Σ is finite, or $n = 1$ and Σ has finitely many predicate symbols.

Example 2. *Let P be a property for the family of theories of abelian groups. Following [10,19], the* RS*-rank for P can be an arbitrary countable ordinal or infinity. Taking a sentence φ for abelian groups, one can obtain, following the results above, an arbitrary at most countable value $\mathrm{RS}_P(\varphi)$ or $\mathrm{RS}_P(\varphi) = \infty$ depending on the information written by φ. This means that all possibilities for spectra described in Theorem 16 can be realized.*

Example 3. *Let P be a property for the family of theories of graphs. Following [10,20], the* RS*-rank for P can again be an arbitrary countable ordinal or infinity. Taking a sentence φ for graphs, one can obtain, in a similar manner to Example 2, an arbitrary—at most countable—value $\mathrm{RS}_P(\varphi)$ or $\mathrm{RS}_P(\varphi) = \infty$ depending on the information written by φ. Thus, all possibilities for spectra described in Theorem 16 can be realized.*

8. Conclusions

In this work, we study the links between formulas and properties and considered and described the characteristics of properties with respect to satisfying formulas, their ranks and degrees. Possibilities of spectra for ranks, degrees and cardinalities of definable properties are shown. Generic formulas and theories are introduced and characterized.

Possibilities for ranks and degrees for formulas and theories with respect to the given properties are described. Their highest values form generic sentences and theories, which are also described and characterized.

There are many natural model-theoretic and other properties that can be studied and described in this context. In this case, the relations $\rhd_{\mathrm{pt}}$ and $\rhd_{\mathrm{tt}}$ are preserved under natural closures. At the same time, the operator of the E-closure does not preserve a series of natural model-theoretic properties. For instance, there are families that are strongly minimal whose accumulation points have strictly ordered properties. This implies that

families of ω-stable, superstable and stable theories cannot be E-closed in these classes. Natural questions arise regarding the characteristics and characterizations of families in classes $\mathcal{T}$ whose E-closures are contained in $\mathcal{T}$.

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